Chapter 31 Electromagnetic Oscillations and Alternating Current

LC Oscillations, Qualitatively

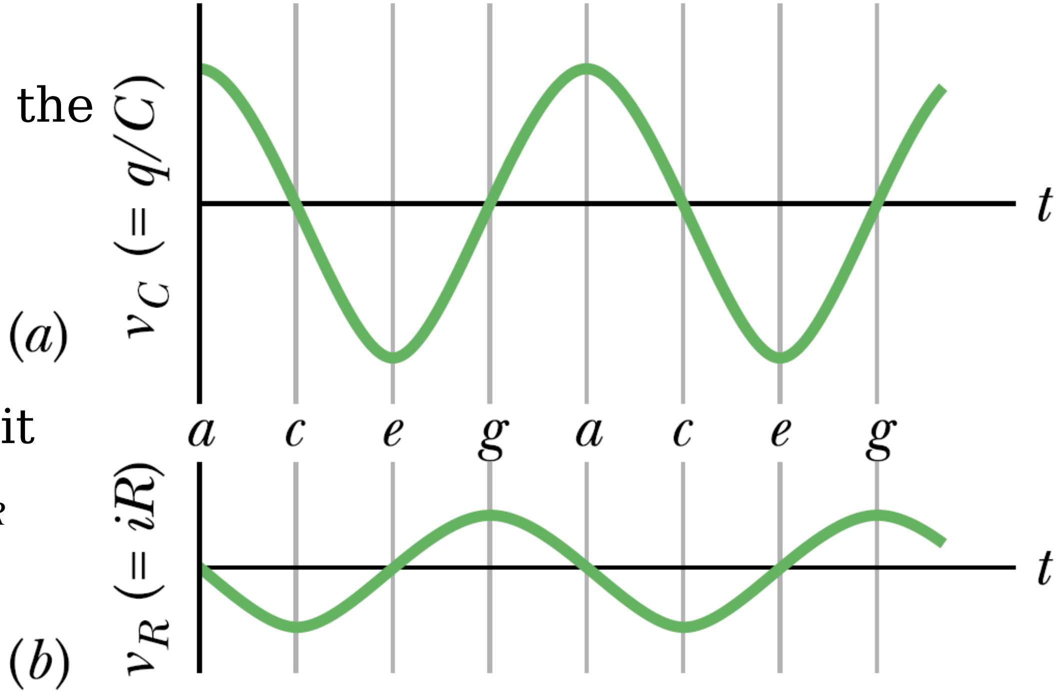
- In the *LC* circuit the charge, current, and potential difference vary sinusoidally (with period *T* and angular frequency ω). The resulting oscillations of the capacitor's electric field and the inductor's magnetic field are said to be **electromagnetic oscillations**.

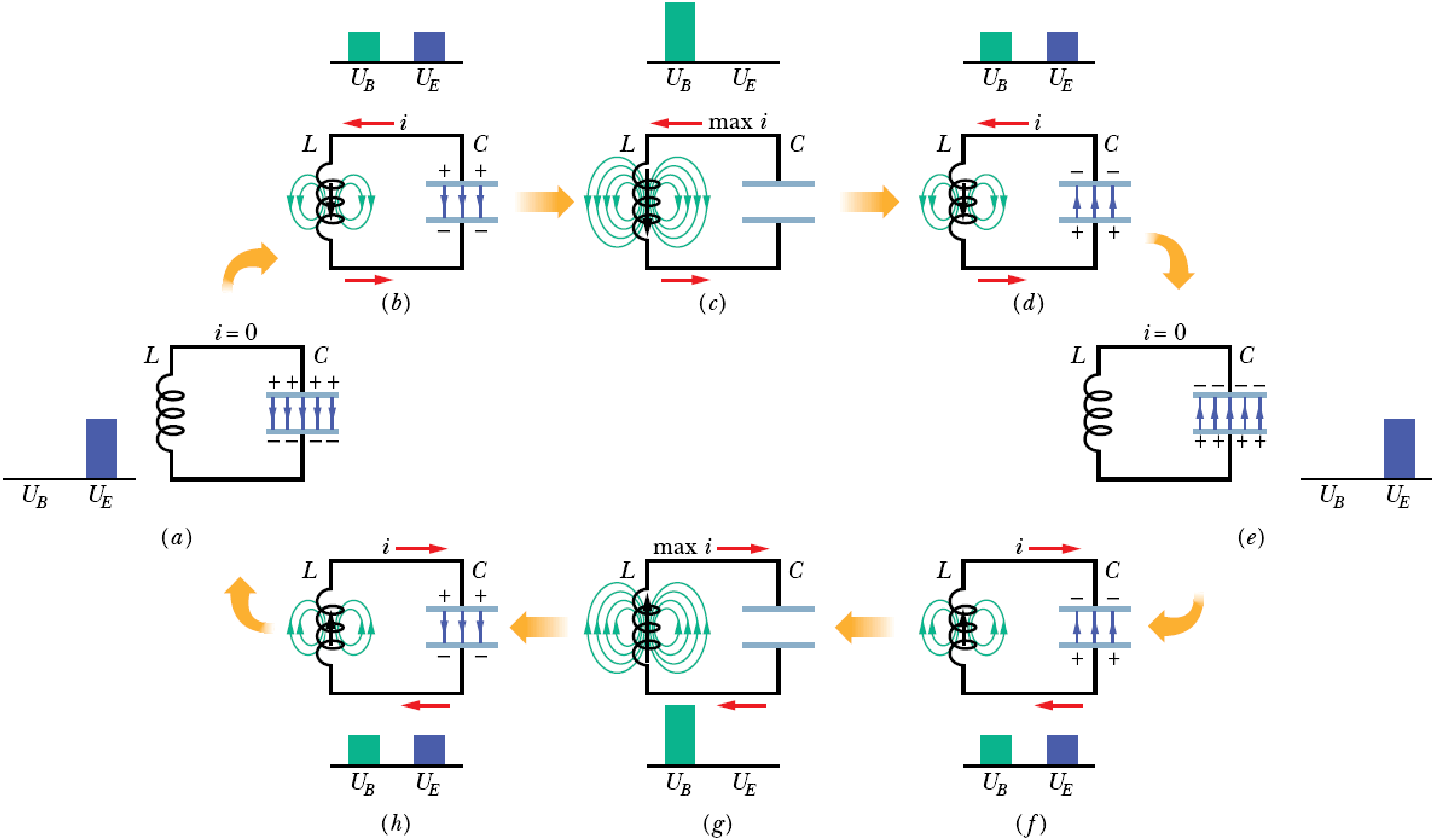
- the energy stored in the electric field of the capacitor is $U_E = \frac{q^2}{2C}$

- the energy stored in the magnetic field of the inductor is $U_B = \frac{L i^2}{2}$

- To determine the charge $q(t)$ on the capacitor,put in a voltmeter to measure the potential difference (or *voltage*) v_C that exists across the capacitor *C*: $v_C = \frac{q}{C}$

- To measure the current, connect a small resistance *R* in series in the circuit and measure the potential difference v_R across it: $v_R = i R$





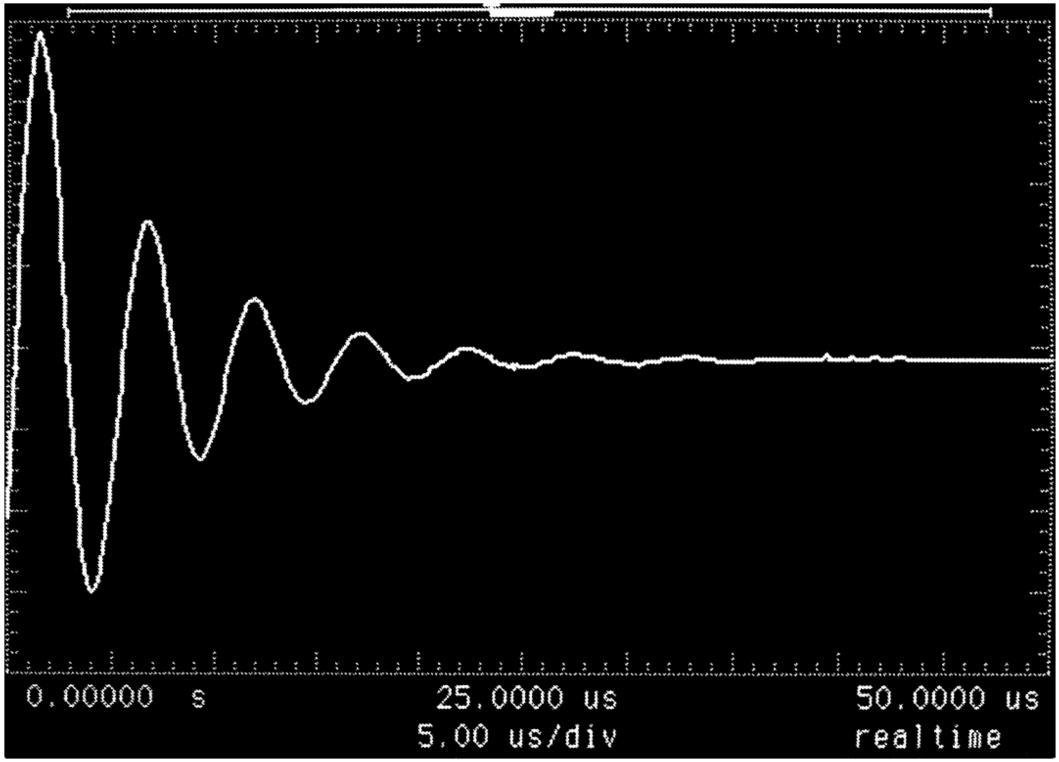
- In an actual LC circuit, the oscillations will not continue indefinitely because there is always some resistance present that will drain energy from the electric and magnetic fields and dissipate it as thermal energy (the circuit may become warmer).

The Electrical– Mechanical Analogy

- the analogy between the oscillating LC system and an oscillating block–spring system:

q corresponds to x , $1/C$ correspond to k

i corresponds to v , L correspond to m



- These correspondences suggest that in an LC oscillator, the capacitor is mathematically like the spring in a block– spring system and the inductor is like the block.

- In a block– spring system: $\omega = \sqrt{\dfrac{k}{m}}$

- The correspondences suggest that to find the angular frequency of oscillation for an ideal LC circuit, k should be replaced by $1/C$ and m by L,

$$\omega = \frac{1}{\sqrt{LC}} \qquad LC \text{ circuit}$$

Block-Spring System		LC Oscillator	
Element	Energy	Element	Energy
Spring	Potential , $k\,x^2/2$	Capacitor	Electrical , $q^2/2C$
Block	Kinetic , $m\,v^2/2$	Inductor	Magnetic , $L\,i^2/2$
$v = \mathrm{d}x/\mathrm{d}t$		$i = \mathrm{d}q/\mathrm{d}t$	

LC Oscillations, Quantitatively

The Block–Spring Oscillator

- the total energy of a block– spring oscillator: $U = U_b + U_s = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$

- Energy conservation, no friction:

$$0 \Leftarrow \frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2} m v^2 + \frac{1}{2} k x^2\right) = m v \frac{dv}{dt} + k x \frac{dx}{dt} \quad \Rightarrow \quad m \frac{d^2 x}{dt^2} + k x = 0$$

$$\Rightarrow \quad x(t) = X \cos(\omega t + \phi) \quad \text{displacement}$$

X is the amplitude of the mechanical oscillations, ω is the angular frequency of the oscillations, and ϕ is a phase constant.

The *LC* Oscillator

- the total energy in an oscillating LC circuit: $U = U_B + U_E = \frac{1}{2} L i^2 + \frac{1}{2} \frac{q^2}{C}$

U_B is the energy stored in the magnetic field of the inductor and U_E is the energy stored in the electric field of the capacitor.

- Energy conservation, no resistance:

$$0 \Leftarrow \frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2} L i^2 + \frac{1}{2} \frac{q^2}{C}\right) = L i \frac{di}{dt} + \frac{q}{C} \frac{dq}{dt} \quad \Rightarrow \quad \boxed{L \frac{d^2 q}{dt^2} + \frac{1}{C} q = 0 \quad LC \text{ oscillation}}$$

$$\Rightarrow \quad \boxed{q(t) = Q \cos(\omega t + \phi) \quad \text{charge}}$$

- The current of the LC oscillator:

$$i(t) = \frac{dq}{dt} = -\omega Q \sin(\omega t + \phi) = -I \sin(\omega t + \phi) \quad \text{current} \quad \Leftarrow \quad I = \omega Q$$

- The angular frequency of the LC oscillator:

$$\frac{d^2 q}{dt^2} = -\omega^2 Q \cos(\omega t + \phi)$$

$$\Rightarrow \quad -L\omega^2 Q \cos(\omega t + \phi) + \frac{Q}{C}\cos(\omega t + \phi) = 0 \quad \Leftarrow \quad L\frac{d^2 q}{dt^2} + \frac{q}{C} = 0 \quad \Rightarrow \quad \omega = \frac{1}{\sqrt{LC}}$$

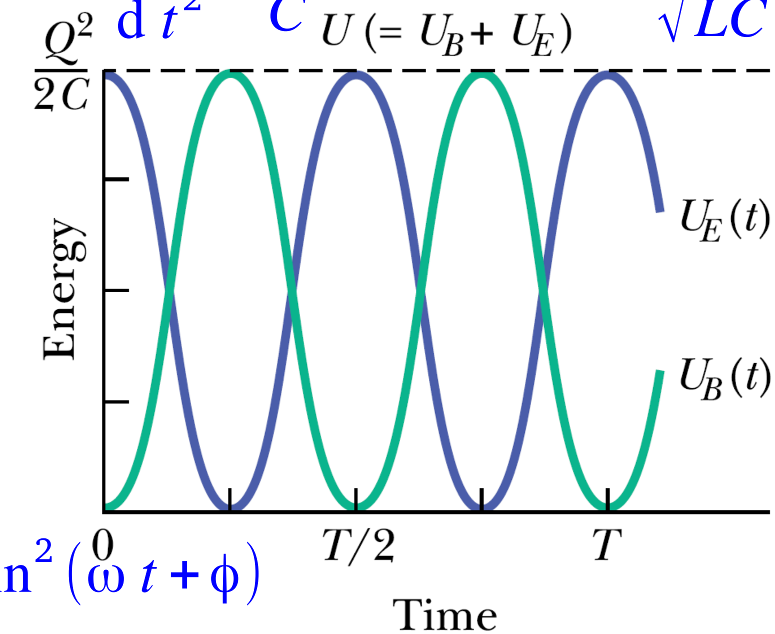
- ϕ is determined by the initial conditions.

- The electrical energy stored in the LC circuit

$$U_E = \frac{q^2}{2C} = \frac{Q^2}{2C}\cos^2(\omega t + \phi)$$

- The magnetic energy stored in the LC circuit

$$U_B = \frac{Q^2}{2C}\sin^2(\omega t + \phi) \quad \Leftarrow \quad U_B = \frac{Li^2}{2} = \frac{L\omega^2 Q^2}{2}\sin^2(\omega t + \phi)$$



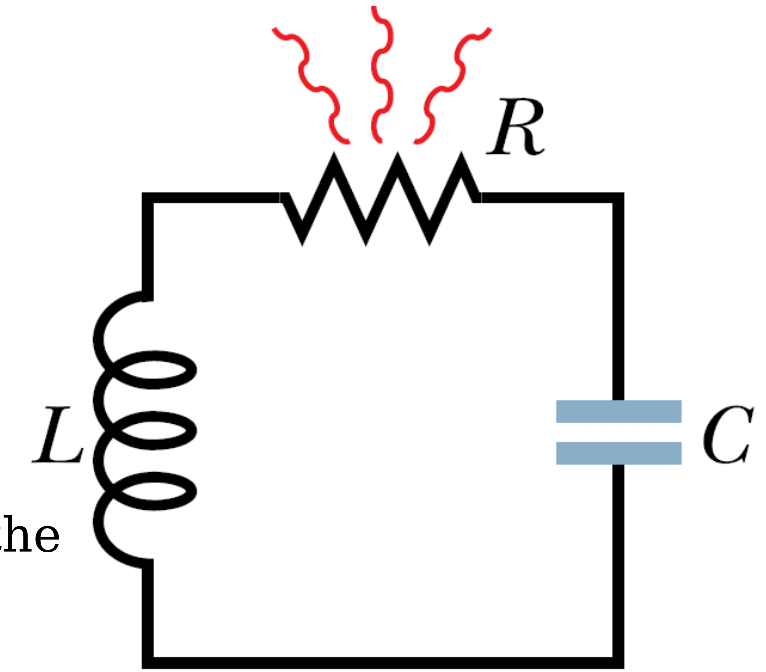
problem 31-1

Note: (1) $(U_E)_{max} = (U_B)_{max} = \frac{Q^2}{2C}$

(2) $U_E + U_B = \frac{Q^2}{2C} = \text{constant}$

(3) $U_E = (U_E)_{max}$ when $U_B = 0$; $U_B = (U_B)_{max}$ when $U_E = 0$

Damped Oscillations in an *RLC* Circuit



- A circuit containing resistance, inductance, and capacitance is called an *RLC circuit*. We shall here discuss only *series RLC circuits*

- With a resistance present, the total EM energy of the circuit is no longer constant; it decreases with time as energy is transferred to thermal energy in the resistance.

- Because of this loss of energy, the oscillations of charge, current, and potential difference decrease in amplitude, and the oscillations are *damped*.

- The rate of energy transferred to thermal energy: $\frac{dU}{dt} = -i^2 R$

$$\Rightarrow \frac{d}{dt}\left(\frac{L i^2}{2} + \frac{q^2}{2C}\right) = L i \frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = -i^2 R \Rightarrow \boxed{L\frac{d^2 q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C} q = 0 \quad RLC \text{ curcuit}}$$

$$\Rightarrow \boxed{q = Q e^{-Rt/2L} \cos(\omega' t + \phi)} \quad \Leftarrow \quad \omega' = \sqrt{\omega^2 - (R/2L)^2} \le \omega \quad \Leftarrow \quad \omega = 1/\sqrt{LC}$$

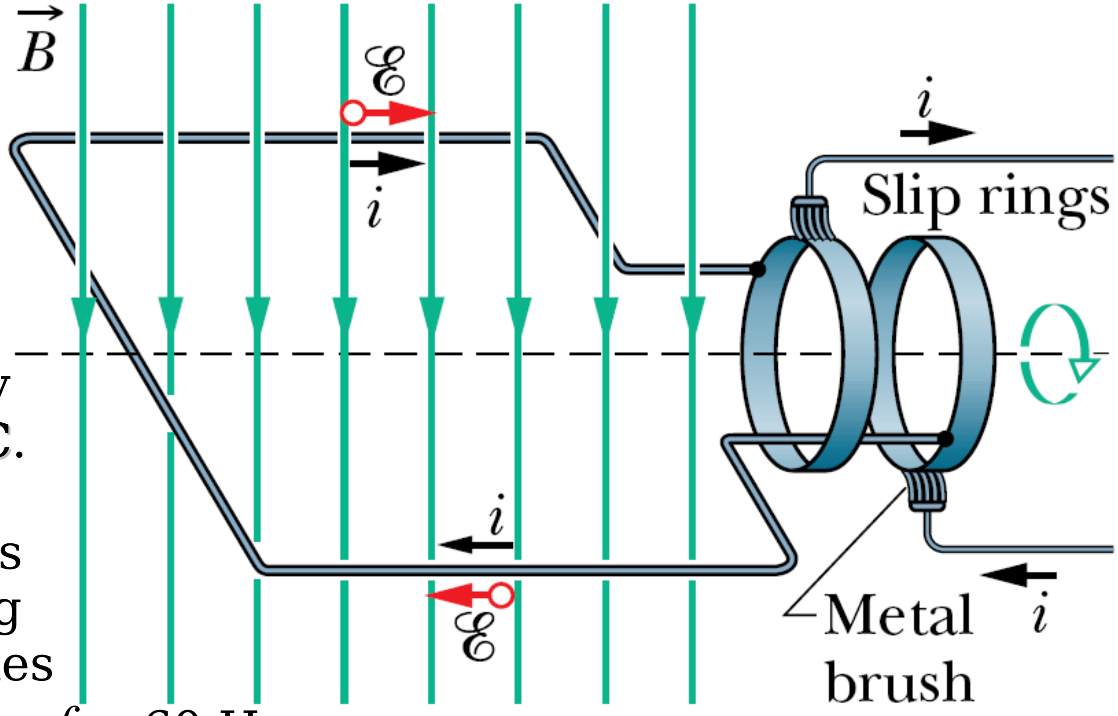
- The electrical energy: $U_E = \frac{q^2}{2C} = \frac{Q^2}{2C} e^{-Rt/L} \cos^2(\omega' t + \phi)$

  the magnetic energy: $U_B = \frac{L i^2}{2} = \frac{Q^2}{2C} e^{-Rt/L} \sin^2(\omega' t + \phi)$

$$\Rightarrow \quad U = \frac{Q^2}{2C} e^{-Rt/L}$$

problem 31-2

Alternating Current



- If the energy is supplied via oscillating emfs and currents, the current is said to be an **alternating current**, or **AC** for short. The nonoscillating current from a battery is said to be a **direct current**, or **DC**.

- These oscillating emfs and currents vary sinusoidally with time, reversing direction (in North America) 120 times per second and thus having frequency f = 60 Hz.

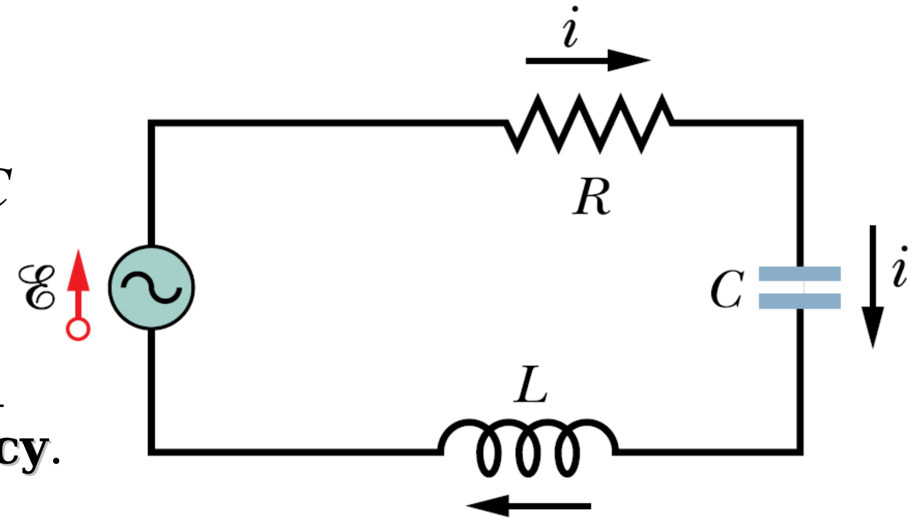
- The advantage of alternating current: *As the current alternates, so does the magnetic field that surrounds the conductor*. This makes possible the use of Faraday's law of induction.

- In a generator: $\mathscr{E} = \mathscr{E}_m \sin \omega_d t$, $i = I \sin(\omega_d t - \phi)$ where ω_d is called the **driving angular frequency**.

- the current may not be in phase with the emf.

- the **driving frequency** $f_d = \dfrac{\omega_d}{2\pi}$

Forced Oscillations



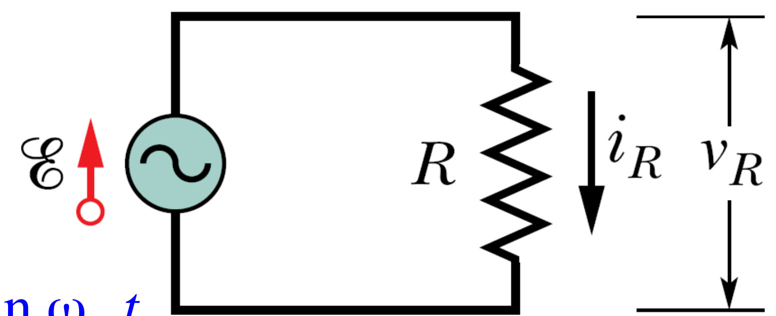
- An undamped *LC* circuits or a damped *RLC* circuits (with small enough *R*) without any external emf are said to be *free oscillations*, and the angular frequency $\omega = 1/\sqrt{LC}$ is said to be the circuit's **natural angular frequency**.

- When the external alternating emf is connected to an *RLC* circuit, the oscillations of charge, potential difference, and current are said to be *driven oscillations* or *forced oscillations*., with the driving angular frequency ω_d :

Whatever the natural angular frequency ω of a circuit may be, forced oscillations of charge, current, and potential difference in the circuit always occur at the driving angular frequency ω_d.

Three Simple Circuits

A Resistive Load



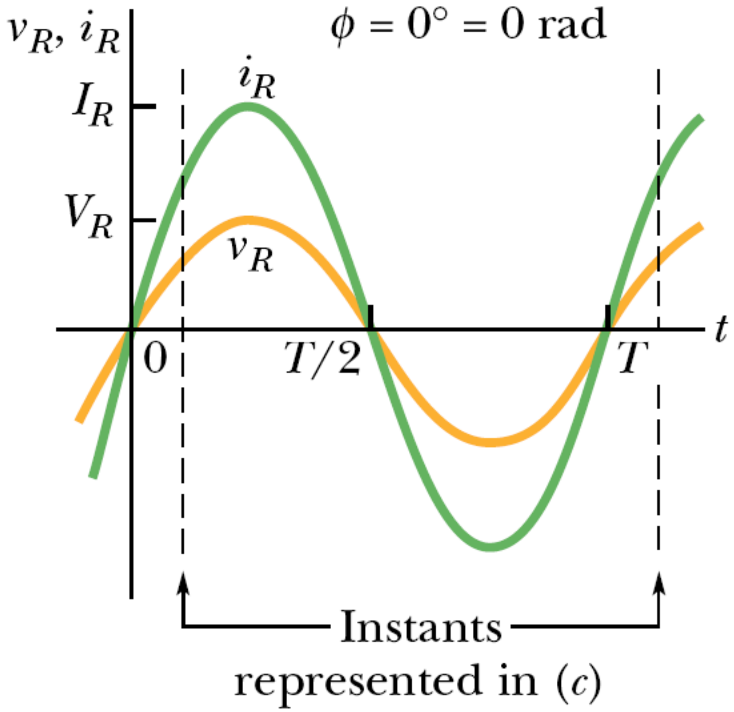
(*a*)

- By the loop rule: $\mathscr{E} - v_R = 0 \;\Rightarrow\; v_R = \mathscr{E}_m \sin \omega_d t$

$$\Rightarrow\; v_R = V_R \sin \omega_d t \;\Leftarrow\; V_R = \mathscr{E}_m \;\Rightarrow\; i_R = \frac{v_R}{R} = \frac{V_R}{R} \sin \omega_d t$$

$$i_R = I_R \sin(\omega_d t - \phi) \;\Rightarrow\; \phi = 0\,, \qquad \boxed{V_R = I_R R \quad \text{resistor}}$$

- v_R and i_R are *in phase*, which means that their corresponding maxima (and minima) occur at the same times.



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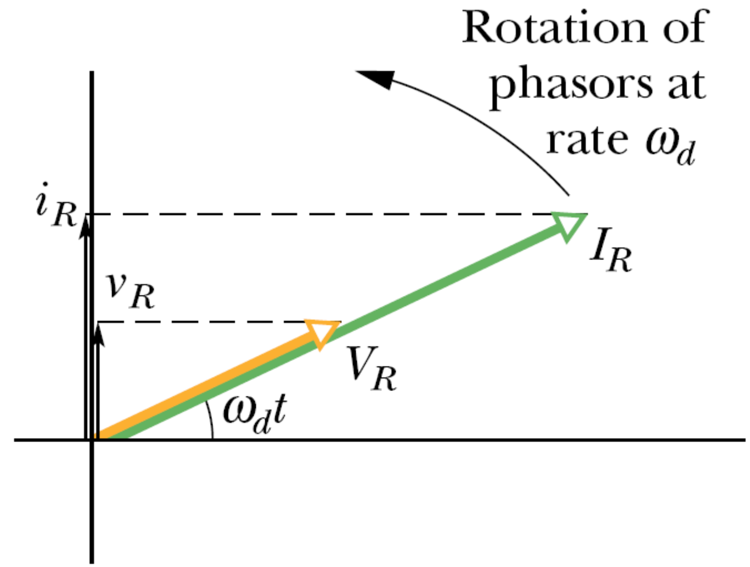

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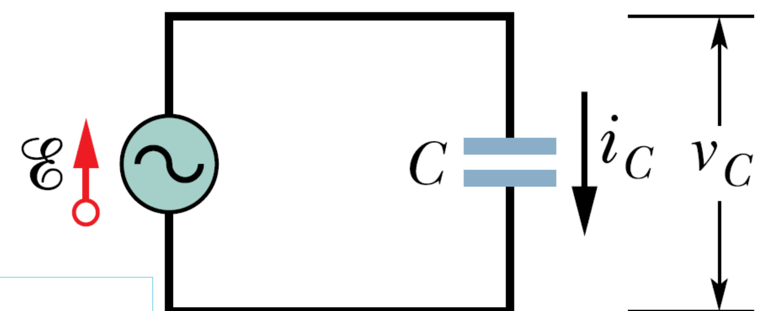
problem 31-3



(a)

A Capacitive Load

- the potential difference across the capacitor

  $v_C = V_C \sin \omega_d t \quad \Rightarrow \quad q_C = C\, v_C = C\, V_C \sin \omega_d t$

- The current: $i_C = \dfrac{d\, q_C}{d\, t} = \omega_d\, C\, V_C \cos \omega_d t$

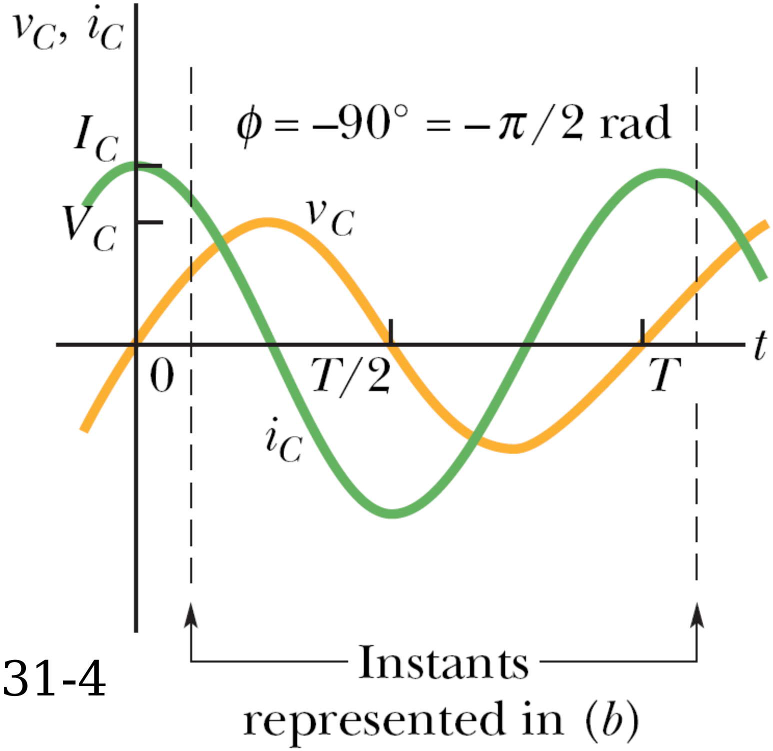
- **capacitive reactance**: $X_C = \dfrac{1}{\omega_d C}$ capacitive reactance

- the SI unit of X_C is the *ohm*, just as for resistance R.

- $\cos \omega_d t = \sin\left(\omega_d t + \dfrac{\pi}{2}\right) \quad \Rightarrow \quad i_C = I_C \sin\left(\omega_d t - \phi\right) = \dfrac{V_C}{X_C} \sin\left(\omega_d t + \dfrac{\pi}{2}\right)$

  $\Rightarrow$ $V_C = I_C X_C$ capacitor $\Leftarrow$ true for any capacitance in any circuit

Problem 31-4



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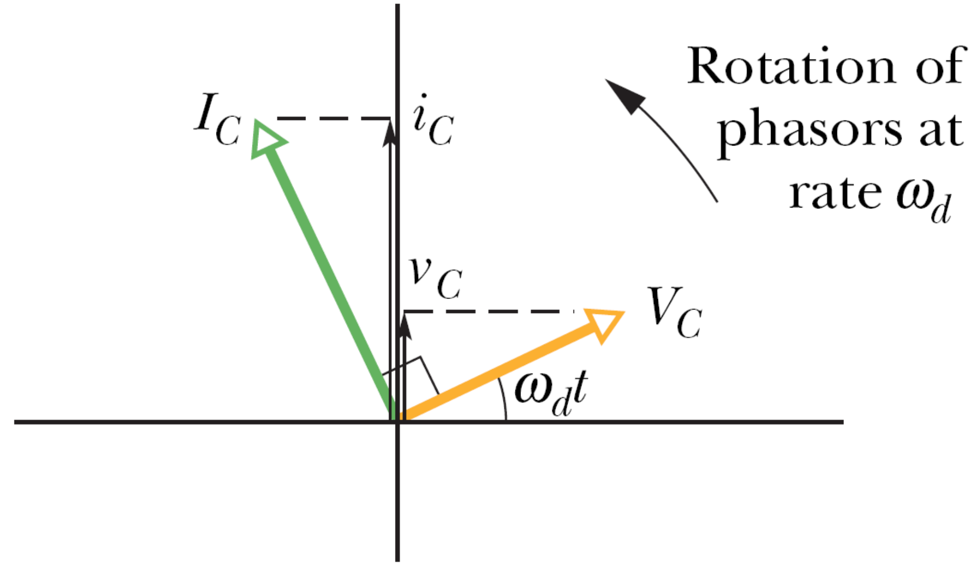

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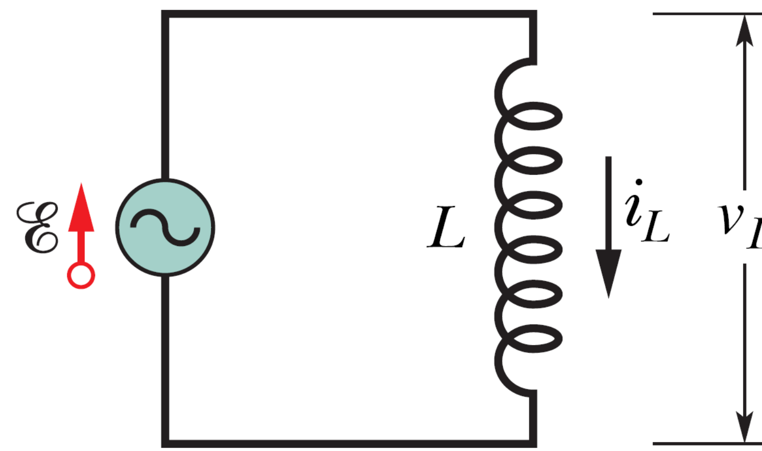


An Inductive Load

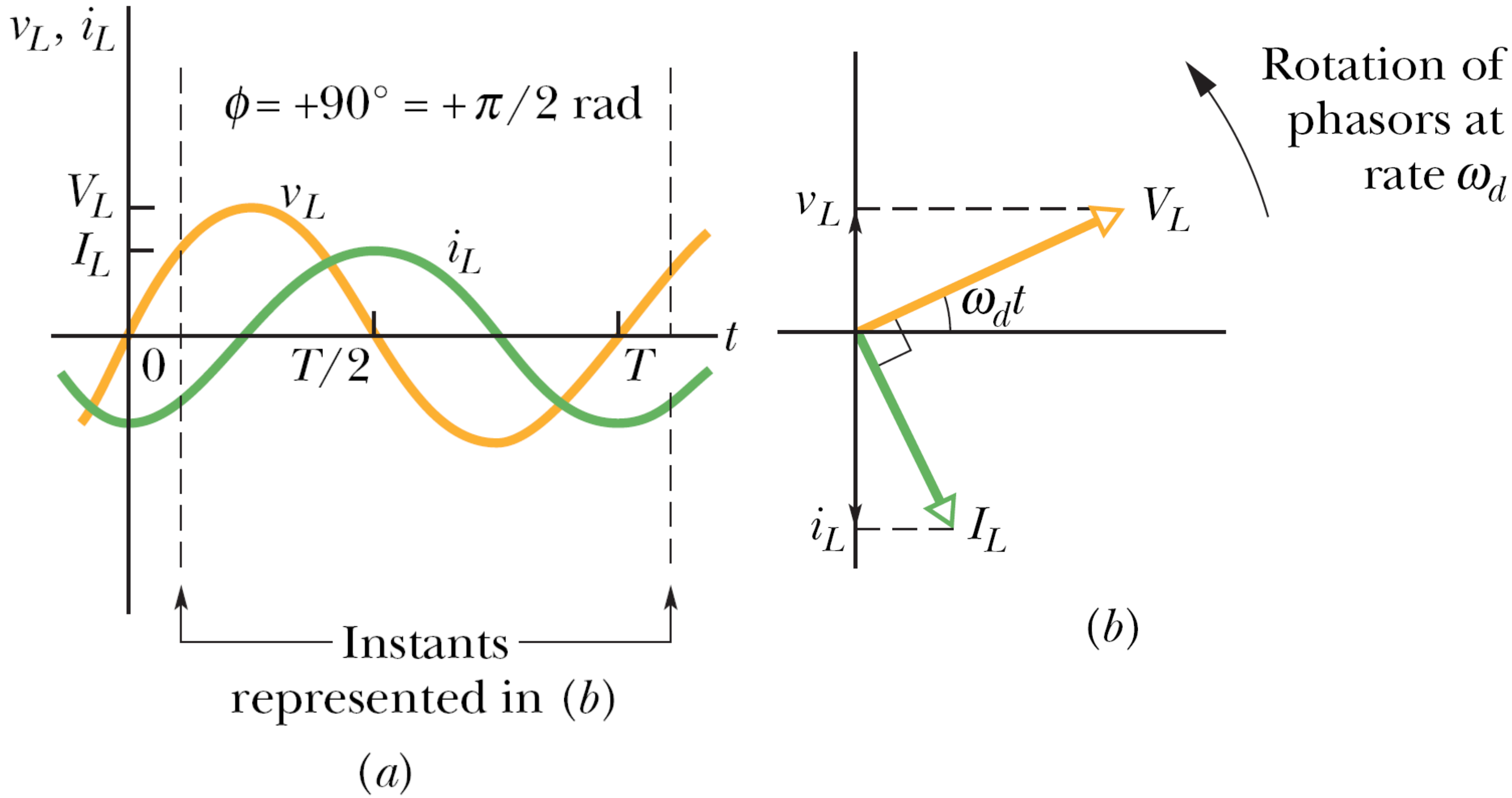
- the potential difference across the inductance

$$v_L = V_L \sin \omega_d t = L \frac{d i_L}{d t} \quad \Rightarrow \quad \frac{d i_L}{d t} = \frac{V_L}{L} \sin \omega_d t$$

- $i_L = \int d i_L = \frac{V_L}{L} \int \sin \omega_d t \, d t = -\frac{V_L}{\omega_d L} \cos \omega_d t$
- **inductive reactance**: $X_L = \omega_d L$ inductive reactance
- the SI unit of X_L is the *ohm*, just as for X_C an for R.
- $-\cos \omega_d t = \sin\left(\omega_d t - \frac{\pi}{2}\right) \Rightarrow i_L = I_L \sin\left(\omega_d t - \phi\right) = \frac{V_L}{X_L} \sin\left(\omega_d t - \frac{\pi}{2}\right)$

$\Rightarrow$ $V_L = I_L X_L$ inductor $\Leftarrow$ true for any inductance in any circuit

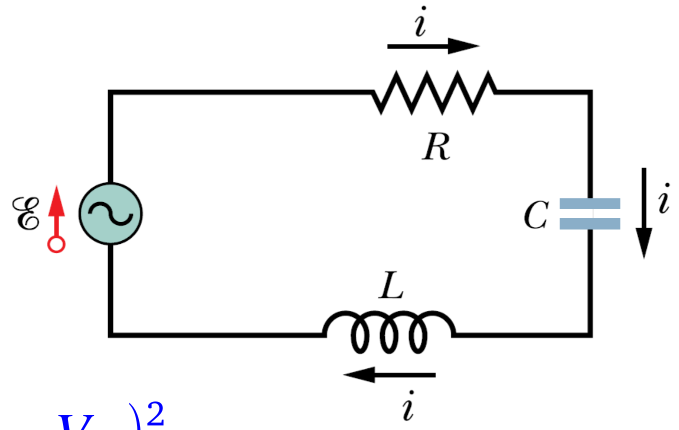
Problem 31-5



Phase and Amplitude Relations for Alternating Currents and Voltages

Circuit Element	Symbol	Resistance or Recactance	Phase of the Current	Phase Constant (or Angle) ϕ	Amplitude Relation
Resistor	R	R	In phase with v_R	0	$V_R = I_R R$
Capacitor	C	$X_C = \frac{1}{\omega_d C}$	Leads v_C by $\frac{\pi}{2}$	$-\frac{\pi}{2}$	$V_C = I_C X_C$
Inductor	L	$X_L = \omega_d L$	Lags v_L by $\frac{\pi}{2}$	$\frac{\pi}{2}$	$V_L = I_L X_L$

The Series *RLC* Circuit



- Apply a *RLC* circuit the alternating emf

$$\mathscr{E} = \mathscr{E}_m \sin \omega_d t \quad \text{applied emf} \quad \Rightarrow \quad i = I \sin(\omega_d t - \phi)$$

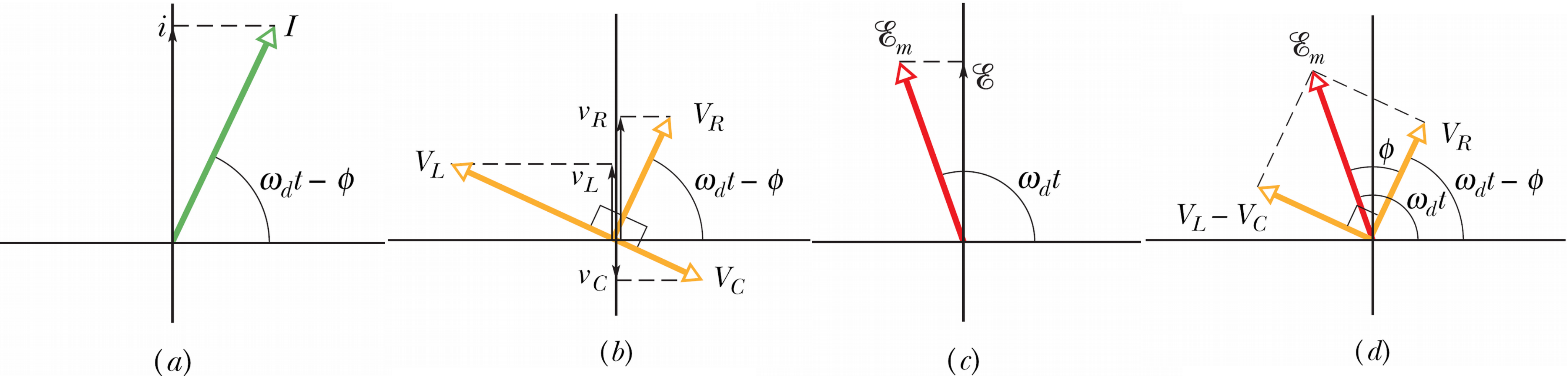
The Current Amplitude

- For the loop rule: $\mathscr{E} = v_R + v_C + v_L \quad \Rightarrow \quad \mathscr{E}_m^2 = V_R^2 + (V_L - V_C)^2$

$$\Rightarrow \quad \mathscr{E}_m^2 = (I\,R)^2 + (I\,X_L - I\,X_C)^2 \quad \Rightarrow \quad I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{\mathscr{E}_m}{Z} \quad \text{where}$$

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad \text{impedance} \quad \Rightarrow \quad I = \frac{\mathscr{E}_m}{\sqrt{R^2 + [\omega_d L - 1/(\omega_d C)]^2}} \quad \text{current amplitude}$$

- The value of I depends on the difference between $\omega_d L$ and $1/\omega_d C$ or, equivalently, the difference between X_L and X_C .



$$\mathscr{E} = \mathscr{E}_m \sin \omega_d t \quad \Rightarrow \quad i = I \sin(\omega_d t - \phi)$$

$$\Rightarrow \quad \begin{aligned} v_R &= i\,R = V_R \sin(\omega_d t - \phi) \\ v_C &= \quad -V_C \cos(\omega_d t - \phi) \\ v_L &= \quad\;\; V_L \cos(\omega_d t - \phi) \end{aligned} \quad \Leftarrow \quad \begin{aligned} V_R &= I\,R \\ V_C &= I\,X_C \\ V_L &= I\,X_L \end{aligned} \quad \Rightarrow \quad \text{Define } V_X = V_L - V_C$$

$$\mathscr{E}_m \sin \omega_d t \Leftarrow \mathscr{E} = v_R + v_C + v_L = V_R \sin(\omega_d t - \phi) + V_X \cos(\omega_d t - \phi)$$

$$= (V_R \cos\phi + V_X \sin\phi) \sin \omega_d t + (V_X \cos\phi - V_R \sin\phi) \cos \omega_d t$$

Coefficient comparison gives

$$\begin{aligned} V_R \cos\phi + V_X \sin\phi &= \mathscr{E}_m \\ V_X \cos\phi - V_R \sin\phi &= 0 \end{aligned}$$

$$\Rightarrow \quad \tan\phi = \frac{V_X}{V_R} = \frac{V_L - V_C}{V_R} = \frac{X_L - X_C}{R}$$

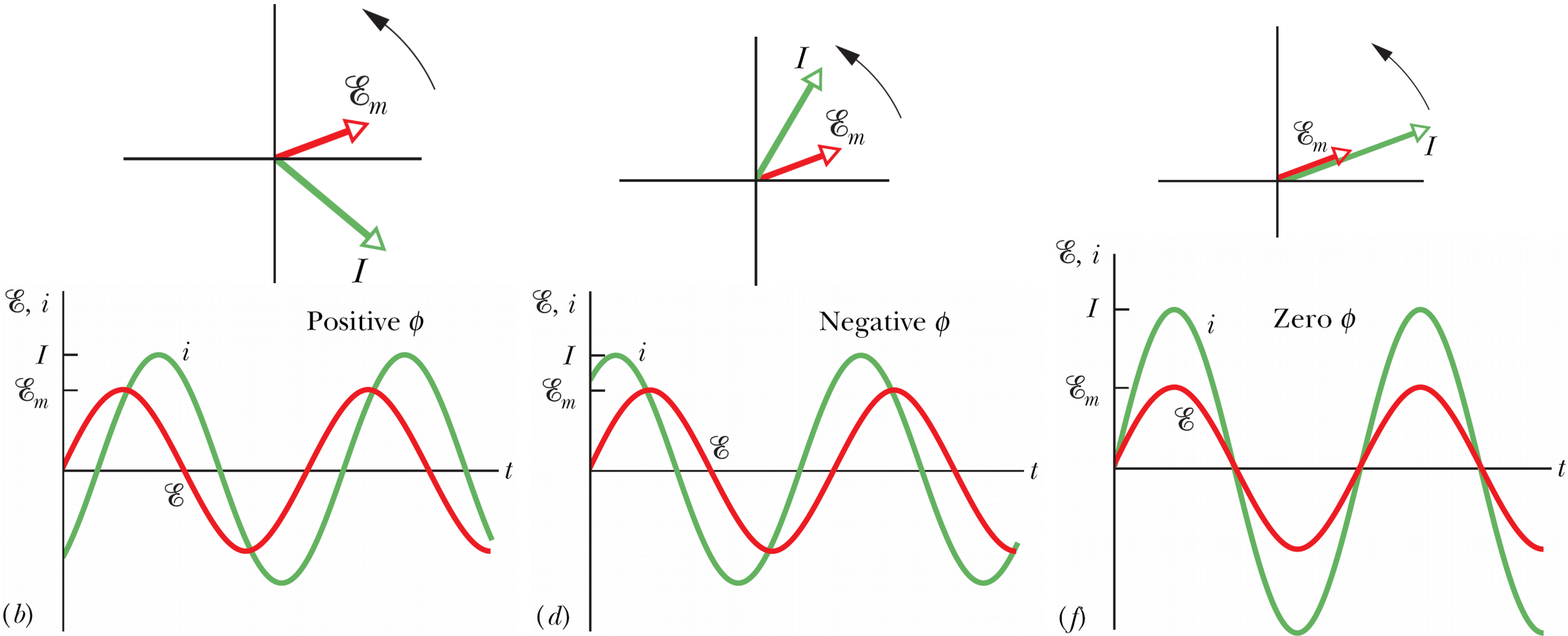
$$V_R = \mathscr{E}_m \cos\phi\,,\; V_X = \mathscr{E}_m \sin\phi \quad \Rightarrow \quad \mathscr{E}_m^2 = V_R^2 + V_X^2 = V_R^2 + (V_L - V_C)^2$$

- The current that we have been describing in this section is the *steady-state current* that occurs after the alternating emf has been applied for some time.

- When the emf is first applied to a circuit, a brief *transient current* occurs. Its duration is determined by the time constants $\tau_L = L/R$ and $\tau_C = RC$ as the inductive and capacitive elements "turn on."

The Phase Constant

- From the plot: $\tan\phi = \dfrac{V_L - V_C}{V_R} = \dfrac{I X_L - I X_C}{I R} \Rightarrow$ $\tan\phi = \dfrac{X_L - X_C}{R}$ phase constant



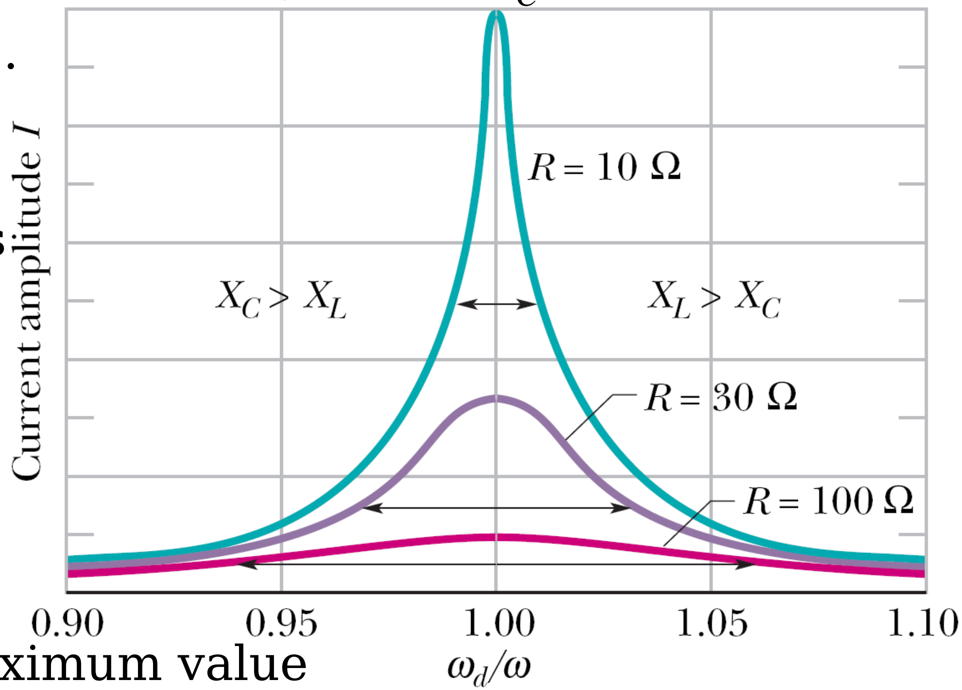
- 3 different results for the phase constant

  $X_L > X_C$: The circuit is said to be *more inductive than capacitive*.

  $X_C > X_L$: The circuit is said to be *more capacitive than inductive*.

  $X_C = X_L$: The circuit is said to be in *resonance*.

- In the *purely inductive circuit*, where X_L is nonzero and $X_C = R = 0$, then $\phi = \pi/2$ (the greatest value of ϕ). In the *purely capacitive circuit*, where X_C is nonzero and $X_L = R = 0$, then $\phi = -\pi/2$ (the least value of ϕ).



Resonance

- For a given resistance R, that amplitude is a maximum when the quantity $\omega_d L - 1/\omega_d C$ in the denominator is zero

$$\omega_d L = \frac{1}{\omega_d C} \quad \Rightarrow \quad \omega_d = \frac{1}{\sqrt{LC}} \quad \text{maximum } I$$

- the natural angular frequency ω of the *RLC* circuit is also equal to $1/\sqrt{LC}$, the maximum value of I occurs when the driving angular frequency matches the natural angular frequency—that is, at resonance.

$$\omega_d = \omega = \frac{1}{\sqrt{LC}} \quad \text{resonance}$$

- The *resonance curves* peak at their maximum current amplitude I ($=\mathscr{E}_m/R$) when $\omega_d=\omega$, but the maximum value of I decreases with increasing R. The curves also increase in width (measuring at half the maximum value of I) with increasing R.

- For small ω_d, $X_L(=\omega_d L)$ is small and $X_C(=1/\omega_d C)$ is large. Thus, the circuit is mainly capacitive and the impedance is dominated by the large X_C, which keeps the current low.

- As ω_d increases, X_C remains dominant but decreases while X_L increases. The decrease in X_C decreases the impedance, allowing the current I to increase. When the increasing X_L and the decreasing X_C reach equal values, the current I is greatest and the circuit is in resonance, with $\omega_d=\omega$.

- As ω_d continue to increase, the increasing X_L becomes more dominant over the decreasing X_C. The impedance increases because of X_L and the current decreases.

- In summary: The low-angular-frequency side of a resonance curve is dominated by the capacitor's reactance, the high-angular frequency side is dominated by the inductor's reactance, and resonance occurs in the middle.

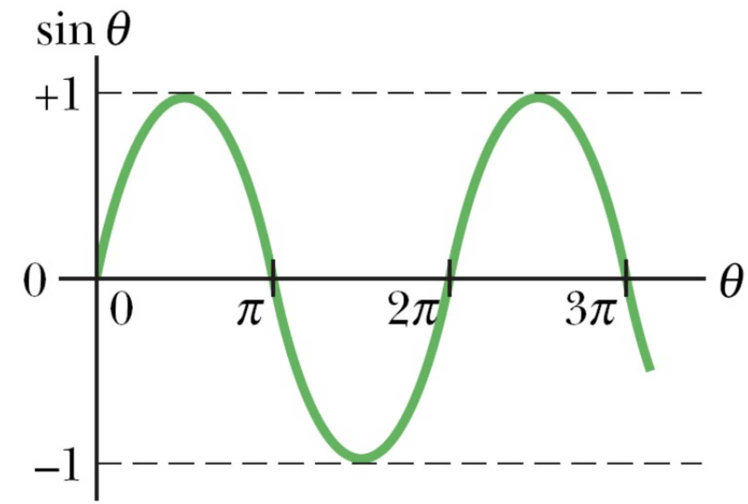
Problem 31-6

Power in Alternating-Current Circuits

- In steady-state operation the average energy stored in the capacitor and inductor together remains constant. The net transfer of energy is thus from the generator to the resistor, where EM energy is dissipated as thermal energy.

- The instantaneous rate at which energy is dissipated in the resistor

$$P = i^2 R = [I \sin(\omega_d t - \phi)]^2 R = I^2 R \sin^2(\omega_d t - \phi)$$



(a)

- The *average* rate at which energy is dissipated

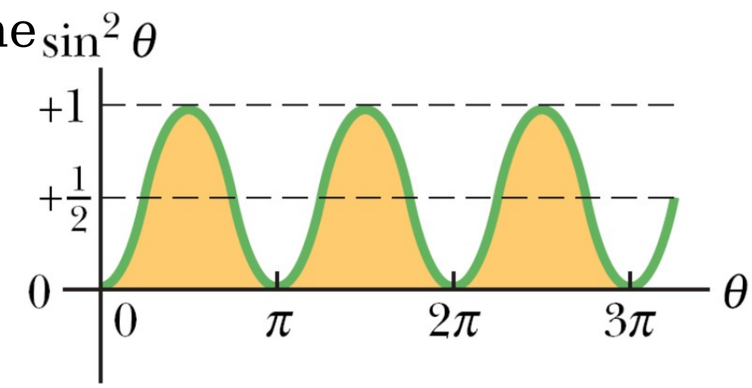
$$P_{avg} = \frac{1}{T}\int_0^T P\,dt = \frac{I^2 R}{T}\int_0^T \sin^2(\omega_d t - \phi)\,dt = \frac{I^2 R}{2}$$

$$= \left(\frac{I}{\sqrt{2}}\right)^2 R \quad \Rightarrow \quad I_{rms} \equiv \frac{I}{\sqrt{2}} \quad \text{rms current}$$

$$\Rightarrow \quad P_{avg} = I_{rms}^2 R \quad \text{average power}$$

- if we switch to the rms current, we can compute the average rate of energy dissipation for alternating-current circuits just as for direct-current circuits.

- $$V_{rms} = \frac{V}{\sqrt{2}}, \quad \mathscr{E}_{rms} = \frac{\mathscr{E}}{\sqrt{2}} \quad \text{rms voltage; rms emf}$$



(b)

- Alternating-current instruments, such as ammeters and voltmeters, are usually calibrated to read I_{rms}, V_{rms}, and $\mathscr{E}_{\text{rms}}$.

- plug an alternating-current voltmeter into a electrical outlet and it reads 120 V, that is an rms voltage. The maximum value of the potential difference at the outlet is $\sqrt{2}\times(120\text{ V})$ or 170 V.

- $$I_{\text{rms}}=\frac{\mathscr{E}_{\text{rms}}}{Z}=\frac{\mathscr{E}_{\text{rms}}}{\sqrt{R^2+(X_L-X_C)^2}} \quad \Rightarrow \quad P_{\text{avg}}=\frac{\mathscr{E}_{\text{rms}}}{Z}I_{\text{rms}}R=\mathscr{E}_{\text{rms}}I_{\text{rms}}\frac{R}{Z}$$

$$\Rightarrow \quad \boxed{P_{\text{avg}}=\mathscr{E}_{\text{rms}}I_{\text{rms}}\cos\phi \quad \text{average power}} \quad \Leftarrow \quad \cos\phi=\frac{V_R}{\mathscr{E}_m}=\frac{IR}{IZ}=\frac{R}{Z} \quad \text{power factor}$$

- The equation is independent of the sign of the phase constant $\phi \Leftarrow \cos\phi=\cos(-\phi)$.

- To maximize the rate at which energy is supplied to a resistive load in an *RLC* circuit, we should keep the power factor $\cos\phi$ as close to 1 as possible $\Leftarrow$ $\phi=0$.

Problem 31-7

Selected problems: 10, 26, 36, 60