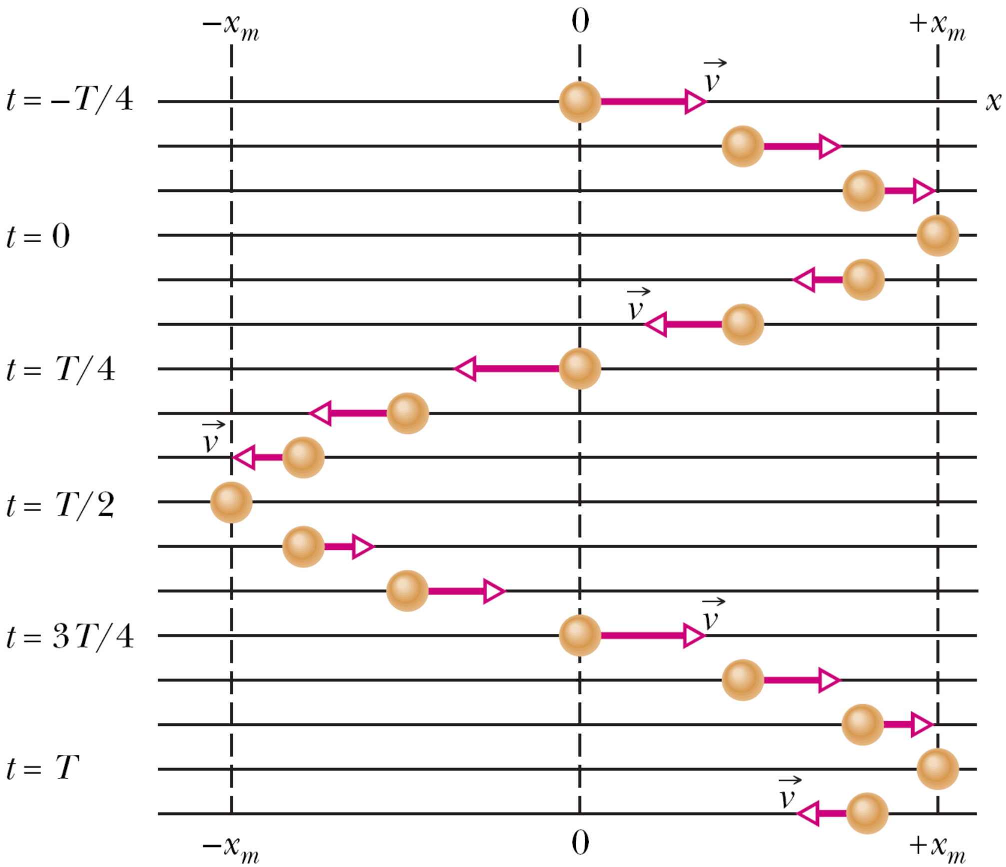
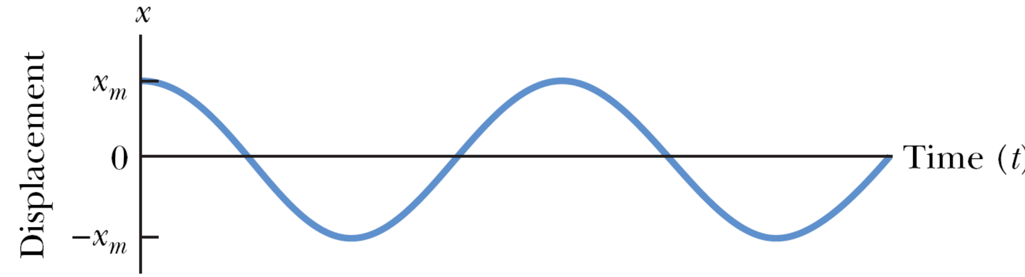
Chapter 15 Oscillations

Simple Harmonic Motion





- **Frequency, f:** number of oscillations that are completed each second.

The SI unit is the **hertz** (Hz),

$$1 \text{ hertz} = 1 \text{ Hz} = 1 \text{ oscillation per second} = 1/\text{s}.$$

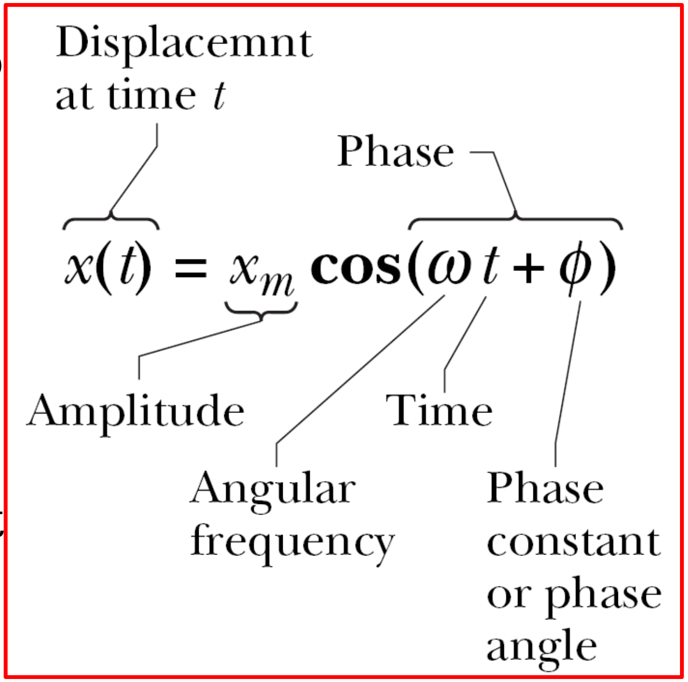
- **Period T** of the motion: the time for one complete oscillation (or **cycle**): $T = \dfrac{1}{f}$

- **Periodic motion** or **harmonic motion**: Any motion that repeats itself at regular intervals.

- We are interested here in **simple harmonic motion** (SHM), the periodic motion is a sinusoidal function of time,

$$x(t) = x_m \cos(\omega t + \phi)$$

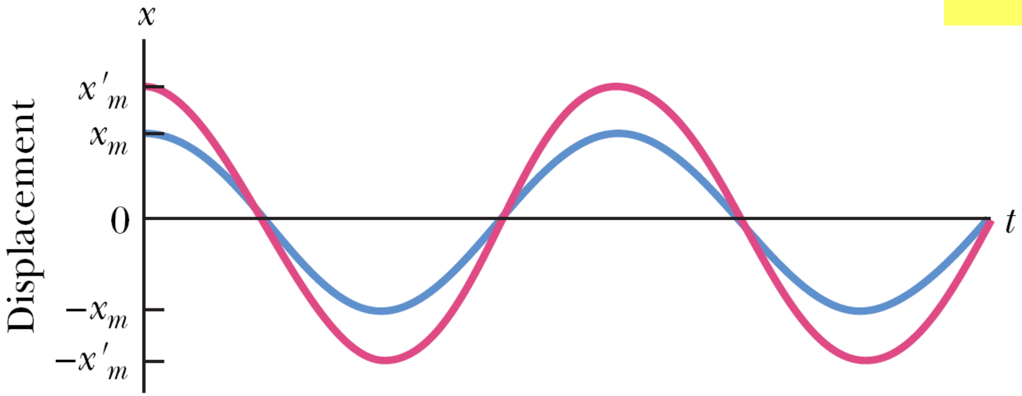
- The cosine function varies between the limits ± 1; so the displacement $x(t)$ varies between the limits $\pm x_m$.

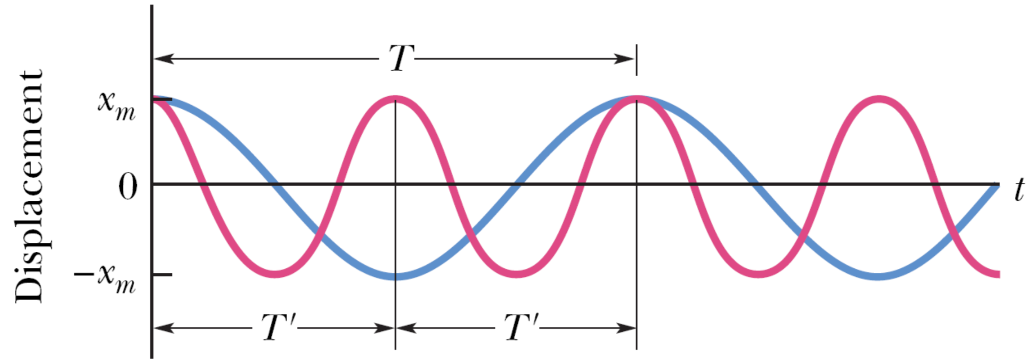


- The displacement $x(t)$ must return to its initial value after one period T of the motion; that is, $x(t) = x(t+T)$ for all t, $x_m \cos \omega t = x_m \cos \omega (t+T)$, assuming $\phi=0$.

- The cosine function repeats itself when its argument (the phase) has increased by 2π rad,

$$\omega(t+T) = \omega t + 2\pi \;\Rightarrow\; \omega T = 2\pi \;\Rightarrow\; \omega = \frac{2\pi}{T} = 2\pi f$$



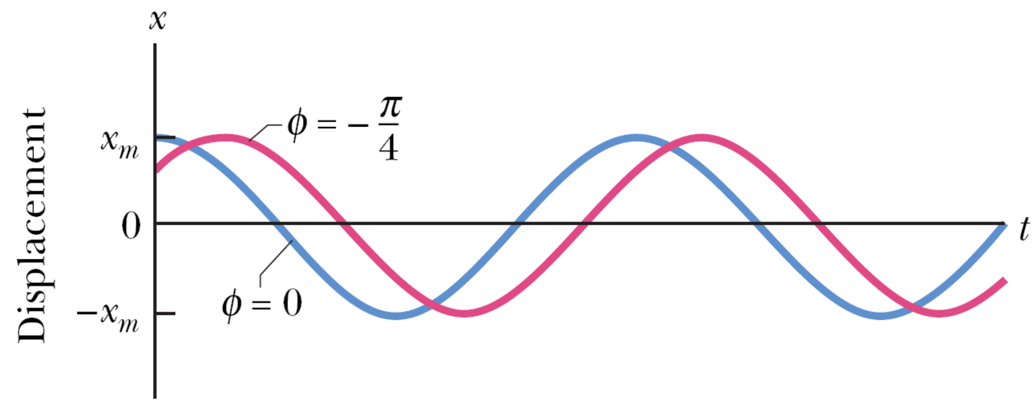


The Velocity of SHM

- The velocity of a particle moving with simple harmonic motion,

$$v(t) = \frac{d}{dt} x(t) = \frac{d}{dt}\left[x_m \cos(\omega t + \phi)\right]$$

$$\Rightarrow \; v(t) = -\omega x_m \sin(\omega t + \phi) \quad \text{velocity}$$



$$x(t) = x_m \cos(\omega t + \phi) = x_m \sin\left(\omega t + \phi + \frac{\pi}{2}\right) = x_m \sin(\omega t + \phi')$$

$$= \underbrace{x_m \cos\phi}\ \cos\omega t + \underbrace{\left(-x_m \sin\phi\right)}\ \sin\omega t$$

$$= \quad A \quad \cos\omega t + \quad B \quad \sin\omega t = A\cos\omega t + B\sin\omega t$$

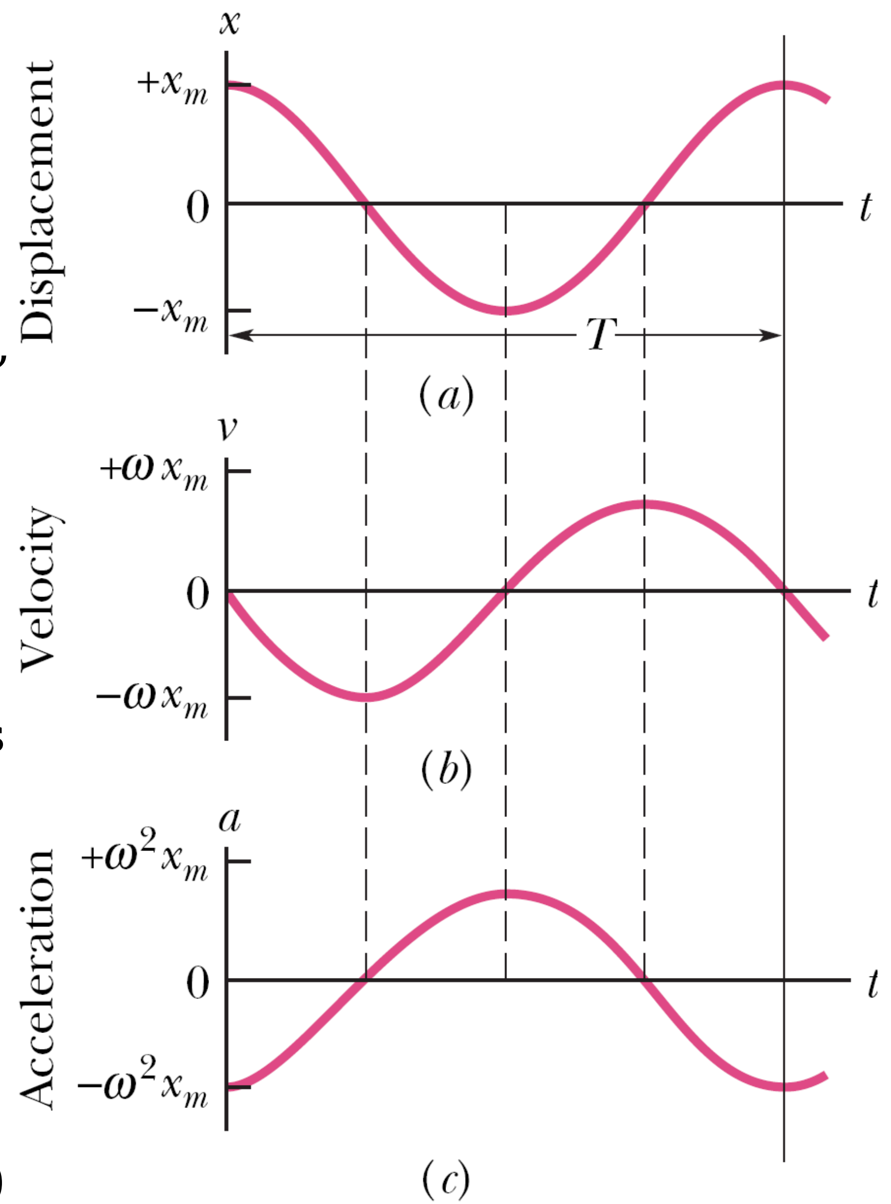
$$= \frac{x_m}{2}\left[e^{i(\omega t+\phi)} + e^{-i(\omega t+\phi)}\right] \qquad \Leftarrow \qquad e^{i\theta} = \cos\theta + i\sin\theta$$

Euler formula

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots, \qquad \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots$$

- The **velocity amplitude**: $v_m = \omega x_m$

- The curve of $v(t)$ is shifted (to the left) from the curve of $x(t)$ by one-quarter period; when the magnitude of the displacement is greatest, the magnitude of the velocity is least (ie, 0). When the magnitude of the displacement is least (ie, 0), the magnitude of the velocity is greatest.



The Acceleration of SHM

- The acceleration of the oscillating particle is

$$a(t) = \frac{d}{dt} v(t) = \frac{d}{dt}\left[-\omega x_m \sin(\omega t + \phi)\right]$$

$$\Rightarrow \quad a(t) = -\omega^2 x_m \cos(\omega t + \phi) \quad \text{acceleration}$$

- The **acceleration amplitude**: $a_m = \omega^2 x_m$

- The acceleration curve is shifted (to the left) by $T/4$ relative to the velocity curve.

- The hallmark of simple harmonic motion: $a(t) = -\omega^2 x(t)$

In SHM, the acceleration is proportional to the displacement but opposite in sign, and the 2 quantities are related by the square of the angular frequency.

The Force Law for Simple Harmonic Motion

- Combine Newton's 2nd law with simple harmonic motion: $F = m\,a = -\left(m\,\omega^2\right) x$

- It is Hooke's law — a restoring force that is proportional to the displacement but opposite in sign, $F = -k\,x$ for a spring, the spring constant being $k = m\,\omega^2$

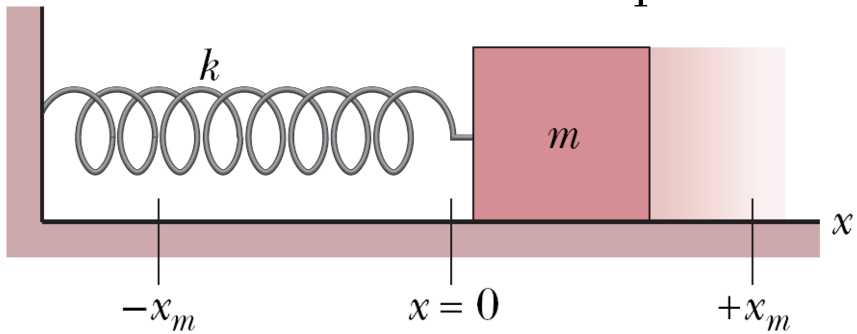
- An alternative definition of simple harmonic motion:

> Simple harmonic motion is the motion executed by a particle subject to a force that is proportional to the displacement of the particle but opposite in sign.

- The block-spring system forms a **linear simple harmonic oscillator**, where *linear* indicates that F is proportional to x rather than to some other power of x.

- The angular frequency is

$$\omega = \sqrt{\frac{k}{m}} \quad \text{angular frequency}$$

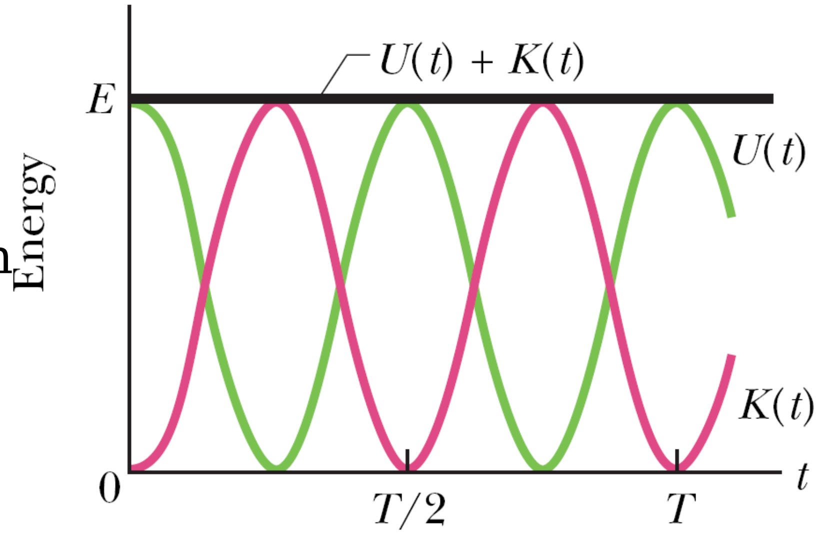


- The period of the linear oscillator is $T = 2\pi\sqrt{\frac{m}{k}}$ period

- A large angular frequency (and thus a small period) goes with a stiff spring (large k) and a light block (small m), and vice versa.

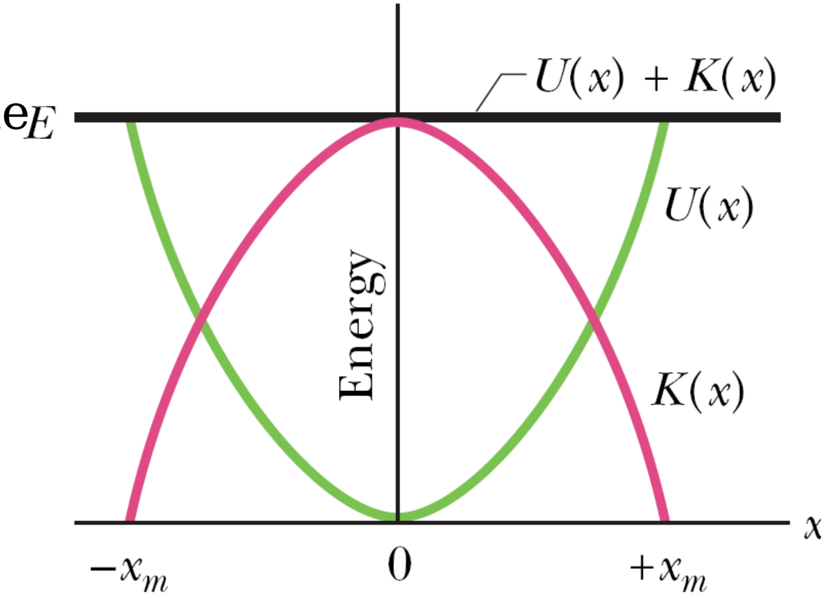
problem 15-1, problem 15-2

Energy in Simple Harmonic Motion



- The potential energy of a linear oscillator is associated entirely with the spring, depending on how much the spring is stretched or compressed

$$U(t)=\frac{1}{2}kx^2=\frac{1}{2}kx_m^2\cos^2(\omega t+\phi)$$



- The kinetic energy of the system is associated entirely with the block, depending on how fast the block is moving

$$K(t)=\frac{1}{2}mv^2=\frac{1}{2}m\omega^2x_m^2\sin^2(\omega t+\phi)$$

$$\Rightarrow K(t)=\frac{1}{2}mv^2=\frac{1}{2}kx_m^2\sin^2(\omega t+\phi)$$

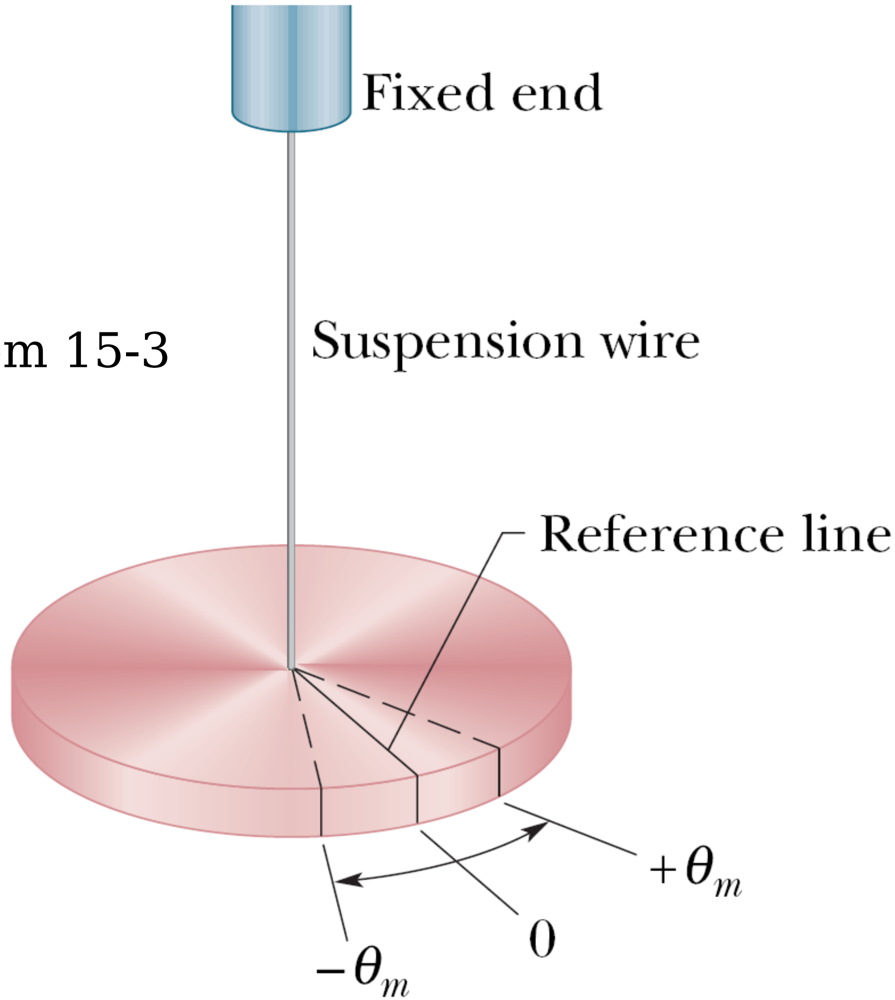
- The mechanical energy follows

$$E=K+U=\frac{1}{2}kx_m^2[\sin^2(\omega t+\phi)+\cos^2(\omega t+\phi)] \Rightarrow E=K+U=\frac{1}{2}kx_m^2$$

- The mechanical energy of a linear oscillator is indeed constant and independent of time.

- The element of springiness stores its potential energy and the element of inertia stores its kinetic energy.

problem 15-3



An Angular Simple Harmonic Oscillator

- The element of springiness is associated with the twisting of a suspension wire.

- The device is called a **torsion pendulum**, with *torsion* referring to the twisting.

- Rotate the disk by some angular displacement from its rest position and release it, it will oscillate about that position in **angular simple harmonic motion.**

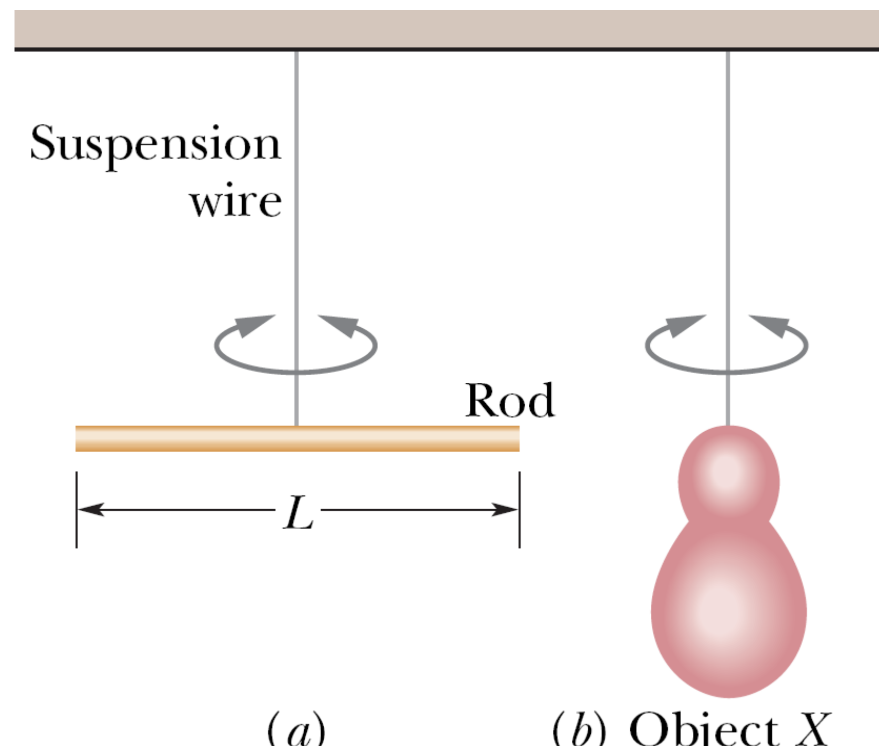
- A restoring torque $\tau = -\kappa\,\theta$, where κ is the **torsion constant**, that depends on the length, diameter, and material of the suspension wire.

- This is the angular form of Hooke's law, and the torsion pendulum is an angular SHM.

- Replace the spring constant with the torsion constant, replace the mass with the rotational inertia of the oscillating disk, then the period of the angular SHM

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \quad \text{torsion pendulum}$$

problem 15-4



Pendulums

- The springiness is associated with the gravitation.

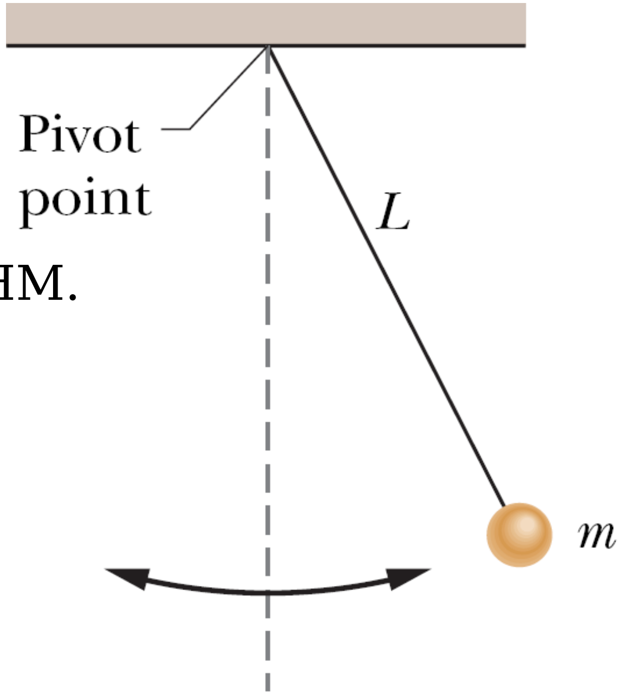
The Simple Pendulum

- The forces acting on the bob are the force from the string and the gravitational force.

- Resolve the gravitational force into a radial component and a component that is tangent to the path taken by the bob.

- The *equilibrium position*: $\theta = 0$.

- This tangential component produces a restoring torque about the pendulum's pivot point $\tau = -L\left(F_g \sin\theta\right) = \mathbb{I}\,\alpha$



- If θ is small, $\sin\theta \approx \theta \;\;\Rightarrow\;\; \alpha \equiv \frac{d^2}{d\,t^2}\theta = -\frac{m\,g\,L}{\mathbb{I}}\,\theta$

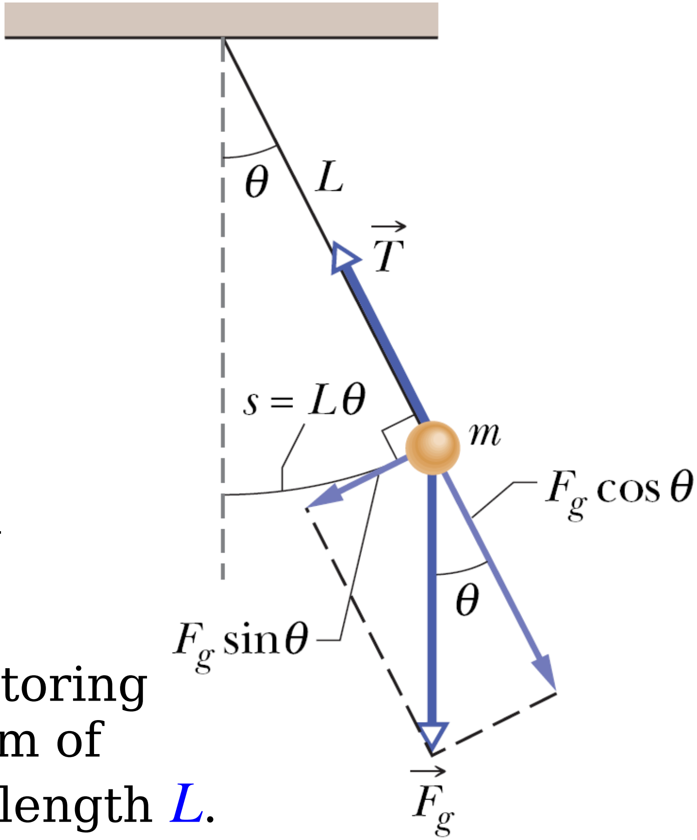
- This eqn is the angular equivalent of the hallmark of SHM.

- The motion of a *simple pendulum swinging through only small angles* is approximately SHM, ie, the **angular amplitude** of the motion must be small.

- The angular frequency of the pendulum is $\omega = \sqrt{\frac{m\,g\,L}{\mathbb{I}}}$

- The period of the pendulum $T = 2\pi\sqrt{\dfrac{\mathbb{I}}{m\,g\,L}}$

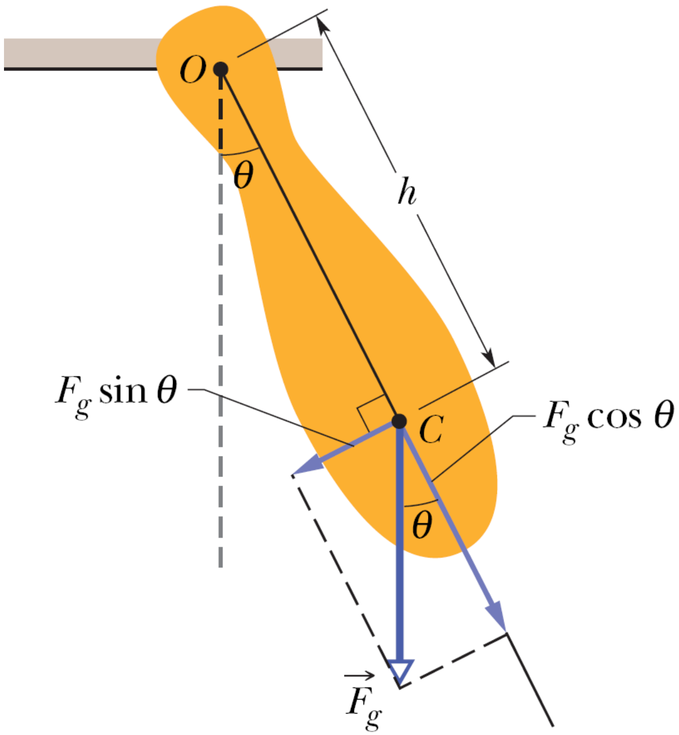
$$\mathbb{I} = m\,L^2 \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{L}{g}}$$ simple pendulum, small amplitude



The Physical Pendulum

- A **physical pendulum**: a real pendulum, can have a complicated distribution of mass.

- The only difference: for a physical pendulum the restoring component of the gravitational force has a moment arm of distance h about the pivot point, rather than of string length L.



- Follow the similar analysis and we would find that the motion is approximately SHM.

- The period of the pendulum is

$$T = 2\pi\sqrt{\frac{\mathbb{I}}{m\,g\,h}}$$ physical pendulum, small amplitude

- A physical pendulum will not swing if it pivots at its center of mass: $h = 0 \quad \Rightarrow \quad T \to \infty$

- We can correspond a simple pendulum of length L_0 with the same period T to any physical physical pendulum. The point along the physical pendulum at distance L_0 from the pivot point is called the *center of oscillation* of the physical pendulum.

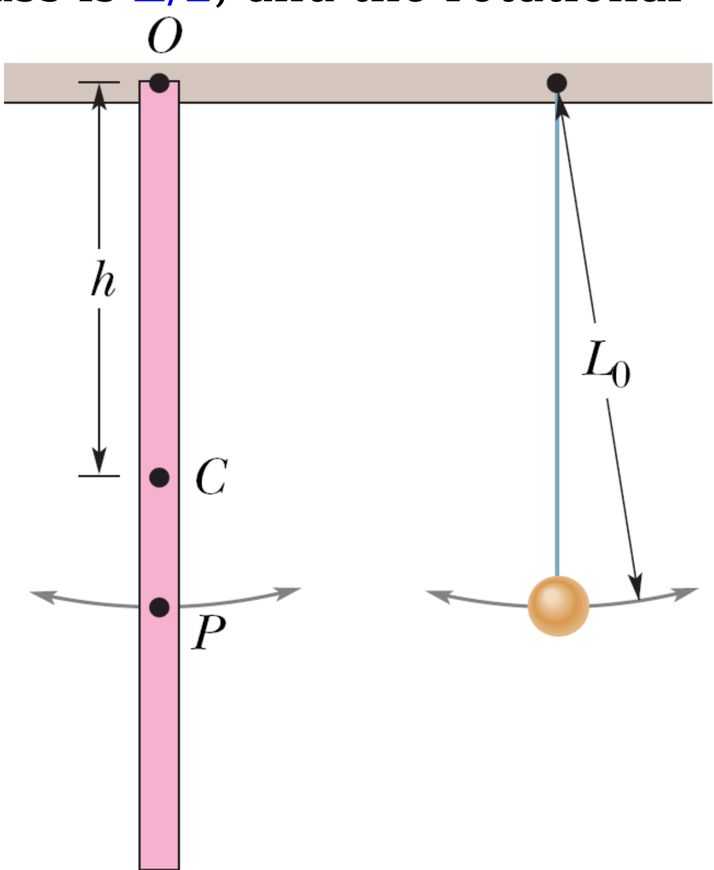
Measuring g

- We can use a physical pendulum to measure the free-fall acceleration g at a particular location on Earth's surface.

- For the pendulum of a uniform rod of length L, suspended from one end, the distance between the pivot point and the center of mass is $L/2$, and the rotational inertia is

$$\mathbb{I} = \mathbb{I}_{\text{com}} + m\,h^2 = \frac{1}{12}\,m\,L^2 + m\left(\frac{1}{2}\,L\right)^2 = \frac{1}{3}\,m\,L^2$$

- From the definition of the period, $g = \dfrac{8\,\pi^2\,L}{3\,T^2}$



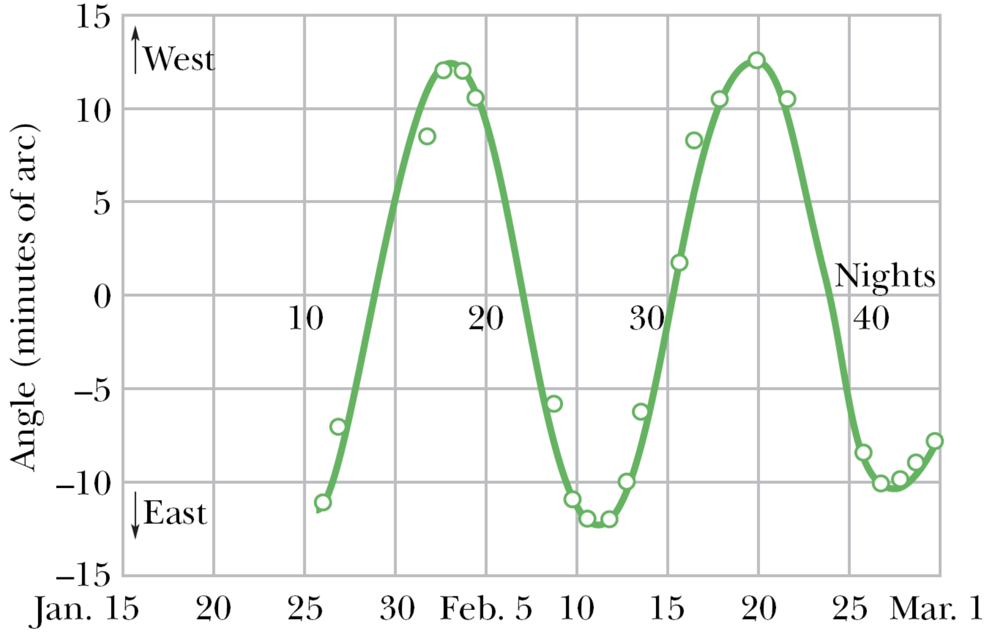
Problem 15-5

Simple Harmonic Motion and Uniform Circular Motion

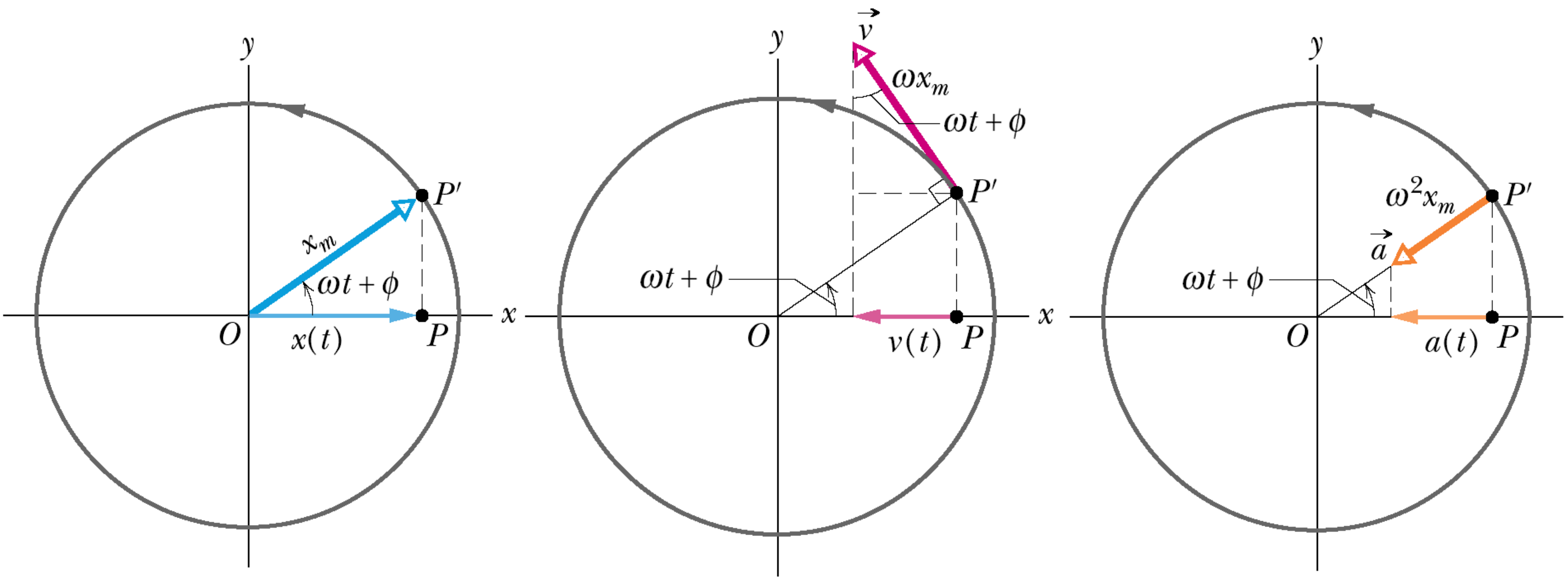
- Galileo's observation and a fitting curve

- The curve suggests the displacement function for SHM.

- In fact it is is the projection of a uniform circular motion on a line in the plane of the motion.



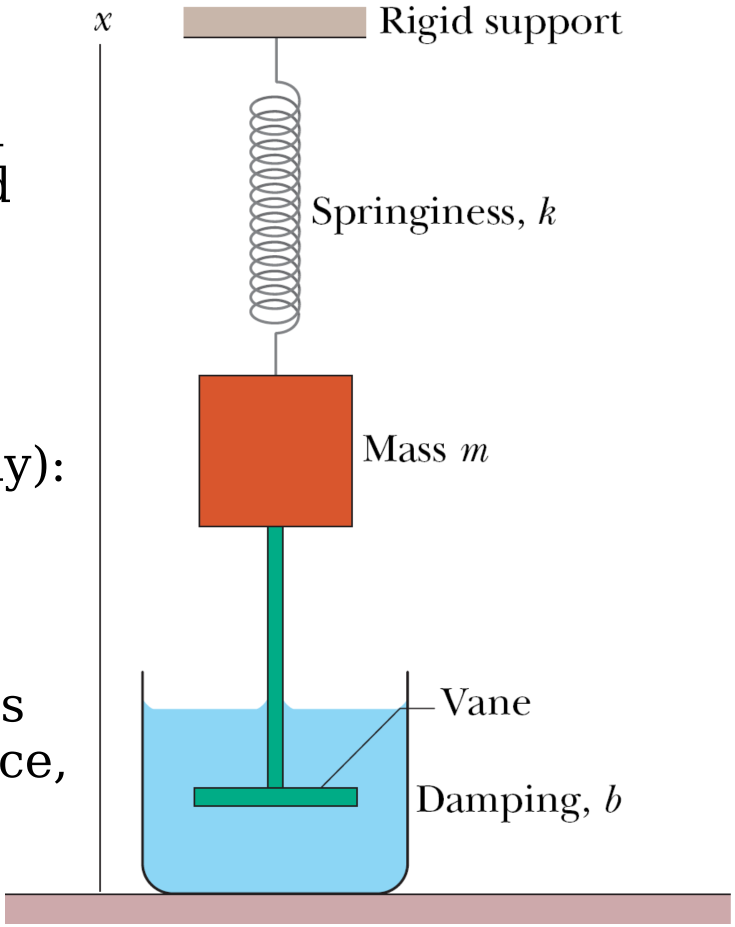
Simple harmonic motion is the projection of uniform circular motion on a diameter of the circle in which the circular motion occurs.



- A *reference particle* P' moving in uniform circular motion with (constant) angular speed in a *reference circle*.

$$\begin{aligned} v &= r\,\omega \\ a &= r\,\omega^2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x(t) &= x_m \cos(\omega t + \phi) \\ v(t) &= -\omega\, x_m \sin(\omega t + \phi) \\ a(t) &= -\omega^2 x_m \cos(\omega t + \phi) \end{aligned}$$

Damped Simple Harmonic Motion



- When the motion of an oscillator is reduced by an external force, the oscillator and its motion are said to be **damped**.

- Assume the liquid exerts a damping force that is proportional to velocity of the vane and block (an assumption that is accurate if the vane moves slowly):

$F_d = -b\,v$ where b is a **damping constant.**

- Assume that the gravitational force on the block is negligible relative to the spring an the damping force, Newton's 2nd law gives

$$-b\,v - k\,x = m\,a \quad \Rightarrow \quad m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + k\,x = 0$$

- The solution of this equation $x(t) = x_m\,e^{-\frac{b\,t}{2\,m}}\cos(\omega' t + \phi) \quad \Leftarrow \quad \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4\,m^2}}$

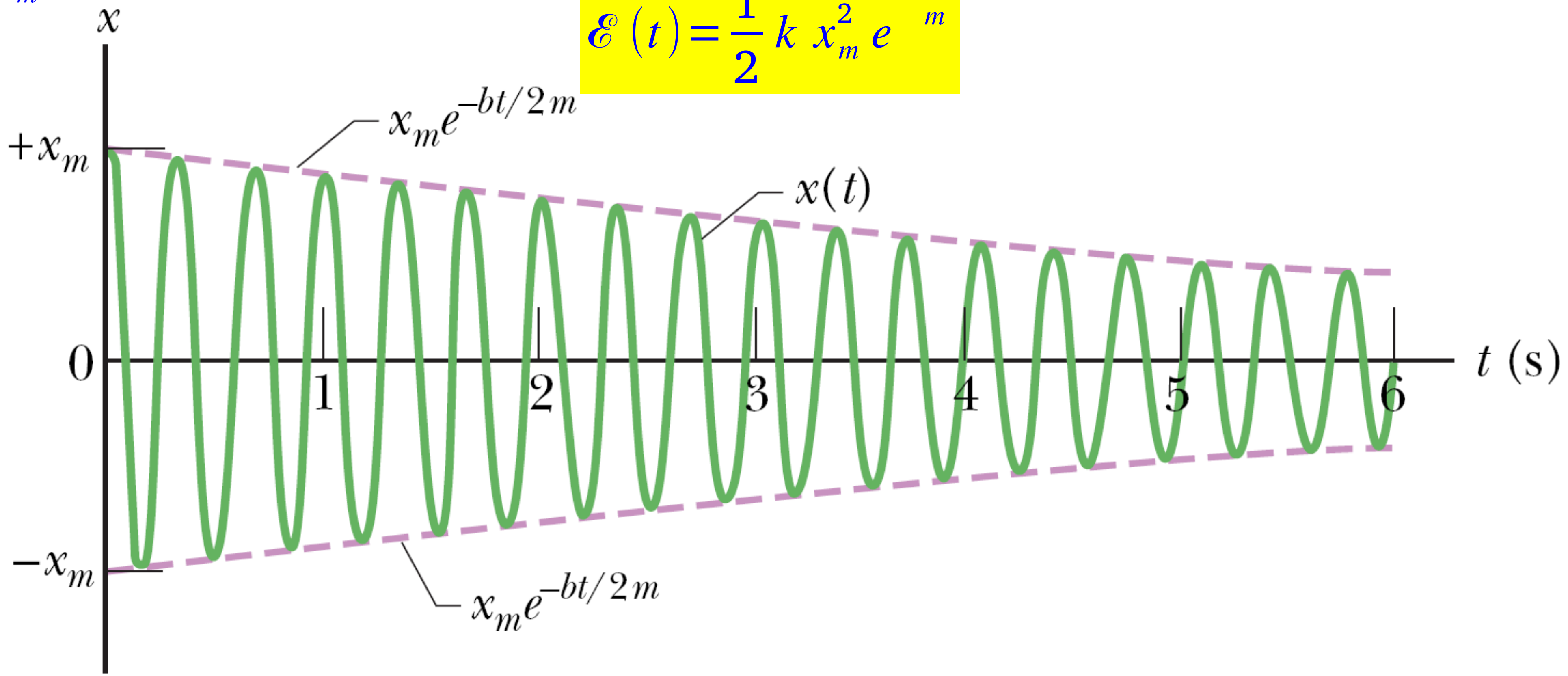
- If $b = 0 \quad \Rightarrow \quad \omega' = \omega = \sqrt{\frac{k}{m}}$

- If b is small, ie, $b \ll \sqrt{k\,m} \quad \Rightarrow \quad \omega' \approx \omega$

- We can regard the solution as a cosine function whose amplitude gradually decreases with time.

- If the oscillator is damped, the mechanical energy is not constant but decreases with time. If the damping is small, we can find $\mathscr{E}$ by replacing x_m with $x_m e^{-bt/2m}$, and

$$\mathscr{E}(t) = \frac{1}{2} k\, x_m^2\, e^{-\frac{bt}{m}}$$



problem 15-6

Forced Oscillations and Resonance

- If someone pushes a swing periodically, the swing has *forced*, or *driven*, *oscillations*.

- 2 angular frequencies are associated with a system undergoing driven oscillations:
(1) the natural angular frequency of the system;
(2) the angular frequency of the external driving force.

- A forced oscillator oscillates at the angular frequency ω_d of the driving force, and its displacement is given

$$x(t) = x_m \cos(\omega_d t + \phi)$$

- The velocity amplitude of the oscillations is greatest

when $\omega_d = \omega$ resonance , a condition called **resonance**.

This is also *approximately* the condition at which the displacement amplitude is greatest.

- The curves show that less damping gives a taller and narrower *resonance peak*.

The chosen problems: 6, 14, 48, 58.