# Chapter 13 Gravitation

## Newton's Law of Gravitation

- In 1665, Newton made a basic contribution to physics when he showed that the force that holds the Moon in its orbit is the same force that makes an apple fall.

- This tendency of bodies to move toward each other is called **gravitation**.

- **Newton's law of gravitation**: Every particle attracts any other particle with a **gravitational force** of magnitude

$$F = G\frac{m_1 m_2}{r^2} \quad \text{Newton's law of gravitaiton}$$

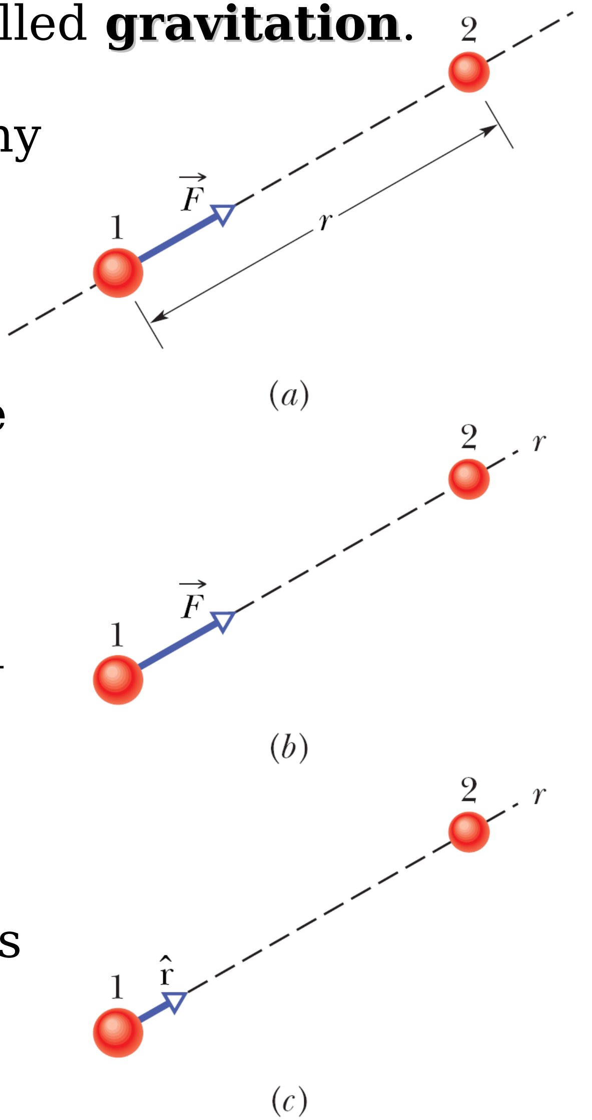
$m_1$ and $m_2$ are the masses of the particles, $r$ is the distance between them, and $G$ is the **gravitational constant**,

$$G = 6.67\times 10^{-11}\ \mathrm{N\cdot m^2/kg^2} = 6.67\times 10^{-11}\ \mathrm{m^3/kg\cdot s^2}$$

- We can describe the gravitational force by using a radial unit vector that is directed away from particle 1 along the

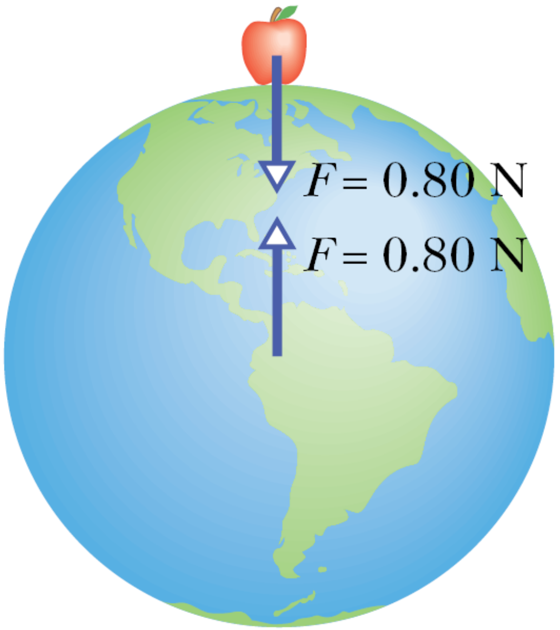
$r$ axis. The force on particle 1 is then $\vec{F} = G\dfrac{m_1 m_2}{r^2}\,\hat{\mathrm{r}}$

- The gravitational force on particle 2 due to particle 1 has the same magnitude as the force on particle 1 but the opposite direction.



- These 2 forces form a 3rd-law force pair.

- This force between 2 particles is not altered by other objects, even if they are located between the particles. No object can shield either particle from the gravitational force due to the other particle.




- The **shell theorem**:

> A uniform spherical shell of matter attracts a particle that is outside the shell as if all the shell's mass were concentrated at its center.

- Although the 3rd-law forces are matched in magnitude, they produce different accelerations on different masses.

# Gravitation and the Principle of Superposition

- **Principle of superposition**, a general principle that says a net effect is the sum of the individual effects.

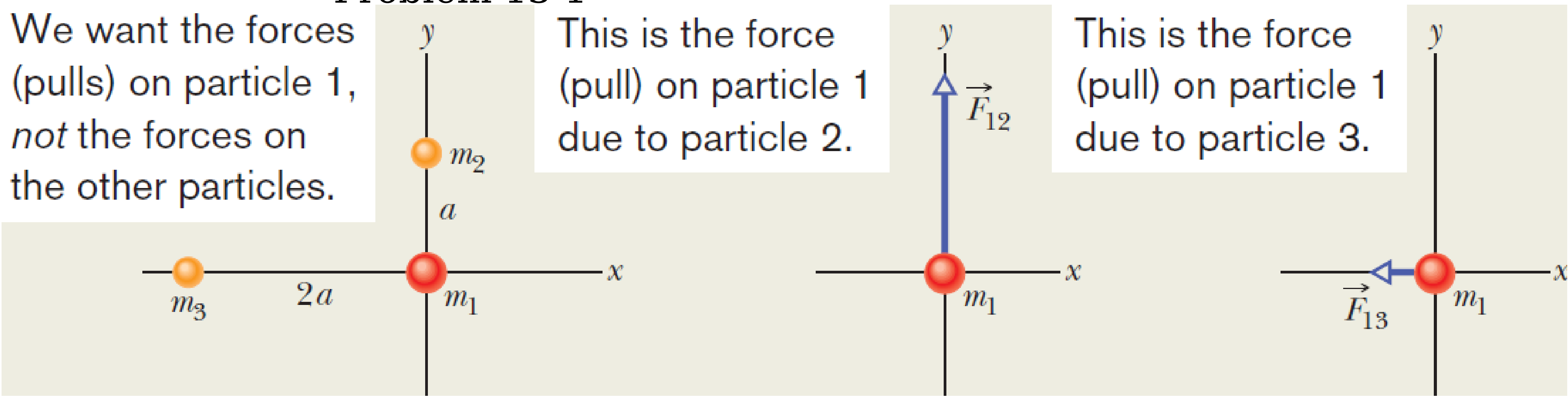
- We can find the net (or resultant) gravitational force on any one of a group of particles from the others by using the principle of superposition.

- For $n$ interacting particles, we can write the principle of superposition for the gravitational forces on particle 1 as $\vec{F}_{1,\text{net}} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \cdots + \vec{F}_{1n}$

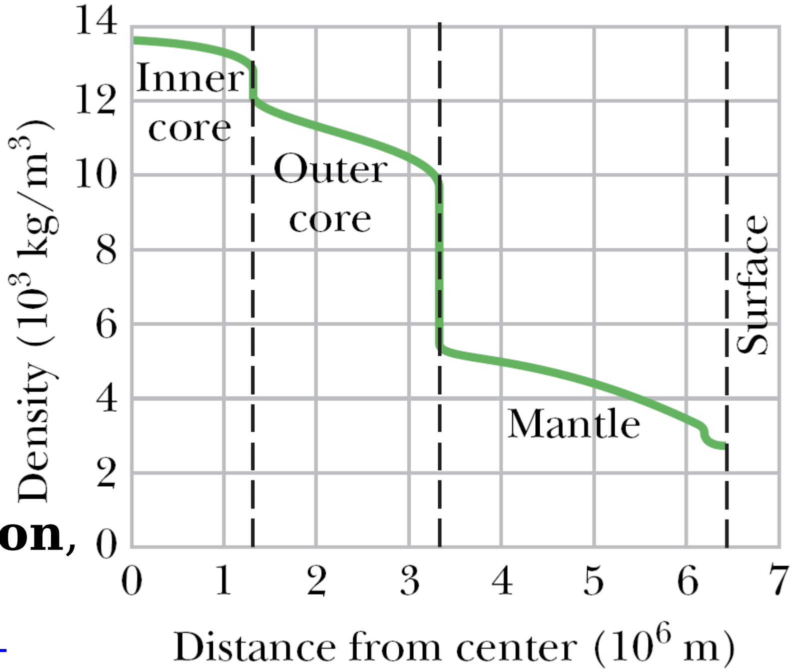
- We can express this equation as a vector sum: $$\vec{F}_{1,\text{net}} = \sum_{i=2}^{n} \vec{F}_{1i}$$

- In the limiting case, we can divide an extended object into differential parts and each producing a differential force on particle 1, $$\vec{F}_1 = \int d\vec{F}$$

Problem 13-1



## Gravitation Near Earth‘s Surface




- Assume that Earth is a uniform sphere of mass $M$. The magnitude of the gravitational force from Earth on a particle of mass $m$, located outside Earth a distance $r$ from Earth's center, $F = G\dfrac{Mm}{r^2}$

- The force will cause a **gravitational acceleration**, and Newton's 2nd law tell us that their relation is $F = m a_g \quad \Rightarrow \quad a_g = \dfrac{GM}{r^2}$

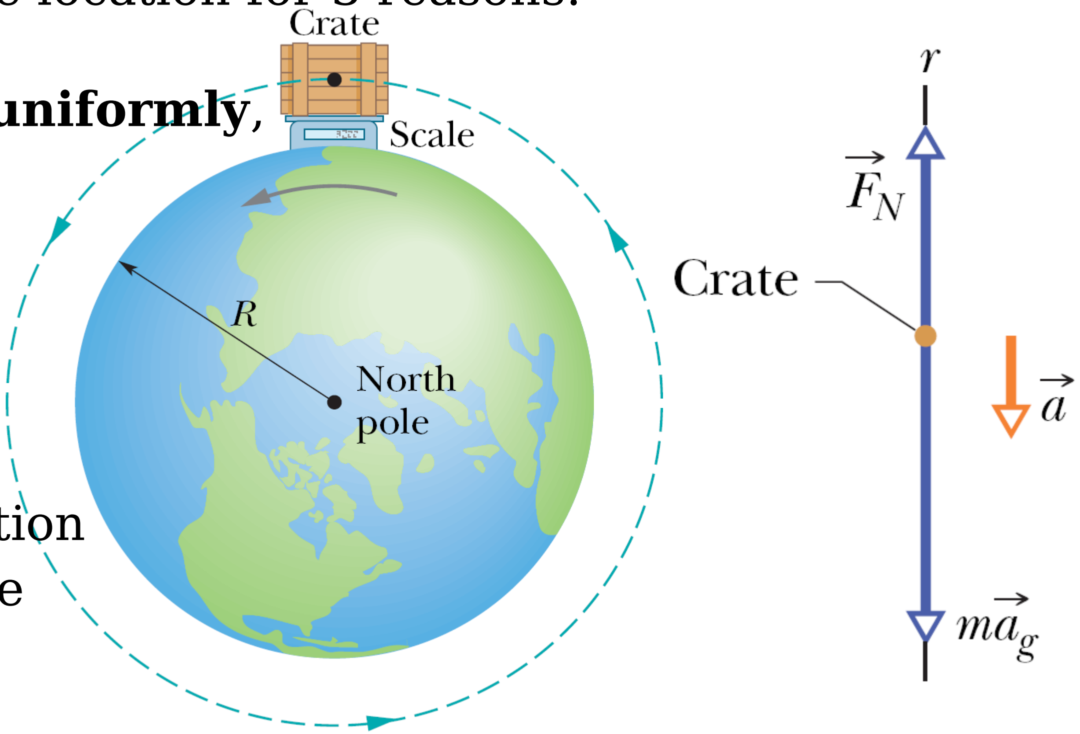
- Any free-fall acceleration $g$ value measured at a given location will differ from the gravitational acceleration $a_g$ for the location for 3 reasons:

(1) **Earth's mass is not distributed uniformly**,

(2) **Earth is not a perfect sphere**,

(3) **Earth rotates**.

- To see how Earth's rotation causes $g$ to differ from $a_g$, let us analyze a situation in which a crate of mass $m$ is on a scale at the equator.



- Because the object travels in a circle about the center of Earth as Earth turns, the crate has a centripetal acceleration directed toward Earth $\vec{a}_r = -R\,\omega^2\,\hat{\mathrm{r}}$

- Newton's 2nd law for forces along the $r$ axis

$$F_{\mathrm{net},\,r} = m\,a_r \quad \Rightarrow \quad F_N - m\,a_g = m\left(-R\,\omega^2\right)$$

- The magnitude $F_N$ of the normal force is equal to the weight $mg$ read on the scale

$$\underset{\Rightarrow\left(\text{measured weight}\right)}{m\,g} = \underset{\left(\text{magnitude of gravitational force}\right)}{m\,a_g} - \underset{\left(\text{mass times centripetal acceleration}\right)}{m\,R\,\omega^2}$$

- Because of Earth's rotation, the measured weight is less than the magnitude of the gravitational force on the crate.

- For a further simplification, $g = a_g - R\,\omega^2$. It reads

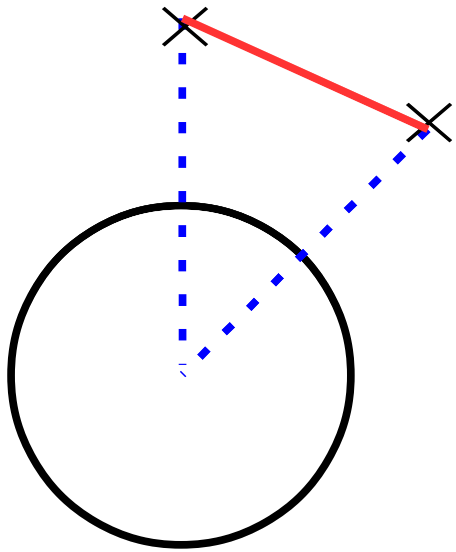
$$\left(\begin{array}{c}\text{free-fall}\\ \text{acceleration}\end{array}\right) = \left(\begin{array}{c}\text{gravitational}\\ \text{acceleration}\end{array}\right) - \left(\begin{array}{c}\text{centripetal}\\ \text{acceleration}\end{array}\right)$$

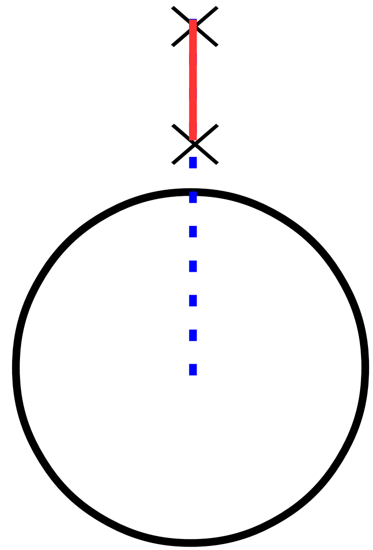
- The measured free-fall acceleration is less than the gravitational acceleration because of Earth's rotation.

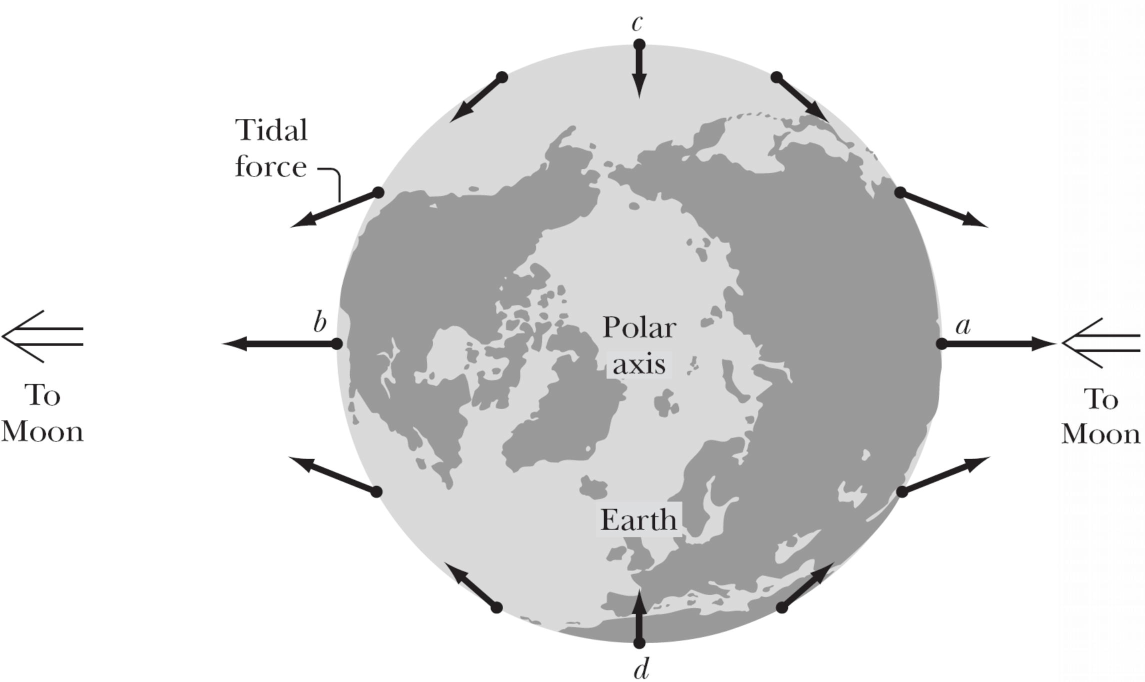
- The greatest difference between accelerations happens on the equator.

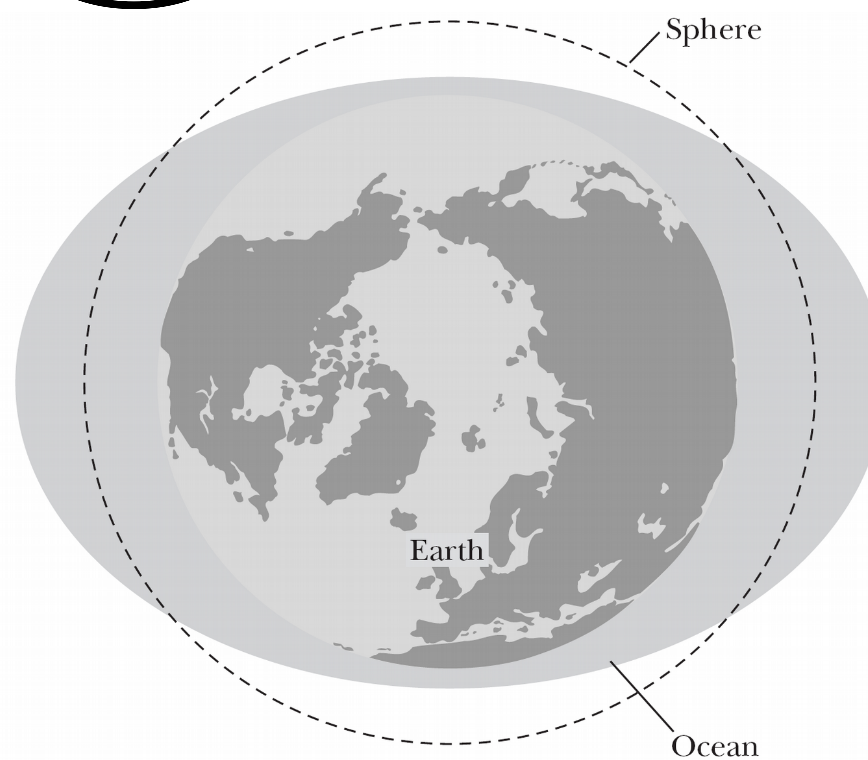
problem 13-2

# Gravity and Tidal Effect







## Gravitation Inside Earth

- Newton's shell theorem can also be applied to a situation in which a particle is *inside* a uniform shell,

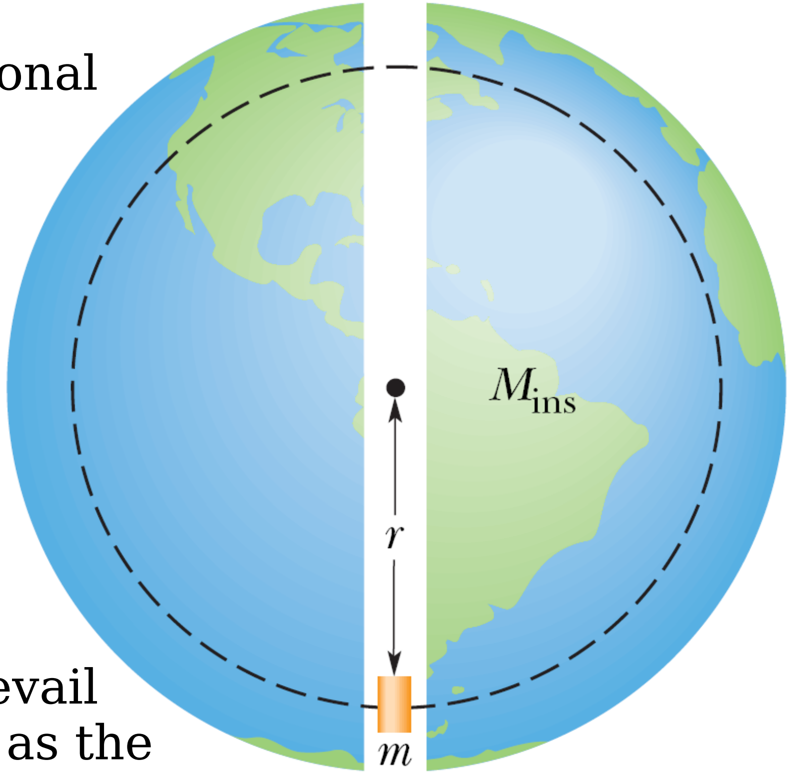
A uniform shell of matter exerts no net gravitational force on a particle located inside it.

- The sum of the force vectors on the particle from all the elements of the shell is 0.

- If a particle move inside the earth, the gravitational force would change for 2 reasons:

(1) It would tend to increase because the particle would be moving closer to the center of Earth;

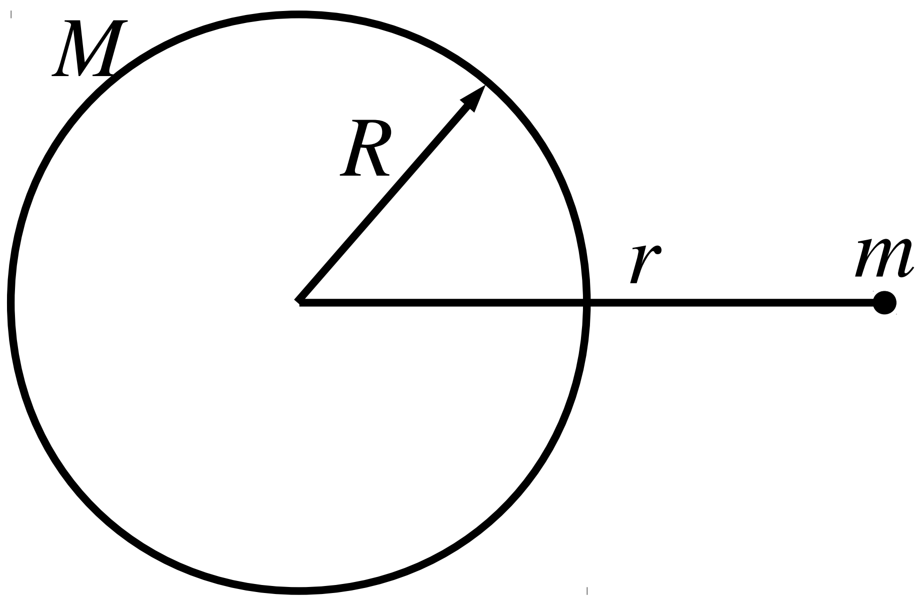
(2) It would tend to decrease because the thickening shell of material lying outside the particle s radial position would not exert any net force on the particle.




- For a uniform Earth, the 2nd influence would prevail and the force on the particle would decrease to 0 as the particle approached the center of Earth.

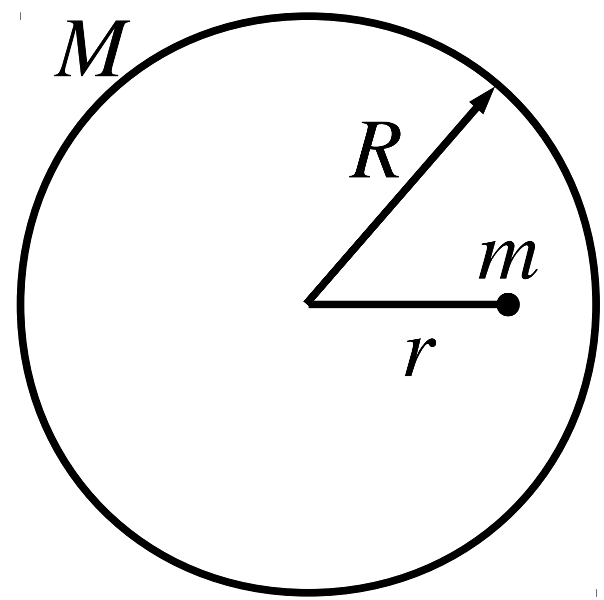
# Shell Theorem

Outside



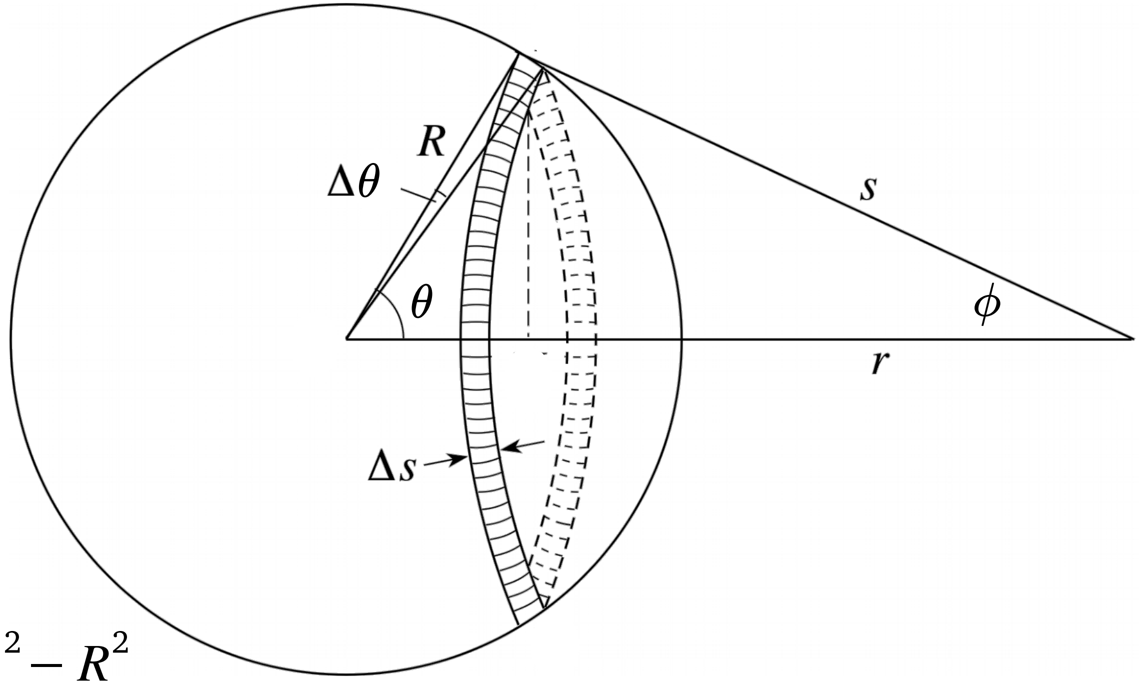

$$F = \frac{G\ M\ m}{r^2}$$

Inside




$$F = 0$$

# Proof of the shell theorem




shell's mass: $M = 4\pi R^2 \sigma \;\Rightarrow\; \sigma = \dfrac{M}{4\pi R^2}$

ring's mass $dM = \sigma \times$ area of ring

$$= \sigma \times 2\pi \times R\sin\theta \times R\,d\theta = \frac{M}{2}\sin\theta\,d\theta$$

law of cosine: $r^2 + R^2 - 2Rr\cos\theta = s^2$

$$\Rightarrow \quad Rr\sin\theta\,d\theta = s\,ds$$

$$s^2 + r^2 - 2sr\cos\phi = R^2 \quad \Rightarrow \quad \cos\phi = \frac{s^2 + r^2 - R^2}{2rs}$$

$$dF_r = dF\cos\phi = \frac{G\,m\,dM}{s^2}\cos\phi \quad \Leftarrow \quad dF = \text{force from the ring} = \frac{G\,m\,dM}{s^2} \quad + \quad \text{symmetry}$$

$$\Rightarrow F_r = \int dF_r = G\,m\int \frac{\cos\phi\,dM}{s^2}$$

$$= \frac{G\,M\,m}{2}\int \frac{\sin\theta\,d\theta}{s^2}\,\frac{s^2 + r^2 - R^2}{2rs} = \frac{G\,M\,m}{4r^2R}\int \frac{s^2 + r^2 - R^2}{s^2}\,ds$$

$$= \begin{cases} \dfrac{G\,M\,m}{4r^2R}\displaystyle\int_{r-R}^{r+R}\left(1 + \frac{r^2 - R^2}{s^2}\right)ds = \frac{G\,M\,m}{r^2} & \text{for } r > R \quad \text{outside} \\ \dfrac{G\,M\,m}{4r^2R}\displaystyle\int_{R-r}^{r+R}\left(1 + \frac{r^2 - R^2}{s^2}\right)ds = 0 & \text{for } r < R \quad \text{inside} \end{cases}$$

- To find an expression for the gravitational force inside a uniform Earth, let's use the plot in *Pole to Pole*.

- Attempt to travel by capsule through a naturally formed (and, of course, fictional) tunnel directly from the south pole to the north pole.

- The *net* gravitational force on the capsule is due to the mass $M_{ins}$ inside the sphere with radius $r$ (the mass enclosed by the dashed outline), not the mass in the outer spherical shell (outside the dashed outline).

- The magnitude of the gravitational force on the capsule: $F = \dfrac{G\,m\,M_{ins}}{r^2}$

- Assume a uniform density

$$\text{density} = \frac{\text{inside mass}}{\text{inside volume}} = \frac{\text{total mass}}{\text{total volume}} \quad \Rightarrow \quad \rho = \frac{M_{ins}}{4\,\pi\,r^3/3} = \frac{M}{4\,\pi\,R^3/3}$$

$$\Rightarrow \quad M_{ins} = \frac{4}{3}\,\pi\,r^3\,\rho = \frac{M}{R^3}\,r^3 \quad \Rightarrow \quad F = \frac{G\,m\,M}{R^3}\,r$$

The force magnitude decreases linearly as the capsule approaches the center, until it is 0 at the center.

- The force in vector form: $\vec{F} = -\dfrac{G\,m\,M}{R^3}\,\vec{r}$

- The capsule would oscillate like a block on a spring, with the center of the oscillation at Earth's center.

## Gravitational Potential Energy

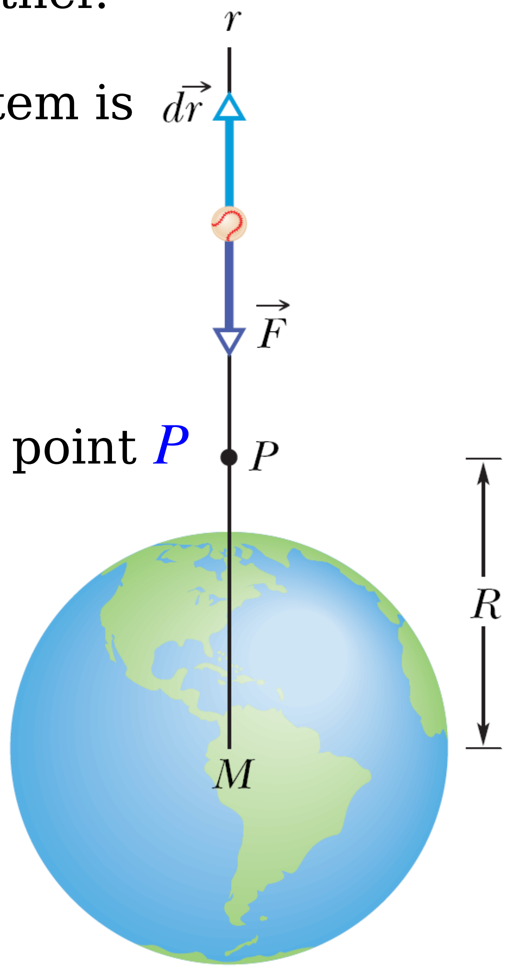
- Consider the gravitational potential energy $U$ of 2 particles, of masses $m$ and $M$, separated by a distance $r$.

- Choose $r \to \infty$ as the reference configuration with $U$ equal to 0, $U(r \to \infty) = 0$

- The potential energy is negative for any finite separation and becomes progressively more negative as the particles move closer together.

- Then the gravitational potential energy of the 2-particle system is

$$U = -\frac{G\,M\,m}{r} \quad \text{gravitational potential energy}$$




**Proof of the above equation**

- The work $W$ done on the ball by the gravitational force from point $P$ to infinity is $W = \int_R^{\infty} \vec{F} \cdot \mathrm{d}\,\vec{r}$

- Since $\vec{F} \cdot \mathrm{d}\,\vec{r} = F(r)\,\mathrm{d}\,r \cos\phi \;\; \& \;\; \phi = \pi$ in this case

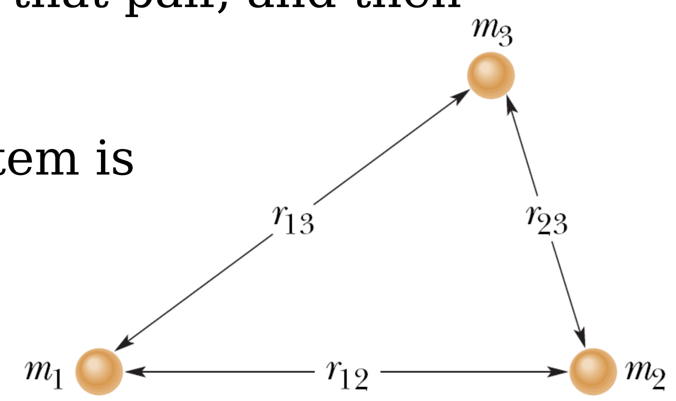
$$\Rightarrow \quad \vec{F} \cdot \mathrm{d}\,\vec{r} = -\frac{G\,M\,m}{r^2}\,\mathrm{d}\,r$$

where $M$ is Earth's mass and $m$ is the mass of the particle.

- Thus $W = -GMm\int_R^{\infty}\frac{dr}{r^2} = \left.\frac{GMm}{r}\right|_R^{\infty} = 0 - \frac{GMm}{R} = -\frac{GMm}{R}$

- Since $\Delta U = -W \Rightarrow U_\infty - U = -W \Rightarrow U = W = -\frac{GMm}{r}$

- If the system contains more than 2 particles, we consider each pair of particles in turn, calculate the gravitational potential energy of that pair, and then algebraically sum the results (ie, **superposition**).




- The gravitational potential energy of the 3-body system is

$$U = -\left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right)$$

## Path Independence

- The gravitational force is a conservative force. Thus, the work done by the gravitational force on a particle moving from an initial point $i$ to a final point $f$ is independent of the path taken between the points.

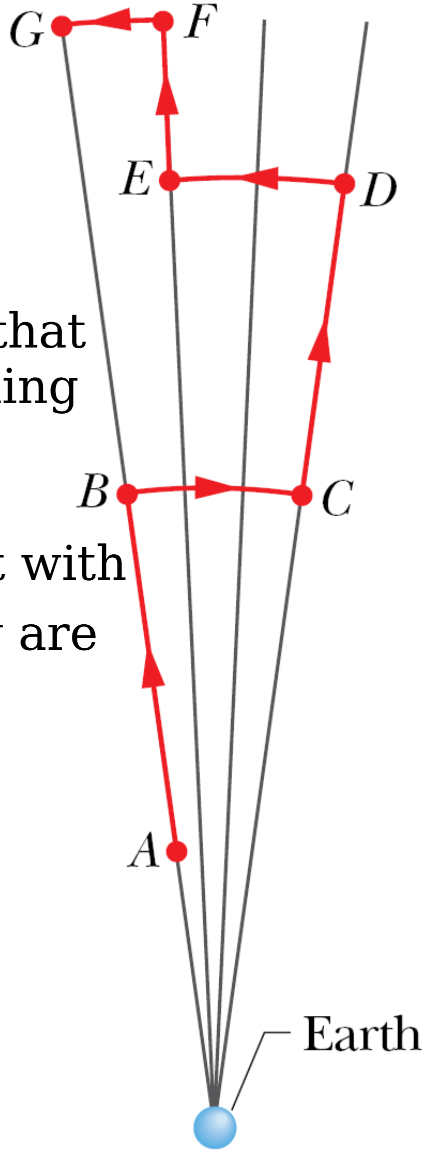
- Since $\Delta U = U_f - U_i = -W$, thus the change in gravitational potential energy is also independent of the path taken.

## Potential Energy and Force

- We can start from the potential energy function and derive the force function,

$$F = -\frac{dU}{dr} = -\frac{d}{dr}\left(-\frac{GMm}{r}\right) = -\frac{GMm}{r^2}$$

- The minus sign indicates that the force on mass $m$ points radially inward, toward mass $M$.




## Escape Speed

- The (Earth) **escape speed**: the certain minimum initial speed that will cause a projectile to move upward forever, theoretically coming to rest only at infinity.

- Consider a projectile of mass $m$, leaving the surface of a planet with escape speed $v$, then the kinetic energy and the potential energy are

$$K = \frac{1}{2} m v^2, \quad U = -\frac{G M m}{R}$$

in which $M$ is the mass of the planet and $R$ is its radius.

- When the projectile reaches infinity, $K = 0$ & $U = 0$, thus

$$K + U = \frac{1}{2} m v^2 + \left( -\frac{G M m}{R} \right) = 0 \quad \Rightarrow \quad v = \sqrt{\frac{2 G M}{R}}$$

- Note that escape speed does not depend on the direction in which a projectile is fired from a planet. However, attaining that speed is easier if the projectile is fired in the direction the site is moving as the planet rotates about its axis.

- Earth's escape speed is 11.2 km/s.

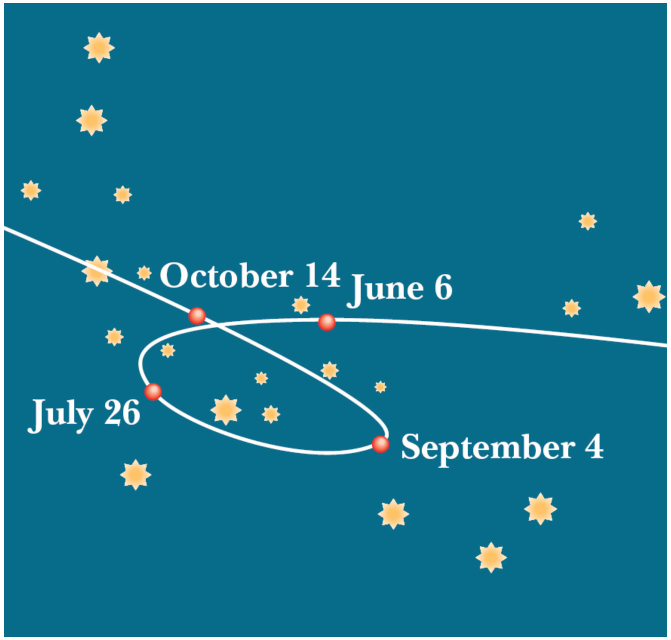
problem 13-3

## Planets and Satellites: Kepler's Laws

- Tycho Brahe (1546-1601) make observations to compile the data;

- Johannes Kepler (1571-1630) derived the famous 3 laws of planetary motion from the data;

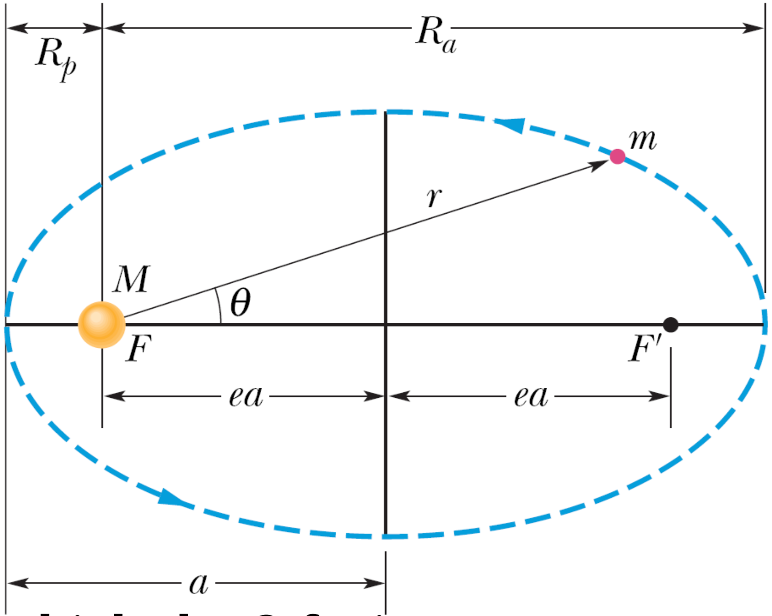
- Newton (1642-1727) showed that his law of gravitation leads to Kepler's laws.




**1. THE LAW OF ORBITS**: All planets move in elliptical orbits, with the Sun at one focus.

- We assume that the mass of the Sun is much greater than the mass of the planet, so that the center of mass of the planet-Sun system is approximately at the center of the Sun.

- The orbit is described by giving its **semimajor axis** a and its **eccentricity** $e$, $e$ defined so that $ea$ is the distance from the center of the ellipse to either focus.



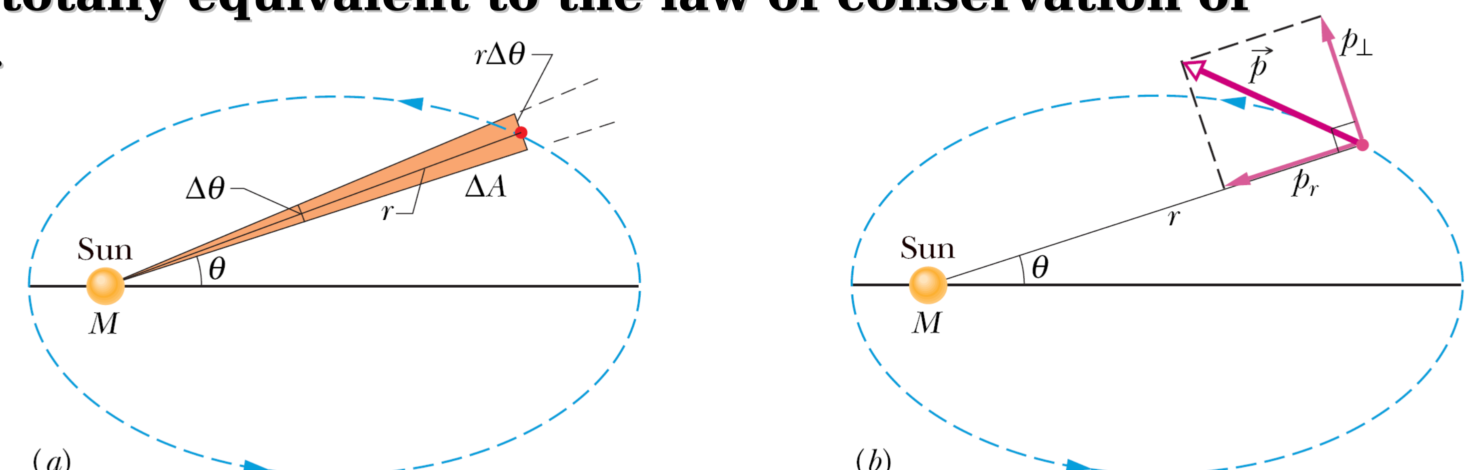

- *An eccentricity of zero corresponds to a circle*, in which the 2 foci merge to a single central point.

- The eccentricity of Earth's orbit is only 0.0167.

**2. THE LAW OF AREAS**: A line that connects a planet to the Sun sweeps out equal areas in the plane of the planet's orbit in equal time intervals; that is, the rate $dA/dt$ at which it sweeps out area $A$ is constant.

- The planet will move most slowly when it is farthest from the Sun and most rapidly when it is nearest to the Sun.

- **Kepler's 2nd law is totally equivalent to the law of conservation of angular momentum**.




**Proof**

- The area

$$\Delta A \approx \frac{1}{2}(r)(r\,\Delta\theta)$$

$$= \frac{1}{2} r^2 \Delta\theta$$

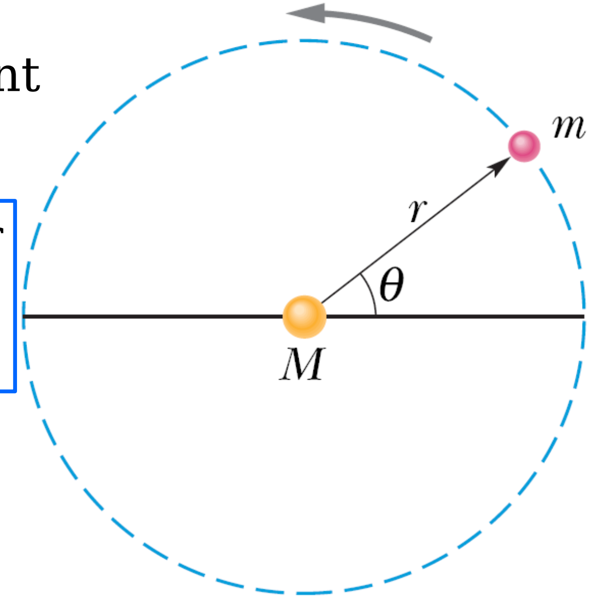
- The instantaneous rate at which area is being swept out $\frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt} = \frac{1}{2} r^2 \omega$

in which $\omega$ is the angular speed of the rotating line connecting Sun and planet.

- The magnitude of the angular momentum of the planet about the Sun is

$$L = r\,p_\perp = (r)(m\,v_\perp) = (r)(m\,r\,\omega) = m\,r^2\omega \quad \Rightarrow \quad \frac{dA}{dt} = \frac{L}{2m}$$

- $\frac{dA}{dt}$ being constant means that $L$ must also be constant — angular momentum is conserved.




**3. THE LAW OF PERIODS**: The square of the period of any planet is proportional to the cube of the semimajor axis of its orbit.

- Applying Newton's 2nd law to the orbiting planet yields

$$\frac{GMm}{r^2} = m\frac{v^2}{r} = (m)(r\omega^2) \Leftarrow \quad v = r\omega$$

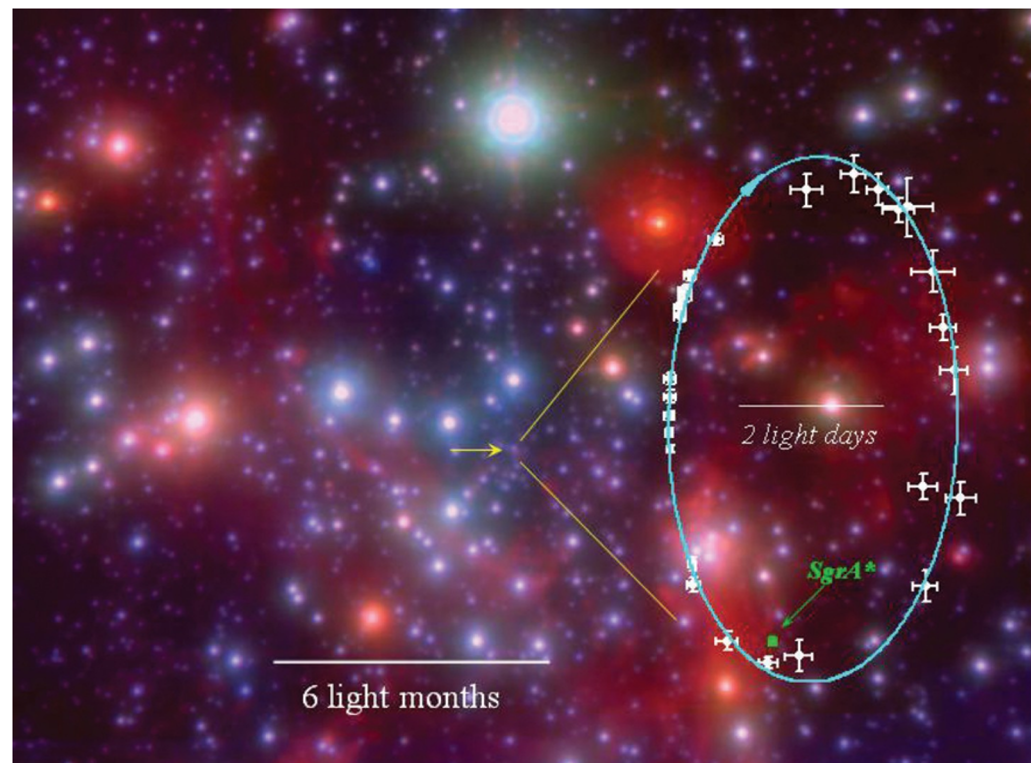
- Since $\omega = 2\pi/T$, where $T$ is the period of motion thus,

$$T^2 = \frac{4\pi^2}{GM} r^3 \quad \text{law of periods}$$

- The equation holds also for elliptical orbits, provided we replace $r$ with $a$, the semimajor axis of the ellipse.

- This law predicts that the ratio $\frac{T^2}{a^3}$ has essentially the same value for every planetary orbit around a given massive body.

problem 13-4

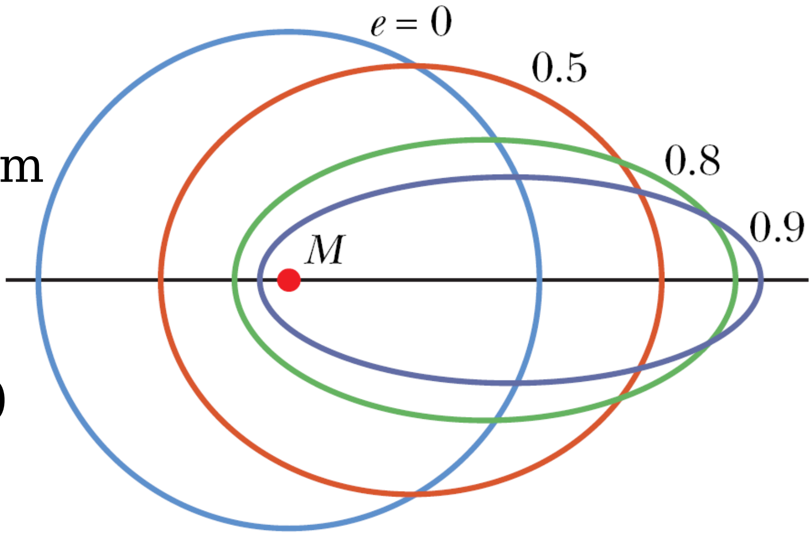


$$T^2 = \frac{4\pi^2}{G\,M}\,r^3$$

$$\Rightarrow \quad \frac{4\pi^2}{\omega^2} = \frac{4\pi^2}{G\,M}\,r^3 \quad \Leftarrow \quad T = \frac{1}{f} = \frac{2\pi}{\omega}$$

$$\Rightarrow \quad G\,M^1 = \omega^2\,r^3 \quad \Leftarrow \quad \text{Kepler's 1-2-3 law}$$

# Satellites: Orbits and Energy




- The potential energy of the Earth-satellite system is

$$U = -\frac{G\,M\,m}{r}$$

- To find the kinetic energy of a satellite in a circular orbit, we write Newton's 2nd law ($F = ma$) as $\frac{G\,M\,m}{r^2} = m\,\frac{v^2}{r}$

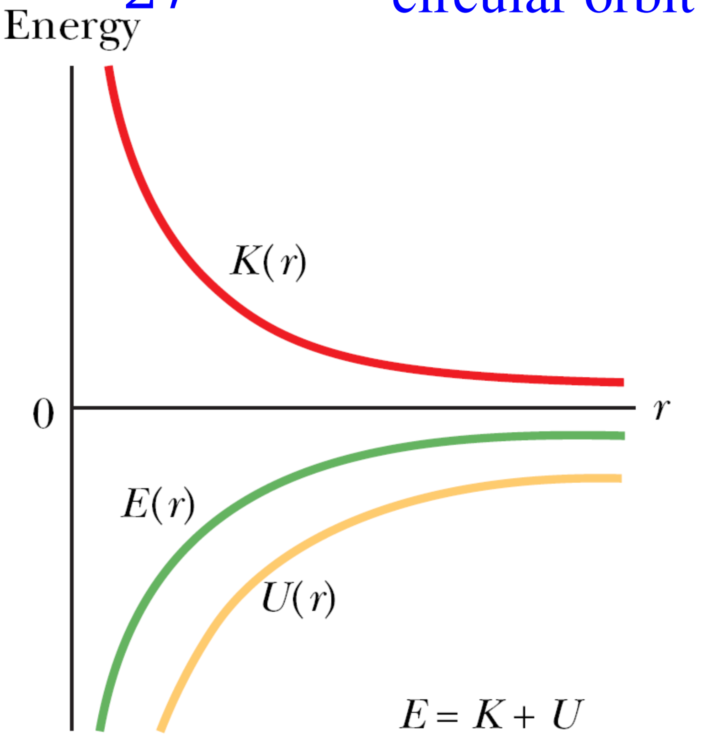
- The kinetic energy is $K = \frac{1}{2}\,m\,v^2 = \frac{G\,M\,m}{2\,r} = -\frac{U}{2}$ circular orbit

- The total energy $\mathscr{E} = K + U = \frac{G\,M\,m}{2\,r} - \frac{G\,M\,m}{r} = -\frac{G\,M\,m}{2\,r}$ $\Rightarrow$ $\mathscr{E} = -K$ circular orbit

- For a satellite in an elliptical orbit of semimajor axis $a$, we can substitute $a$ for $r$ to find the mechanical energy: $\mathscr{E} = -\frac{G\,M\,m}{2\,a}$ elliptical orbit

- The total energy of an orbiting satellite depends only on the semimajor axis of its orbit and not on its eccentricity.

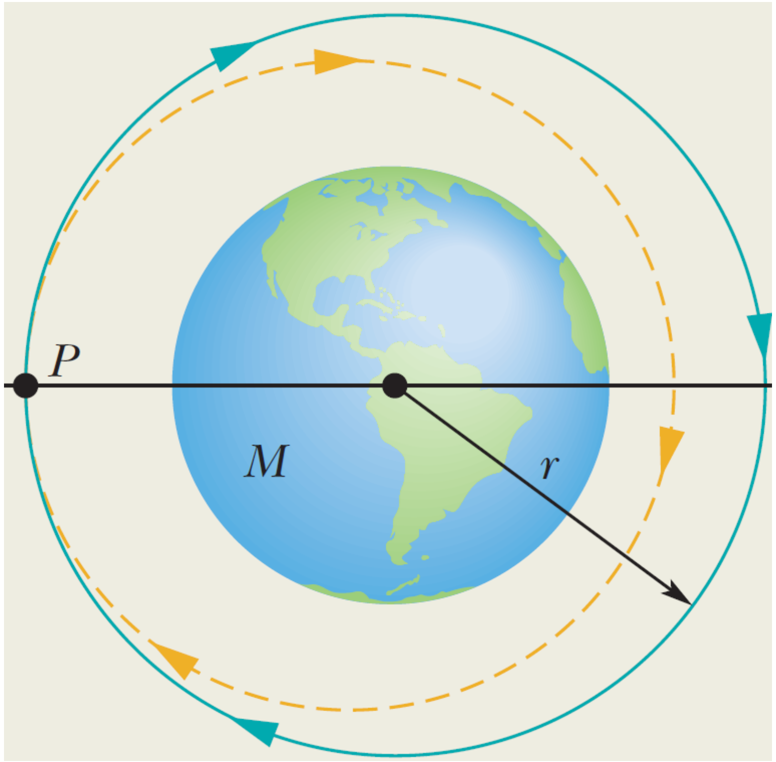
- The same satellite would have the same total mechanical energy $\mathscr{E}$ in all 4 orbits.



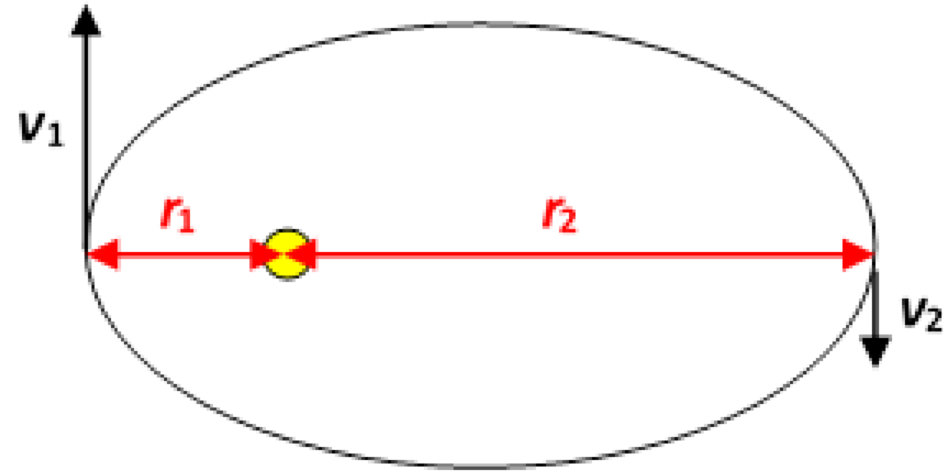
- The variation of $K$, $U$, and $\mathscr{E}$ with $r$ for a satellite moving in a circular orbit about a massive central body.

Problem 13-5

problem 13-6



**Proof:** $\mathcal{E} = \frac{m}{2} v_1^2 - \frac{G\,M\,m}{r_1} = \frac{m}{2} v_2^2 - \frac{G\,M\,m}{r_2}$




$$\Rightarrow \quad \frac{m}{2} v_1^2 - \frac{G\,M\,m}{r_1} - \left( \frac{m}{2} v_2^2 - \frac{G\,M\,m}{r_2} \right) = 0$$

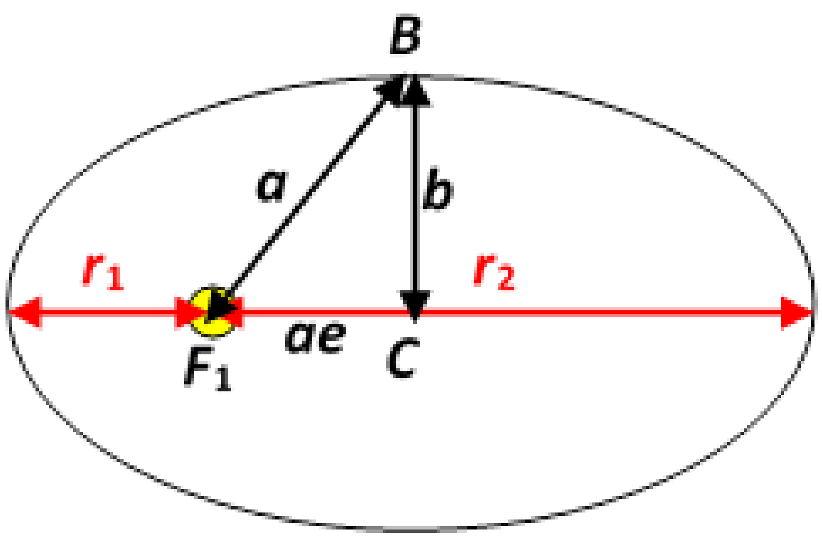
$$\Rightarrow \quad \frac{v_1^2 - v_2^2}{2} = G\,M \left( \frac{1}{r_1} - \frac{1}{r_2} \right)$$

$$L = m\,v_1\,r_1 = m\,v_2\,r_2 \quad \Rightarrow \quad v_2 = \frac{r_1}{r_2} v_1 \quad \Rightarrow \quad \frac{v_1^2}{2} \left( 1 - \frac{r_1^2}{r_2^2} \right) = \frac{G\,M}{r_1\,r_2} (r_2 - r_1)$$

$$\Rightarrow \quad \frac{m}{2} v_1^2 = \frac{G\,M\,m}{r_1} \frac{r_2}{r_1 + r_2} \quad \Rightarrow \quad \mathcal{E} = \frac{G\,M\,m}{r_1} \left( \frac{r_2}{r_1 + r_2} - 1 \right) = -\frac{G\,M\,m}{2\,a} \quad \Leftarrow \quad r_1 + r_2 = 2\,a$$

$$v_1 = \frac{L}{m\,r_1}, \quad v_2 = \frac{L}{m\,r_2} \quad \Rightarrow \quad \frac{L^2}{2\,m^2} \left( \frac{1}{r_1^2} - \frac{1}{r_2^2} \right) = G\,M \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \quad \Rightarrow \quad \frac{L^2}{2\,m^2} = \frac{G\,M}{1/r_1 + 1/r_2}$$

$$A = \pi\,a\,b\,, \quad \frac{\mathrm{d}\,A}{\mathrm{d}\,t} = \frac{L}{2\,m} = \text{const}$$




$$\Rightarrow \quad T^2 = \left( \frac{A}{\mathrm{d}\,A/\mathrm{d}\,t} \right)^2 = \frac{(\pi\,a\,b)^2}{L^2/4\,m^2}$$

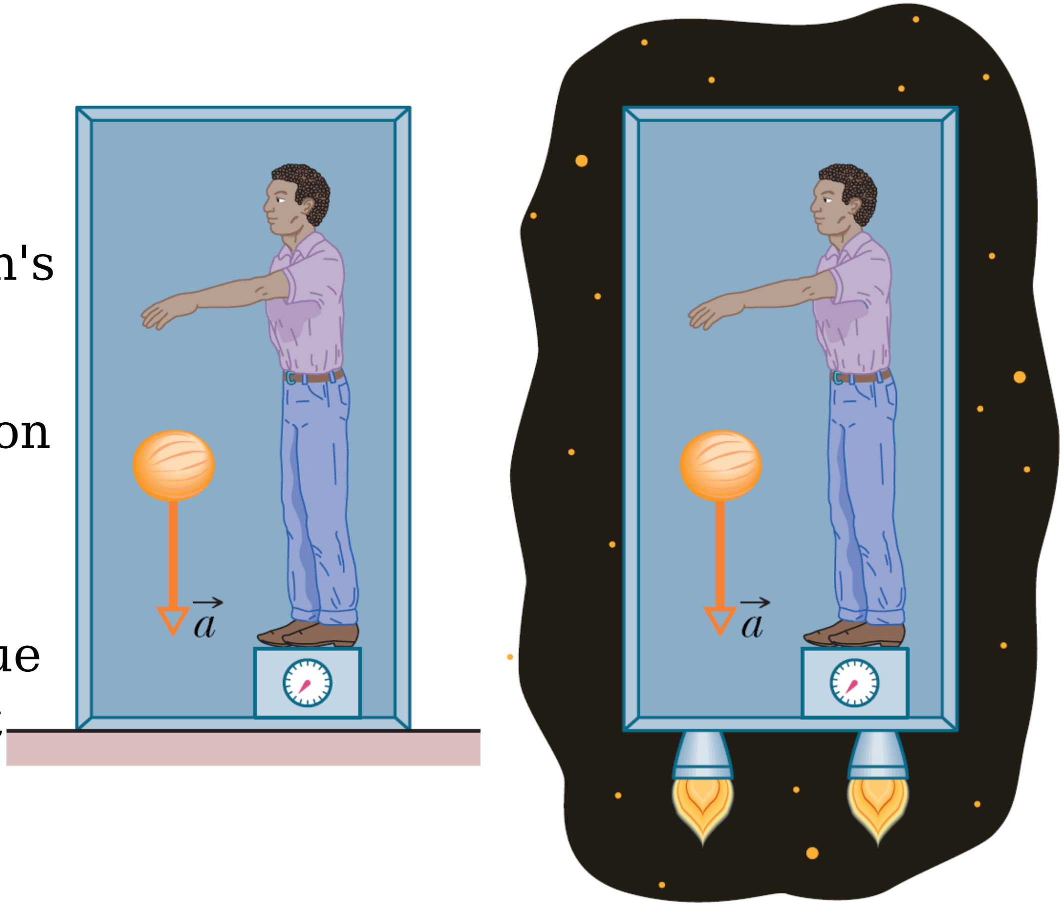
$$= \frac{2\,(\pi\,a\,b)^2}{G\,M} \left( \frac{1}{r_1} + \frac{1}{r_2} \right) = \frac{2\,(\pi\,a\,b)^2}{G\,M} \frac{2\,a}{r_1\,r_2}$$

$$= \frac{4\,\pi^2\,a^3}{G\,M} \quad \Leftarrow \quad r_1\,r_2 = b^2$$

# Einstein and Gravitation

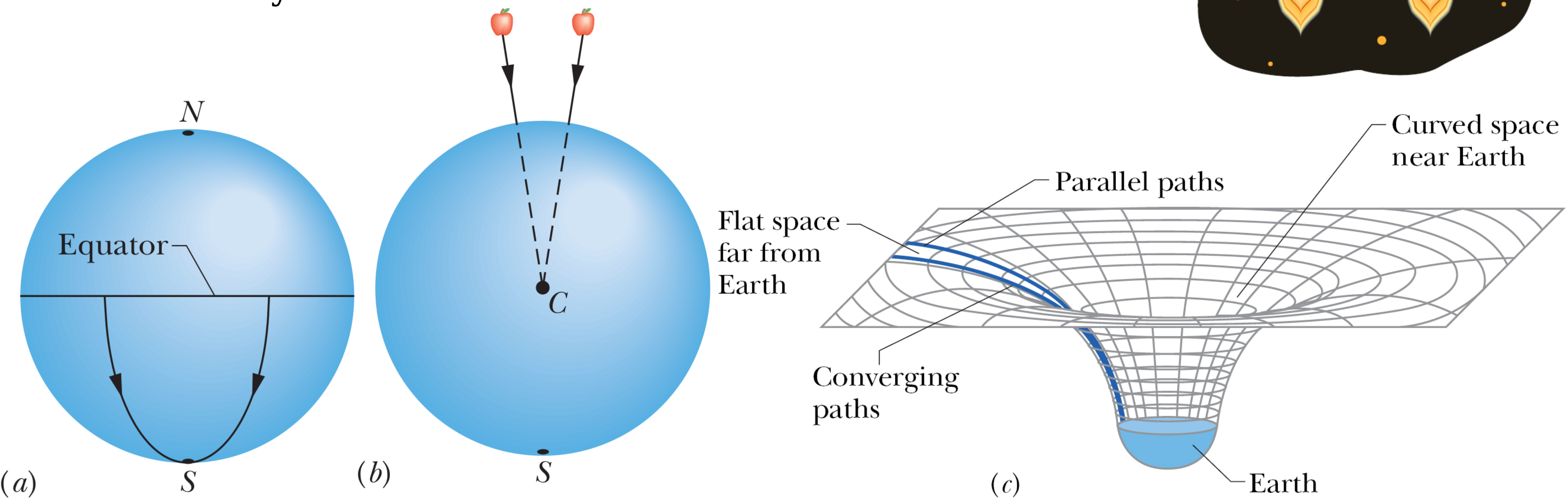
## Principle of Equivalence

- The fundamental postulate of Einstein's **general theory of relativity** about gravitation is called the **principle of equivalence**, which says that gravitation and acceleration are equivalent.

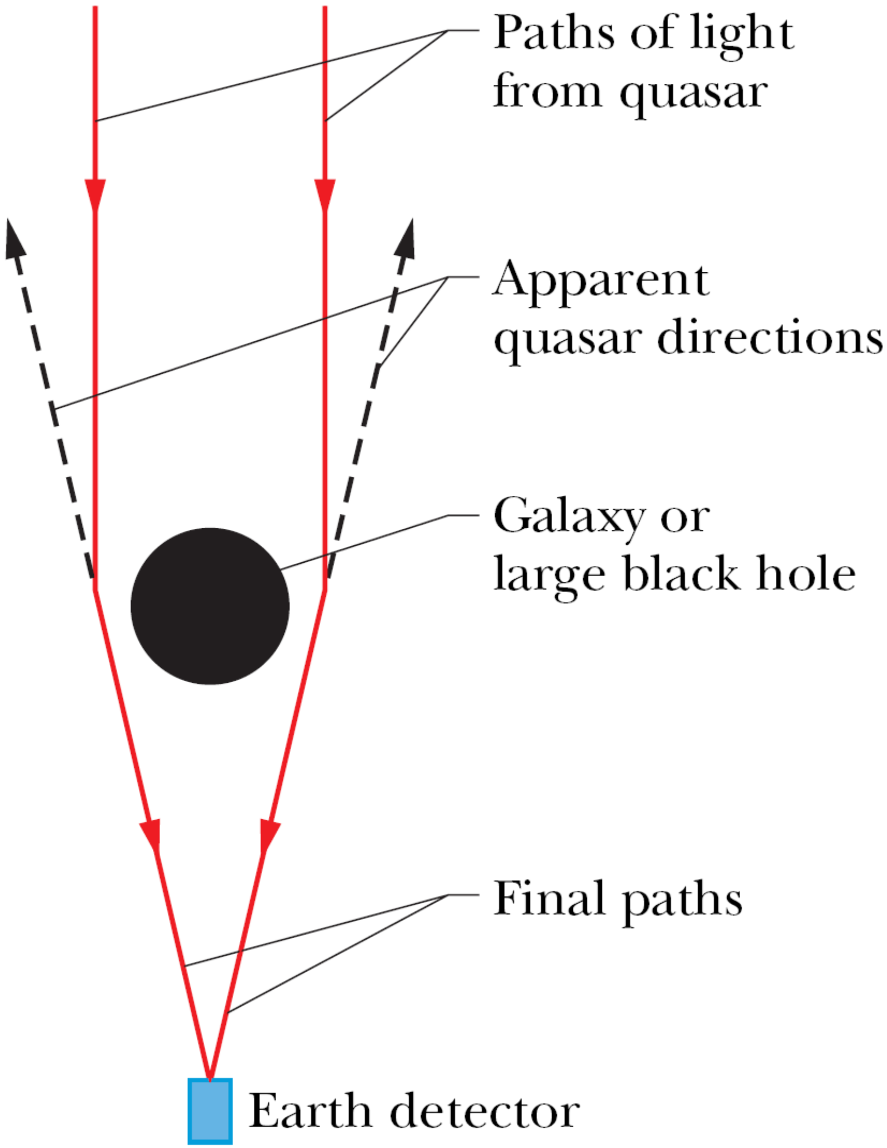


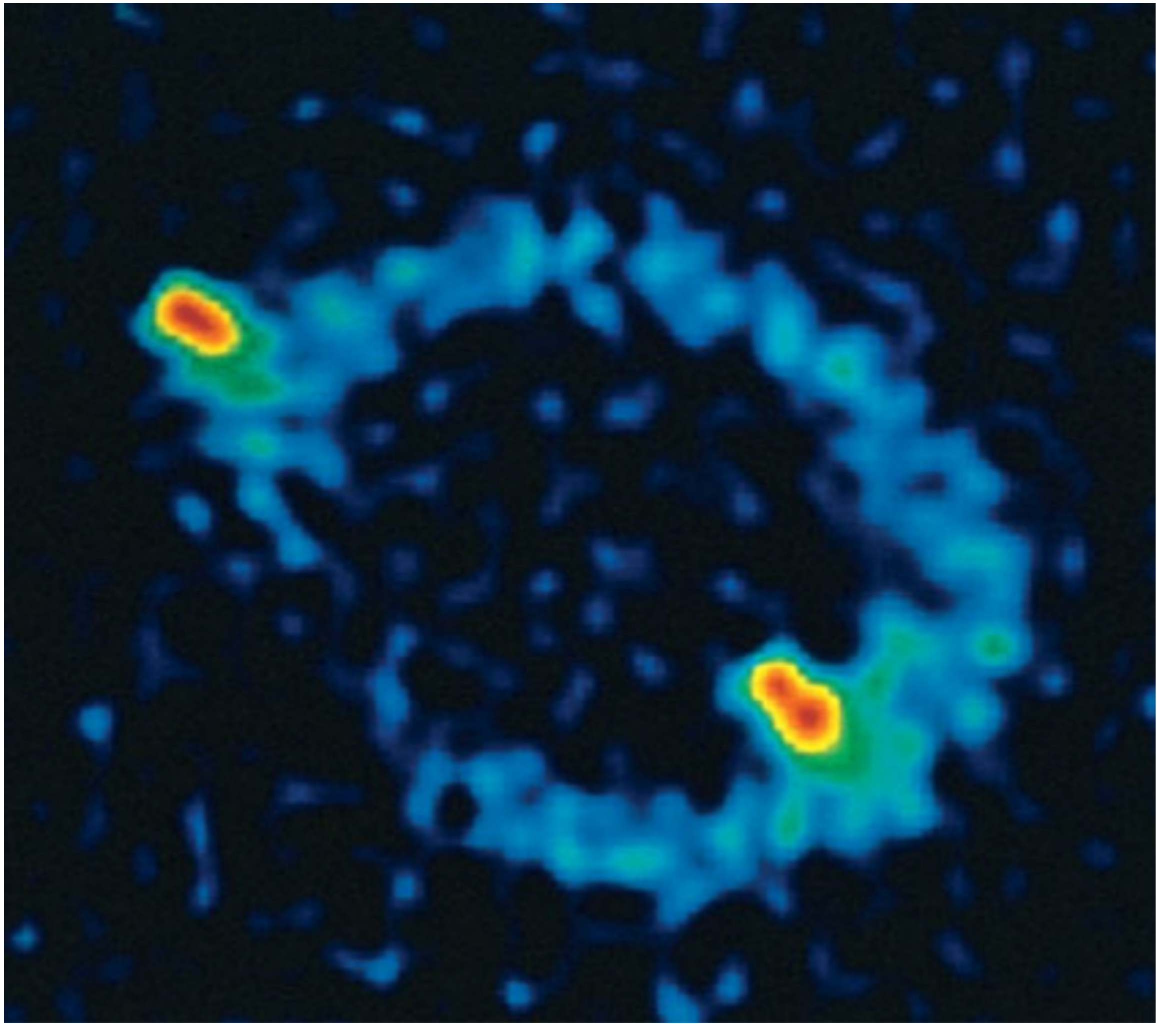

## Curvature of Space

- Einstein showed that gravitation is due to a curvature of space (*spacetime*) that is caused by the masses.



- When light passes near a massive object, the path of the light bends because of the curvature of space there, an effect called *gravitational lensing*.



The chosen problems: 6, 14, 42, 52.