Chapter 12 Equilibrium and Elasticity

Equilibrium

- We call an object is in equilibrium if
1. The linear momentum of its center of mass is constant;
2. Its angular momentum about its center of mass, or about any other point, is also constant; i.e.,

$$\vec{P} = \text{constant} \quad \text{and} \quad \vec{L} = \text{constant}$$

- An object is in **static equilibrium** if the constant is 0.

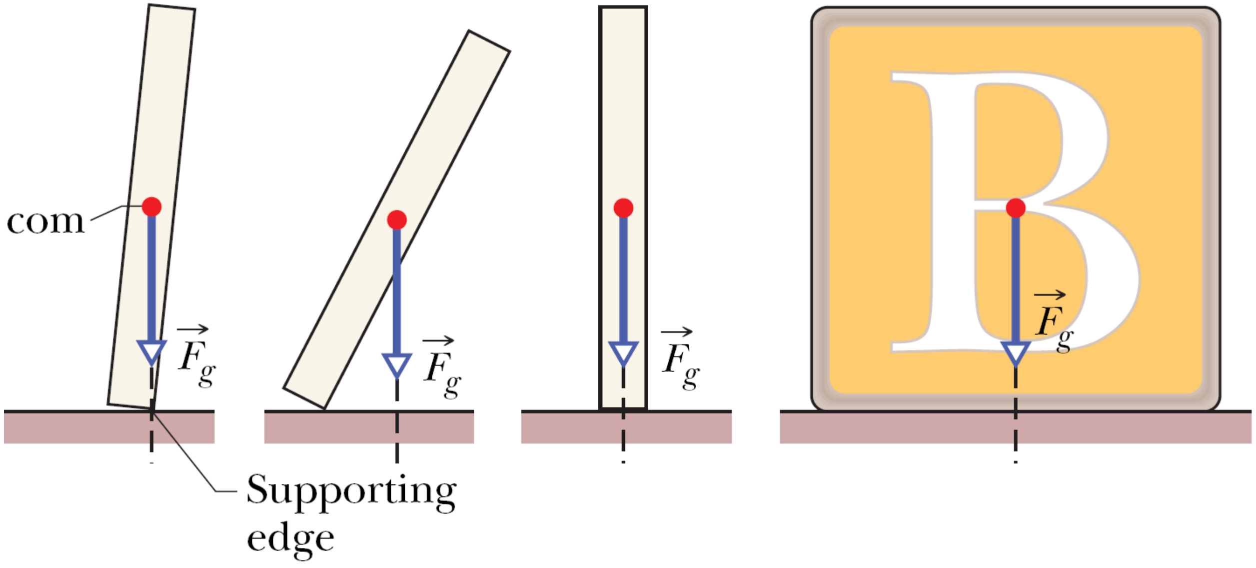
- If a body returns to a state of static equilibrium after having been perturbed, the body is said to be in *stable* static equilibrium; otherwise, the body is said to be in *unstable* static equilibrium.

The Requirements of Equilibrium

- Newton's 2nd law in its linear momentum form, $\vec{F}_{\text{net}} = \dfrac{d\vec{P}}{dt}$

- If the body is in translational equilibrium,

$$\vec{F}_{\text{net}} = 0 \quad \text{balance of forces}$$



- Newton's 2nd law in its angular momentum form, $\vec{\tau}_{\text{net}} = \dfrac{d\vec{L}}{dt}$
- If the body is in rotational equilibrium, $\vec{\tau}_{\text{net}} = 0$ balance of torques
- Thus, the 2 requirements for a body to be in equilibrium are:

1. The vector sum of all the external forces that act on the body must be 0.

2. The vector sum of all the external torques that act on the body, measured about any possible point, must also be 0.

- Its component forms are:

balance of forces $F_{\text{net},x} = 0, \quad F_{\text{net},y} = 0, \quad F_{\text{net},z} = 0$

balance of torques $\tau_{\text{net},x} = 0, \quad \tau_{\text{net},y} = 0, \quad \tau_{\text{net},z} = 0$

- If forces acting on the body only lie in the xy plane, then

$$F_{\text{net},x} = 0, \quad F_{\text{net},y} = 0, \quad \tau_{\text{net},z} = 0$$

- For static equilibrium, there are other requirements:

3. The linear momentum of the body must be 0.

4. The angular momentum of the body must be 0.

The Center of Gravity

The gravitational force on a body effectively acts at a single point, called the **center of gravity** (cog) of the body.

- We would like to prove that

If $\vec{g}$ is the same for all elements of a body, then the body's center of gravity (cog) is coincident with the body's center of mass (com).

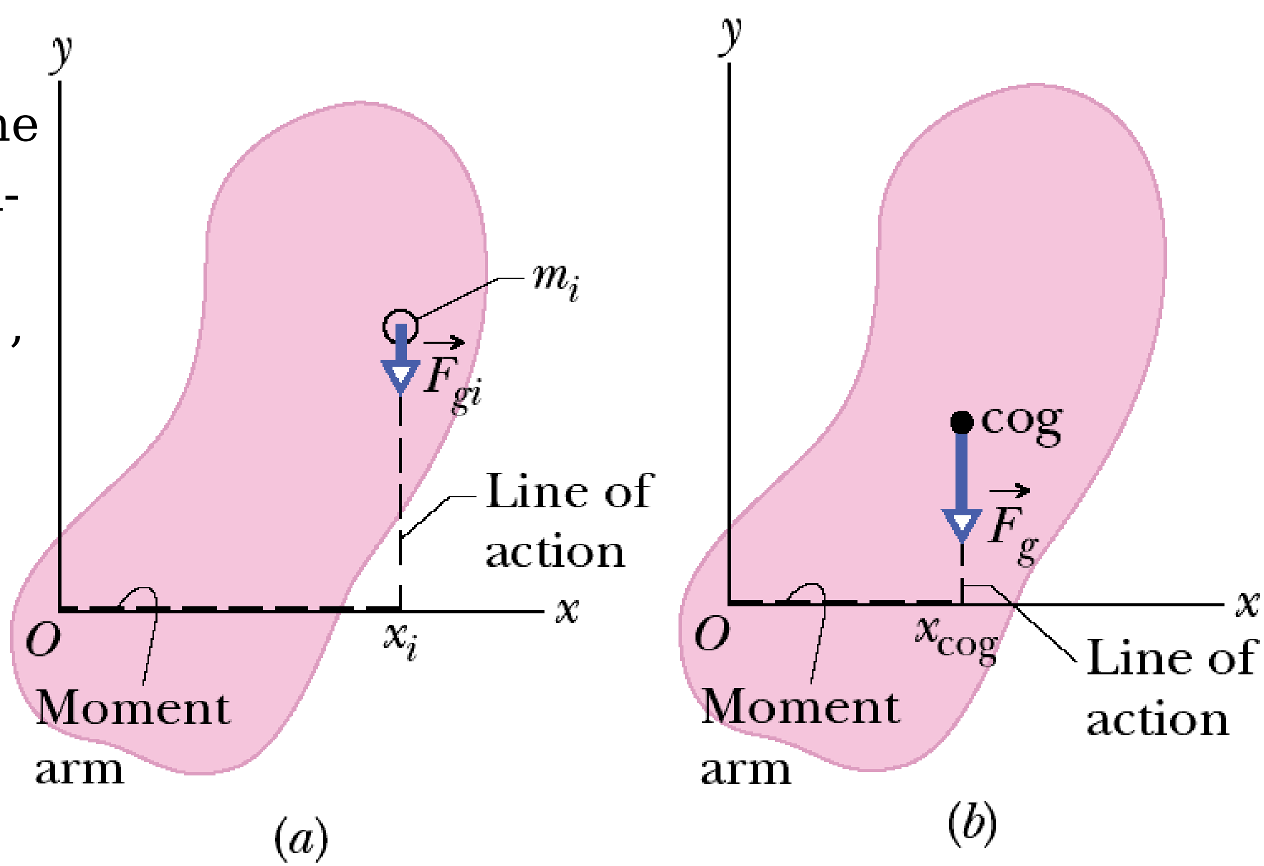
Proof

- Assume the total mass of a body is M, and the mass of one of its element is m_i, the gravitational acceleration at the location of the element i is $\vec{g}_i$, then each gravitational force produces a torque

$$\vec{\tau}_i = \vec{r}_i \times \vec{F}_{gi}$$

and the net torque on all the element of the body is

$$\vec{\tau}_{\text{net}} = \sum \vec{\tau}_i = \sum \vec{r}_i \times \vec{F}_{gi}$$



- Considering the body as a whole, the gravitational force acting at the body's center of gravity produces a torque $\vec{\tau} = \vec{r}_{\text{cog}} \times \vec{F}_g$

Since the gravitational force on the body is equal to the sum of the gravitational forces on all its elements,

$$\vec{\tau} = \vec{r}_{\text{cog}} \times \vec{F}_g = \vec{r}_{\text{cog}} \times \sum \vec{F}_{g\,i}$$

- Since the torque due to the gravitational force acting at the center of gravity is equal to the net torque due to all the gravitational forces acting on all the elements, then

$$\vec{r}_{\text{cog}} \times \sum \vec{F}_{g\,i} = \sum \vec{r}_i \times \vec{F}_{g\,i} \quad \Rightarrow \quad \vec{r}_{\text{cog}} \times \sum m_i \vec{g}_i = \sum \vec{r}_i \times m_i \vec{g}_i$$

- If the accelerations at all the locations of the elements are the same and assume $\vec{g}_i = -g_i \hat{\mathrm{j}}$ then

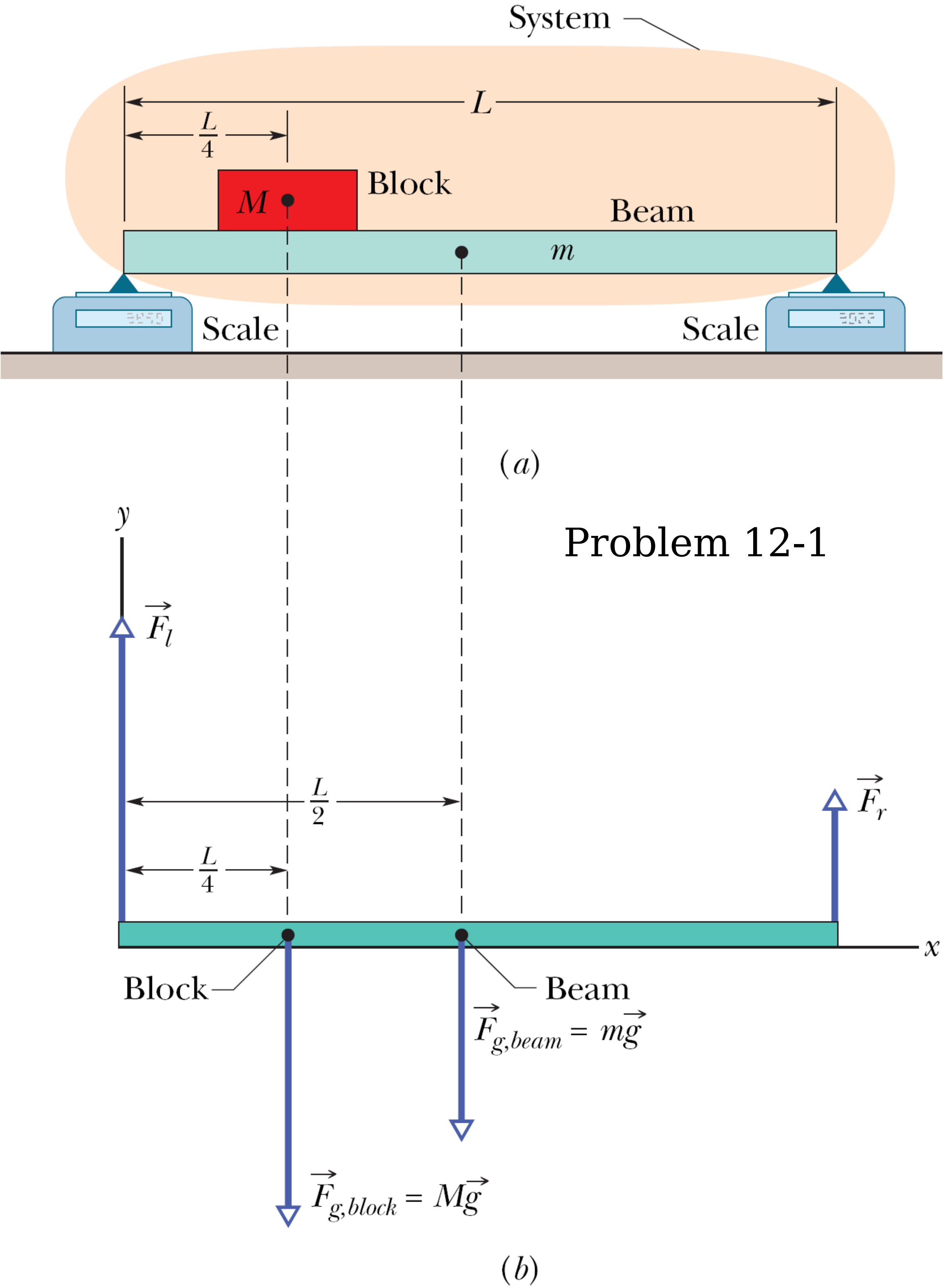
$$\vec{r}_{\text{cog}} \sum m_i = \sum m_i \vec{r}_i + \sum m_i y_i \hat{\mathrm{j}}$$

- By rotating the body with different angles we can argue $y_i = 0$ for each element
- Thus, $\vec{r}_{\text{cog}} \sum m_i = \sum m_i \vec{r}_i \quad \Rightarrow \quad \vec{r}_{\text{cog}} = \frac{1}{M} \sum m_i \vec{r}_i \quad \Rightarrow \quad \vec{r}_{\text{cog}} = \vec{r}_{\text{com}}$

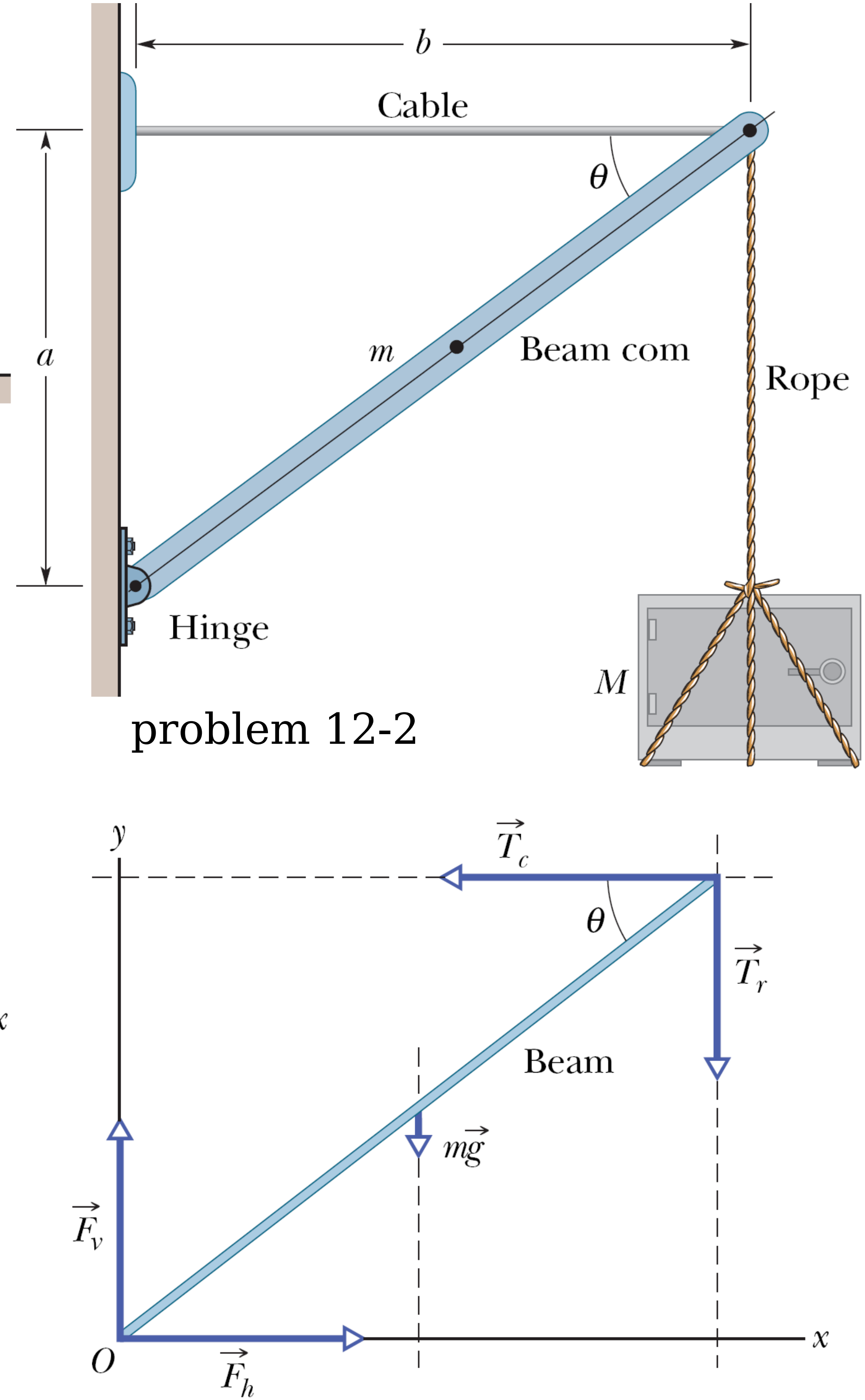
Some Examples of Static Equilibrium

- In the examples, the force involved in the equilibrium are all in the xy plane, which means that the torques involved are parallel to the z axis.

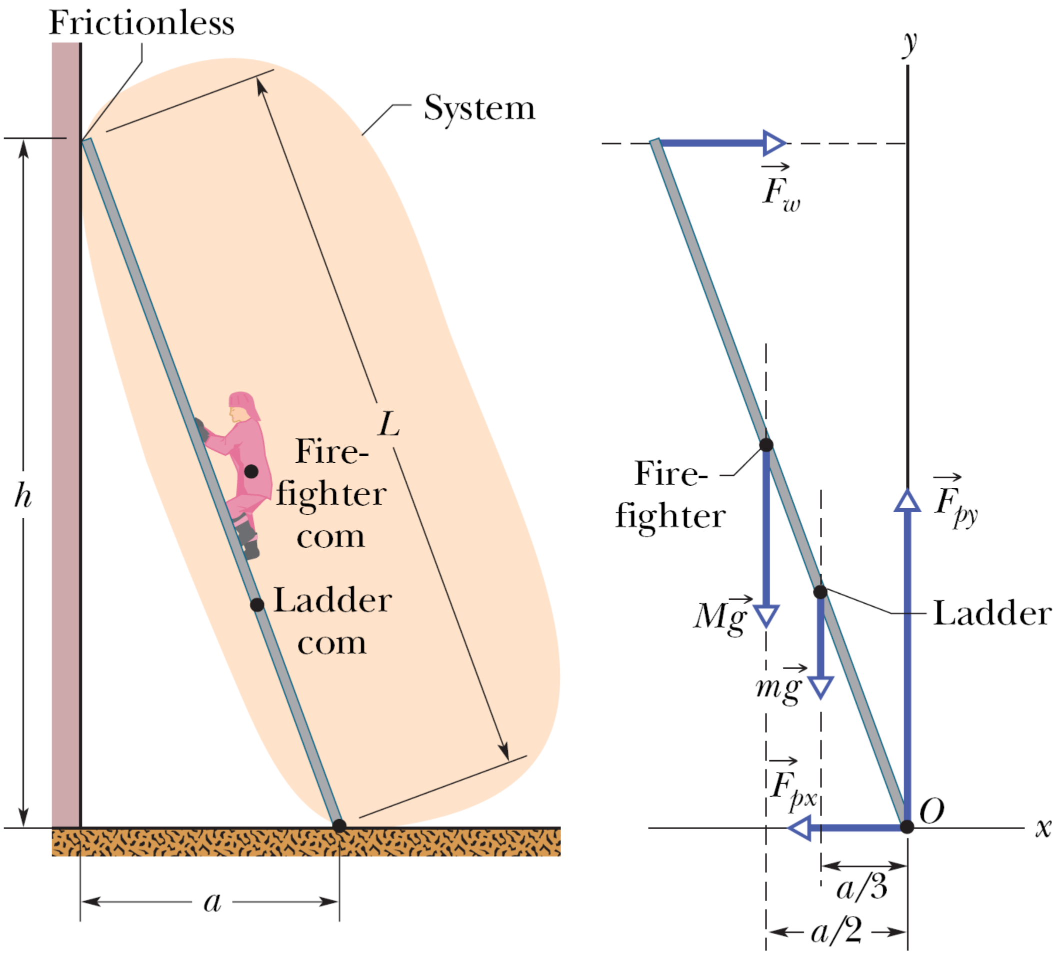
Problem 12-1



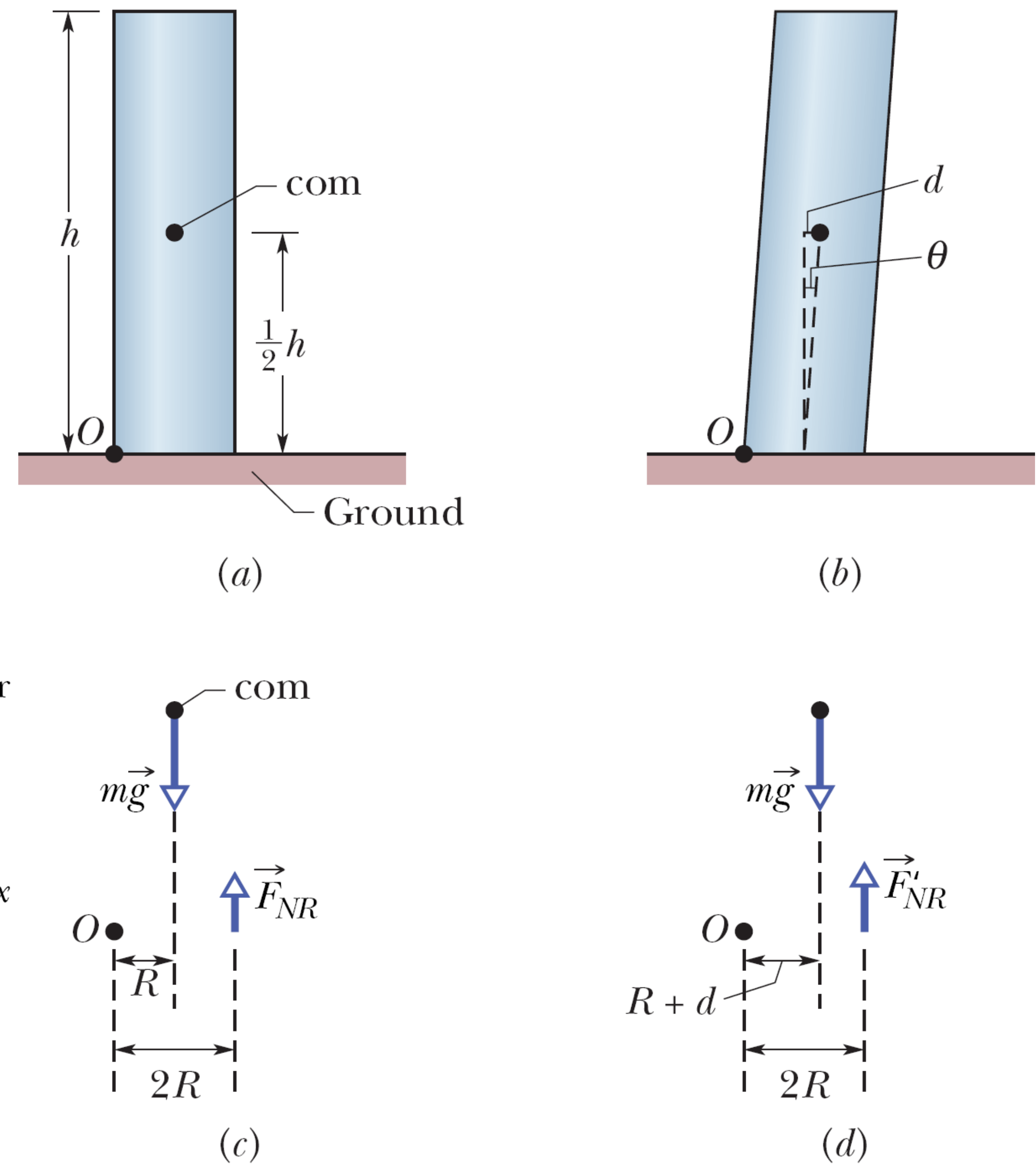
problem 12-2



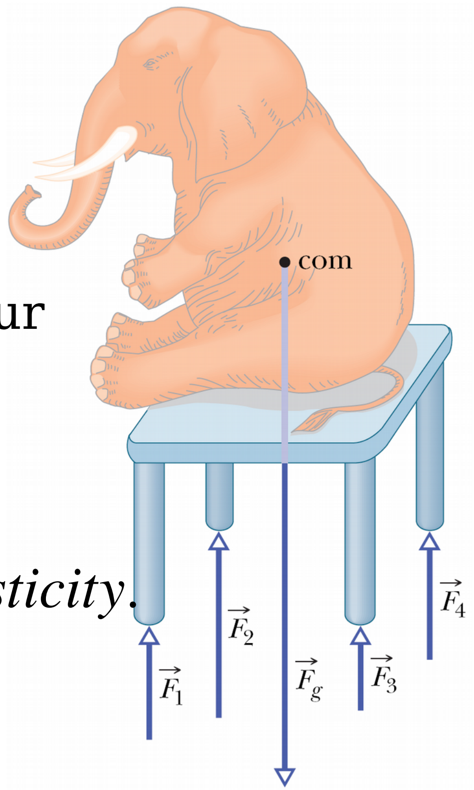
problem 12-3



problem 12-4



Indeterminate Structures



- Problems in which there are more unknowns than equations, are called **indeterminate**.

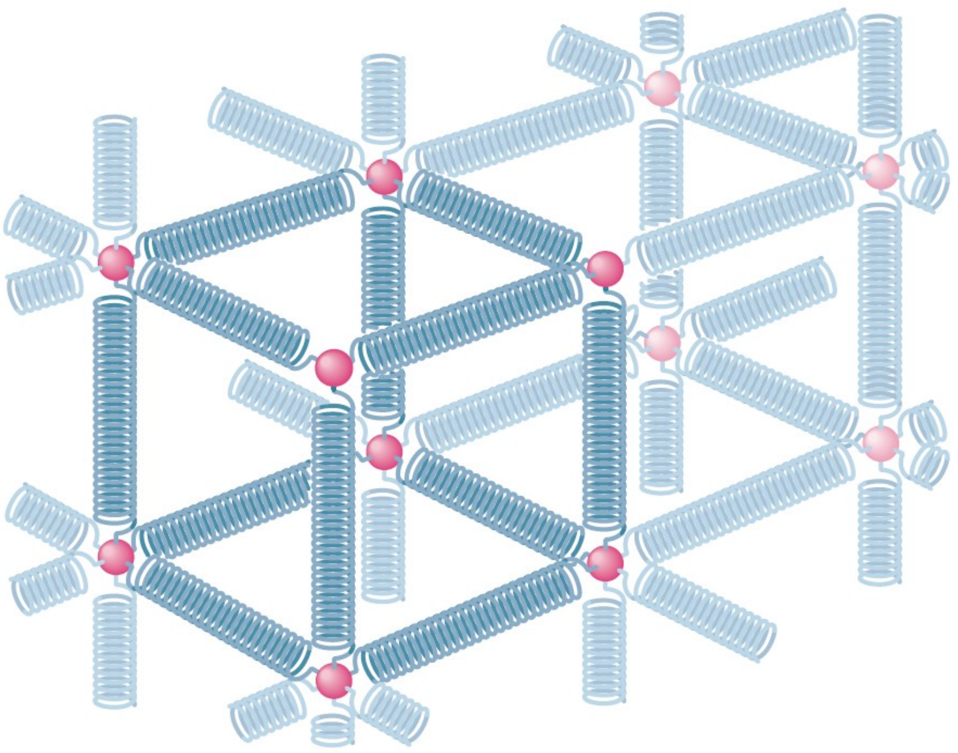
- Up to now, the possible indeterminate problems come from our assumption that the bodies to which we apply the equations of static equilibrium are perfectly rigid.

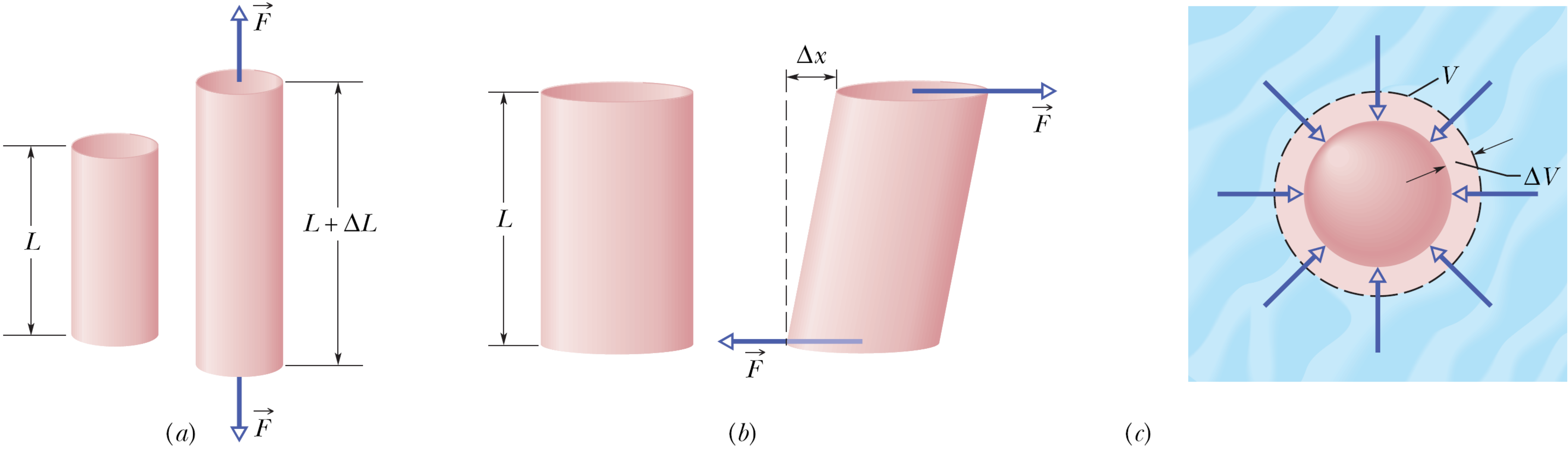
- To solve such indeterminate equilibrium problems, we must supplement equilibrium equations with some knowledge of *elasticity*.

Elasticity

- When a large number of atoms come together to form a metallic solid, they settle into equilibrium positions in a 3-dim *lattice*. The atoms are held together by interatomic forces that are modeled as tiny springs.

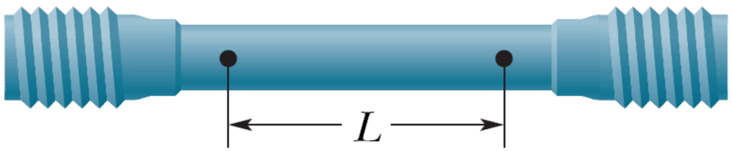
- All real "rigid" bodies are to some extent elastic, which means that we can change their dimensions slightly by pulling, pushing, twisting, or compressing them.





- **stress** = deforming force per unit area

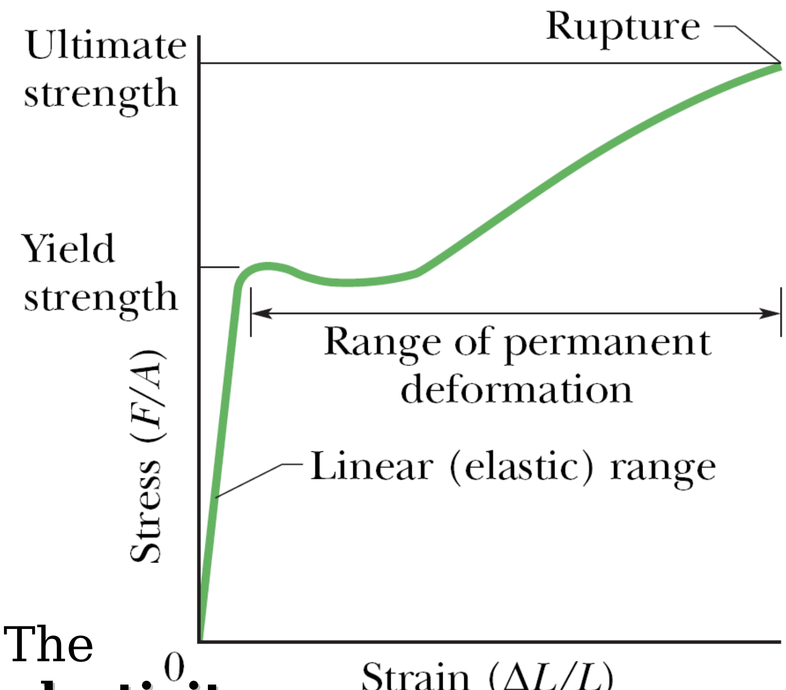
  (a) tensile stress



  (b) shearing stress

  (c) hydraulic stress

- **strain** = unit deformation



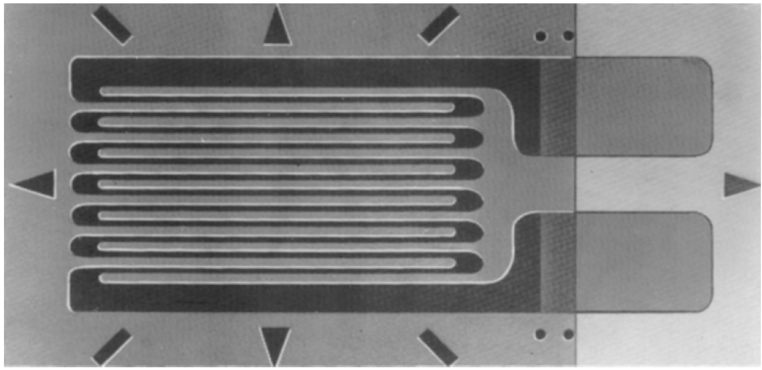
- Stress and strain are proportional to each other. The constant of proportionality is called a **modulus of elasticity**

$$\text{stress} = \text{modulus} \times \text{strain}$$

- For a substantial range of applied stresses, the stress-strain relation is linear, and the specimen recovers its original dimensions when the stress is removed.

- If the stress is increased beyond the **yield strength** S_y of the specimen, the specimen becomes permanently deformed.

- If the stress continues to increase, the specimen eventually ruptures, at a stress called the **ultimate strength** S_u.



Tension and Compression

- For simple tension or compression: $\text{tension} \equiv \frac{F}{A}$
where F is the magnitude of the force applied perpendicularly to an area A on the object.

- The strain, or unit deformation, is then the fractional (or sometimes percentage) change in a length of the specimen, a dimensionless quantity,

$$\text{strain} \equiv \frac{\Delta L}{L}$$

- Because the strain is dimensionless, the modulus has the same dimensions as the stress namely, force per unit area.

- The modulus for tensile and compressive stresses is the **Young's modulus**, E

$$\frac{F}{A} = E \frac{\Delta L}{L}$$

Shearing

- In the case of shearing, the stress is also a force per unit area, but the force vector lies in the plane of the area rather than perpendicular to it.
- $\text{strain} \equiv \dfrac{\Delta x}{L}$
- The corresponding modulus is called the **shear modulus**, G: $\dfrac{F}{A} = G\dfrac{\Delta x}{L}$

Hydraulic Stress

- the stress is the fluid pressure p on the object.
- $\text{strain} \equiv \dfrac{\Delta V}{V}$ where V is the original volume of the specimen and ΔV is the absolute value of the change in volume.
- The corresponding modulus is called the **bulk modulus** B.
- The object is said to be under *hydraulic compression*, and the pressure can be called the *hydraulic stress* $p = B\dfrac{\Delta V}{V}$

problem 12-5 problem 12-6

The chosen problems: 4, 28, 46.