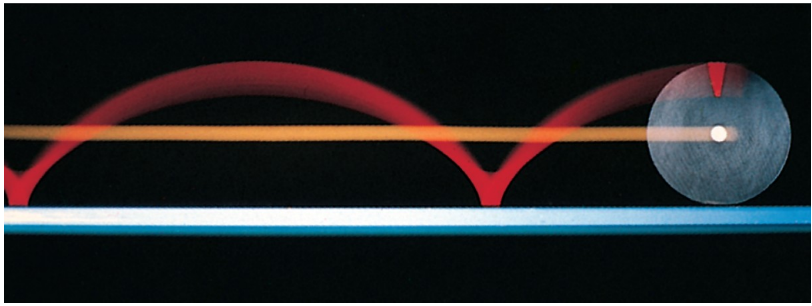
Chapter 11 Rolling, Torque, and Angular Momentum

Rolling as Translation and Rotation Combined

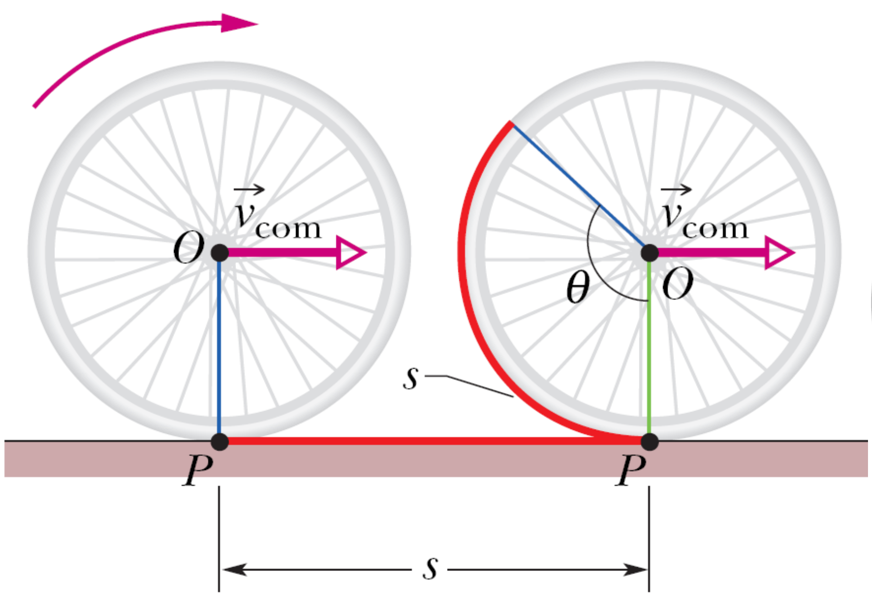
- Treat **rolling** as a combination of **translation** of the center of mass and **rotation** of the rest of the object around that center.

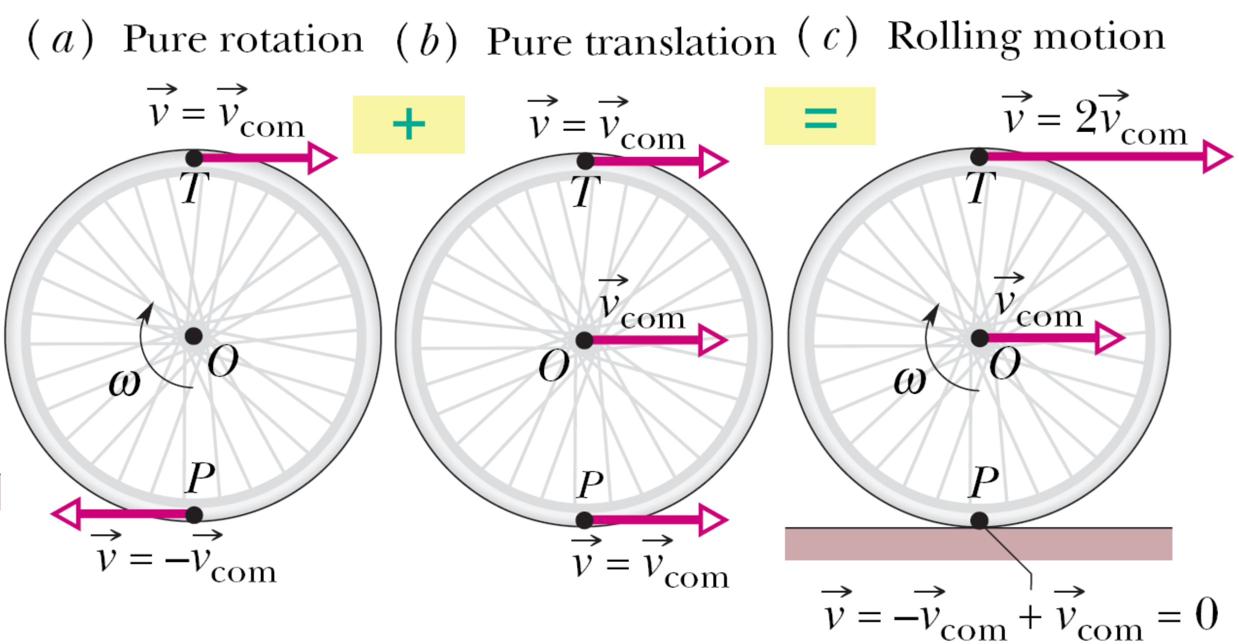
- consider only objects that roll smoothly along a surface: no slipping or bouncing on the surface.



- During a time interval t, the wheel moves forward by a distance s, $s = R\,\theta$, where R is the radius of the wheel.

- The linear speed of the center of the wheel is $v_{com} = \frac{d\,s}{d\,t}$; The angular speed of the wheel is $\omega = \frac{d\,\theta}{d\,t}$, thus $v_{com} = R\,\omega$ smooth rolling motion





- In the combination of translation and rotation, the portion of the wheel at the bottom is stationary and the portion at the top is moving at speed $2v_{com}$, faster than any other portion of the wheel.

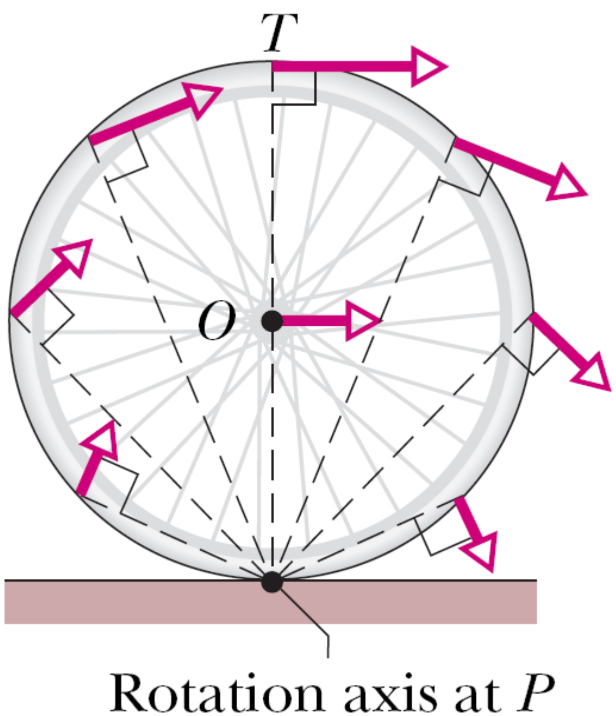
- The motion of any round body rolling smoothly over a surface can be separated into purely rotational and purely translational motions.

Rolling as Pure Rotation

- Another way to look at the rolling motion of a wheel as pure rotation about an axis that extends through the point where the wheel contacts the street as the wheel moves.

- The angular velocity is the same as one observes it in pure rotation about an axis through its center of mass. Verify this

$$v_{top} = (2\ R)\ (\omega) = 2\ (R\ \omega) = 2\ v_{com}$$



The Kinetic Energy of Rolling

- If we view the rolling as pure rotation about an axis through P (the bottom of a wheel), then $K = \frac{1}{2}\mathbb{I}_p\,\omega^2$ where $\mathbb{I}_p$ is the rotational inertia of the wheel about the axis through P.

- From the parallel-axis theorem $\mathbb{I}_p = \mathbb{I}_{\text{com}} + M\,R^2$ where $\mathbb{I}_{\text{com}}$ is its rotational inertia about an axis through its center of mass, the kinetic energy becomes

$$K = \frac{1}{2}\mathbb{I}_{\text{com}}\,\omega^2 + \frac{1}{2}M\,R^2\,\omega^2$$

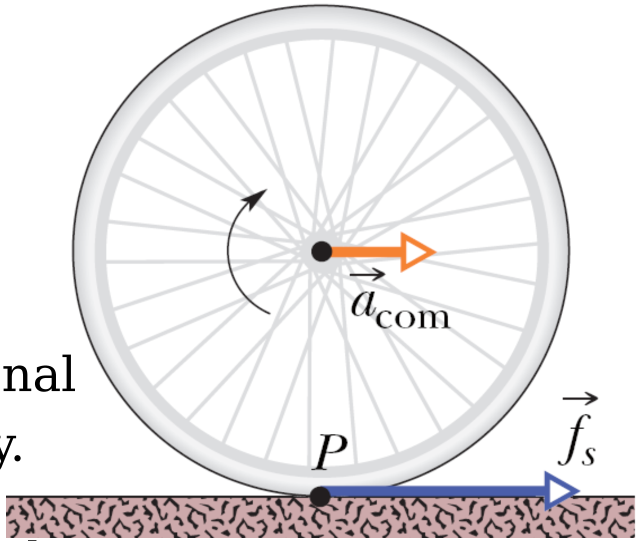
- Using $v_{\text{com}} = R\,\omega$ gives $K = \frac{1}{2}\mathbb{I}_{\text{com}}\,\omega^2 + \frac{1}{2}M\,v_{\text{com}}^2$

- Consider the 1st term of RHS as the kinetic energy associated with the rotation of the wheel about an axis through its center of mass, and the 2nd term as the kinetic energy associated with the translational motion of the wheel's center of mass.

A rolling object has 2 types of kinetic energy: a rotational kinetic energy due to its rotation about its center of mass and a translational kinetic due to translation of its center of mass.

The Forces of Rolling

Friction and Rolling



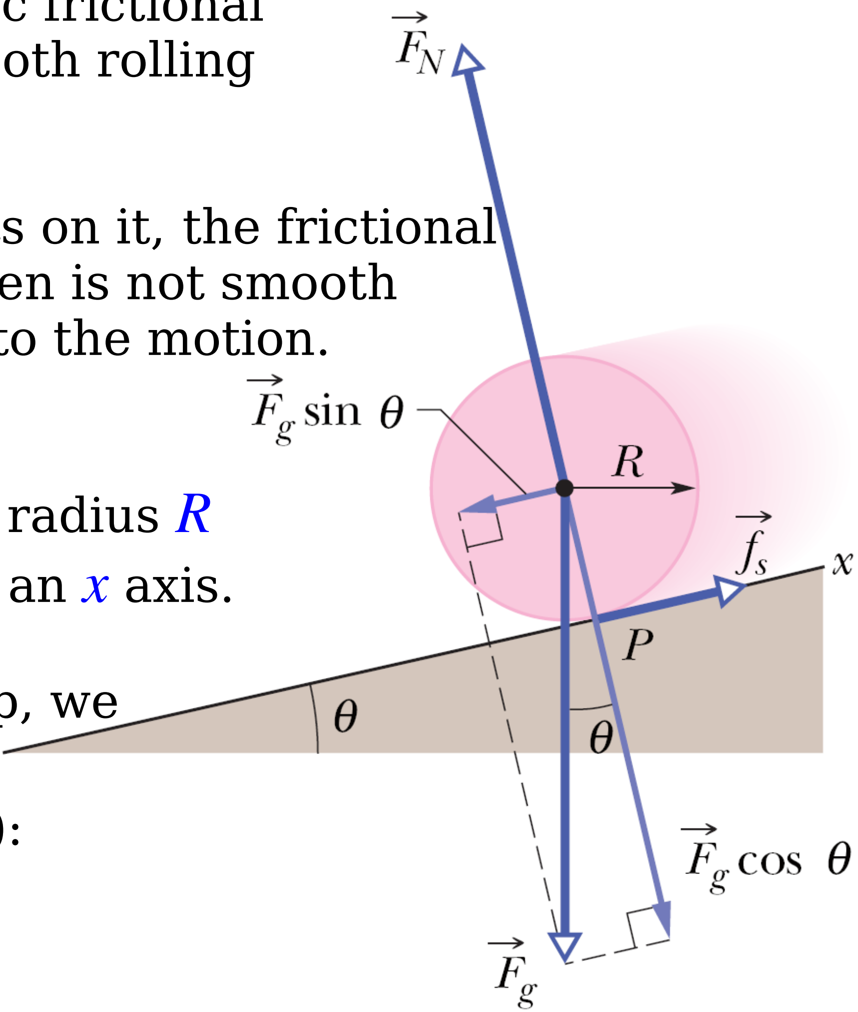
- If a net force acts on the rolling wheel, then that net force causes acceleration of the center of mass, and the accelerations tend to the wheel slide at P. Thus, a frictional force must act on the wheel at P to oppose that tendency.

- If the wheel does not slide, the force is a static frictional force and the motion is smooth rolling. For smooth rolling

$$a_{com} = R\,\alpha \quad \text{smooth rolling motion}$$

- If the wheel does slide when the net force acts on it, the frictional force is a kinetic frictional force. The motion then is not smooth rolling, and the above equation does not apply to the motion.

Rolling Down a Ramp



- Assume a round uniform body of mass M and radius R rolling smoothly down a ramp at angle θ, along an x axis.

- To find the body's acceleration down the ramp, we use Newton's 2nd law in its linear version ($F_{net} = M a$) and its angular version($\tau_{net} = \mathbb{I}\alpha$):

1: $\vec{F}_g = F_{g,x}\,\hat{\mathrm{i}} + F_{g,y}\,\hat{\mathrm{j}} = -M\,g\sin\theta\,\hat{\mathrm{i}} - M\,g\cos\theta\,\hat{\mathrm{j}}$

2: $\vec{F}_N = F_N\,\hat{\mathrm{j}}$

3: $\vec{f}_s = f_s\,\hat{\mathrm{i}}$, the frictional force opposing the sliding must be *up* the ramp.

- Write Newton's 2nd law along for the x-component

$$f_s + F_{g,x} = f_s - M\,g\sin\theta = M\,a_{\mathrm{com},x}$$

- Apply Newton's 2nd law in angular form to the body’s rotation about its center of mass

$$\tau_{\mathrm{net}} = R\,f_s + F_g\cdot 0 + F_N\cdot 0 = \mathbb{I}_{\mathrm{com}}\,\alpha$$

- Here $a_{\mathrm{com},x}$ is negative when α is positive, thus

$$\alpha = -\frac{a_{\mathrm{com},x}}{R},\quad f_s = -\mathbb{I}_{\mathrm{com}}\frac{a_{\mathrm{com},x}}{R^2}\quad\Rightarrow\quad a_{\mathrm{com},x} = -\frac{g\sin\theta}{1+\dfrac{\mathbb{I}_{\mathrm{com}}}{M\,R^2}}$$

Problem 11-1

The Yo-Yo

- If a yo-yo rolls down its string for a distance, it loses potential energy but gains kinetic energy in both translational and rotational forms. As it climbs back up, it loses kinetic energy and regains potential energy.

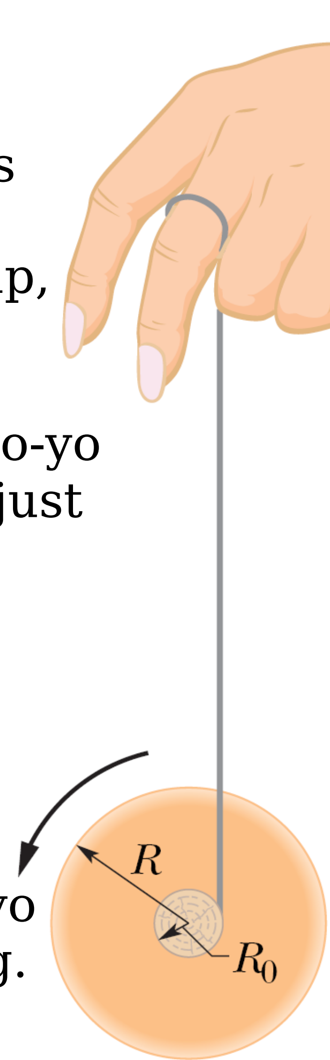
- Find an expression for the linear acceleration of a yo-yo rolling down a string, we could use Newton s 2^{nd} law just as we did for the body rolling down a ramp:

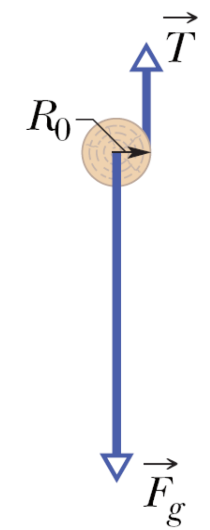
1: $\theta = \dfrac{\pi}{2}$

2: Use the radius of the axle, R_0.

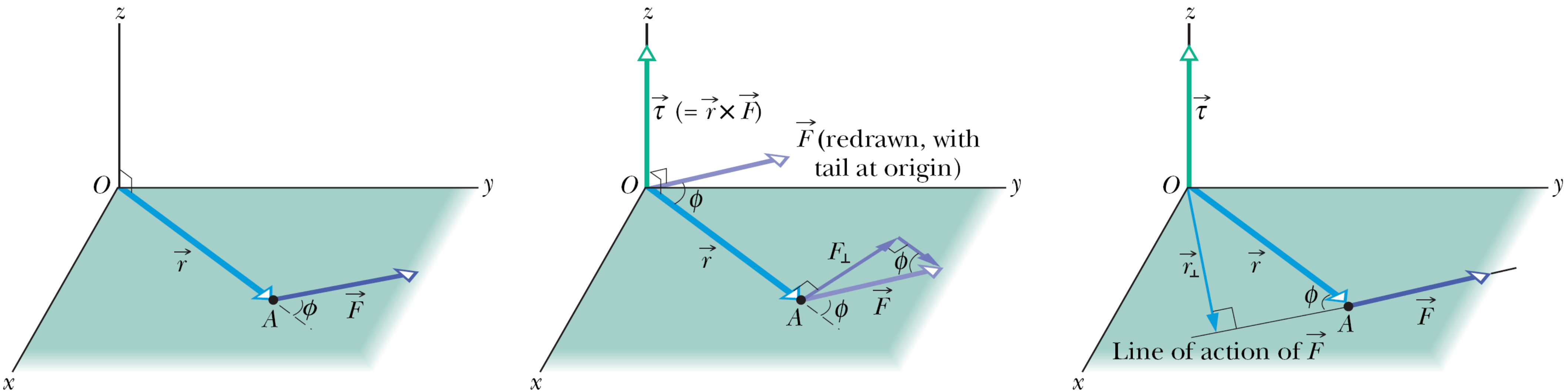
3: Instead of being slowed by frictional force, the yo-yo is slowed by the tension forces on it from the string.

- A similar analysis gives $a_{com} = -\dfrac{g}{1 + \dfrac{I_{com}}{M R_0^2}}$





Torque Revisited



- We now expand the definition of torque to apply it to an individual particle that moves along any path relative to a *fixed* point (rather than a *fixed* axis).

- The torque acting on the particle relative to the fixed point is a vector quantity defined as

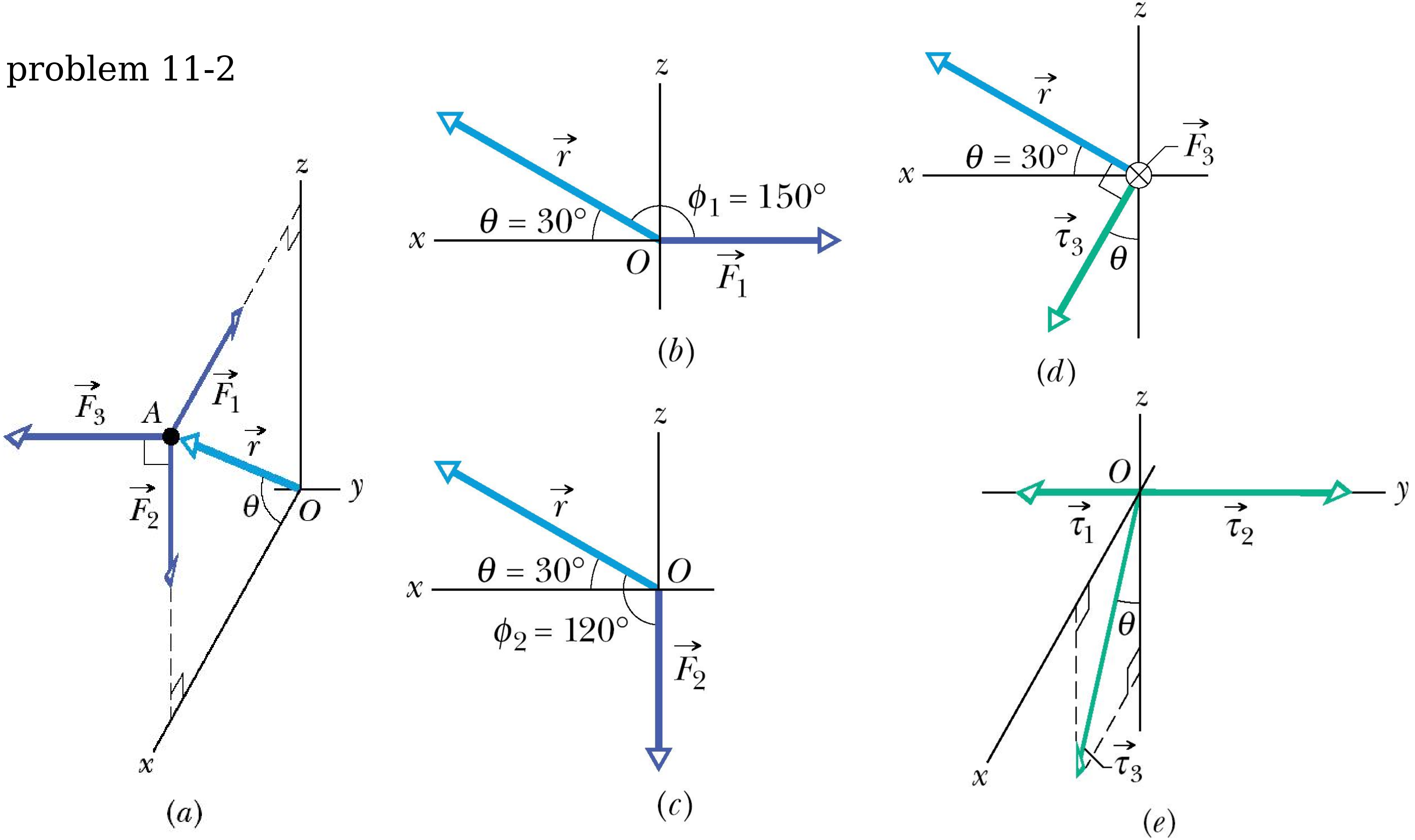
$$\vec{\tau} = \vec{r} \times \vec{F} \quad \text{torque defined}$$

- The magnitude of the torque $\tau = r\, F \sin \phi$

- Define $r_\perp = r \sin \phi$ and $F_\perp = F \sin \phi$, then the definition returns to the old form:

$$\tau = r\, F_\perp = r_\perp\, F$$

problem 11-2



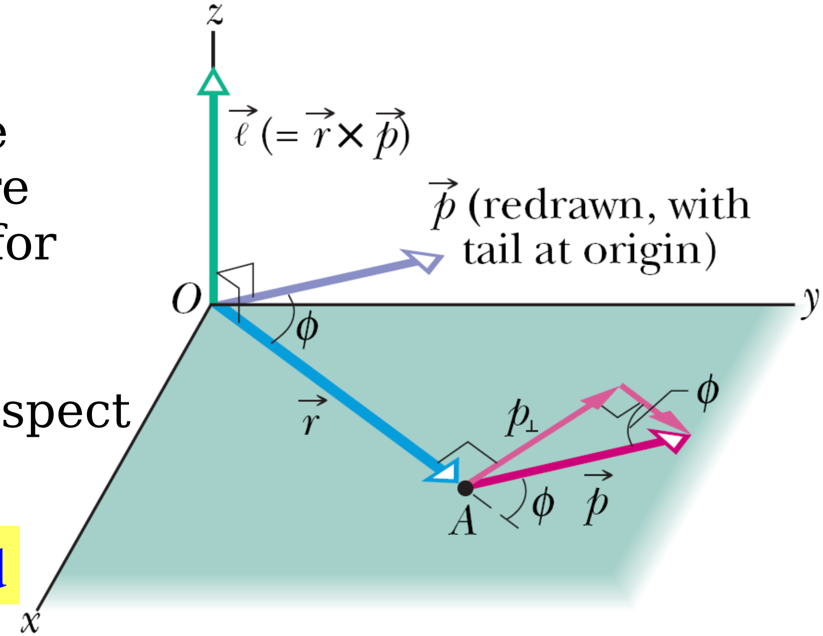
z
$\vec{F}_1$
$\vec{F}_3$
A
$\vec{r}$
y
$\vec{F}_2$
θ
O
x
(a)
z
$\vec{r}$
$\phi_1 = 150°$
$\theta = 30°$
x
O
$\vec{F}_1$
(b)
z
$\vec{r}$
$\theta = 30°$
O
x
$\phi_2 = 120°$
$\vec{F}_2$
(c)
z
$\vec{r}$
$\vec{F}_3$
$\theta = 30°$
x
$\vec{\tau}_3$
θ
(d)
z
O
y
$\vec{\tau}_1$
$\vec{\tau}_2$
θ
$\vec{\tau}_3$
x
(e)

Angular Momentum

- Since the concept of linear momentum and the principle of conservation of linear momentum are extremely powerful tools, we would like to look for the similar characters in rotation.

- The **angular momentum** of a particle with respect to the origin is a vector quantity defined as

$$\vec{\ell} = \vec{r} \times \vec{p} = m\left(\vec{r} \times \vec{v}\right) \quad \text{angular momentum defined}$$



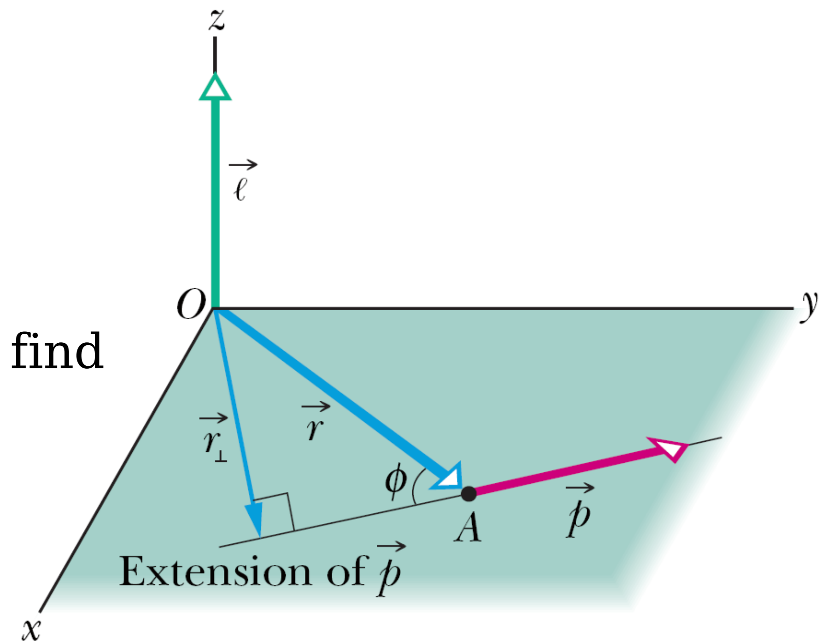
- Angular momentum bears the same relation to linear momentum that torque does to force.

- The SI unit of angular momentum is the kilogram-meter-squared per second ($kg \cdot m^2/s$), equivalent to the joule-second ($J \cdot s$).

- Use the right-hand rule for vector products to find the direction of the angular momentum vector.

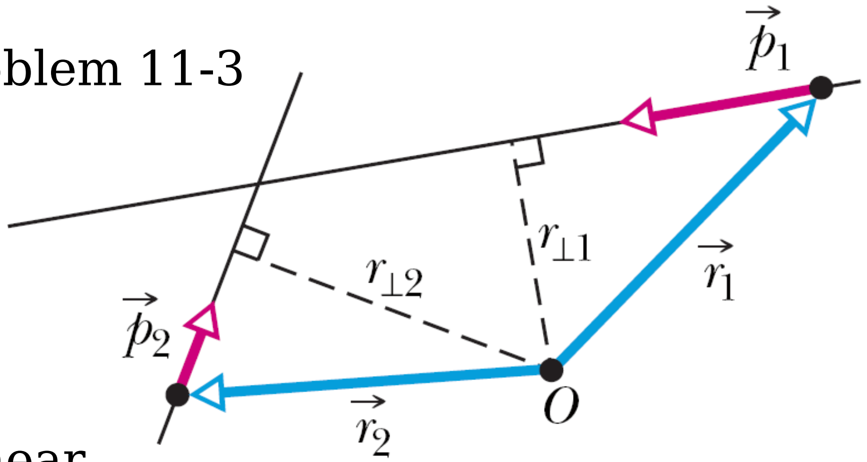
- the magnitude of the angular momentum is

$$\ell = r\, m\, v \sin\phi = r\, p_\perp = r\, m\, v_\perp = r_\perp\, p = r_\perp\, m\, v$$



- Just as is true for torque, angular momentum has meaning only with respect to a specified origin.

problem 11-3



Newton's 2nd Law in Angular Form

- Newton's 2nd law

$$\vec{F}_{\text{net}} = \frac{d\,\vec{p}}{d\,t} \quad \text{single particle}$$

expresses the relation between force and linear momentum for a single particle.

- The relation between torque and angular momentum $\vec{\tau}_{\text{net}} = \dfrac{d\,\vec{\ell}}{d\,t}$ single particle

The (vector) sum of all the torques acting on a particle is equal to the time rate of change of the angular momentum of that particle.

Proof of the above equation

- The definition of the angular momentum of a particle: $\vec{\ell} = m\left(\vec{r} \times \vec{v}\right)$

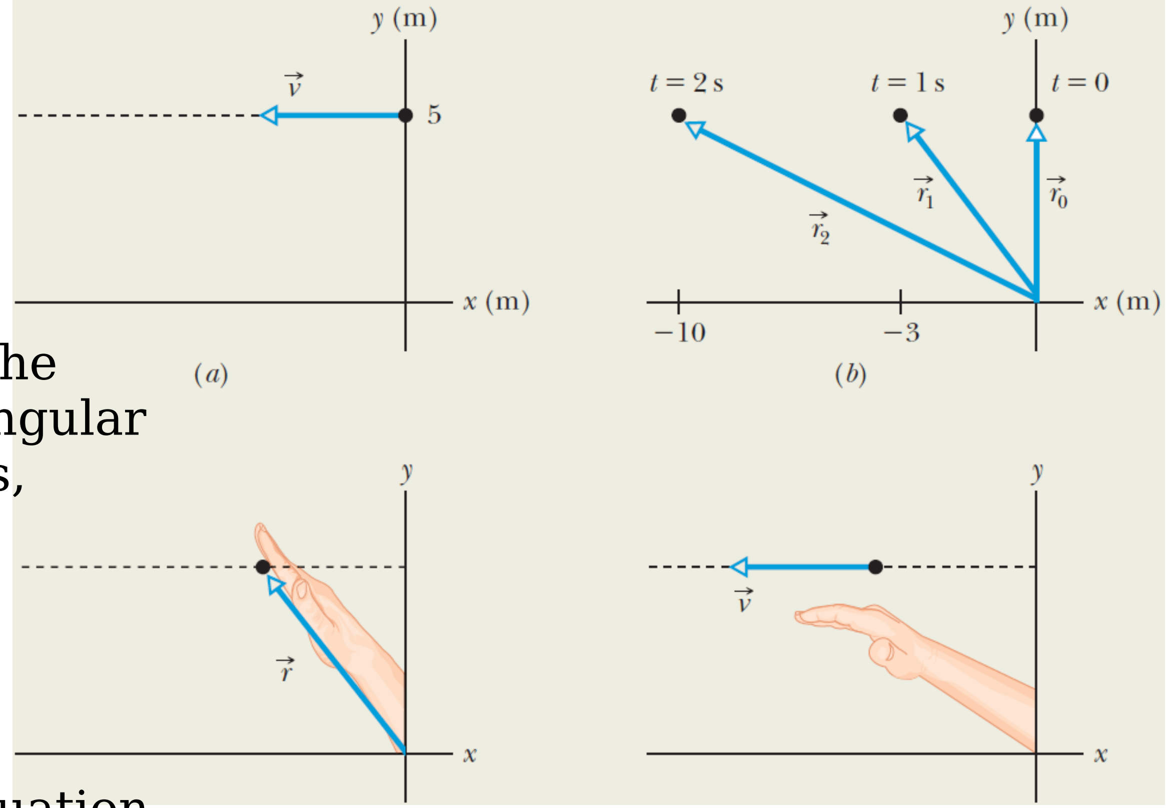
- Differentiating each side with respect to time t yields

$$\frac{d\,\vec{\ell}}{d\,t} = m\left(\vec{r} \times \frac{d\,\vec{v}}{d\,t} + \frac{d\,\vec{r}}{d\,t} \times \vec{v}\right) = m\left(\vec{r} \times \vec{a} + \vec{v} \times \vec{v}\right) = m\left(\vec{r} \times \vec{a}\right) = \vec{r} \times m\,\vec{a}$$

- Thus $\dfrac{d}{d\,t}\,\vec{\ell} = \vec{r} \times \vec{F}_{\text{net}} = \sum\left(\vec{r} \times \vec{F}\right)$ and $\vec{\tau}_{\text{net}} = \dfrac{d}{d\,t}\,\vec{\ell}$

problem 11-4

The Angular Momentum of a System of Particles



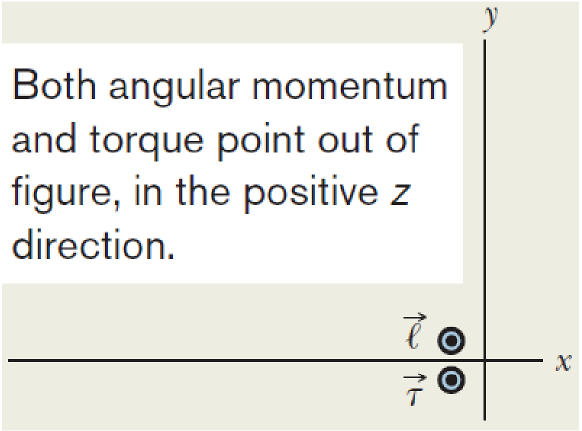
- The total angular momentum of the system is the (vector) sum of the angular momenta of the individual particles,

$$\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + \vec{\ell}_3 + \cdots + \vec{\ell}_n = \sum_{i=1}^{n} \vec{\ell}_i$$

- Find the resulting change in the total angular momentum by taking the time derivative of the above equation

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \frac{d\vec{\ell}_i}{dt} = \sum_{i=1}^{n} \vec{\tau}_{\text{net},\,i}$$

- The rate of change of the system's angular momentum is equal to the vector sum of the (*internal* and *external*) torques on its individual particles.



- Only the external torques can change the total angular momentum of a system

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt} \quad \text{system of particles}$$

- Newton's 2nd law in angular form:

The net external torque acting on a system of particles is equal to the time rate of change of the system's total angular momentum.

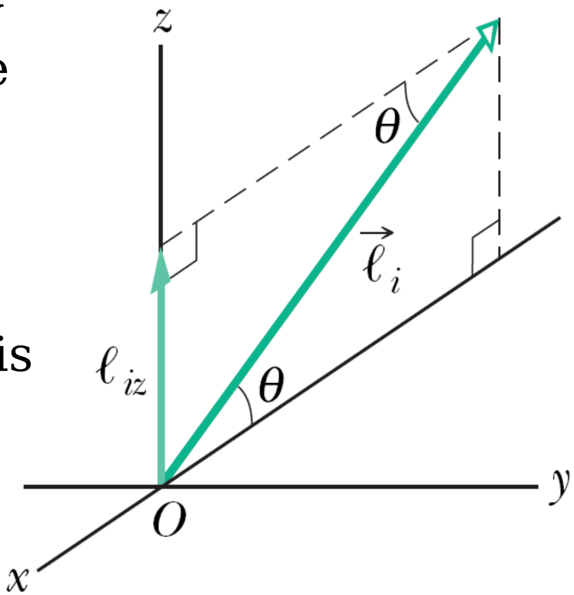
- Torques and the system's angular momentum must be measured relative to the same origin.

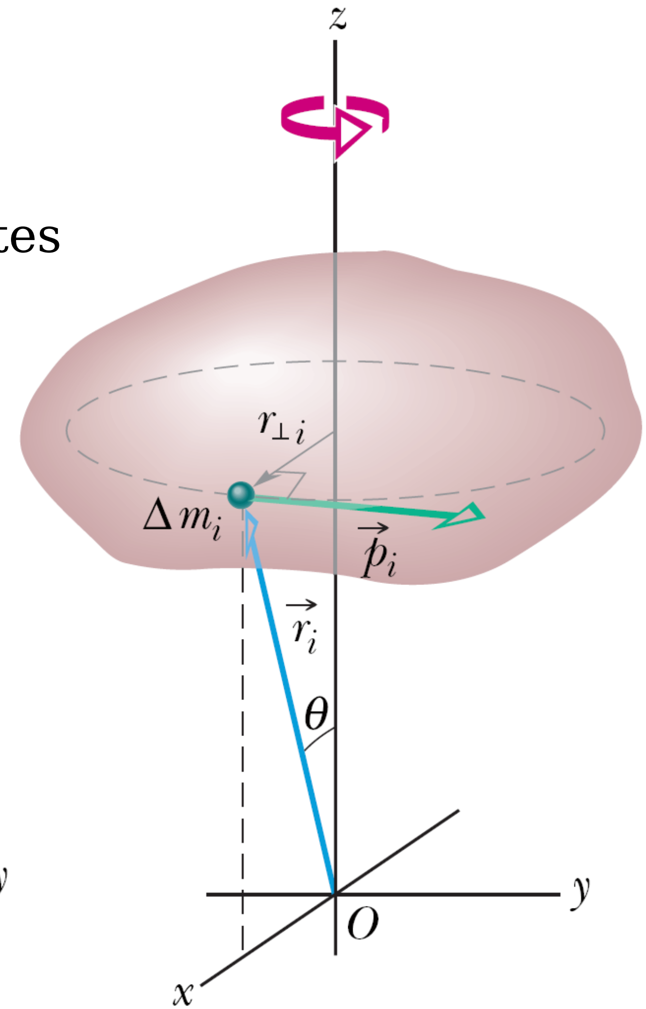
- If the center of mass of the system is not accelerating relative to an inertial frame, that origin can be any point. However, if the center of mass of the system is accelerating, the origin can be only at that center of mass.

The Angular Momentum of a Rigid Body Rotating About a Fixed Axis

- The fixed axis of rotation is a z axis, and the body rotates about it with constant angular speed.

- Find the angular momentum by summing the z components of the angular momenta of the mass elements in the body.





- The magnitude of the angular momentum of one mass element is

$$\ell_i = (r_i)(p_i)\sin\frac{\pi}{2} = (r_i)(\Delta m_i v_i)$$

- We are interested in the component of that is parallel to the rotation axis, the z axis:

$$\ell_{iz} = \ell_i \sin\theta = (r_i \sin\theta)(\Delta m_i v_i) = r_{\perp i}\, \Delta m_i v_i$$

- The z component of the total angular momentum for the rotating rigid body is

$$L_z = \sum_{i=1}^{n} \ell_{iz} = \sum_{i=1}^{n} \Delta m_i v_i r_{\perp i} = \sum_{i=1}^{n} \Delta m_i (\omega r_{\perp i}) r_{\perp i} = \omega \left(\sum_{i=1}^{n} \Delta m_i r_{\perp i}^2 \right)$$

- The quantity in the parentheses is the rotational inertia of the body about the fixed axis, thus

$$L = \mathbb{I}\,\omega \quad \text{rigid body, fixed axis}$$

More Corresponding Variables and Relations for Translational and Rotational Motion

Translational		Rotational	
Force	$\vec{F}$	Torque	$\vec{\tau}\ (=\vec{r}\times\vec{F})$
Linear momentum	$\vec{p}$	Angular momentum	$\vec{\ell}\ (=\vec{r}\times\vec{p})$
Linear momentum	$\vec{P}\ (=\sum\vec{p}_i)$	Angular momentum	$\vec{L}\ (=\sum\vec{\ell}_i)$
Newton's 2nd law	$\vec{F}_{\text{net}}=\dfrac{d\vec{P}}{dt}$	Newton's 2nd law	$\vec{\tau}_{\text{net}}=\dfrac{d\vec{L}}{dt}$
Conservation law	$\vec{P}=\text{constant}$	Conservation law	$\vec{L}=\text{constant}$

Conservation of Angular Momentum

- If no net external torque acts on a system, then $\frac{d\vec{L}}{dt} = 0$ or

$$\vec{L} = \text{constant} \quad \text{isolated system}$$

law of conservation of angular momentum

$$\begin{pmatrix} \text{net angular momentum} \\ \text{at some inital time } t_i \end{pmatrix} = \begin{pmatrix} \text{net angular momentum} \\ \text{at some later time } t_f \end{pmatrix}$$

or

$$\vec{L}_i = \vec{L}_f \quad \text{isolated system}$$

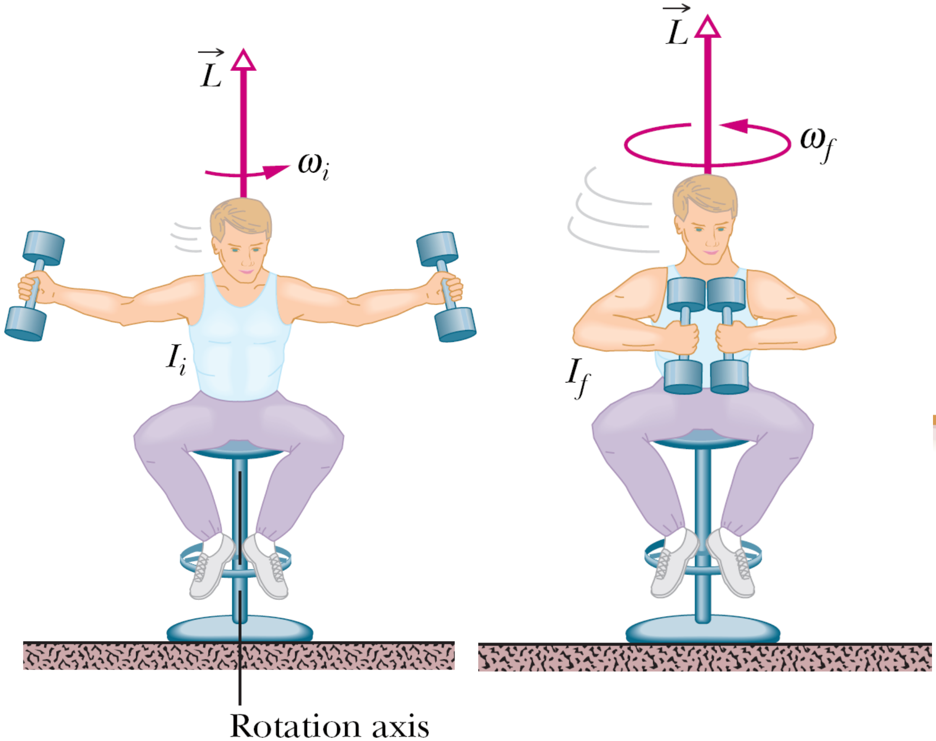
If the net external torque acting on a system is 0, the angular momentum of the system remains constant, no matter what changes take place within the system.

and its component form:

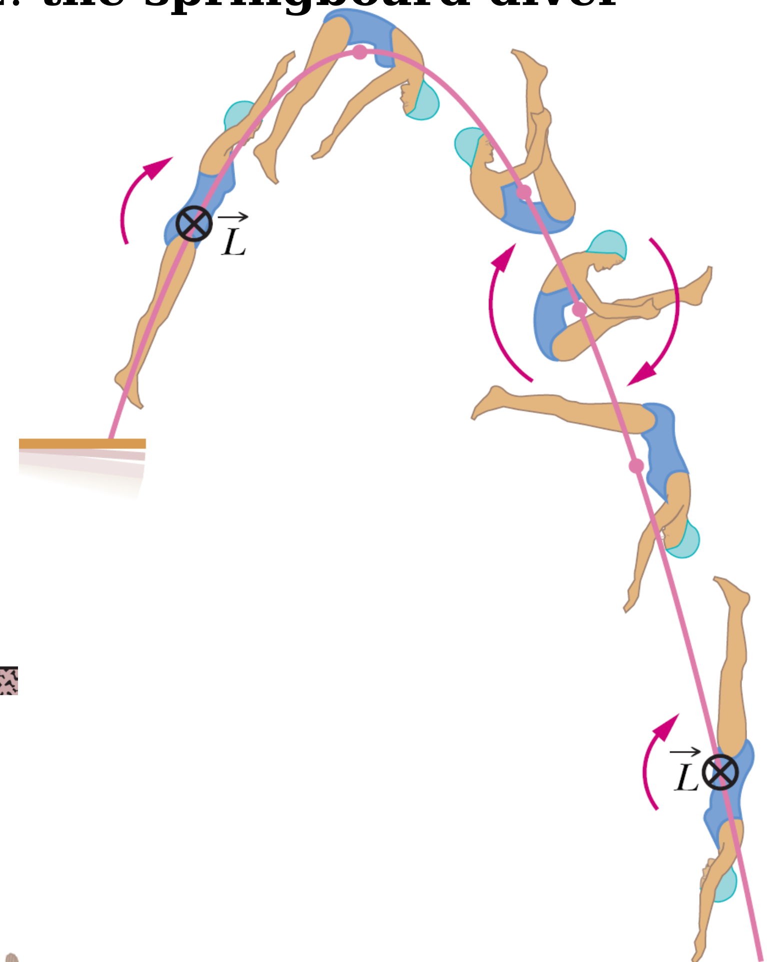
If the component of the net *external* torque on a system along a certain axis is 0, then the component of the angular momentum of the system along that axis cannot change, no matter what changes take place within the system.

- If the initially rigid body somehow redistributes its mass relative to the rotation axis, changing its rotational inertia about that axis. However, the angular momentum of the body cannot change, then $\mathbb{I}_i \omega_i = \mathbb{I}_f \omega_f$

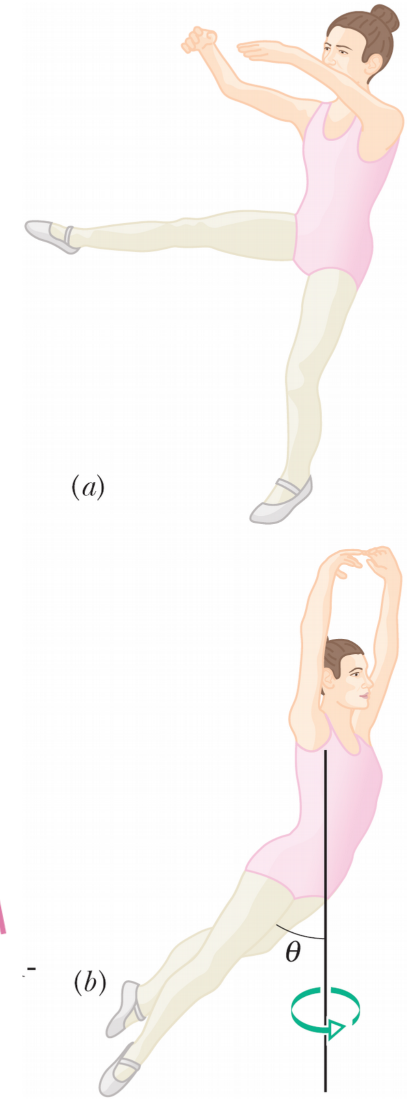
1. **The spinning volunteer**



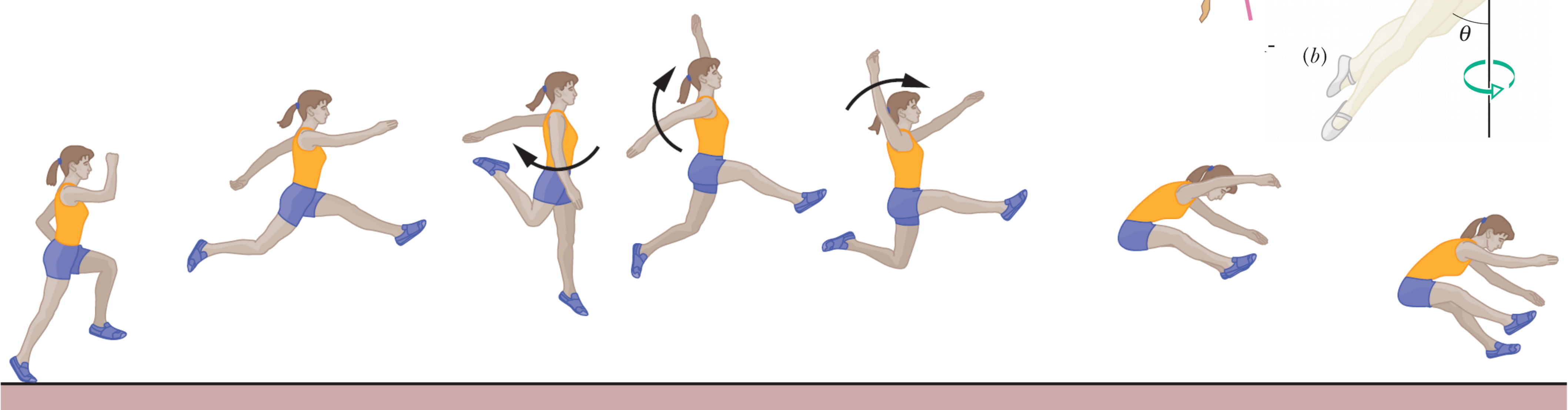
2. **the springboard diver**



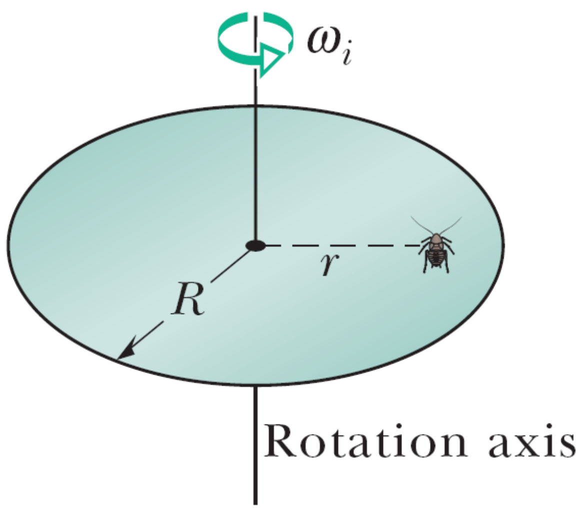
4. **Tour jeté**



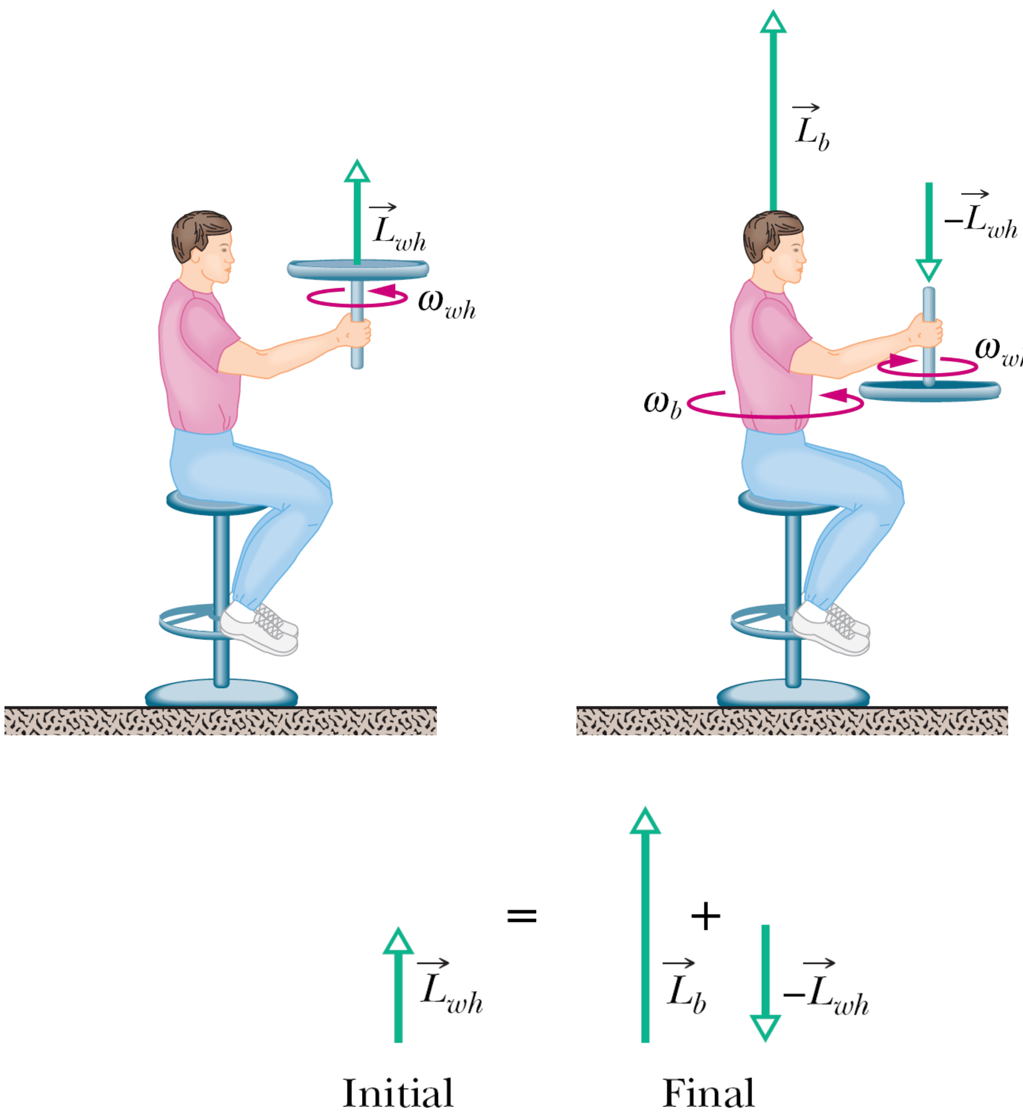
3. **Long jump**



problem 11-6



problem 11-5



Precession of a Gyroscope

- Assume a rapidly spinning gyroscope is released with the shaft. It first rotates slightly downward but then it begins to rotate horizontally about a vertical axis through support point in a motion called **precession** (進動).

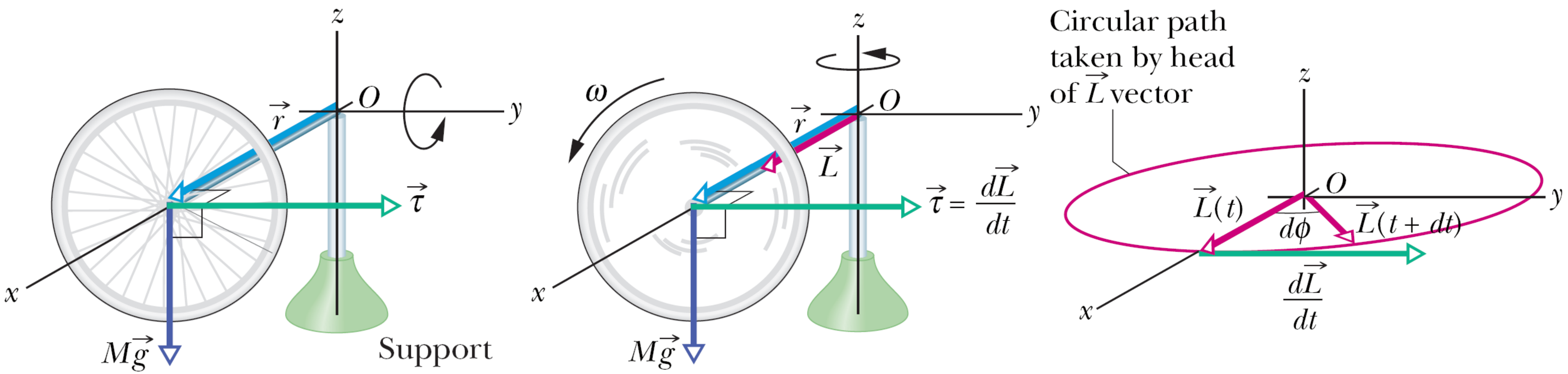
- The clue is that when the spinning gyroscope is released, the torque due to the gravitational force must change not an initial angular momentum of 0 but rather some already existing nonzero angular momentum due to the spin.

- The magnitude the angular momentum is $L = \mathbb{I}\,\omega$

- The incremental change $\mathrm{d}\vec{L} = \vec{\tau}\,\mathrm{d}t$ and $\tau = M\,g\,r$

- Since $\mathrm{d}\vec{L} \perp \vec{L} \;\Rightarrow\; L = \text{constant}$

Proof: $\vec{L}^2 = \vec{L}\cdot\vec{L} = L^2 \;\Rightarrow\; \dfrac{\mathrm{d}L^2}{\mathrm{d}t} = 2\vec{L}\cdot\dfrac{\mathrm{d}\vec{L}}{\mathrm{d}t} \;\Rightarrow\; \dfrac{\mathrm{d}L^2}{\mathrm{d}t} = 0$ for $\mathrm{d}\vec{L} \perp \vec{L} \;\Rightarrow\; L = \text{const}$



- Therefore, the magnitude of the angular momentum does not change, only the direction changes.

- To find the **precession rate** $\mathrm{d}\,L = \tau\,\mathrm{d}\,t = M\,g\,r\,\mathrm{d}\,t$

- The incremental angle due to an incremental amount of change of the angular momentum in an incremental time interval is

$$\mathrm{d}\,\phi = \frac{\mathrm{d}\,L}{L} = \frac{M\,g\,r\,\mathrm{d}\,t}{\mathbb{I}\,\omega} \quad \Rightarrow \quad \Omega \equiv \frac{\mathrm{d}\,\phi}{\mathrm{d}\,t} = \frac{M\,g\,r}{\mathbb{I}\,\omega} \quad \text{precession rate}$$

The chosen problem: 14, 41, 66, 67.