

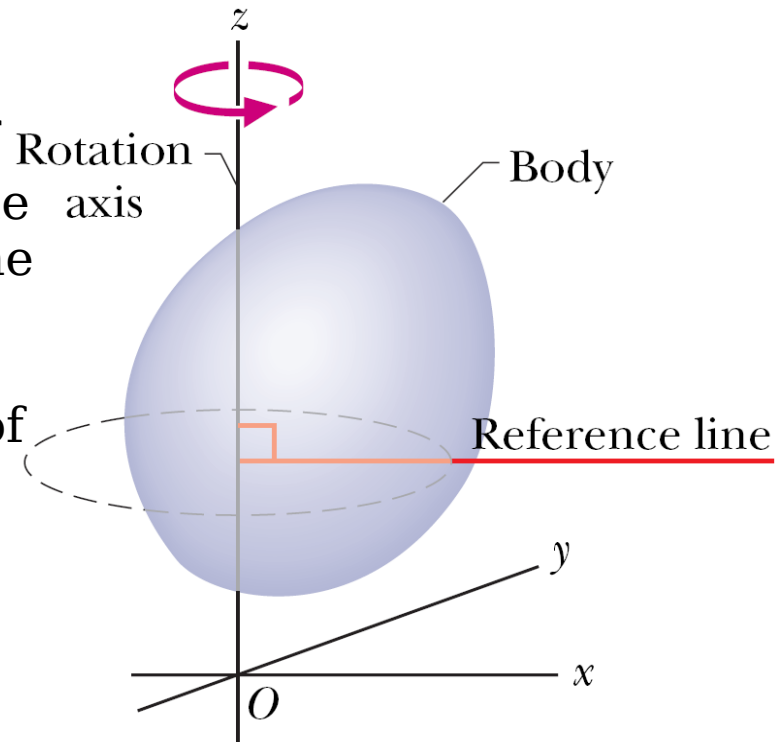
Chapter 10 **Rotation**

Translation: an object moves along a straight or curved line;

Rotation: an object turns about an axis.

The Rotational Variables

- A **rigid body** is a body that can rotate with all its parts locked together and without any change in its shape.
- A **fixed axis** means that the rotation occurs about an axis that does not move. And the fixed axis is called the **axis of rotation** or the **rotation axis**.
- In pure rotation (*angular motion*), every point of the body moves in a circle whose center lies on the axis of rotation, and every point moves through the same angle during a particular time interval.
- In pure translation (*linear motion*), every point of the body moves in a straight line, and every point moves through the same *linear distance* during a particular time interval.



Angular Position

- A *reference line*, fixed in the body, perpendicular to the rotation axis and rotating with the body.

- The **angular position** of this line is the angle of the line relative to a fixed direction, which we take as the **zero angular position**,

$$\theta = \frac{s}{r} \text{ radian measure}$$

- Angle, being the ratio of 2 lengths, is a pure number and thus has no dimension.

- The units of angle:
radian (rad), revolution (rev), and degree,

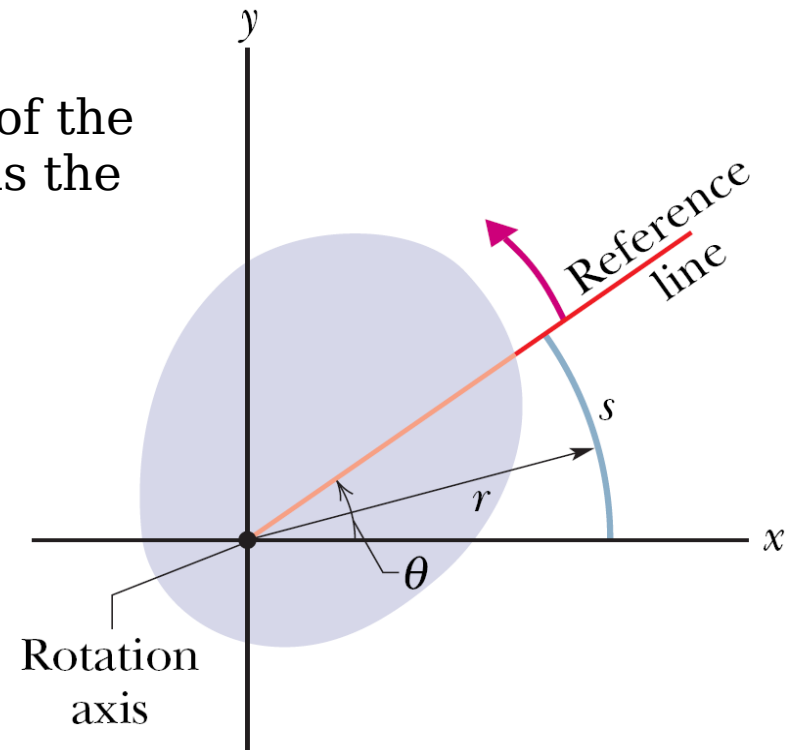
$$1 \text{ rev} = 360^\circ = \frac{2\pi r}{r} = 2\pi \text{ rad}, \quad 1 \text{ rad} = 57.3^\circ = 0.159 \text{ rev}$$

- For pure rotation, we can know all about a rotating body if we know $\theta(t)$, the angular position of the body's reference line as a function of time.

Angular Displacement

- If the body rotates about the rotation axis, changing the angular position of the reference line from θ_1 to θ_2 , the body undergoes an **angular displacement** $\Delta\theta$ given by

$$\Delta\theta = \theta_2 - \theta_1$$



An angular displacement in the counterclockwise direction is positive, and one in the clockwise direction is negative.

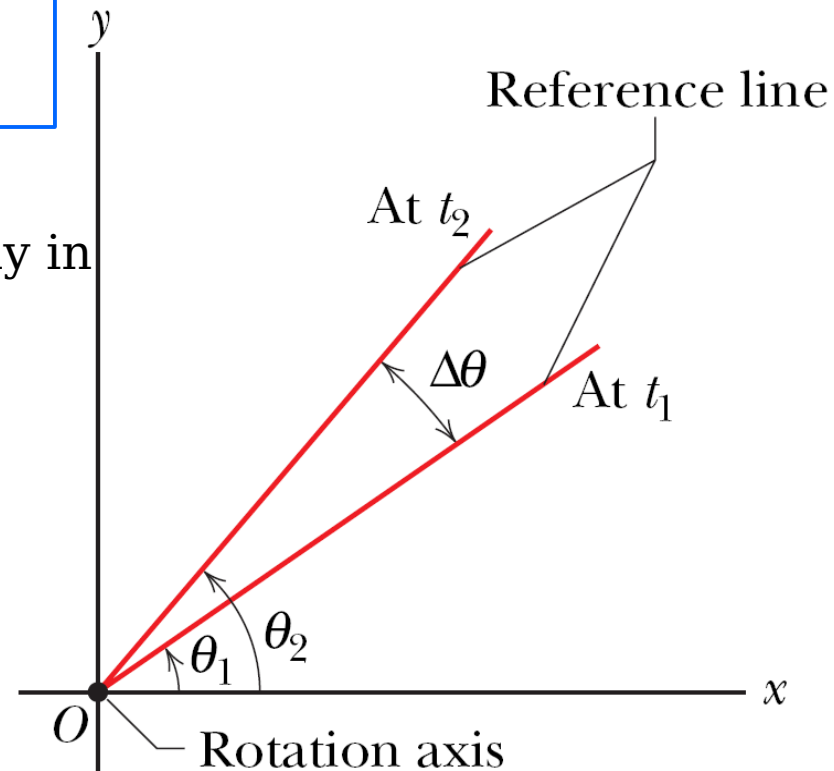
Angular Velocity

● Define the average angular velocity of the body in the time interval Δt from t_1 to t_2 to be

$$\omega_{\text{avg}} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta \theta}{\Delta t}$$

then the **(instantaneous) angular velocity** is

$$\omega = \lim_{\Delta t \rightarrow 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}$$



● All the equations hold not only for the rotating rigid body as a whole but also for *every particle of the body* because the particles are all locked together.

● The unit of angular velocity is **rad/s** or **rev/s**. And **rpm** means **rev/min**.

● The angular velocity of a rotating rigid body is either positive or negative, depending on whether the body is rotating counterclockwise (positive) or clockwise (negative). The magnitude of an angular velocity is called the **angular speed**.

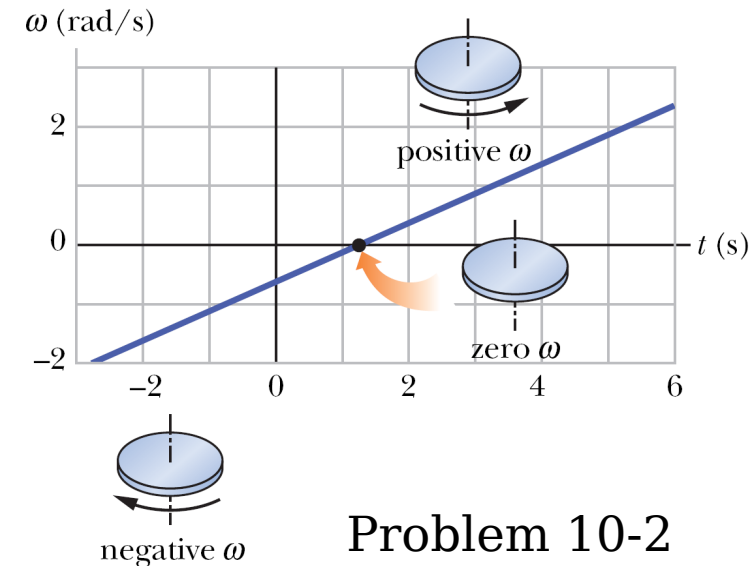
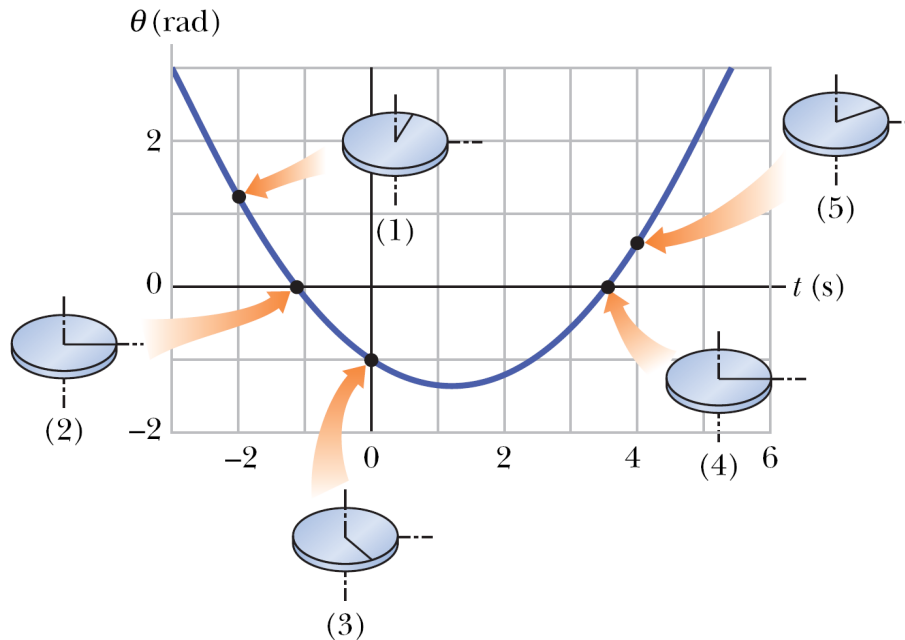
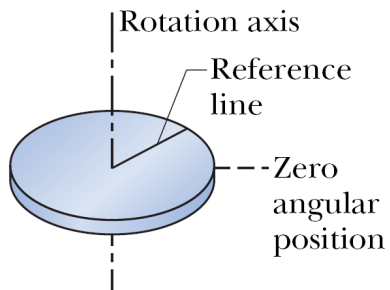
Angular Acceleration

- The **average** and **instantaneous angular accelerations** are defined as

$$\alpha_{\text{avg}} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta \omega}{\Delta t}, \quad \alpha = \lim_{\Delta t \rightarrow 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}$$

and the unit is rad/s^2 or rev/s^2 .

problem 10-1

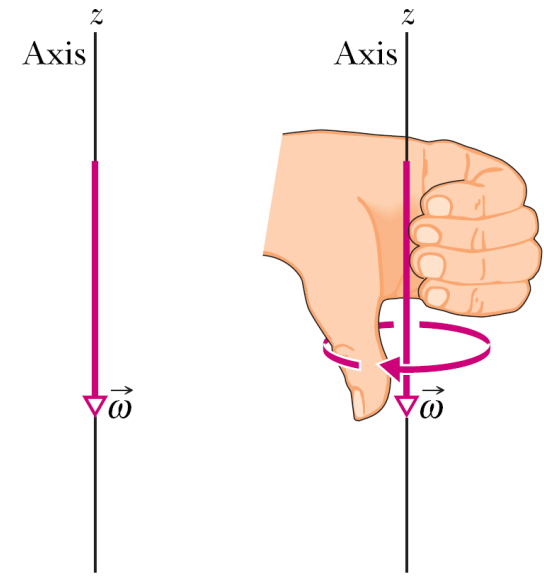
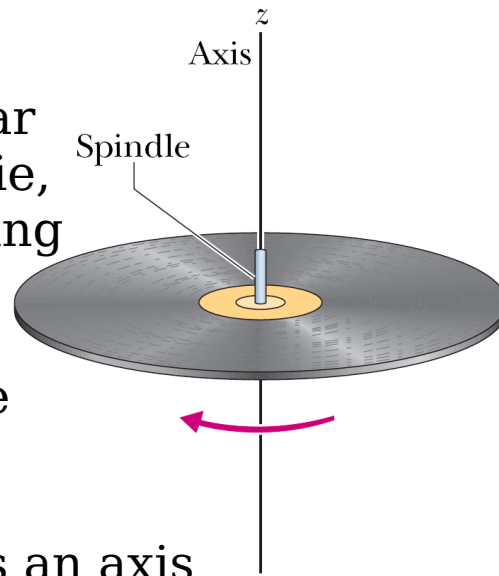


Problem 10-2

Are Angular Quantities Vectors ?

- "Can we treat the angular displacement, velocity, and acceleration of a rotating body as vectors?" The answer is a qualified "yes".
- We can represent its angular velocity as a vector pointing along the axis of rotation.

- establish a direction for the angular vector by using a **right-hand rule**, ie, curl your right hand about the rotating disk, your fingers pointing in the direction of rotation. Your thumb will then point in the direction of the angular velocity vector.

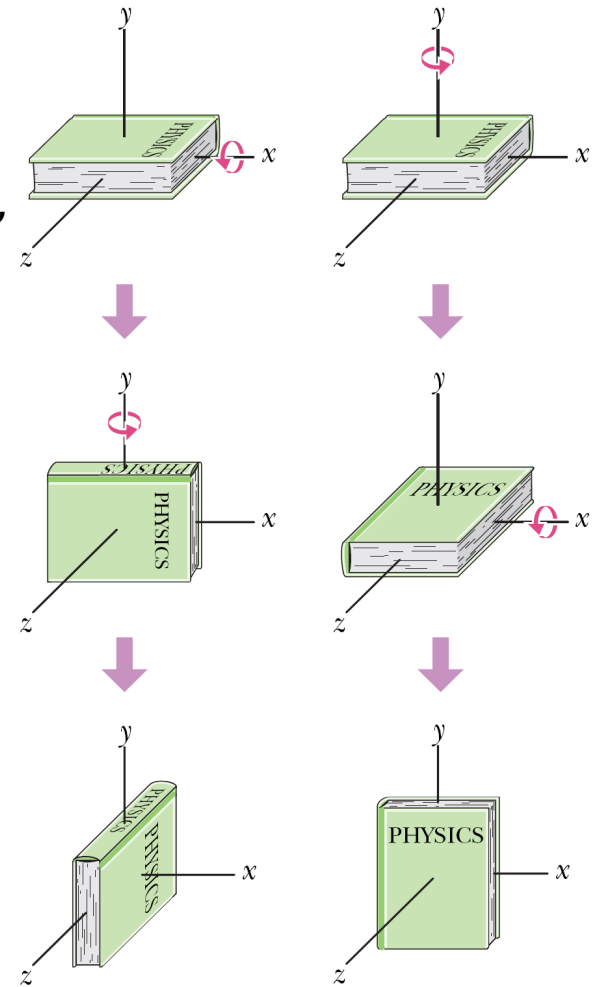


- In a pure rotation, a vector defines an axis of rotation and the motion, not a direction in which something moves.

- Considering only rotations that are about a fixed axis, the angular quantities obey all the rules for vector manipulation.

- Caution: Angular *displacements* (unless they are extremely small) *cannot* be treated as vectors. Why not? Because they won't obey the rules of vector addition.

- Although the 2 angular displacements are identical, their order is not, and the book ends up with different orientations. **Not commute!**



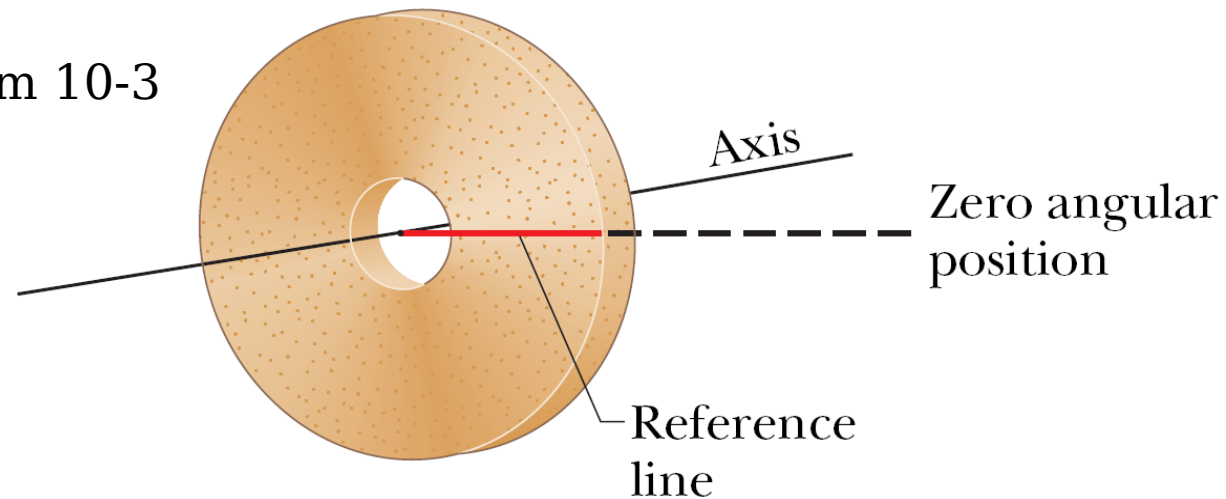
Rotation with Constant Angular Acceleration

Equations of Motion for Constant Linear Acceleration and for Constant Angular Acceleration

Linear Equation	Missing Variable	Angular Equation
$v = v_0 + a t$	$x - x_0$	$\omega = \omega_0 + \alpha t$
$x - x_0 = v_0 t + \frac{a}{2} t^2$	v	$\theta - \theta_0 = \omega_0 t + \frac{\alpha}{2} t^2$
$v^2 = v_0^2 + 2 a (x - x_0)$	t	$\omega^2 = \omega_0^2 + 2 \alpha (\theta - \theta_0)$
$x - x_0 = \frac{v_0 + v}{2} t$	a	$\theta - \theta_0 = \frac{\omega_0 + \omega}{2} t$
$x - x_0 = v t - \frac{a}{2} t^2$	v_0	$\theta - \theta_0 = \omega t - \frac{\alpha}{2} t^2$

problem 10-3

problem 10-4



Relating the Linear and Angular Variables

● When a rigid body rotates around an axis, each particle in the body moves in its own circle around that axis, and all the particles make one revolution in the same amount of time; ie, they all have the same angular speed.

● However, the farther a particle is from the axis, the greater the circumference of its circle is, and so the faster its linear speed must be.

The Position

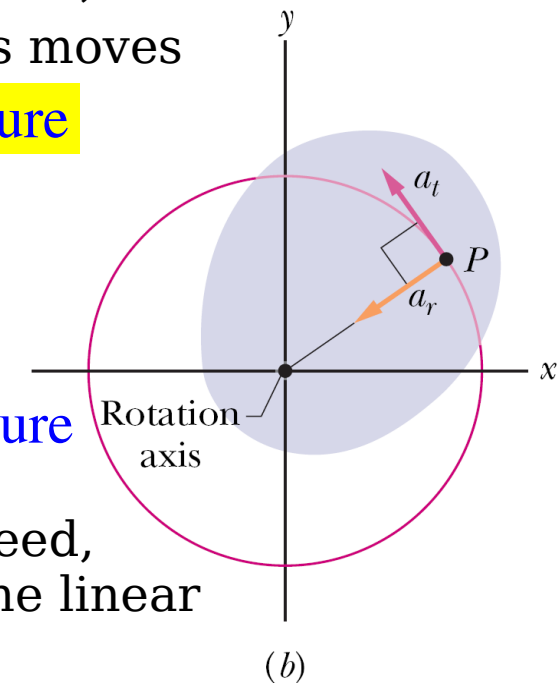
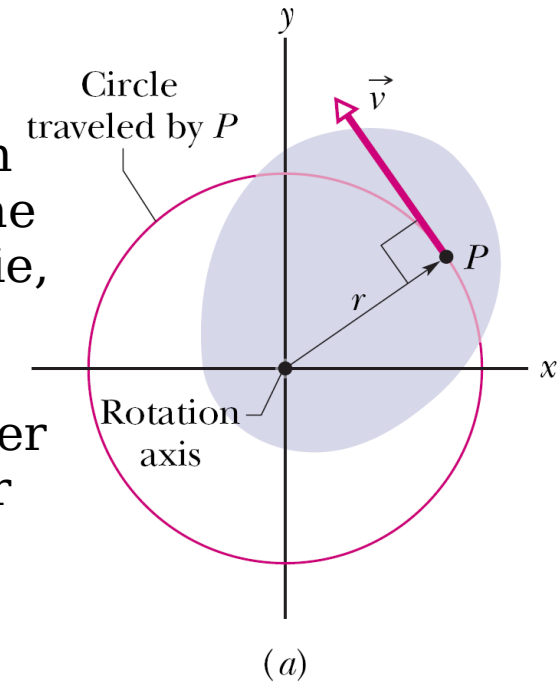
● If a reference line on a rigid body rotates through an angle θ , a point within the body at a position r from the rotation axis moves a distance s along a circular arc, then $s = \theta r$ radian measure

● *Caution:* The angle here must be measured in **radians**.

The Speed

● With r held constant, $\frac{ds}{dt} = r \frac{d\theta}{dt} \Rightarrow v = \omega r$ radian measure

● All points within the rigid body have the same angular speed, points with greater radius have greater linear speed, and the linear velocity is always tangent to the circular path of the point.



● For a constant angular speed, the period of revolution for the motion of each point and for the rigid body itself is $T = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$ radian measure

● The time for one revolution is the distance traveled in one revolution divided by the speed, also means the angular distance traveled in one revolution divided by the angular speed.

The Acceleration

● with r held constant, $\frac{dv}{dt} = r \frac{d\omega}{dt}$

● However, dv/dt represents only the part of the linear acceleration that is responsible for changes in the *magnitude* v of the linear velocity.

● We call it the *tangential component* of the linear acceleration of the point,

$$a_t = r \alpha \quad \text{radian measure}$$

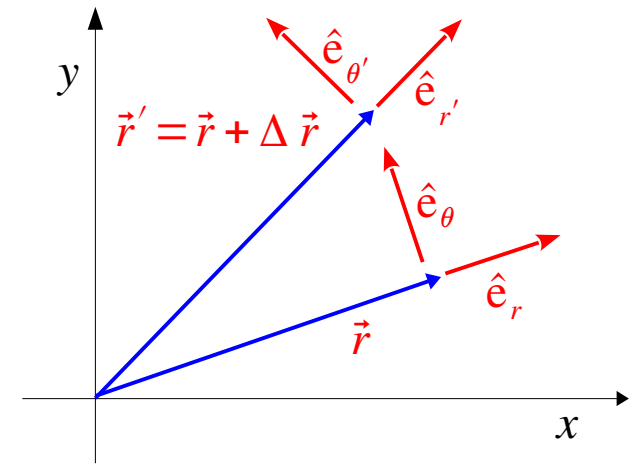
● A particle (or point) moving in a circular path has a *radial component* of linear acceleration, $a_r = v^2/r$ (directed radially inward), that is responsible for changes in

the *direction* of the linear velocity, $a_r = \frac{v^2}{r} = r \omega^2$ radian measure

● The linear acceleration of a point on a rotating rigid body has, in general, 2 components: the radially inward component is present whenever the angular velocity of the body is not 0; the tangential component is present whenever the angular acceleration is not 0.

(Advanced) Derivation for a polar coordinate

- In a polar coordinate, the unit vectors are \hat{e}_r (radial direction) and \hat{e}_θ (tangential direction)
- The position vector can be written as $\vec{r} = r \hat{e}_r$

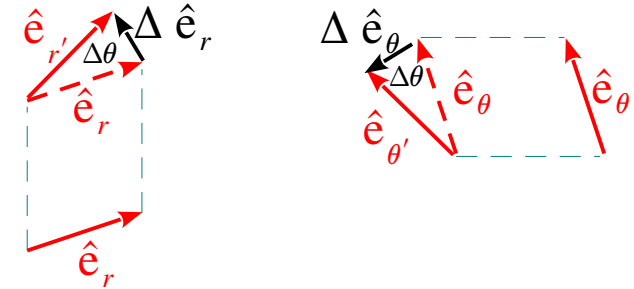


and the velocity is $\vec{u} = \frac{d}{dt} \vec{r} = \hat{e}_r \frac{dr}{dt} + r \frac{d}{dt} \hat{e}_r$

- Since the derivatives of the unit vectors in a polar coordinate are

$$\frac{d}{dt} \hat{e}_r = \omega \hat{e}_\theta, \quad \frac{d}{dt} \hat{e}_\theta = -\omega \hat{e}_r$$

therefore, $\vec{u} = \dot{r} \hat{e}_r + r \omega \hat{e}_\theta \iff \dot{r} = \frac{dr}{dt}$



- With r held constant $\vec{u} = r \omega \hat{e}_\theta = v \hat{e}_\theta$

- The acceleration is
$$\begin{aligned} \vec{a} &= \frac{d}{dt} \vec{u} = \ddot{r} \hat{e}_r + \dot{r} \frac{d}{dt} \hat{e}_r + (\dot{r} \omega + r \dot{\omega}) \hat{e}_\theta + r \omega \frac{d}{dt} \hat{e}_\theta \\ &= \ddot{r} \hat{e}_r + \dot{r} \omega \hat{e}_\theta + (\dot{r} \omega + r \dot{\omega}) \hat{e}_\theta - r \omega^2 \hat{e}_r \end{aligned}$$

$$\Rightarrow \vec{a} = \left(\ddot{r} - \frac{v^2}{r} \right) \hat{e}_r + (2 \dot{r} \omega + r \alpha) \hat{e}_\theta$$

- With r held constant $\vec{a} = -\frac{v^2}{r} \hat{e}_r + r \alpha \hat{e}_\theta = a_r \hat{e}_r + a_t \hat{e}_\theta$

Kinetic Energy of Rotation

- We treat any rotating rigid body as a collection of particles with different speeds. Then add up the kinetic energies of all the particles to find the total kinetic energy of the body,

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \dots = \sum \frac{1}{2} m_i v_i^2$$

$$= \sum \frac{1}{2} m_i (\omega r_i)^2 = \frac{1}{2} \left(\sum m_i r_i^2 \right) \omega^2$$

in which the angular velocity is the same for all particles.

- The **rotational inertia** (or **moment of inertia**) of the body with respect to the axis of rotation tells us how the mass of the rotating body is distributed about its axis of rotation, and is a constant,

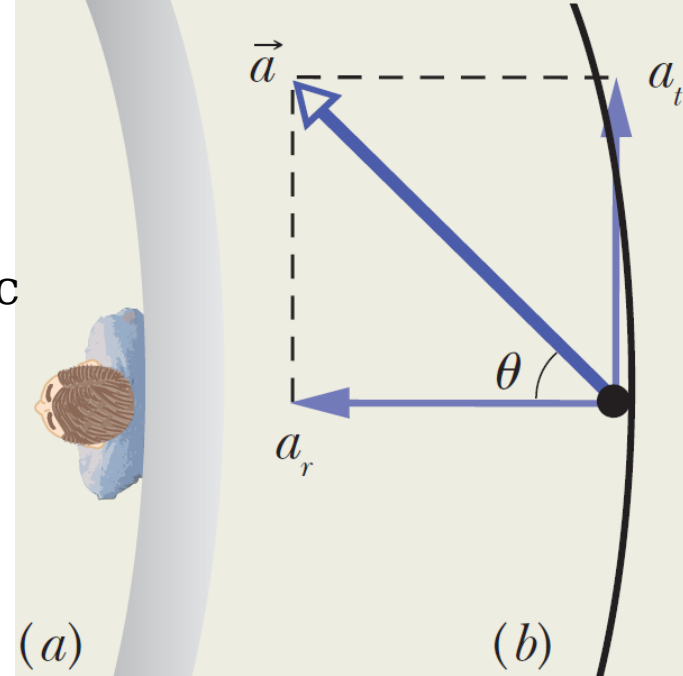
$$\mathbb{I} = \sum m_i r_i^2 \quad \text{rotational inertia}$$

- An axis must always be specified if the value of I is to be meaningful.

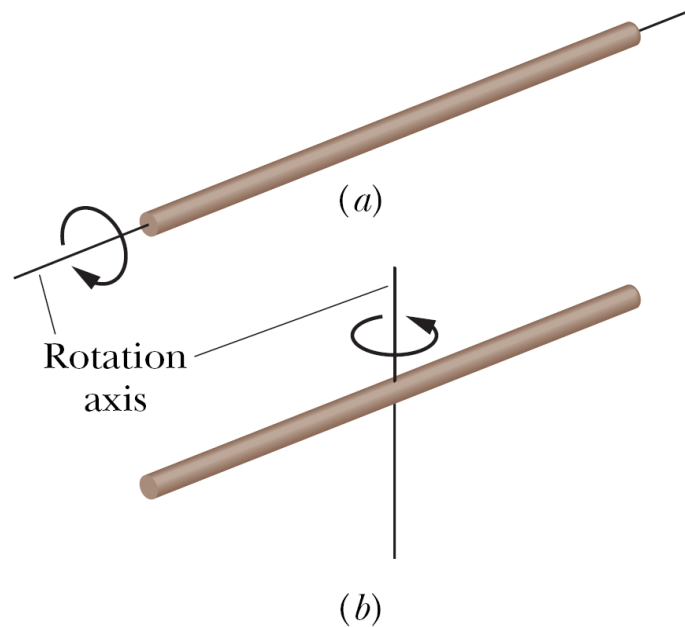
- The kinetic energy becomes $K = \frac{1}{2} \mathbb{I} \omega^2$ radian measure

and the SI unit for \mathbb{I} is the kilogram-square-meter ($\text{kg}\cdot\text{m}^2$).

- The rotational inertia of a rotating body involves not only its mass but also how that mass is distributed.



- It is easier to rotate a long, fairly heavy rod around its central (longitudinal) axis than around an axis perpendicular to the rod and through the center.
- The reason is that the mass is distributed much closer to the rotation axis in the first rotation. As a result, the rotational inertia of the rod is much smaller.
- In general, smaller rotational inertia means easier rotation.



Calculating the Rotational Inertia

● If a rigid body consists of a few particles, we can calculate its rotational inertia about a given rotation axis with $\mathbb{I} = \sum m_i r_i^2$.

● For a *continuous* rigid body, we replace the sum with an integral and define the rotational inertia of the body as $\mathbb{I} = \int r^2 dm$ rotational inertia, continuous body

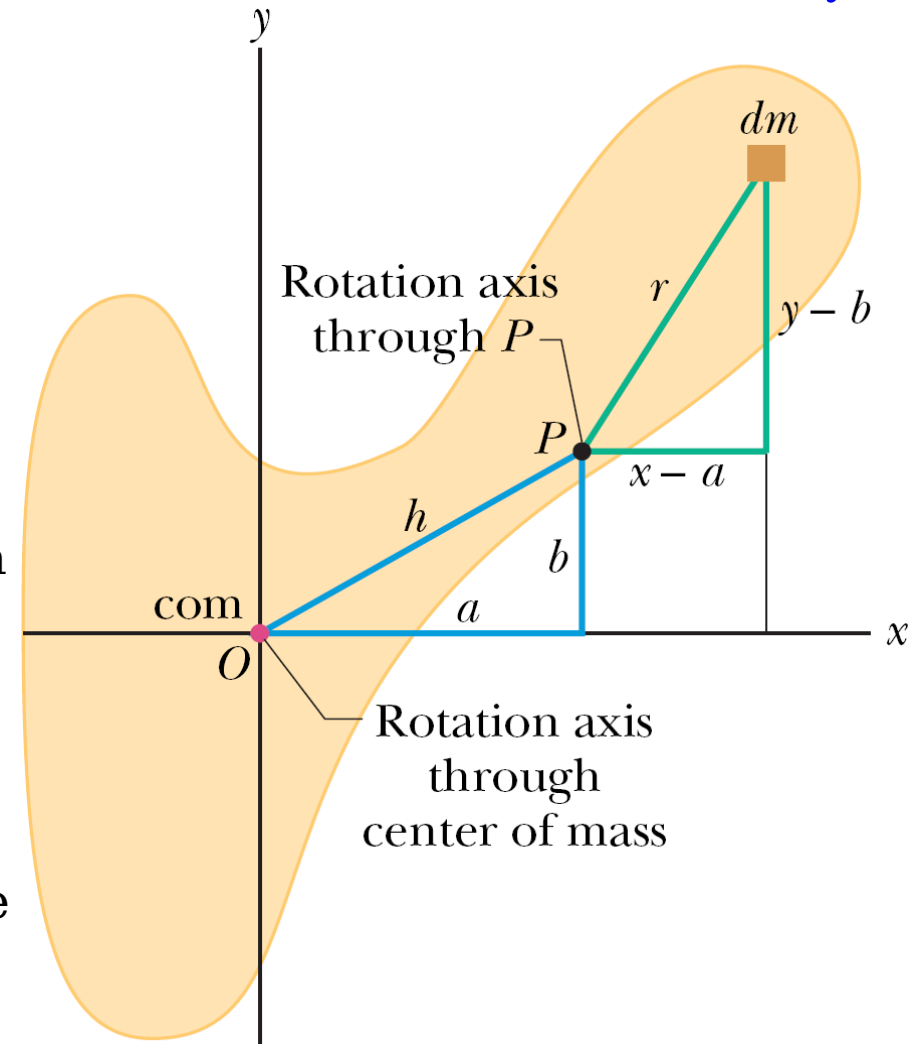
Parallel-Axis Theorem

● To find the rotational inertia \mathbb{I} of a body of mass M about a given axis: let \mathbb{I}_{com} is the rotational inertia of the body about a *parallel* axis that extends through the body's center of mass. Let h be the perpendicular distance between the given axis and the axis through the center of mass, then the rotational inertia about the given axis is

$$\mathbb{I} = \mathbb{I}_{\text{com}} + M h^2 \quad \text{parallel-axis theorem}$$

Proof of the Parallel-Axis Theorem

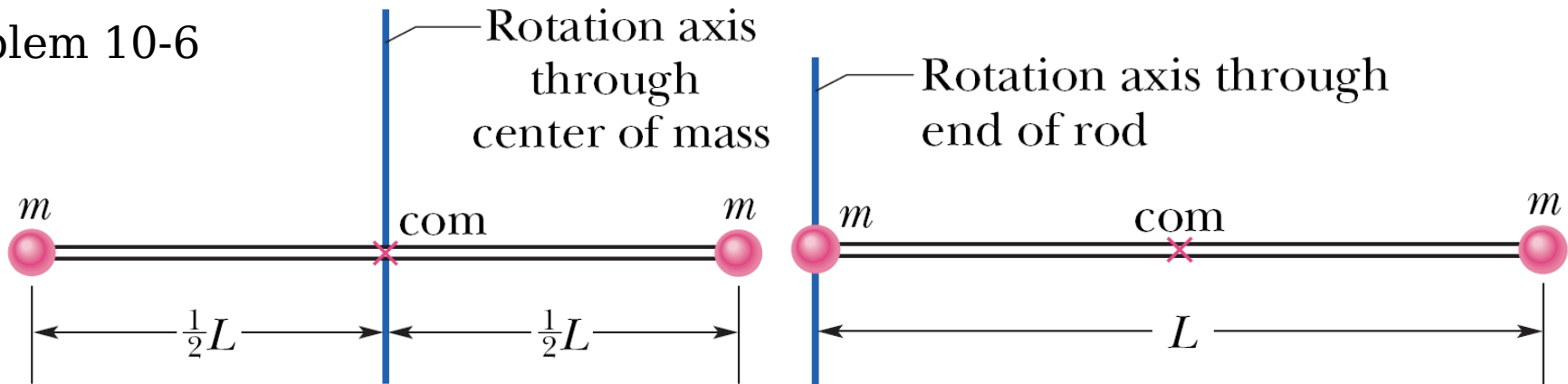
● Let O be the center of mass of a body. Place the origin of the coordinates at O .



- Consider an axis through $O \perp$ the plane of the figure, and another axis through point $P \parallel$ the first axis. And $(x,y)_P = (a,b)$. Then the rotational inertia of the body about the axis through P is

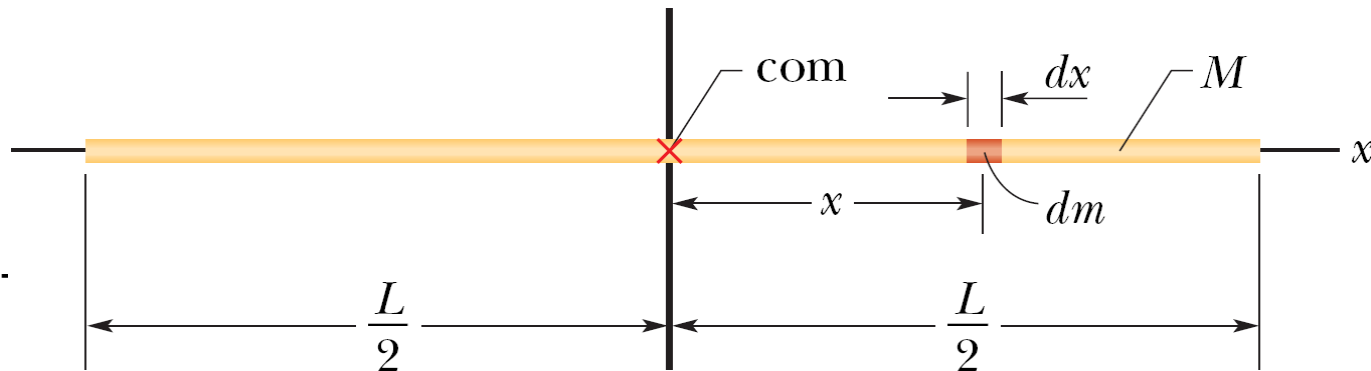
$$\begin{aligned}
 \mathbb{I} &= \int r^2 \, d m = \int [(x - a)^2 + (y - b)^2] \, d m \\
 &= \int (x^2 + y^2) \, d m - 2 a \int x \, d m - 2 b \int y \, d m + \int (a^2 + b^2) \, d m \\
 &= \mathbb{I}_{\text{com}} + 0 + 0 + M h^2
 \end{aligned}$$

Problem 10-6



problem 10-

Rotation axis



problem 10-

Axis

Hoop about central axis

$I = MR^2$ (a)

Axis

Annular cylinder (or ring) about central axis

$I = \frac{1}{2}M(R_1^2 + R_2^2)$ (b)

Axis

Solid cylinder (or disk) about central axis

$I = \frac{1}{2}MR^2$ (c)

Axis

Solid cylinder (or disk) about central diameter

$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$ (d)

Axis

Thin rod about axis through center perpendicular to length

$I = \frac{1}{12}ML^2$ (e)

Axis

Solid sphere about any diameter

$I = \frac{2}{5}MR^2$ (f)

Axis

Thin spherical shell about any diameter

$I = \frac{2}{3}MR^2$ (g)

Axis

Hoop about any diameter

$I = \frac{1}{2}MR^2$ (h)

Axis

Slab about perpendicular axis through center

$I = \frac{1}{12}M(a^2 + b^2)$ (i)

Appendix

(a) assume the line density of the ring is s , then $M = 2 \pi s R$.

$$\mathbb{I} = \int r^2 \, d m = \int R^2 s \, d \ell = s R^3 \int_0^{2 \pi} d \theta = 2 \pi s R^3 = M R^2$$

(b) assume the surface density of the thick ring is σ , then $M = \sigma \pi (R_2^2 - R_1^2)$

$$\begin{aligned} \mathbb{I} &= \int r^2 \, d m = \int_{R_1}^{R_2} \int_0^{2 \pi} \sigma r^2 (r \, d \theta \, d r) = 2 \pi \sigma \int_{R_1}^{R_2} r^3 \, d r = 2 \pi \sigma \left. \frac{r^4}{4} \right|_{R_1}^{R_2} \\ &= \pi \sigma \frac{R_2^4 - R_1^4}{2} = M \frac{R_2^2 + R_1^2}{2} \end{aligned}$$

(c) use (b)'s result and set $R_1 = 0$: $\mathbb{I} = \frac{M R^2}{2}$

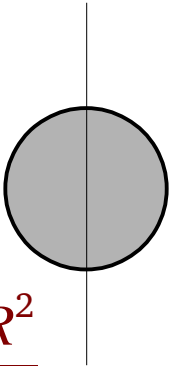
(h) assume the line density of the ring is s , then $M = 2 \pi s R$.

$$\mathbb{I} = \int r^2 \, d m = \int_0^{2 \pi} R^2 \sin^2 \theta s R \, d \theta = 2 s R^3 \int_0^{\pi} \sin^2 \theta \, d \theta = \pi s R^3 = \frac{M R^2}{2}$$

For a disk with a rotating axis like the figure, assume the surface density is σ , then $M = \pi \sigma R^2$. Use the result of (h),

$$\mathbb{I} = \int r^2 \, d m = \int_0^R d r \int_0^{2 \pi} (r \sin \theta)^2 \sigma r \, d \theta = \pi \sigma \int_0^R r^3 \, d r = \pi \sigma \frac{R^4}{4} = \frac{M R^2}{4}$$

(d) assume the volume density of the bar is ρ , then $M = \pi \rho R^2 L$. By using the above result and the parallel-axis theorem,



$$\begin{aligned}
 I &= \int_{-L/2}^{L/2} \frac{\pi \rho R^4}{4} d\ell + \int_{-L/2}^{L/2} \pi \rho R^2 \ell^2 d\ell = \frac{\pi \rho R^4}{4} \int_{-L/2}^{L/2} d\ell + \pi \rho R^2 \int_{-L/2}^{L/2} \ell^2 d\ell \\
 &= \frac{\pi \rho R^4 L}{4} + \frac{\pi \rho R^2 L^3}{12} = \frac{M R^2}{4} + \frac{M L^2}{12}
 \end{aligned}$$

(e) assume the line density of the rod is s , then $M = sL$.

$$I = \int_{-L/2}^{L/2} \ell^2 d m = \int_{-L/2}^{L/2} s \ell^2 d\ell = s \left. \frac{\ell^3}{3} \right|_{-L/2}^{L/2} = \frac{s L^3}{12} = \frac{M L^2}{12}$$

(g) assume the surface density of the sphere is σ , then $M = 4 \pi R^2 \sigma$.

$$\begin{aligned}
 I &= \int r^2 d m = \int_0^\pi \int_0^{2\pi} (R \sin \theta)^2 \sigma (R d\theta) (R \sin \theta d\phi) = \sigma R^4 \int_0^\pi \sin^3 \theta d\theta \int_0^{2\pi} d\phi \\
 &= 2 \pi \sigma R^4 \int_0^\pi \sin^3 \theta d\theta = \frac{8}{3} \pi \sigma R^4 = \frac{2}{3} M R^2
 \end{aligned}$$

(f) assume the volume density of the ball is ρ , then $M = \frac{4 \pi}{3} \rho R^3$

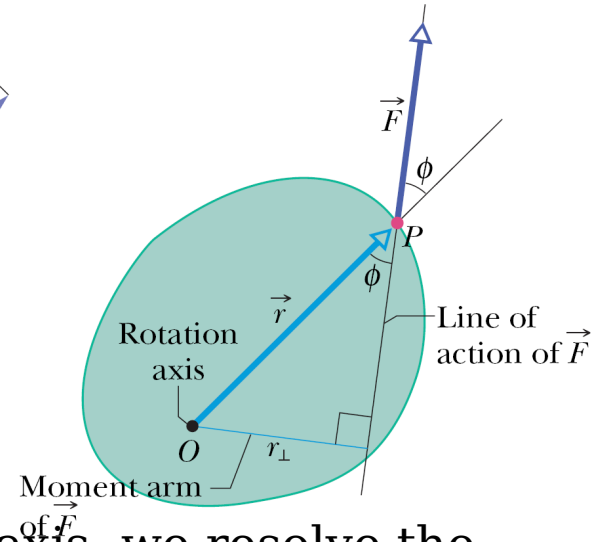
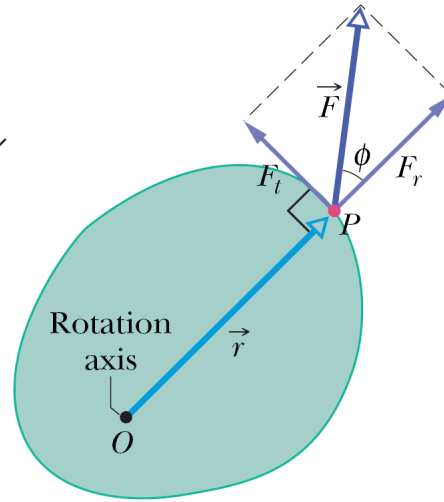
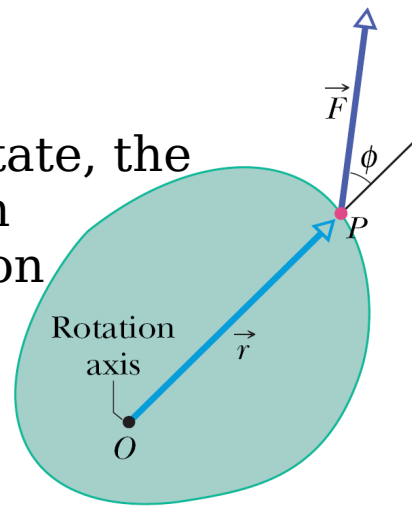
$$\begin{aligned}
 I &= \int r^2 d m = \int_0^R \int_0^\pi \int_0^{2\pi} (r \sin \theta)^2 \rho (r \sin \theta d\phi) (r d\theta) d r \\
 &= \rho \int_0^R r^4 d r \int_0^\pi \sin^3 \theta d\theta \int_0^{2\pi} d\phi = \frac{2}{5} \pi \rho R^5 \int_0^\pi \sin^3 \theta d\theta = \frac{8}{15} \pi \rho R^5 = \frac{2}{5} M R^2
 \end{aligned}$$

(i) assume the surface density of the sphere is σ , then $M = a b \sigma$.

$$I = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} (x^2 + y^2) \sigma d x d y = \sigma \int_{-a/2}^{a/2} \left(b x^2 + \frac{b^3}{12} \right) d x = \sigma a b \frac{a^2 + b^2}{12} = M \frac{a^2 + b^2}{12}$$

Torque

● To make an object rotate, the magnitude, the position applied and the direction of the force are all important.



● To determine how a force results in a rotation of the body around the rotation axis, we resolve the force into 2 components:

— *radial component*: points along radius, not cause rotation;

— *tangential component*: perpendicular to radius, $F_t = F \sin \theta$, cause rotation.

● The ability of a force to rotate the body depends not only on the magnitude of its tangential component, but also on just how far from the rotation axis.

● Define **torque** as the product of the 2 factors $\tau = (r) (F \sin \phi)$

● 2 equivalent ways of computing the torque are

$$\tau = r F_t = (r) (F \sin \phi) = (r \sin \phi) (F) = r_{\perp} F$$

where r_{\perp} is the perpendicular distance between the rotation axis and an extended line running through the force. This extended line is called the **line of action** of the force, and r_{\perp} is called the **moment arm** of the force.

● The SI unit of torque is the newton-meter ($\text{N} \cdot \text{m}$), but torque is not a work.

Newton's 2nd Law for Rotation

● A torque can cause rotation of a rigid body, therefore, we want to relate the net torque on a rigid body to the angular net acceleration that torque causes about a rotation axis,

$$\tau_{\text{net}} = \mathbb{I} \alpha \quad \text{Newton's 2nd law for rotation}$$

Proof of the Equation

Relate F_t to the particle's tangential acceleration at along the path with Newton's 2nd law, $F_t = m a_t$

Then the torque acting on the particle is

$$\tau = r F_t = r m a_t$$

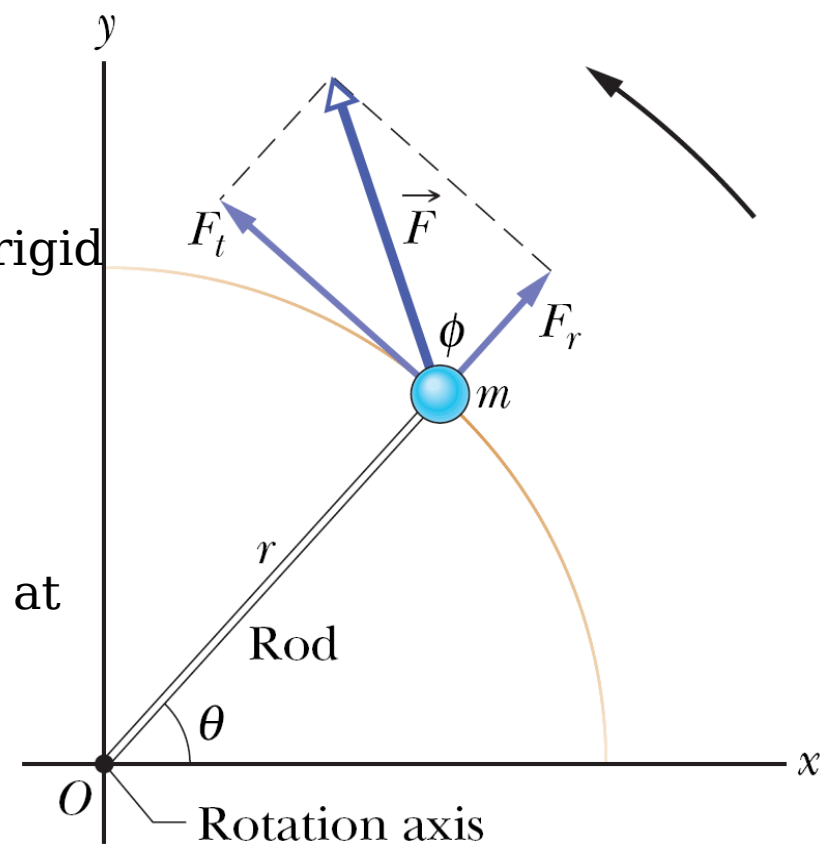
$$\text{Since } a_t = r \alpha \Rightarrow \tau = r m (r \alpha) = (m r^2) \alpha$$

The quantity in parentheses on the right is the rotational inertia of the particle about the rotation axis, thus

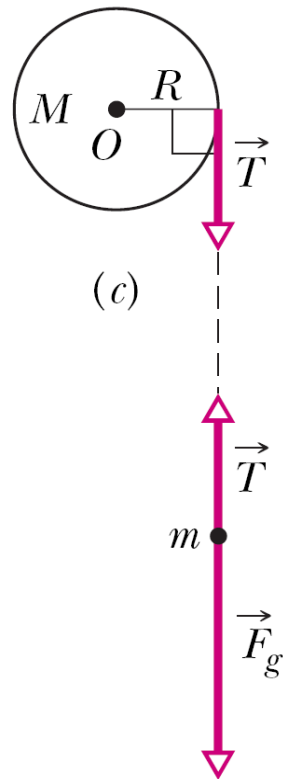
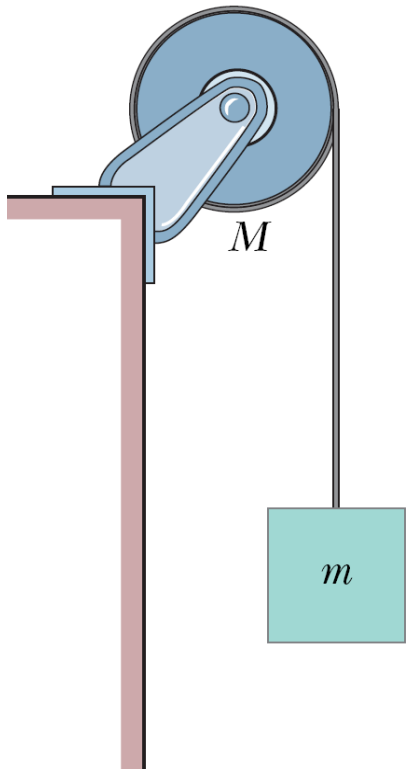
$$\tau = \mathbb{I} \alpha \quad \text{radian measure}$$

For the situation in which more than one force is applied to the particle,

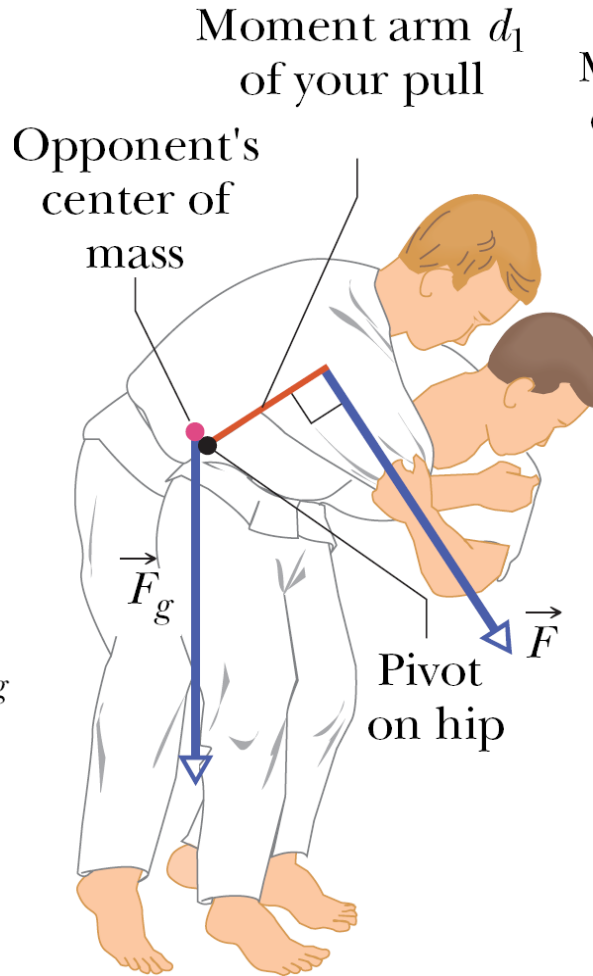
$$\tau_{\text{net}} = \mathbb{I} \alpha \quad \text{radian measure}$$



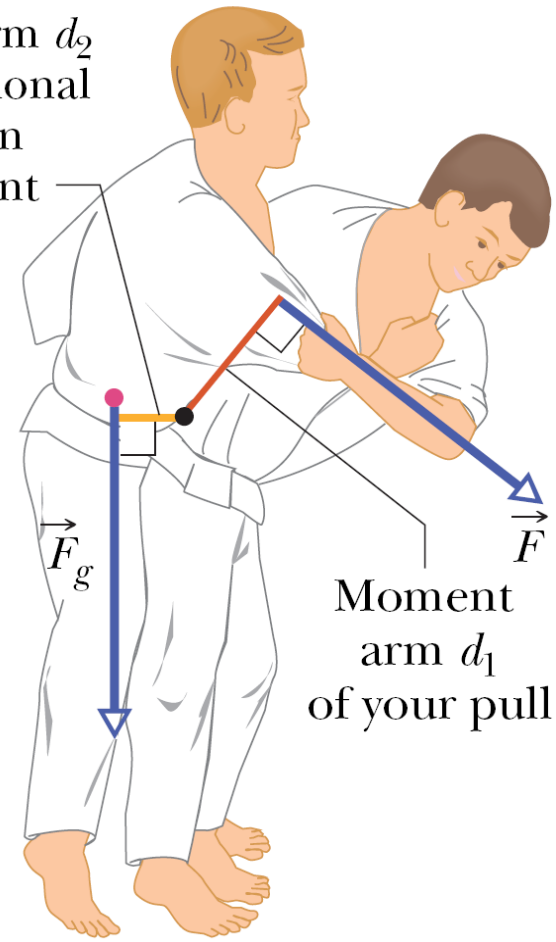
problem 10-10



problem 10-9



Moment arm d_2 of gravitational force on opponent



The chosen problems: 7, 42, 66.

Work and Rotational Kinetic Energy

● When a torque accelerates a rigid body in rotation about a fixed axis, the torque does work on the body.

● Relate the change in kinetic energy to the work with the work-kinetic energy

theorem, $\Delta K = K_f - K_i = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2 = W$ work-kinetic energy theorem

● The work is $W = \int_{\theta_i}^{\theta_f} \tau d\theta$ work, rotation about fixed axis

● If torque is constant $W = \tau (\theta_f - \theta_i)$ work, constant torque

● The power is $P = \frac{dW}{dt} = \tau \omega$ power, rotation about fixed axis

Proof of these equations

since $\Delta K = K_f - K_i = W$ and $K = \frac{1}{2} m v^2 = \frac{1}{2} m r^2 \omega^2$

it gives $\Delta K = \frac{1}{2} (m r^2) \omega_f^2 - \frac{1}{2} (m r^2) \omega_i^2 = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2 = W$

Although We derived it for a rigid body with one particle, but it holds for any rigid body rotated about a fixed axis.

For the work, $dW = \vec{F} \cdot d\vec{r} = F_t r d\theta = \tau d\theta$

thus the work done during a finite angular displacement is $W = \int_{\theta_i}^{\theta_f} \tau d\theta$

For the power $P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau \omega$

Problem 10-11

Some Corresponding Relations for Translational and Rotational Motion

Pure Translation (Fixed Direction)		Pure Rotation (Fixed Axis)	
Position	x	Angular position	θ
Velocity	$v = dx/dt$	Angular velocity	$\omega = d\theta/dt$
Acceleration	$a = dv/dt$	Angular acceleration	$\alpha = d\omega/dt$
Mass	m	Rotational inertia	\mathbb{I}
Newton's 2nd law	$F_{\text{net}} = ma$	Newton's 2nd law	$\tau_{\text{net}} = \mathbb{I} \alpha$
Work	$W = \int F dx$	Work	$W = \int \tau d\theta$
Kinetic energy	$K = \frac{1}{2} m v^2$	Kinetic energy	$K = \frac{1}{2} \mathbb{I} \omega^2$
Power (constant force)	$P = F v$	Power (constant torque)	$P = \tau \omega$
W-K theorem	$W = \Delta K$	W-K theorem	$W = \Delta K$