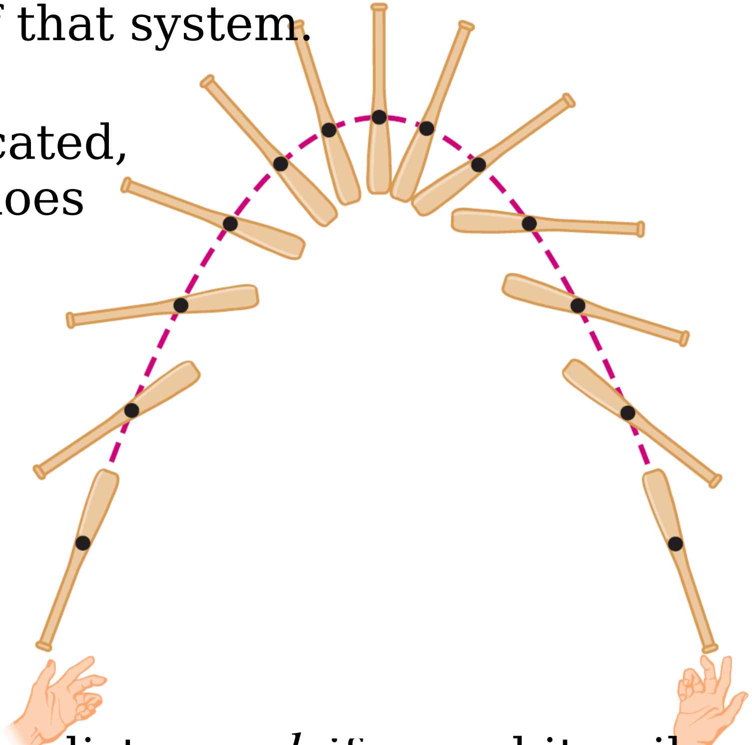
Chapter 9 Center of Mass and Linear Momentum

- we can simplify the complicated motion of a system of objects by determining a special point of the system — the *center of mass* of that system.

- Although the motion of an object could be complicated, it has one special point – the center of mass – that does move in a simple parabolic path.



The Center of Mass

The **center of mass** of a system of particles is the point that moves as though (1) all of the system's mass were concentrated there and (2) all external forces were applied there.

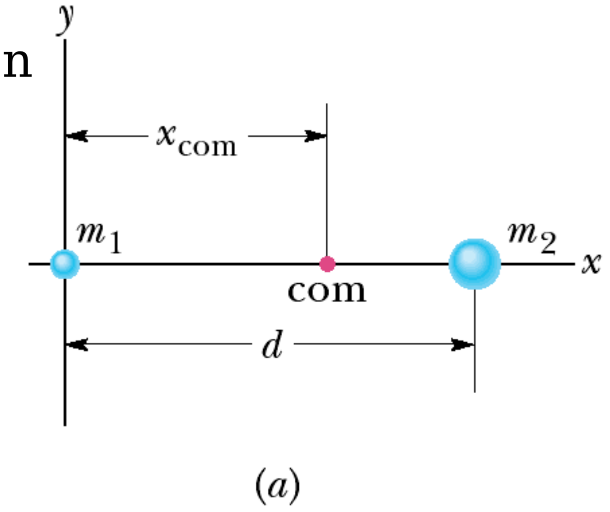
Systems of Particles

- For two particles of masses m_1 and m_2 separated by distance d, if we arbitrarily choose the origin of an x axis to coincide with the particle of m_1, then we *define* the position of the center of mass (com) of this two-particle system to be

$$x_{com} = \frac{m_2}{m_1 + m_2} d$$

For example, if $m_2 = 0$, then $x_{\text{com}} = 0$;
if $m_1 = 0$, then $x_{\text{com}} = d$;
if $m_1 = m_2$, then $x_{\text{com}} = d/2$.

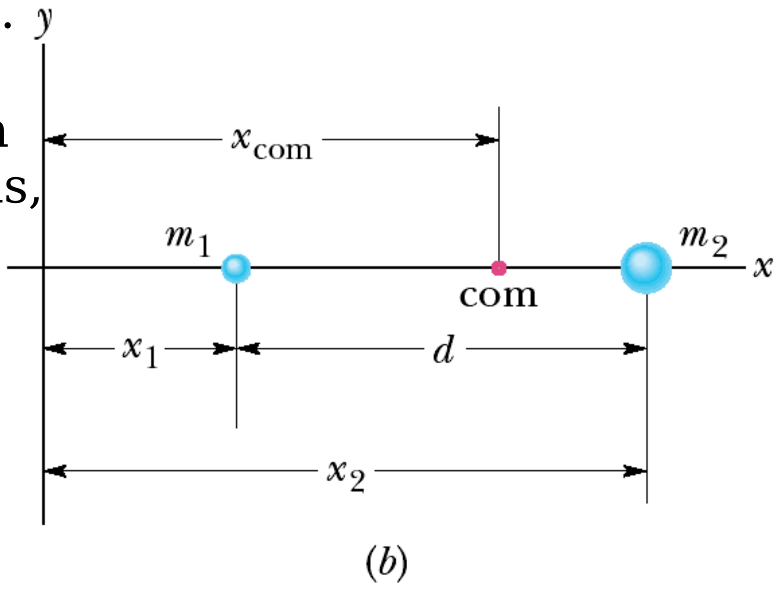
- the center of mass must lie somewhere between the 2 particles.



- For a more generalized situation, in which the coordinate system has been shifted, then the position of the center of mass is

$$x_{\text{com}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_1 x_1 + m_2 x_2}{M}$$

where $M = m_1 + m_2$, the total mass of that system.



- extend this equation to a more general situation in which n particles are strung out along the x axis, then the location of the center of mass is

$$x_{\text{com}} = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{M} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i$$

where $M = m_1 + m_2 + \ldots + m_n$

- In 3-D situations, the center of mass is identified by 3 coordinates as

$$\boxed{x_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i , \quad y_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i y_i , \quad z_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i z_i}$$

- With the language of vector, the position of a particle is $\vec{r}_i = x_i\,\hat{\mathrm{i}} + y_i\,\hat{\mathrm{j}} + z_i\,\hat{\mathrm{k}}$ then the position of the center of mass of a system of particles is

$$\vec{r}_{\mathrm{com}} = x_{\mathrm{com}}\,\hat{\mathrm{i}} + y_{\mathrm{com}}\,\hat{\mathrm{j}} + z_{\mathrm{com}}\,\hat{\mathrm{k}}$$

combining the equations above gives

$$\boxed{\vec{r}_{\mathrm{com}} = \frac{1}{M}\sum_{i=1}^{n} m_i\,\vec{r}_i}$$

Solid Bodies

- An ordinary object contains so many particles that we can best treat it as a continuous distribution of matter.

- the coordinates of the center of mass are defined as

$$\boxed{x_{\mathrm{com}} = \frac{1}{M}\int x\,\mathrm{d}m\,, \quad y_{\mathrm{com}} = \frac{1}{M}\int y\,\mathrm{d}m\,, \quad z_{\mathrm{com}} = \frac{1}{M}\int z\,\mathrm{d}m}$$

- For a uniform object, the density, or mass per unit volume, is uniform,

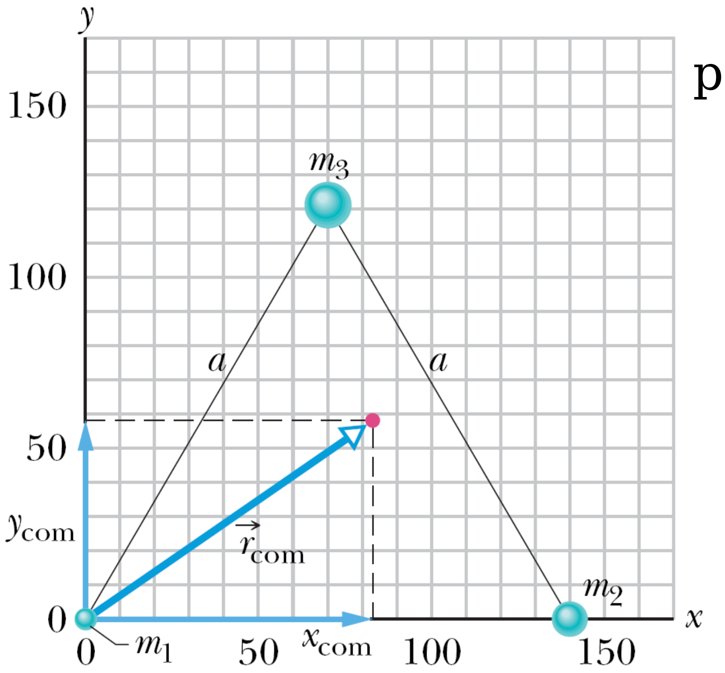
$$\rho = \frac{\mathrm{d}m}{\mathrm{d}V} = \frac{M}{V}$$

Using dm = (M/V) dV,

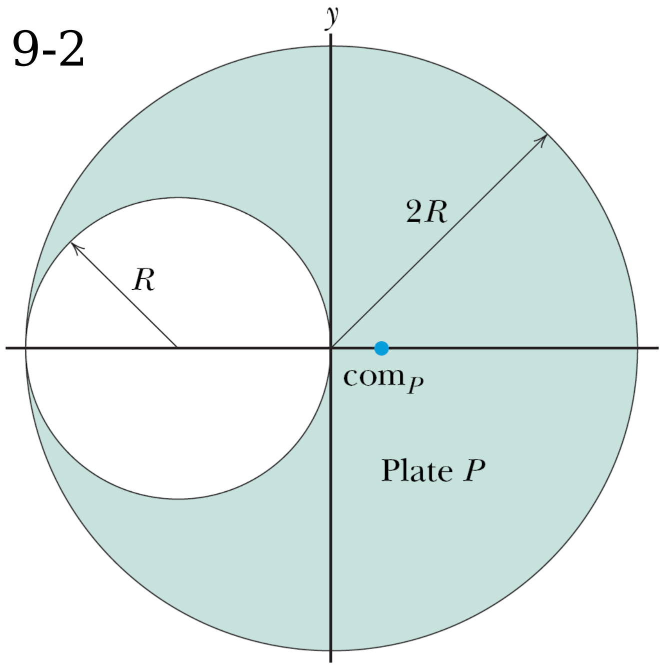
$$\boxed{x_{\mathrm{com}} = \frac{1}{V}\int x\,\mathrm{d}V\,, \quad y_{\mathrm{com}} = \frac{1}{V}\int y\,\mathrm{d}V\,, \quad z_{\mathrm{com}} = \frac{1}{V}\int z\,\mathrm{d}V}$$

- The center of mass of an object need not lie within the object.

problem 9-1



problem 9-2

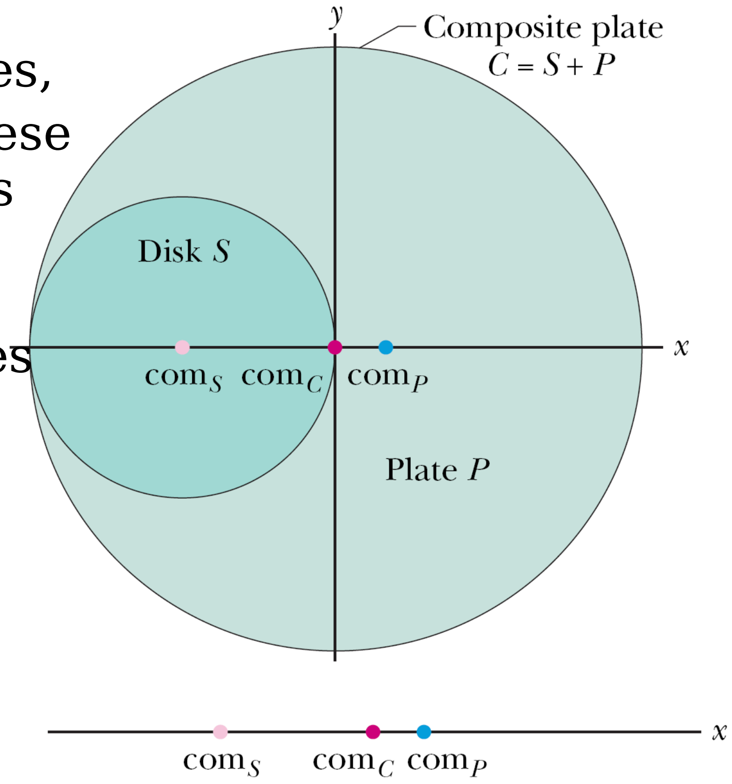


Newton's 2^{nd} Law for a System of Particles

- For an assemblage of *n* particles of different masses, we are interested not in the individual motions of these particles but only in the motion of the center of mass of the assemblage.

- Although the center of mass is just a point, it moves like a particle whose mass is equal to the total mass of the system; we can assign a position, a velocity, and an acceleration to it.

- The Newton‘s 2^{nd} law for the motion of the center of mass of a system



$$\boxed{\vec{F}_{\text{net}} = M\,\vec{a}_{\text{com}} \qquad \text{system of particles}}$$

(1) $\vec{F}_{\text{net}}$ is the net force of *all external forces* that act on the system;
(2) M is the *total mass* of the system;
(3) $\vec{a}_{\text{com}}$ is the acceleration of the *center of mass* of the system.

- We can decompose the above equation into

$$\boxed{F_{\text{net},x} = M\,a_{\text{com},x}, \qquad F_{\text{net},y} = M\,a_{\text{com},y}, \qquad F_{\text{net},z} = M\,a_{\text{com},z}}$$

Proof of the Equation

For a system of *n* particles: $M\,\vec{r}_{\text{com}} = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots + m_n\vec{r}_n$

Differentiating the above equation with respect time gives

$$M\,\vec{v}_{\text{com}} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots + m_n\vec{v}_n$$

Differentiating the above equation with respect time gives

$$M\,\vec{a}_{\text{com}} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots + m_n\vec{a}_n$$

Although the center of mass is just a geometrical point, it has a position, a velocity, and an acceleration, as if it were a particle.

From Newton's 2nd law, $M\,\vec{a}_{\text{com}} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots + \vec{F}_n$

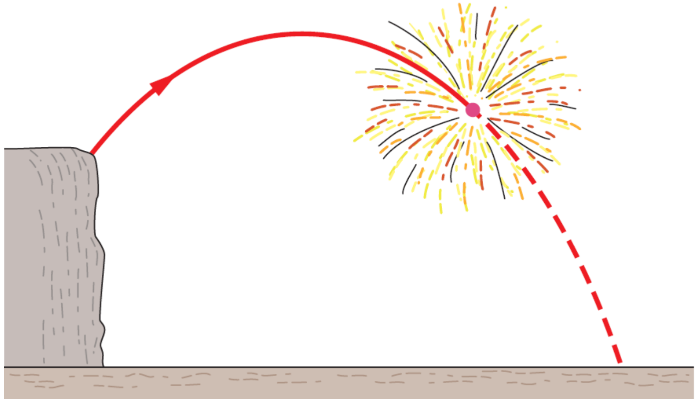
- the forces in the right side will be either the internal forces or the external forces.

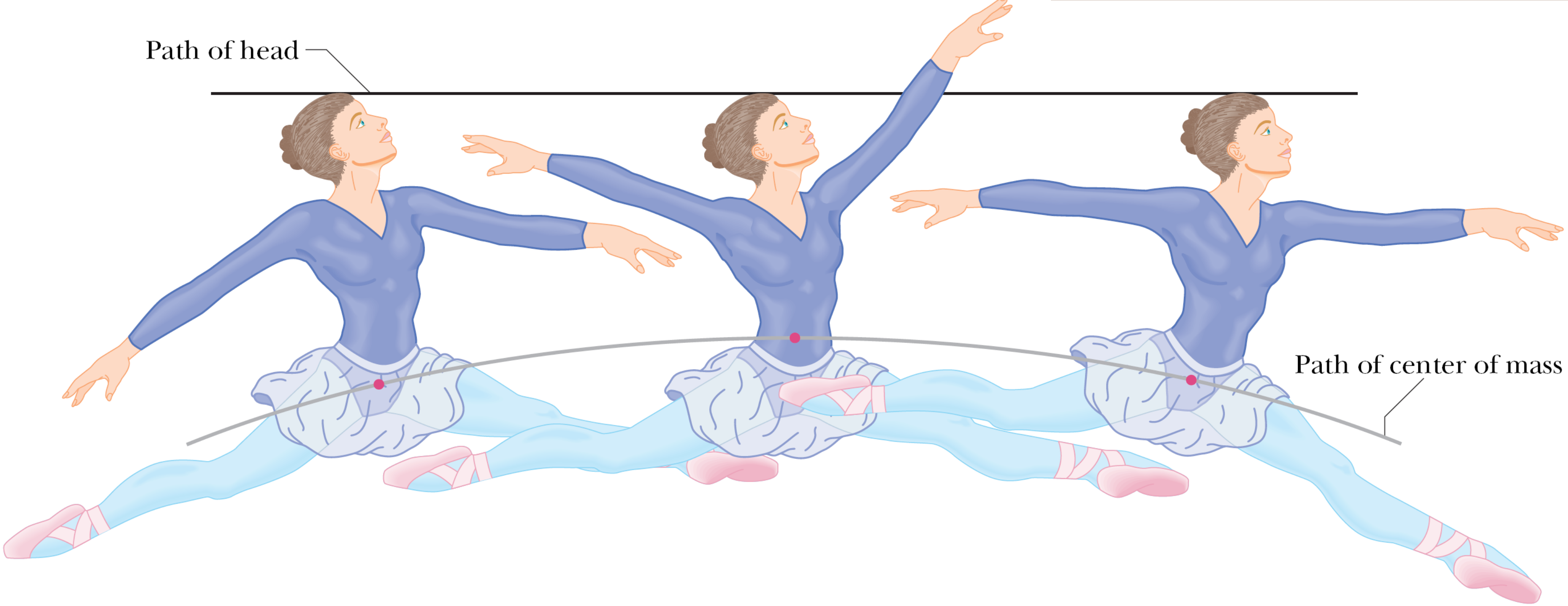
- By Newton's 3rd law, the internal forces form third-law force pairs and cancel out in the sum.

- The left is the vector sum of all the external forces that act on the system, thus

$$\vec{F}_{\text{net}} = M\vec{a}_{\text{com}}$$

- The forces of the explosion of a rocket are *internal* to the system; that is, they are forces on parts of the system from other parts.



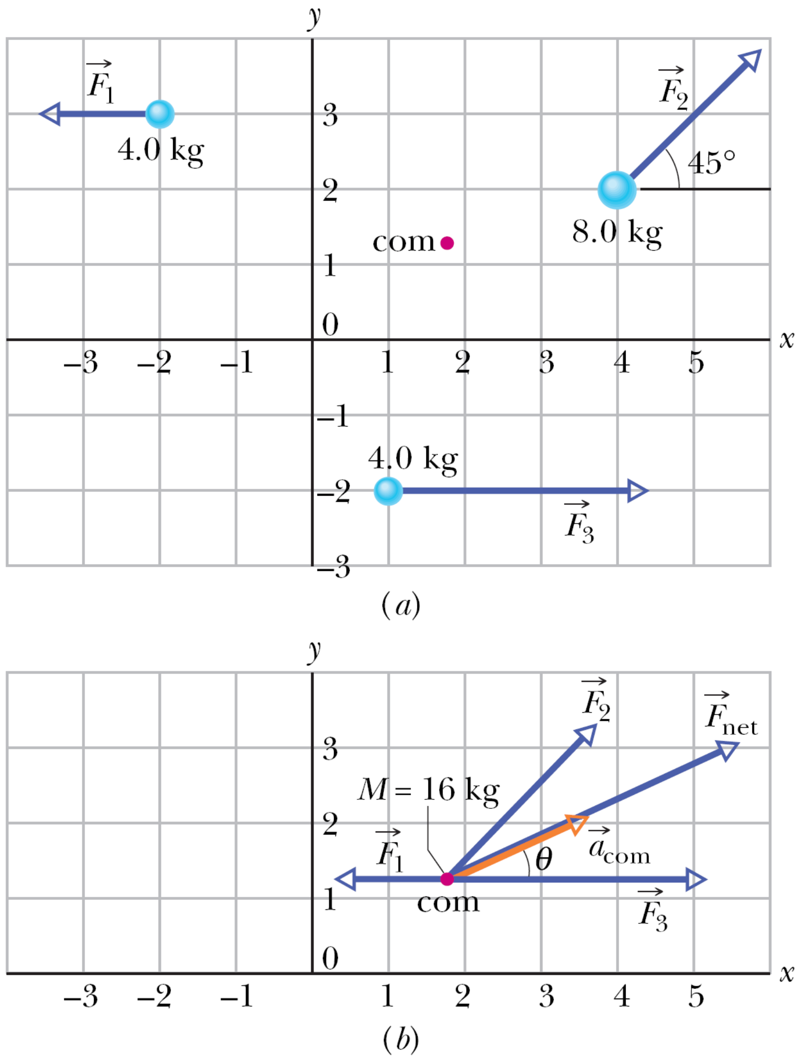


- The net *external* force acting on the system is the gravitational force on the system, regardless of whether the rocket explodes. Thus, $\vec{a}_{\text{com}} = \vec{g}$

- The center of mass of the fragments follows the same parabolic trajectory that the rocket would have followed had it not exploded.

- Although a ballet dancer raises her arms and stretches her legs out to shift her center of mass upward through her body, the center of mass still follows a parabolic path across the stage.

problem 9-3



Linear Momentum

- The **linear momentum** of a particle is a vector quantity that is defined as

$$\boxed{\vec{p} = m\,\vec{v} \qquad \text{linear momentum of a particle}}$$

- Since m is always a positive scalar quantity, the momentum and the velocity have the same direction.

- the SI unit for momentum is the kilogram-meter per second (kg · m/s).

- expressed Newton's 2nd law of motion in terms of momentum:

The time rate of change of the momentum of a particle is equal to the net force acting on the particle and is in the direction of that force.

In equation form this becomes $\boxed{\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt}}$

- the linear momentum can be changed only by a net external force. If there is no net external force, the momentum *cannot* change.

- the equation can be manipulated as $\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = m\vec{a}$

- the relations $\vec{F}_{\text{net}} = d\vec{p}/dt$ and $\vec{F}_{\text{net}} = m\vec{a}$ are equivalent expressions of Newton's 2nd law of motion for a particle.

The Linear Momentum of a System of Particles

- Consider a system of n particles, each with its own mass, velocity, and linear momentum. The system as a whole has a total linear momentum, which is defined to be the vector sum of the individual particles linear momenta,

$$\begin{aligned}\vec{P} &= \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots + \vec{p}_n \\ &= m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots + m_n\vec{v}_n\end{aligned}$$

thus, $\boxed{\vec{P} = M\vec{v}_{\text{com}} \qquad \text{linear momentum, system of particles}}$

The linear momentum of a system of particles is equal to the product of the total mass M of the system and the velocity of the center of mass.

- Taking the time derivative of the above equation gives

$$\frac{d\vec{P}}{dt} = M\frac{d\vec{v}_{\mathrm{com}}}{dt} = M\vec{a}_{\mathrm{com}}$$

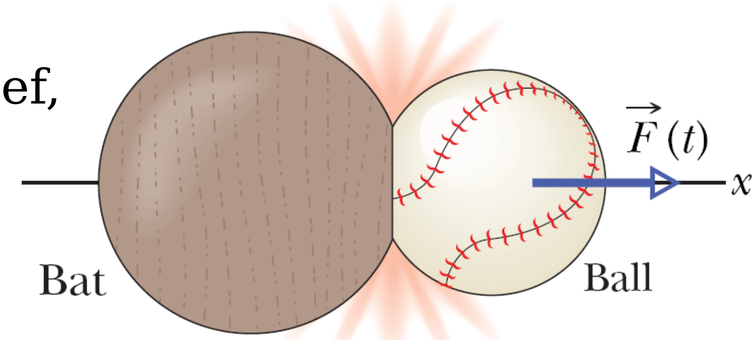
- thus, Newton's 2nd law for a system of particles is

$$\boxed{\vec{F}_{\mathrm{net}} = \frac{d\vec{P}}{dt} \qquad \text{system of particles}}$$

- The net external force on a system of particles changes the linear momentum of the system. And the linear momentum can be changed only by a net external force. If there is no net external force, the linear momentum *cannot* change.

Collision and Impulse

- In a collision, the external force on a body is brief, has large magnitude, and suddenly changes the body's momentum.



Single Collision

- A object experiences a force that varies during the collision and changes the linear momentum of the object. Thus, in time interval dt, the change in the object's momentum is

$$d\vec{p} = \vec{F}(t)\,dt$$

- The net change in the object's momentum due to the collision form a time t_i to t_f is

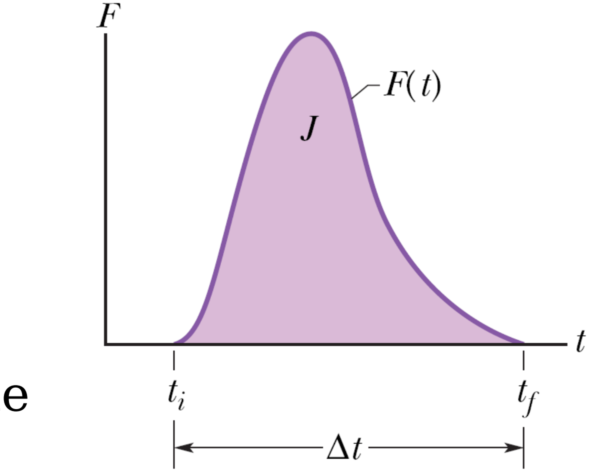
$$\vec{J} \Leftarrow \Delta\vec{p} \Leftarrow \vec{p}_f - \vec{p}_i \Leftarrow \int_{t_i}^{t_f} \mathrm{d}\,\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)\,\mathrm{d}\,t$$

- The impulse of of the collision is a measure of both the magnitude and the duration of the collision force:

$$\boxed{\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)\,\mathrm{d}\,t \quad \text{impulse defined}}$$

- the change in an object's momentum is equal to the impulse on the object:

$$\boxed{\Delta\vec{p} = \vec{J} \quad \text{linear momentum-impulse theorem}}$$

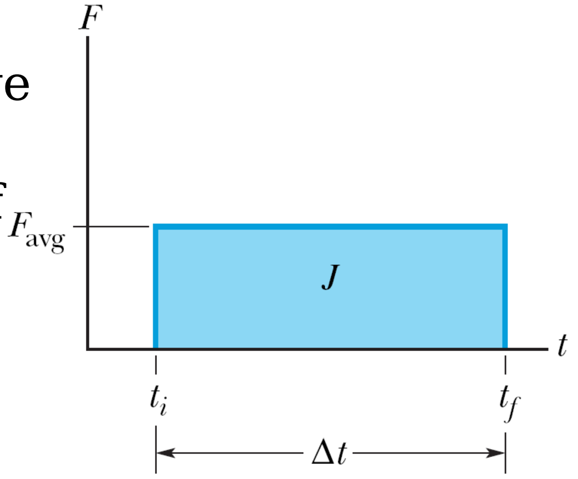


- In a component form, eg, for x axis,

$$J_x \Leftarrow \Delta p_x \Leftarrow p_{f\,x} - p_{i\,x} \Leftarrow \int_{t_i}^{t_f} \mathrm{d}\,p_x = \int_{t_i}^{t_f} F_x\,\mathrm{d}\,t$$

- If the function of the force is known, we can evaluate the impulse by integrating the force function.

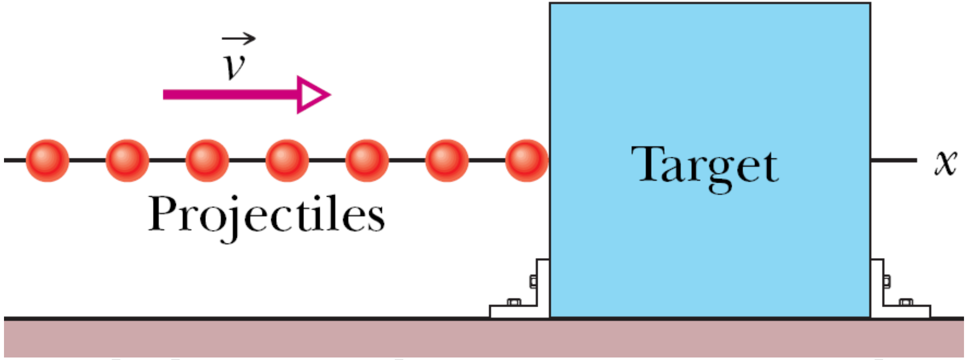
- In many situations, the force function is unknown, but we do know the average magnitude F_{avg} of the force and the durationΔt (= $t_f - t_i$) of the collision, then the magnitude of the impulse can be



$$J = F_{\text{avg}}\,\Delta t$$

- For a bat hitting a ball, Newton's 3rd law tells us that the force on the bat has the same magnitude but the opposite direction as the force on the ball.

Series of Collisions



- We would like to know the average force during a large number of collisions.

- Let n be the number of projectiles that collide in a time interval Δt. Thus each projectile has initial momentum mv and undergoes a change Δp in linear momentum because of the collision.

- The total change in linear momentum for n projectiles during interval Δt is $n\Delta p$, thus the impulse on the target is $J = -n\,\Delta p$

the minus sign indicates that J and Δp have opposite directions.

- The average force acting on the target is $F_{\text{avg}} = \frac{J}{\Delta t} = \frac{-n}{\Delta t}\Delta p = \frac{-n}{\Delta t} m\,\Delta v$

- If the projectiles stop upon impact, then $\Delta v = v_f - v_i = 0 - v = -v$

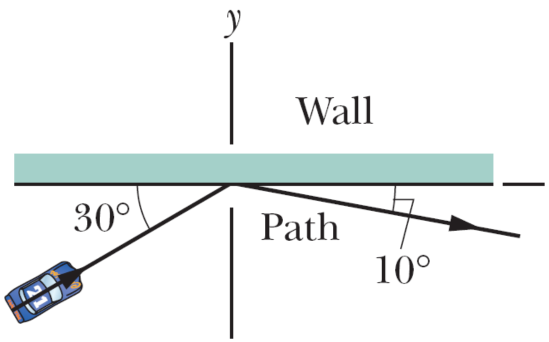
- If the projectiles bounce directly backward from the target with no change in speed, then

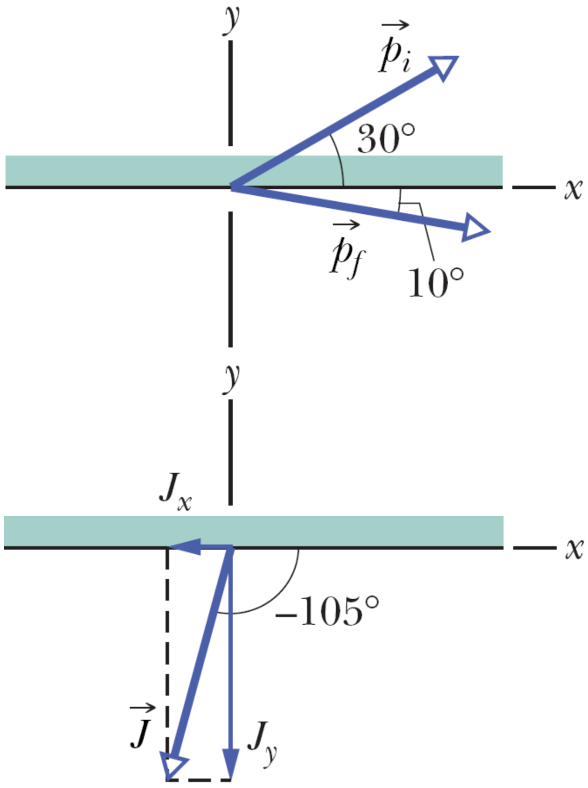
$$\Delta v = v_f - v_i = -v - v = -2v$$

- In time interval Δt, an amount of mass $\Delta m = nm$ collides with the target, then we can rewrite the equation

$$F_{\text{avg}} = -\frac{\Delta m}{\Delta t}\Delta v$$

problem 9-4





Conservation of Linear Momentum

- If the net external force (and thus the net impulse) acting on a system of particles is 0 (the system is isolated) and no particles leave or enter the system (the system is closed) , then

$$\frac{d\vec P}{dt} = 0 \quad \Rightarrow \quad \boxed{\vec P = \text{constant} \qquad \text{closed, isolated system}}$$

If no net external force acts on a system of particles, the total linear momentum of the system cannot change.

This result is called the **law of conservation of linear momentum**.

- The law can also be written as $\vec p_i = \vec p_f \qquad \text{closed, isolated system}$

and

$$\begin{pmatrix}\text{total linear momentum} \\ \text{at some initial time } t_i\end{pmatrix} = \begin{pmatrix}\text{total linear momentum} \\ \text{at some later time } t_f\end{pmatrix}$$

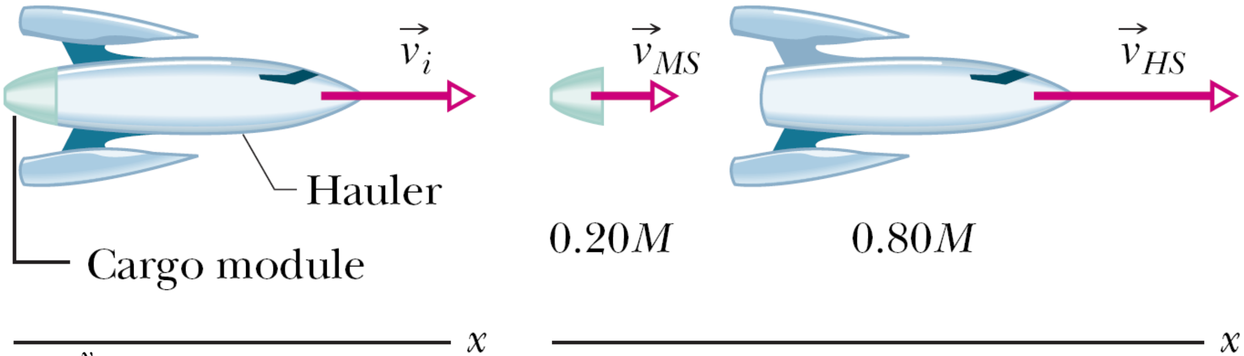
- *Caution*: Momentum should not be confused with energy. In some cases, momentum is conserved but energy is definitely not.

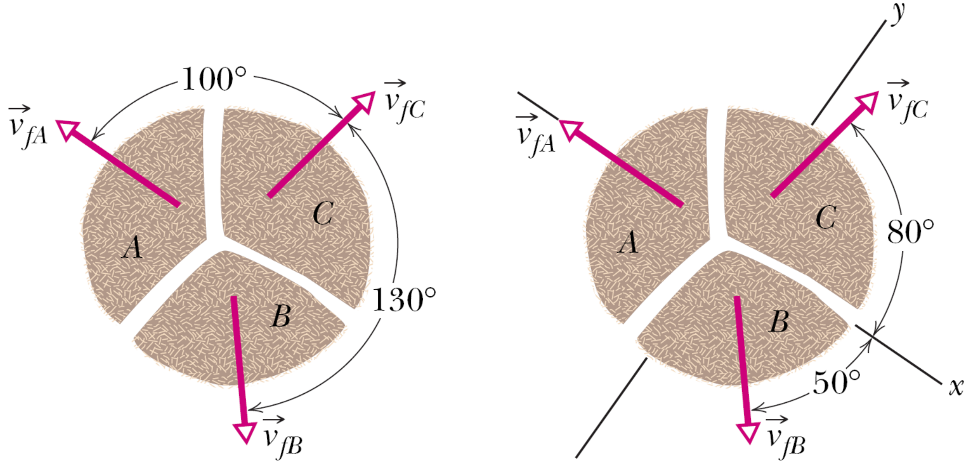
- In components

> If the component of the net *external* force on a closed system is 0 along an axis, then the component of the linear momentum of the system along that axis cannot change.

- Although internal forces can change the linear momentum of portions of the system, they cannot change the total linear momentum of the entire system.

problem 9-5





problem 9-6

Momentum and Kinetic Energy in Collisions

- Assume the system is closed and isolated, then the law of conservation of linear momentum is a very powerful rule because it can allow us to determine the results of a collision *without* knowing the details of the collision.

- For the total kinetic energy of a system of 2 colliding bodies,

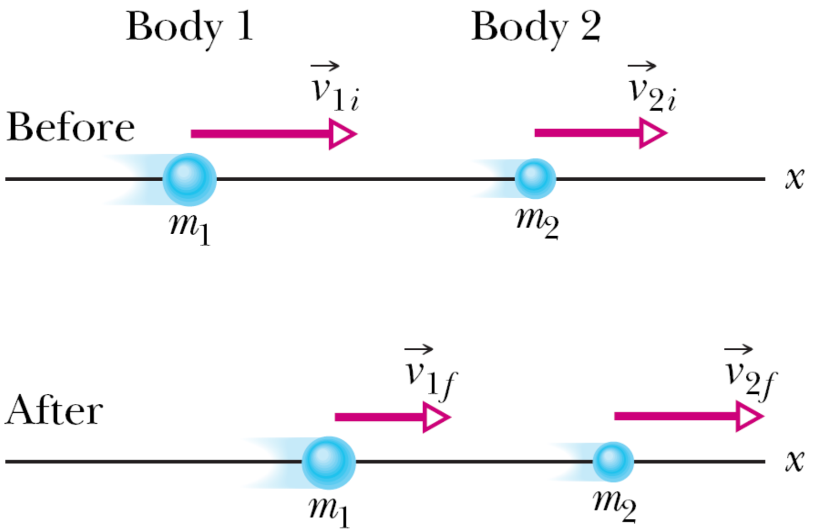
elastic collision: the kinetic energy of the system is *conserved* during the collision.

inelastic collision: the kinetic energy of the system is *not* conserved during the collision.

- In an inelastic collision, some energy is transferred from kinetic energy to other forms of energy, eg, thermal energy, energy of sound, etc.

- In an inelastic collision, the greatest loss occurs if the bodies stick together, in which case the collision is called a **completely inelastic collision**.

Inelastic Collisions in One Dimension



One-Dimensional Inelastic Collision

- the law of conservation of linear momentum for this 2-body system is

$$\begin{pmatrix}\text{total momentum } \vec{P}_i \\ \text{before the collision}\end{pmatrix} = \begin{pmatrix}\text{total momentum } \vec{P}_f \\ \text{after the collision}\end{pmatrix}$$

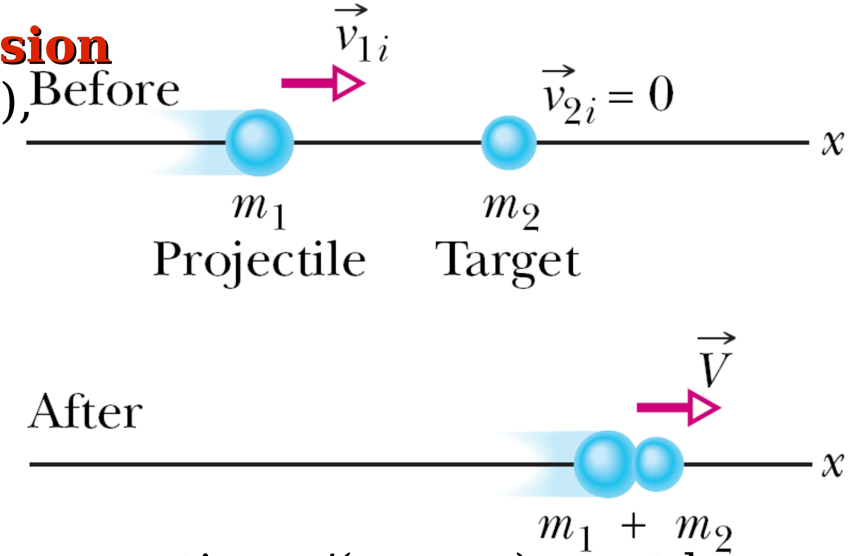
$$\boxed{\vec{p}_{1i}+\vec{p}_{2i}=\vec{p}_{1f}+\vec{p}_{2f} \quad \text{conservation of linear momentum}}$$

Since the motion is 1-dim, we can rewrite the equation as

$$m_1 v_{1i}+m_2 v_{2i}=m_1 v_{1f}+m_2 v_{2f}$$

- If we know values for, say, the masses, the initial velocities, and one of the final velocities, we can find the other final velocity with this equation.

One-Dimensional Completely Inelastic Collision



- For the body with mass m_2 being at rest (v_{2i}=0), we refer to that body as the *target* and to the incoming body as the *projectile*, and

$$m_1 v_{1i}=(m_1+m_2)V$$

or
$$V=\frac{m_1}{m_1+m_2}v_{1i}.$$

- Note that V must be less than v_{1i} because the mass ratio $m_1/(m_1+m_2)$ must be less than unity.

Velocity of the Center of Mass

- In a closed, isolated system, the velocity of the center of mass of the system cannot be changed by a collision because there is no net external force to change it.

- relate the velocity to the total linear momentum by

$$\vec{P} = M\,\vec{v}_{\mathrm{com}} = (m_1 + m_2)\,\vec{v}_{\mathrm{com}}$$

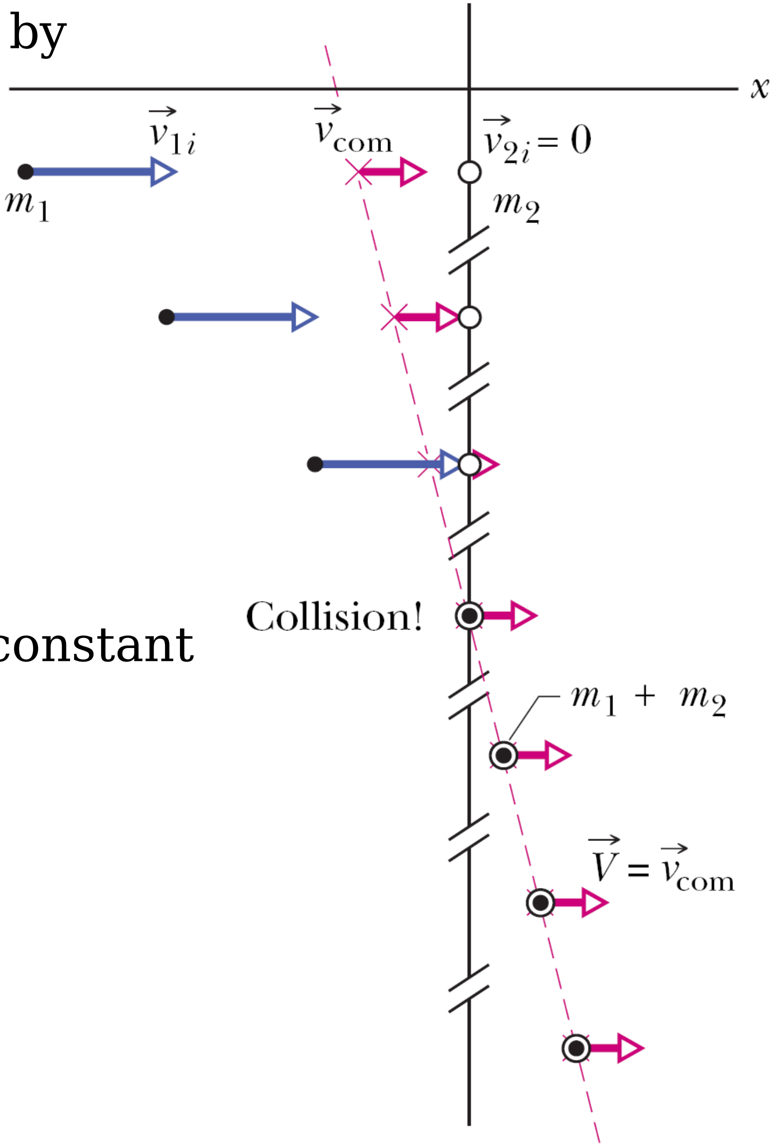
- The total linear momentum is conserved during the collision; and it is given from LHS

$$\vec{P} = \vec{p}_{1i} + \vec{p}_{2i}$$

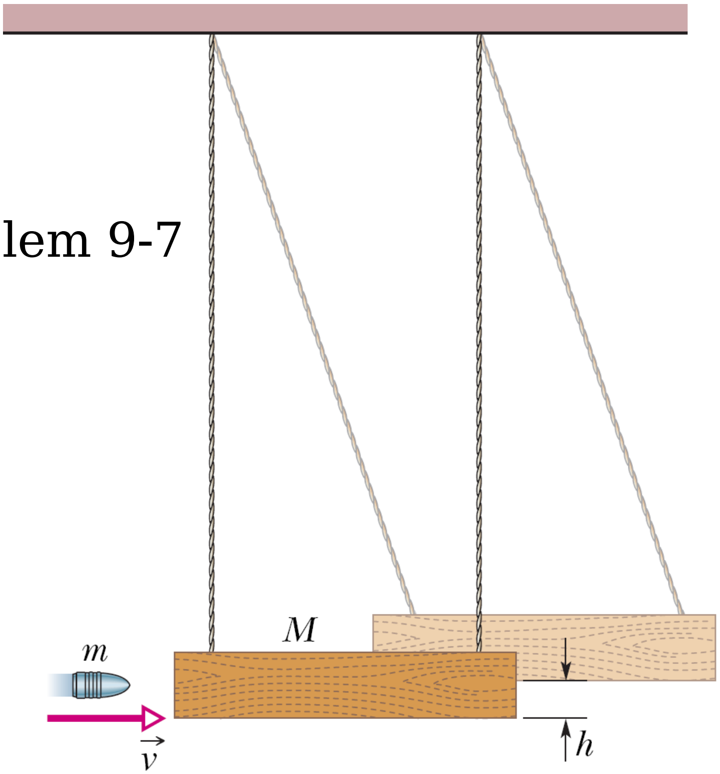
then

$$\boxed{\vec{v}_{\mathrm{com}} = \frac{\vec{P}}{m_1 + m_2} = \frac{\vec{p}_{1i} + \vec{p}_{2i}}{m_1 + m_2}}$$

- the velocity of the center of mass has the same constant value before and after the collision.



problem 9-7

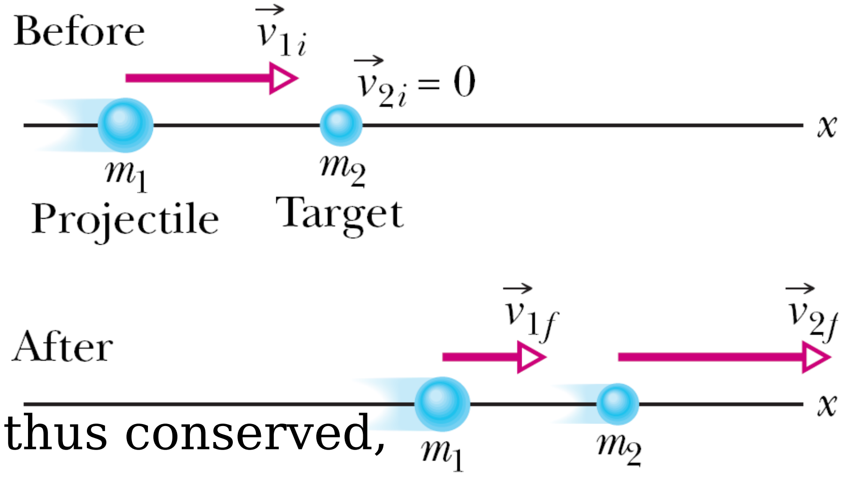


Elastic Collisions in One Dimension

- In an elastic collision, $\begin{pmatrix}\text{total kinetic energy}\\ \text{before the collision}\end{pmatrix} = \begin{pmatrix}\text{total kinetic energy}\\ \text{after the collision}\end{pmatrix}$

and it means

In an elastic collision, the kinetic energy of each colliding body may change, but the total kinetic energy of the system does not change.



Stationary Target

- A projectile body of mass m_1 and initial velocity v_{1i} moves toward a target body of mass m_2 that is initially at rest ($v_{2i} = 0$).

- If the 2-body system is closed and isolated, and the net linear momentum of the system is thus conserved,

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f} \qquad \text{linear momentum}$$

- If the collision is also elastic, then the total kinetic energy is conserved,

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \qquad \text{kinetic energy}$$

- If we know the masses of the bodies and if we also know v_{1i}, the only unknown quantities are v_{1f} and v_{2f}, the final velocities of the 2 bodies, then

$$m_1\left(v_{1i}-v_{1f}\right)=m_2 v_{2f}$$

and $$m_1\left(v_{1i}-v_{1f}\right)\left(v_{1i}+v_{1f}\right)=m_2 v_{2f}^2$$

we can obtain

$$\boxed{\begin{aligned} v_{1f}&=\frac{m_1-m_2}{m_1+m_2}v_{1i} \\ v_{2f}&=\frac{2m_1}{m_1+m_2}v_{1i} \end{aligned}}$$

- v_{2f} is always positive (the initially stationary target body with mass m_2 always moves forward).

- v_{1f} is positive if $m_1 > m_2$, is negative if $m_1 < m_2$.

- some special cases:

1. ***Equal masses***: ie, $m_1 = m_2$, then $v_{1f} = 0$ and $v_{2f} = v_{1i}$.

this means in head-on collisions, bodies of equal mass simply exchange velocities. This is true even if body 2 is not initially at rest.

2. ***A massive target***: ie, $m_1 \ll m_2$, then $v_{1f} \approx -v_{1i}$ and $v_{2f} \approx \frac{2m_1}{m_2}v_{1i}$

- It tells that the projectile simply bounces back along its incoming path, its speed essentially unchanged. And the target moves forward at a low speed.

3. ***A massive projectile***: ie, $m_1 \gg m_2$, then $v_{1f} \approx v_{1i}$ and $v_{2f} \approx 2\, v_{1i}$

- It tells that the projectile simply keep on going, scarcely slowed by the collision; the target moves forward at twice speed of the projectile.

Moving target

- examine the situation in which both bodies are moving before they undergo an elastic collision.

- the conservation of linear momentum gives $m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$

- the conservation of kinetic energy gives $\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_1 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$

- To solve these equation for v_{1f} and v_{2f},

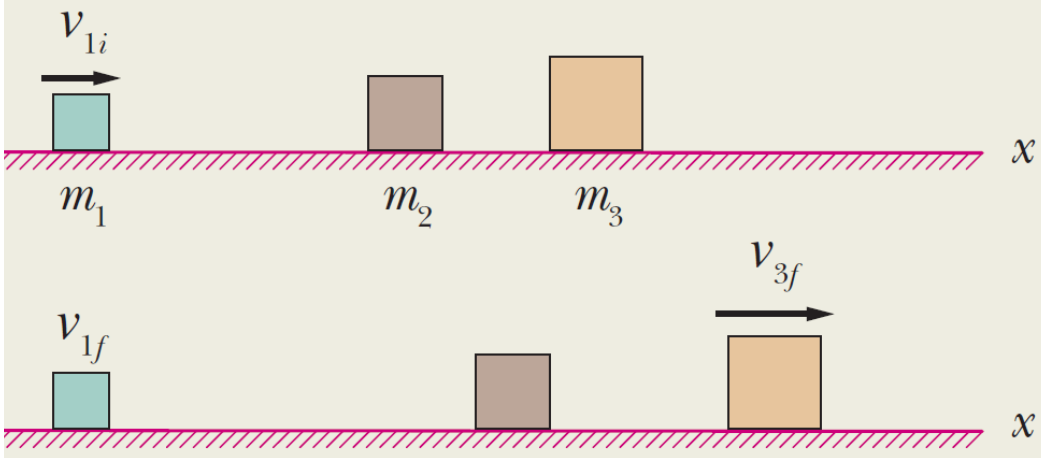
$$m_1 (v_{1i} - v_{1f}) = -m_2 (v_{2i} - v_{2f})$$

$$m_1 (v_{1i} - v_{1f})(v_{1i} + v_{1f}) = -m_2 (v_{2i} - v_{2f})(v_{2i} + v_{2f})$$

then we obtain

$$\boxed{\begin{aligned} v_{1f} &= \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2\, m_2}{m_1 + m_2} v_{2i} \\ v_{2f} &= \frac{2\, m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i} \end{aligned}}$$

problem 9-8



Collisions in Two Dimensions

- When a collision is not head-on, the bodies do not end up traveling along their initial axis.
- For a 2-dim collisions in a closed, isolated system, the conservation of linear momentum gives

$$\vec{P}_{1i}+\vec{P}_{2i}=\vec{P}_{1f}+\vec{P}_{2f}$$

- If the collision is elastic, the conservation of kinetic energy gives

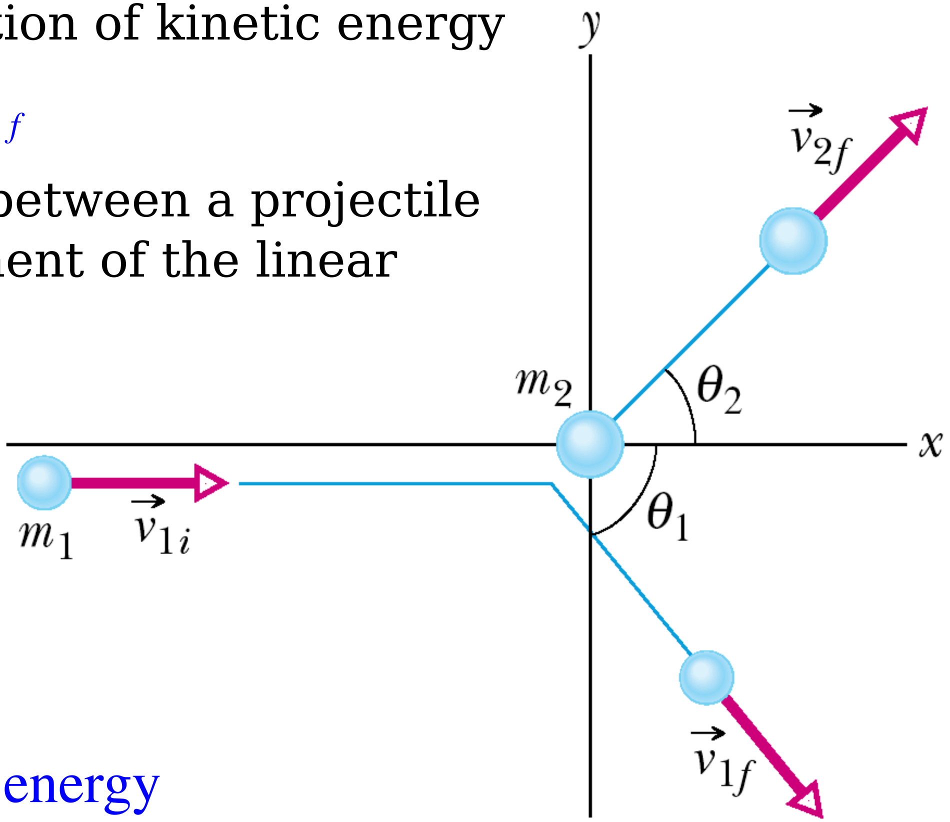
$$K_{1i}+K_{2i}=K_{1f}+K_{2f}$$

- For a *glancing collision* (not head-on) between a projectile and a target initially at rest, the component of the linear momentum along *x*-axis is

$$m_1 v_{1i}=m_1 v_{1f}\cos\theta_1+m_2 v_{2f}\cos\theta_2$$

and along the *y* axis is

$$0=-m_1 v_{1f}\sin\theta_1+m_2 v_{2f}\sin\theta_2$$



- For an elastic collision,

$$\frac{1}{2}m_1 v_{1i}^2=\frac{1}{2}m_1 v_{1f}^2+\frac{1}{2}m_2 v_{2f}^2$$ kinetic energy

- There are 7 variables: m_1, m_2, v_{1i}, v_{1f}, v_{2f}, θ_1, θ_2. If we know any 4 of these quantities, we can solve the 3 equations for the remaining 3 quantities.

System with varying Mass: A Rocket

- We handle the variation of the mass of the rocket as the rocket accelerates by applying Newton's 2nd law, not to the rocket alone but to the rocket and its ejected combustion products taken together.

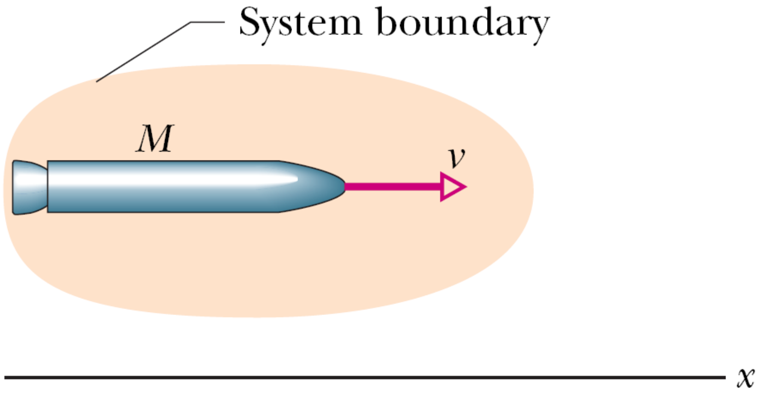
Finding the Acceleration

- For the one-dimensional motion of a rocket, let M bet the mass of the rocket and v its velocity at an arbitrary time t.

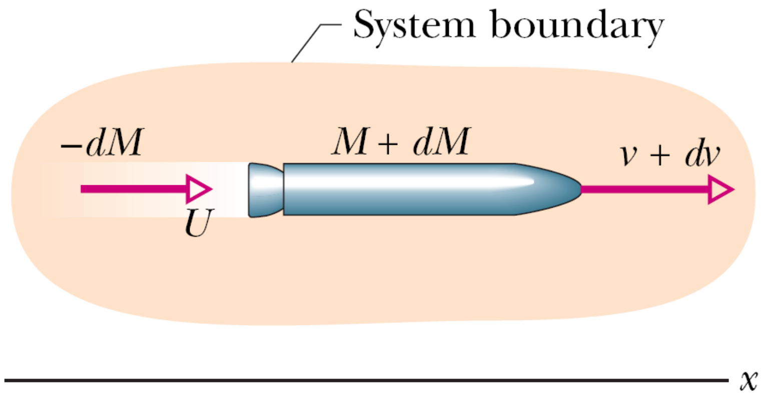
- After a time interval d t, the rocket has velocity v+dv and mass M+dM, where the change in mass dM is a negative quantity. The exhaust products has mass – dM and velocity U.

- The system, consisting of the rocket and its exhaust products, is closed and isolated, thus the linear momentum must be conserved during dt,

$$P_i = P_f$$



(a)



(b)

Then $$M\,v = -U\,\mathrm{d}M + (M + \mathrm{d}M)(v + \mathrm{d}v)$$

where the 1st term in RHS is the linear momentum of the exhaust products and the 2nd term is the linear momentum of the rocket.

- We can simplify the equation by using the relative speed between the rocket and the exhaust products,

$$\begin{pmatrix}\text{velocity of rocket}\\ \text{relative to frame}\end{pmatrix} = \begin{pmatrix}\text{velocity of rocket}\\ \text{relative to products}\end{pmatrix} + \begin{pmatrix}\text{velocity of products}\\ \text{relative to frame}\end{pmatrix}$$

That is, $$(v + \mathrm{d}v) = v_{\mathrm{rel}} + U \quad \text{or} \quad U = v + \mathrm{d}v - v_{\mathrm{rel}}$$

This gives $$-v_{\mathrm{rel}}\,\mathrm{d}M = M\,\mathrm{d}v \quad \Rightarrow \quad -\frac{\mathrm{d}M}{\mathrm{d}t}\,v_{\mathrm{rel}} = M\,\frac{\mathrm{d}v}{\mathrm{d}t}$$

- define R to be the (positive) mass rate of fuel consumption, then $\mathrm{d}M/\mathrm{d}t = -R$,

$$\boxed{R\,v_{\mathrm{rel}} = M\,a \qquad \text{1st rocket equation}}$$

- the LHS of the equation has the dimension of force (kg/s m/s = kg m/s^2 = N) and depends only on design characteristics of the rocket engine.

- We call the term Rv_{rel} the thrust of the rocket engine and represents it with T, then $T = Ma$.

Finding the Velocity

- since $$\mathrm{d}v = -v_{\mathrm{rel}} \frac{\mathrm{d}M}{M} \quad \Rightarrow \quad \int_{v_i}^{v_f} \mathrm{d}v = -v_{\mathrm{rel}} \int_{M_i}^{M_f} \frac{\mathrm{d}M}{M}$$

Evaluating the integrals gives

$$\boxed{v_f - v_i = v_{\mathrm{rel}} \ln \frac{M_i}{M_f} \qquad \text{2nd rocket equation}}$$

- The advantage of multistage rockets is that M_f is reduced by discarding successive stages when their fuel is depleted.

Problem 9.9

The chosen problems: 13, 44, 68