Chapter 4 motion in two and three dimensions

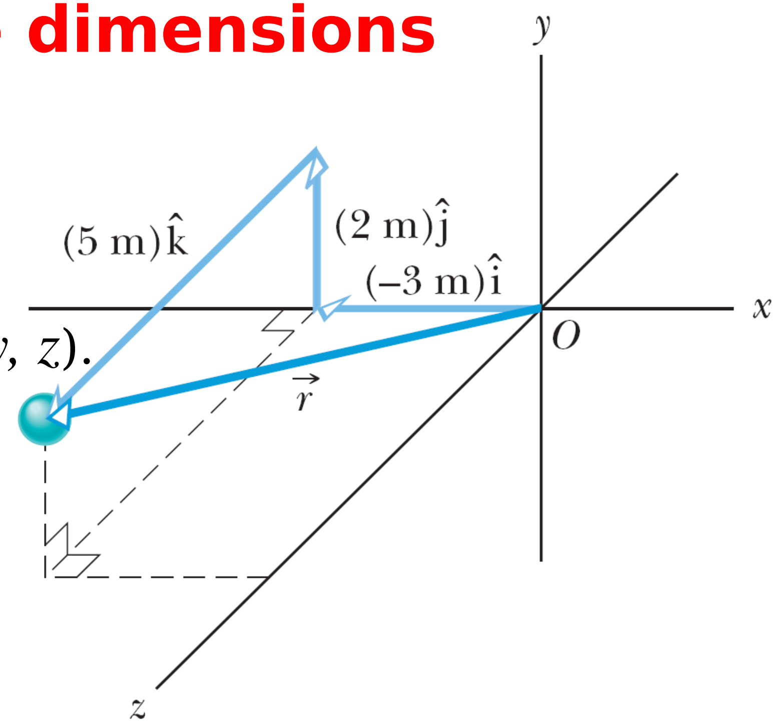
Position and Displacement

- **position vector**: a vector that extends from a reference point to the particle, $\vec{r} = x\,\hat{\mathrm{i}} + y\,\hat{\mathrm{j}} + z\,\hat{\mathrm{k}}$

- The particle has the rectangular coordinates (*x*, *y*, *z*).

  In the figure $\vec{r} = (-3\ \mathrm{m})\,\hat{\mathrm{i}} + (2\ \mathrm{m})\,\hat{\mathrm{j}} + (5\ \mathrm{m})\,\hat{\mathrm{k}}$

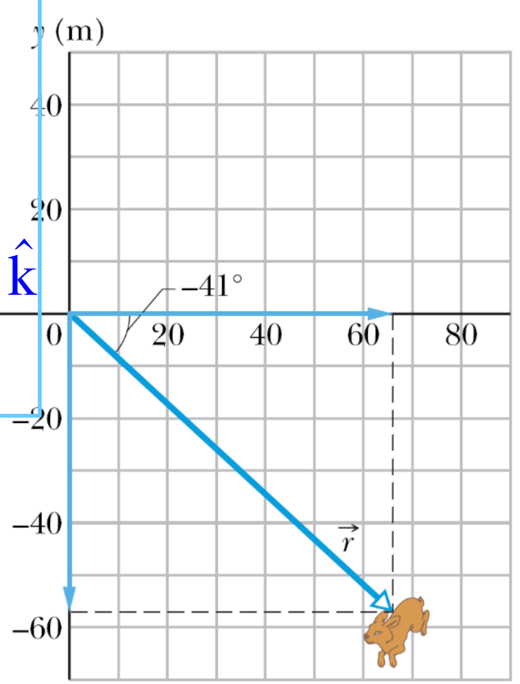
  and rectangular coordinates (-3m, 2m, 5m).

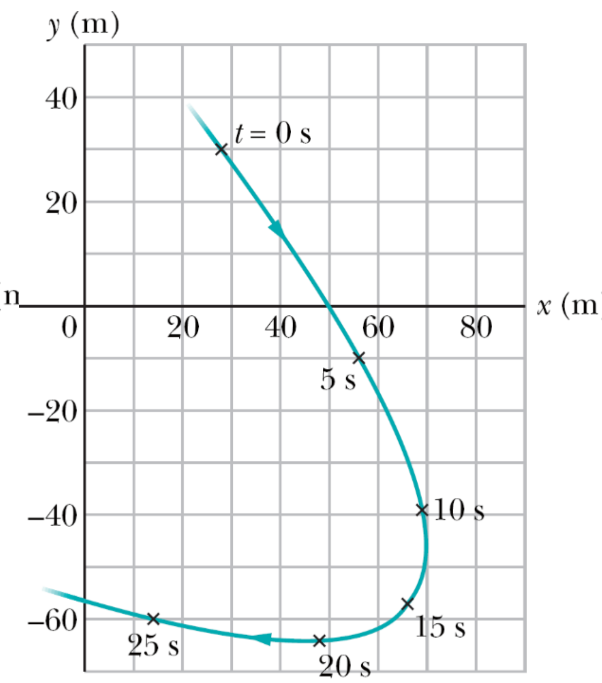


Problem 4.1

- **displacement**: the position vector changes during a certain time interval,

$$
\begin{aligned}
\Delta\vec{r} &= \vec{r}_2 - \vec{r}_1 \\
&= (x_2\,\hat{\mathrm{i}} + y_2\,\hat{\mathrm{j}} + z_2\,\hat{\mathrm{k}}) \\
&\quad - (x_1\,\hat{\mathrm{i}} + y_1\,\hat{\mathrm{j}} + z_1\,\hat{\mathrm{k}}) \\
&= (x_2 - x_1)\,\hat{\mathrm{i}} + (y_2 - y_1)\,\hat{\mathrm{j}} + (z_2 - z_1)\,\hat{\mathrm{k}} \\
&= \Delta x\,\hat{\mathrm{i}} + \Delta y\,\hat{\mathrm{j}} + \Delta z\,\hat{\mathrm{k}}
\end{aligned}
$$





Average Velocity and Instantaneous Velocity

- **average velocity**:

$$\text{average velocity} = \frac{\text{displacement}}{\text{time interval}} \quad \text{or} \quad \boxed{\vec{v}_{\text{avg}} = \frac{\Delta \vec{r}}{\Delta t}}$$

- The direction of the average velocity must be the the same as that of the displacement.

- Use the previous equation to get

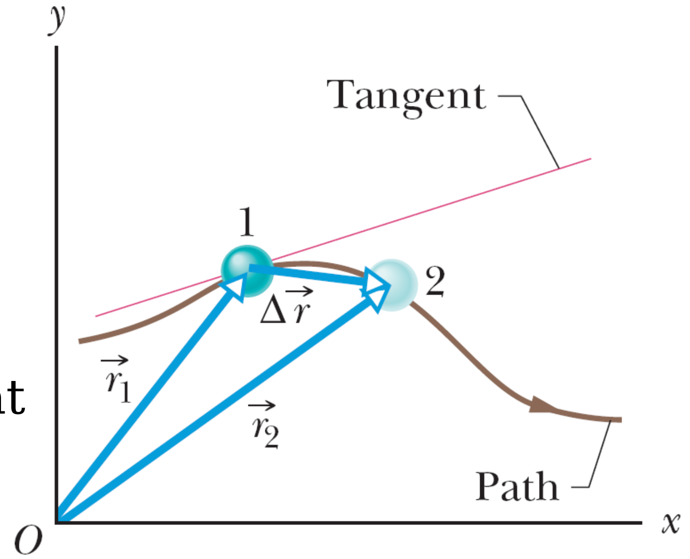
$$\vec{v}_{\text{avg}} = \frac{\Delta x\, \hat{\text{i}} + \Delta y\, \hat{\text{j}} + \Delta z\, \hat{\text{k}}}{\Delta t} = \frac{\Delta x}{\Delta t}\, \hat{\text{i}} + \frac{\Delta y}{\Delta t}\, \hat{\text{j}} + \frac{\Delta z}{\Delta t}\, \hat{\text{k}}$$

- If the time interval during the motion is 2.0s, then

$$\vec{v}_{\text{avg}} = \frac{\Delta \vec{r}}{\Delta t} = \frac{(12.0\text{ m})\,\hat{\text{i}} + (3.0\text{ m})\,\hat{\text{k}}}{2.0\text{ s}}$$
$$= (6.0\text{ m/s})\,\hat{\text{i}} + (1.5\text{ m/s})\,\hat{\text{k}}$$

- **Instantaneous velocity** (or **velocity**): $\vec{v} = \frac{d\vec{r}}{dt}$

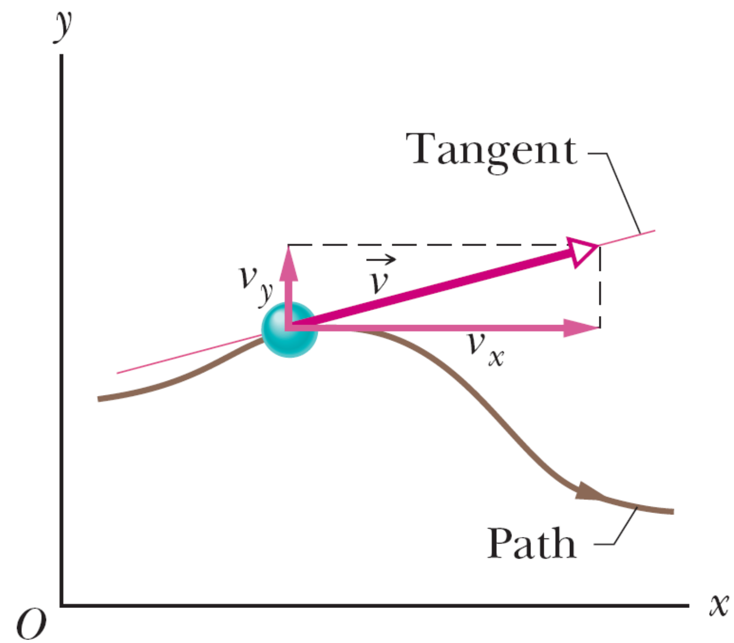
- To find the velocity of the particle at instant t_1, we shrink interval Δt to 0 about t_1:



(1) Position vector $\vec{r}_2$ in Fig. move toward $\vec{r}_1$ so that $\Delta\vec{r}$ shrinks toward 0.

(2) The direction of $\Delta\vec{r}/\Delta t$ (and thus of $\vec{v}_{avg}$) approaches the direction of the line tangent to the particle's path at position 1.

(3) The average velocity $\vec{v}_{avg}$ approaches the instantaneous velocity $\vec{v}$ at t_1.



- The direction of the instantaneous velocity $\vec{v}$ of a particle is always tangent to the particle's path at the particle's position.

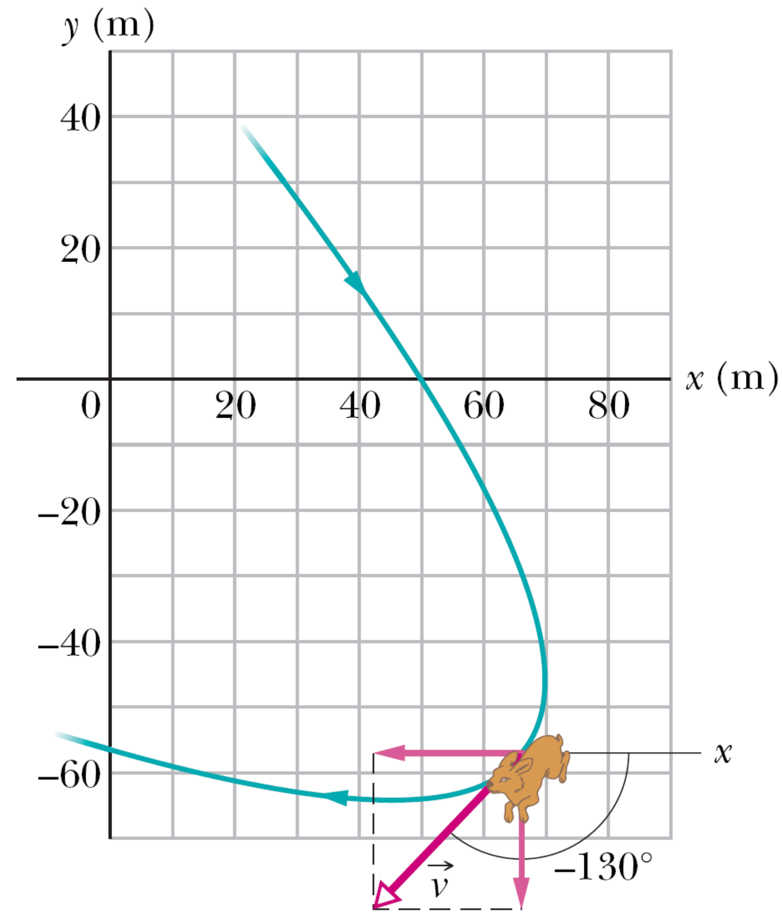
- In unit-vector form

$$\vec{v} = \frac{d}{dt}(x\,\hat{i} + y\,\hat{j} + z\,\hat{k}) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \quad \Rightarrow$$

$$\vec{v} = v_x\,\hat{i} + v_y\,\hat{j} + v_z\,\hat{k}$$

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt}$$

Example 4-2



Average Acceleration and Instantaneous Acceleration

- **average acceleration**:

$$\text{average acceleration} = \frac{\text{change in velocity}}{\text{time interval}}$$

or

$$\vec{a}_{avg} = \frac{\Delta \vec{v}}{\Delta t}$$

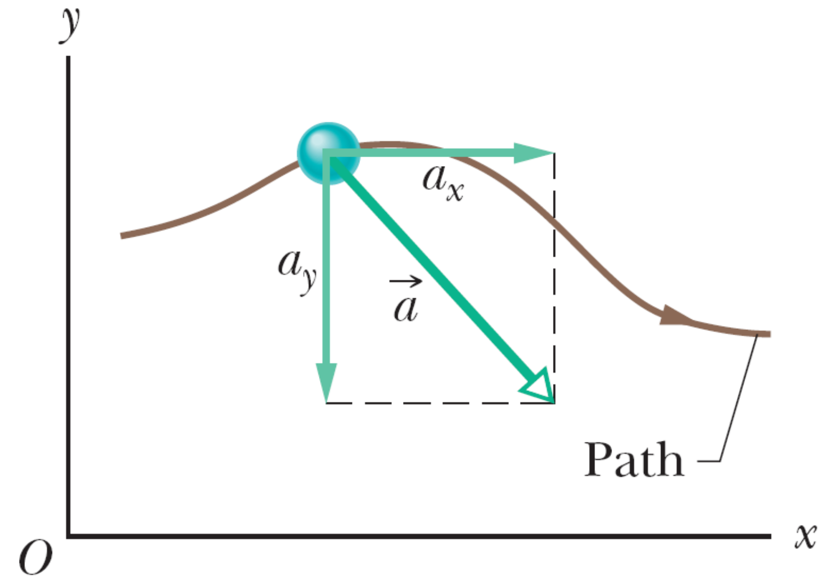
- **instantaneous acceleration** (or **acceleration**):

$$\vec{a} = \frac{d\vec{v}}{dt}$$

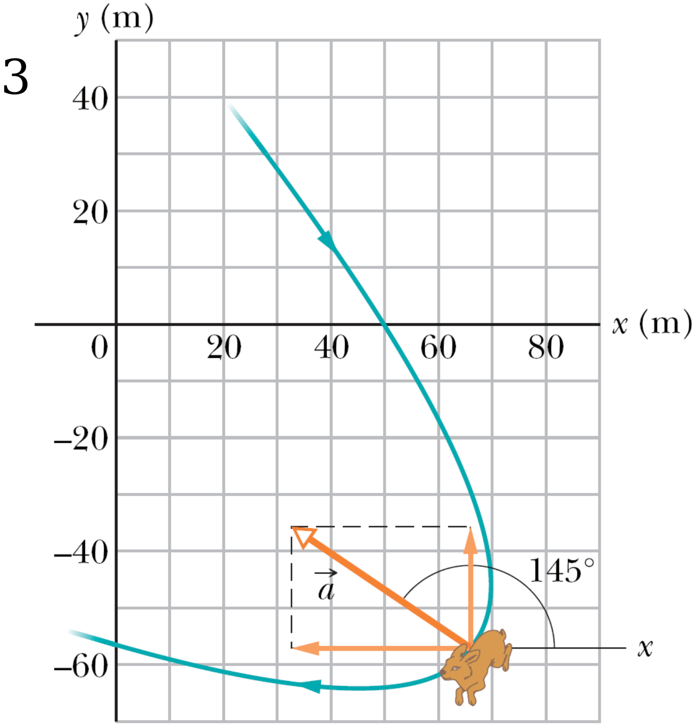
$$\vec{a} = \frac{d}{dt}(v_x \hat{i} + v_y \hat{j} + v_z \hat{k}) = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}$$

$$\Rightarrow \quad \vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

$$a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt}, \quad a_z = \frac{dv_z}{dt}$$



problem 4-3



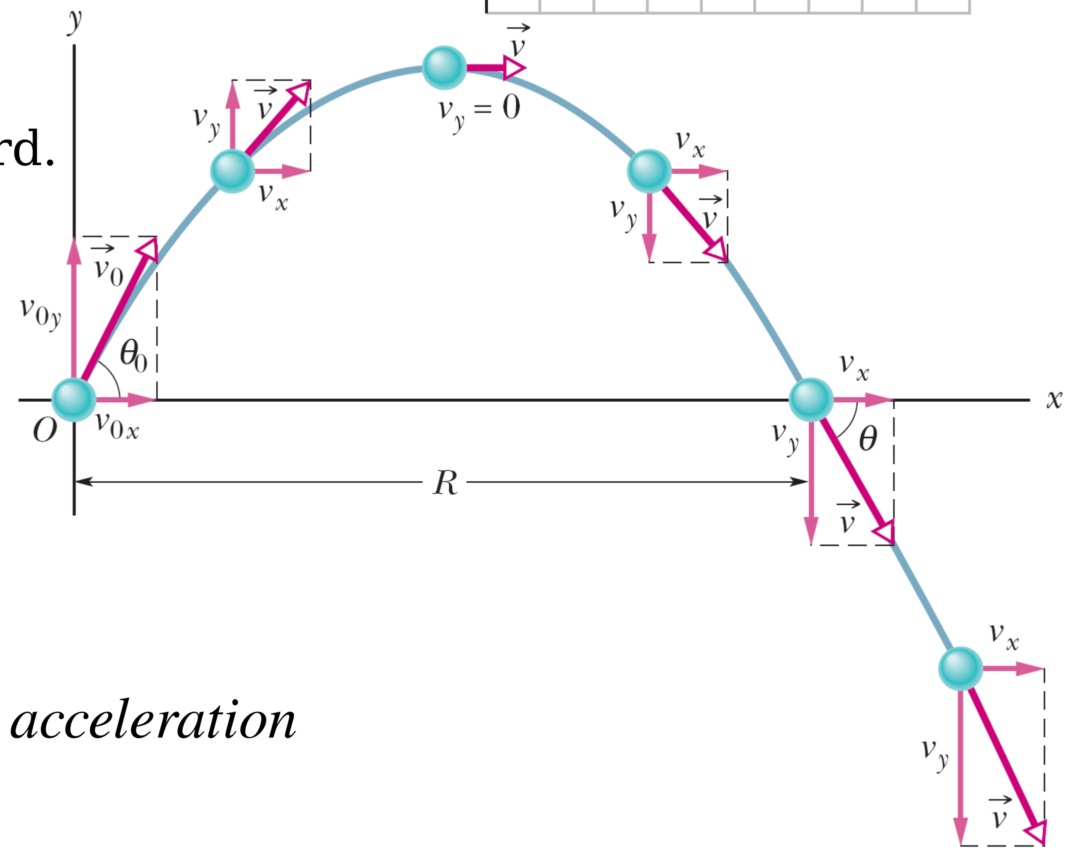
Projectile Motion

- **projectile**: a particle moves in a vertical plane with some initial velocity but its acceleration is always the free-fall acceleration, which is downward.

initial velocity: $\vec{v}_0 = v_{0x}\,\hat{\mathrm{i}} + v_{0y}\,\hat{\mathrm{j}}$

$$v_{0x} = v_0 \cos\theta_0 , \qquad v_{0y} = v_0 \sin\theta_0$$

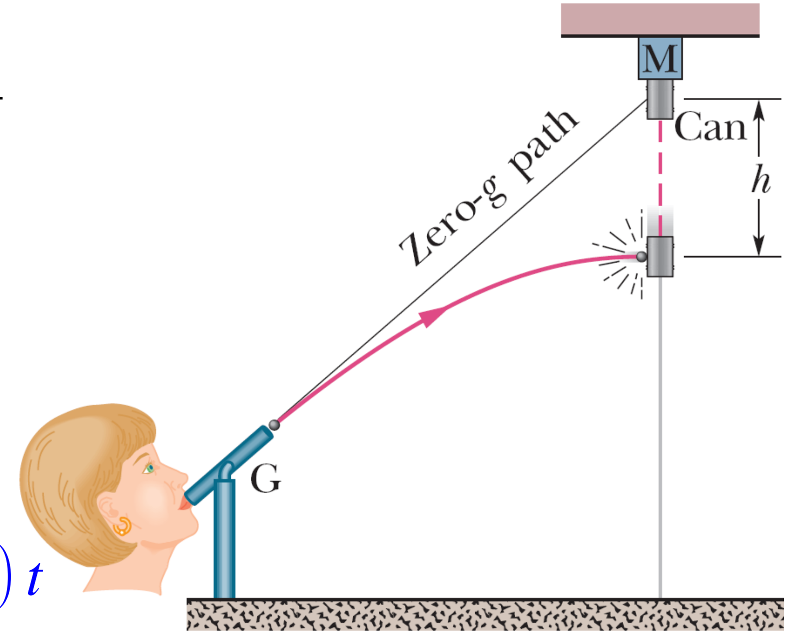
- During its 2D motion, the projectile's position and velocity change continuously, but its acceleration is constant and *always* directed vertically downward.



> In projectile motion, the horizontal motion and the vertical motion are independent of each other; neither motion affects the other.

- We can break up the problem of a 2D motion into two 1D problems:
  horizontal motion: *zero acceleration*
  vertical motion: *constant downward acceleration*

- During the time of flight of the ball, both ball and can fall the same distance h from their zero-g location, and the ball hits the can.



Projectile Motion Analyzed

The Horizontal Motion: no acceleration

$$v_x = v_{0x} = v_0 \cos\theta_0 , \qquad x - x_0 = v_{0x} t = (v_0 \cos\theta_0)\, t$$

The Vertical Motion: constant acceleration – g

$$y - y_0 = v_{0y} t - \frac{1}{2} g t^2 = (v_0 \sin\theta_0)\, t - \frac{1}{2} g t^2$$

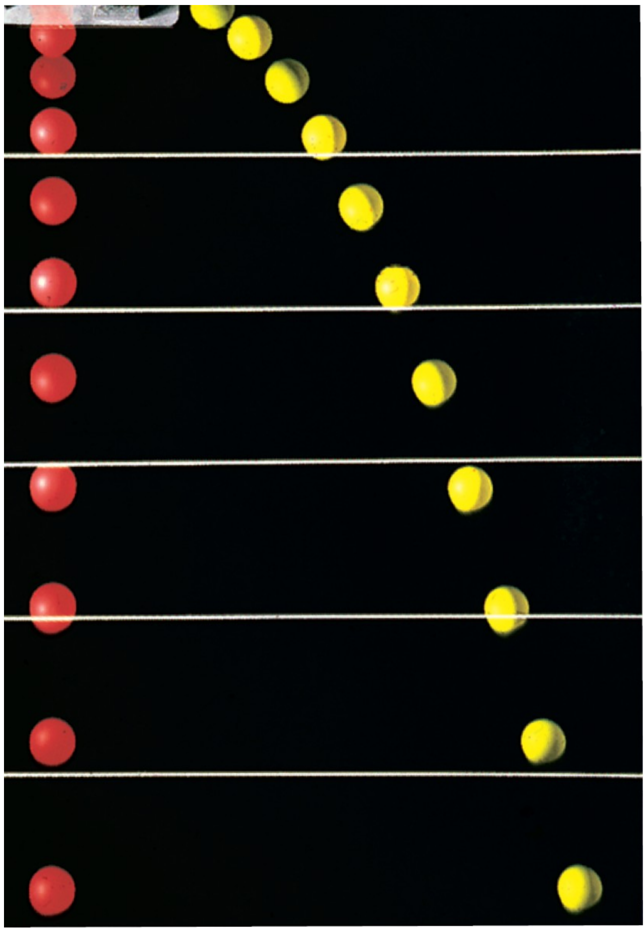
$$v_y = v_0 \sin\theta_0 - g t$$

$$v_y^2 = (v_0 \sin\theta_0)^2 - 2 g (y - y_0)$$

The Equation of the Path: the equation of the projectile's **trajectory**: assuming $x_0 = 0$, and $y_0 = 0$,

$$y = (\tan\theta_0)\, x - \frac{g x^2}{2 (v_0 \cos\theta_0)^2} \qquad \text{trajectory}$$

- This is the equation of parabola (of the form $y = ax + bx^2$) so the path is *parabolic*.



The Horizontal Range R: the horizontal distance the projectile has traveled when it returns to its initial height

$$R = (v_0 \cos\theta_0)\, t\,, \qquad 0 = (v_0 \sin\theta_0)\, t - \frac{1}{2} g t^2$$

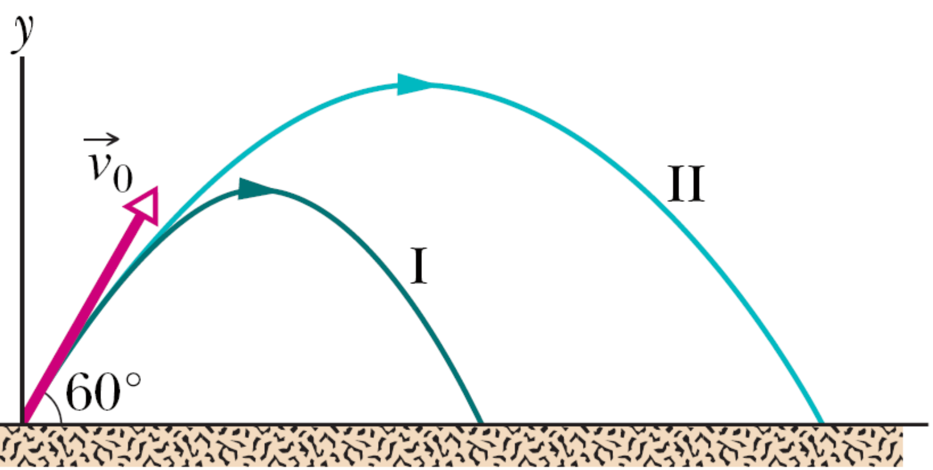
eliminating t yield

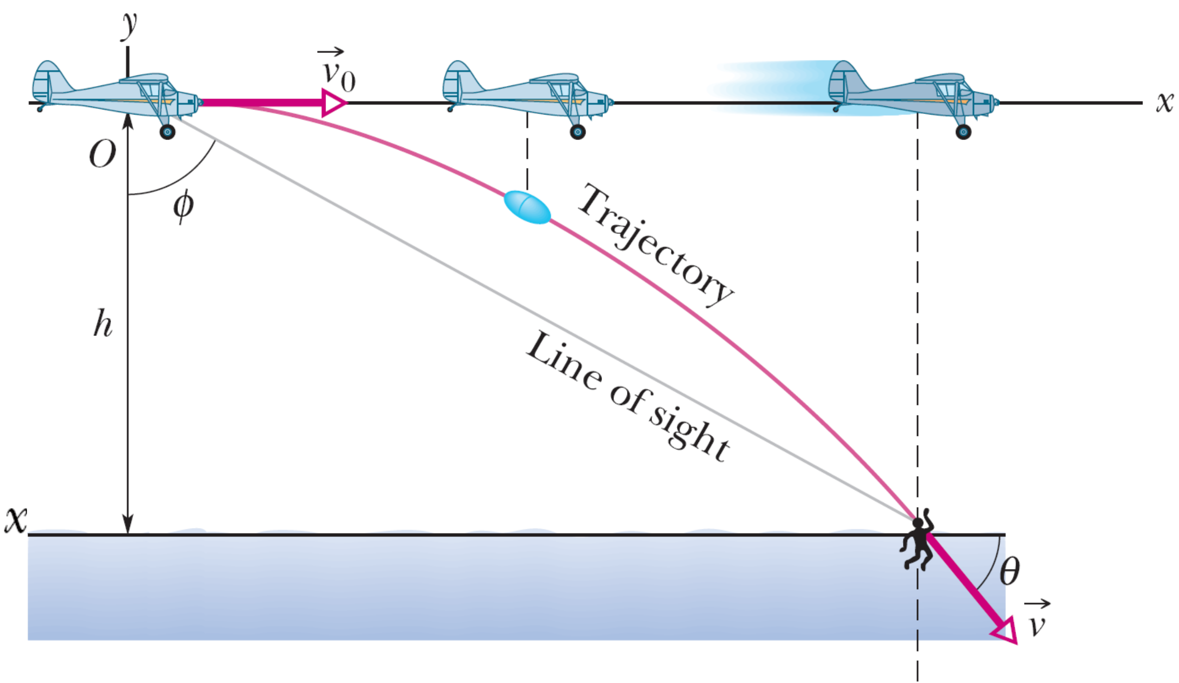
$$R = \frac{2 v_0^2}{g} \sin\theta_0 \cos\theta_0 = \frac{v_0^2}{g} \sin 2\theta_0$$

R has its maximum value when $\sin 2\theta_0 = 1$, which corresponds to $\theta_0 = 45°$.

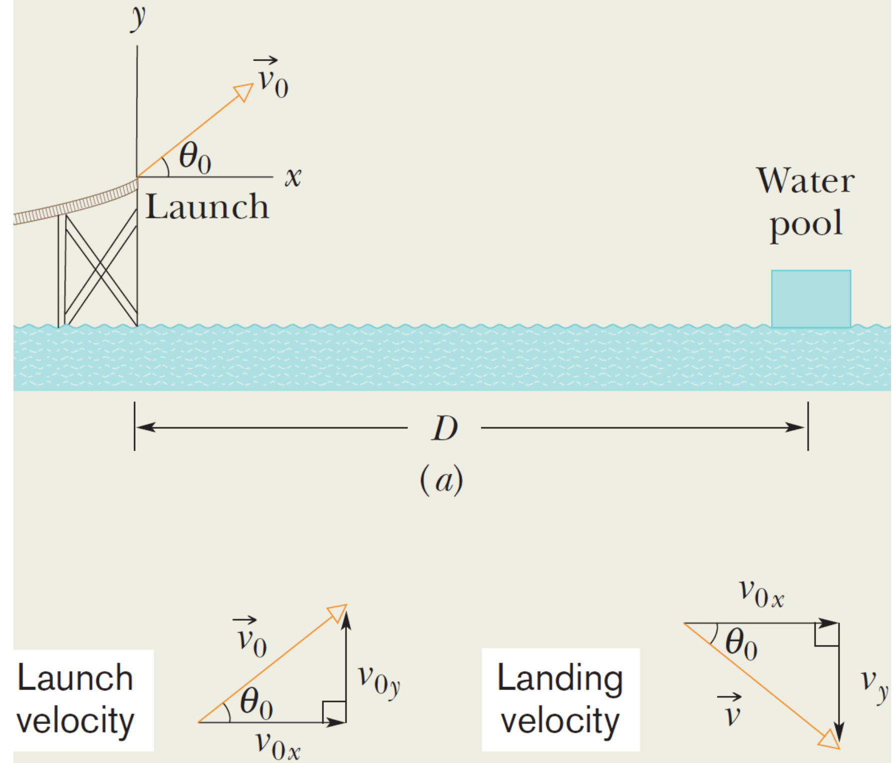
problem 4-4

The Effects of the Air





Problem 4.5



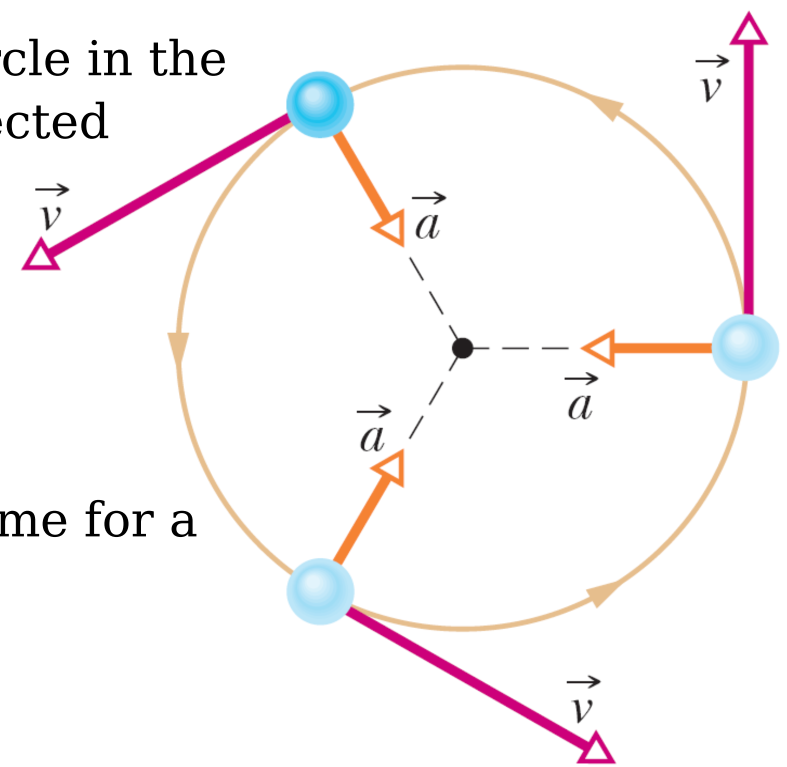
Uniform Circular Motion

- **uniform circular motion**:
a particle travels around a circle or a circular arc at constant (*uniform*) speed.

- *The particle is accelerating* because its velocity changes (only) in direction.

- Both the velocity and the acceleration have constant magnitude as the motion progresses, but their directions change continuously.

- The velocity is always directed tangent to the circle in the direction of motion, the acceleration is always directed *radially inward* (**centripetal acceleration**).

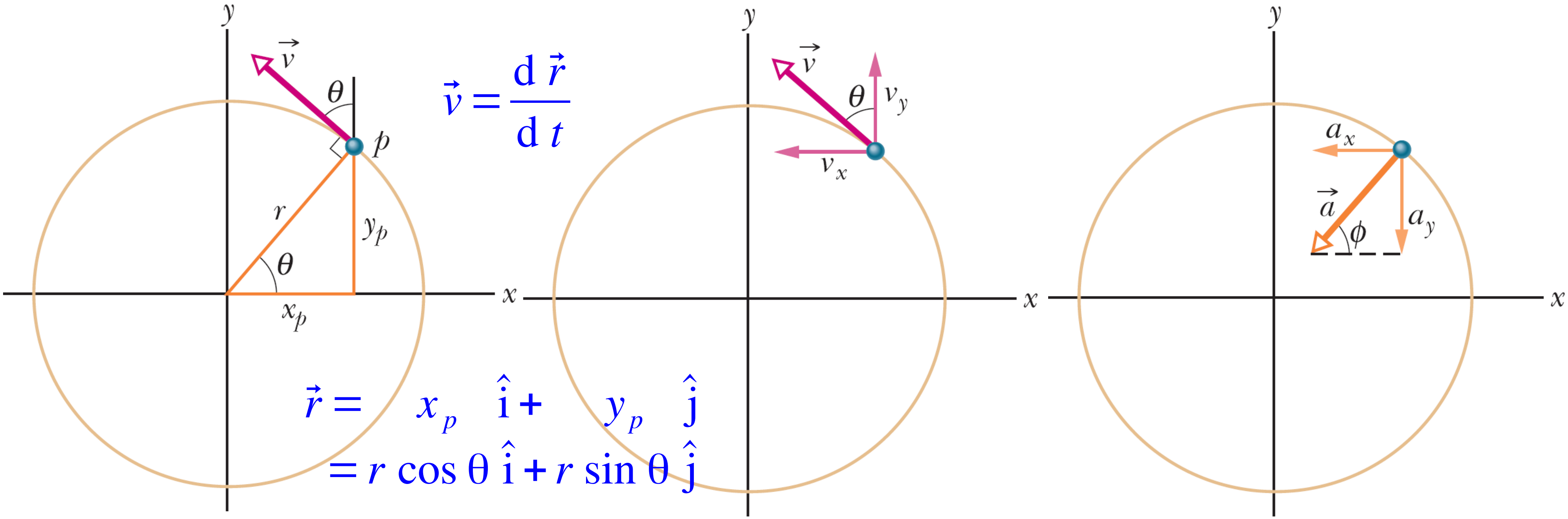


- The magnitude of this acceleration is

$$a = \frac{v^2}{r} \quad \text{centripetal acceleration}$$

- The *period of revolution* (or simply *period*): the time for a particle to go around a closed path exactly once,

$$T = \frac{2\pi r}{v} \quad \text{period}$$



$$\vec{v} = \frac{d\,\vec{r}}{d\,t}$$

$$\vec{r} = x_p\,\hat{i} + y_p\,\hat{j} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$$

- the velocity of a moving particle is always tangent to the particle's path at its position.

$$\vec{v} = v_x\,\hat{i} + v_y\,\hat{j} = \left(-v\sin\theta\right)\hat{i} + \left(v\cos\theta\right)\hat{j} = -\frac{v\,y_p}{r}\,\hat{i} + \frac{v\,x_p}{r}\,\hat{j}$$

$$\Rightarrow \quad \vec{a} = \frac{d\,\vec{v}}{d\,t} = -\frac{v}{r}\,\frac{d\,y_p}{d\,t}\,\hat{i} + \frac{v}{r}\,\frac{d\,x_p}{d\,t}\,\hat{j}$$

since $v_x = \frac{d\,x_p}{d\,t} = -v\sin\theta, \quad v_y = \frac{d\,y_p}{d\,t} = v\cos\theta \quad \Rightarrow \quad \vec{a} = -\frac{v^2}{r}\cos\theta\,\hat{i} - \frac{v^2}{r}\sin\theta\,\hat{j}$

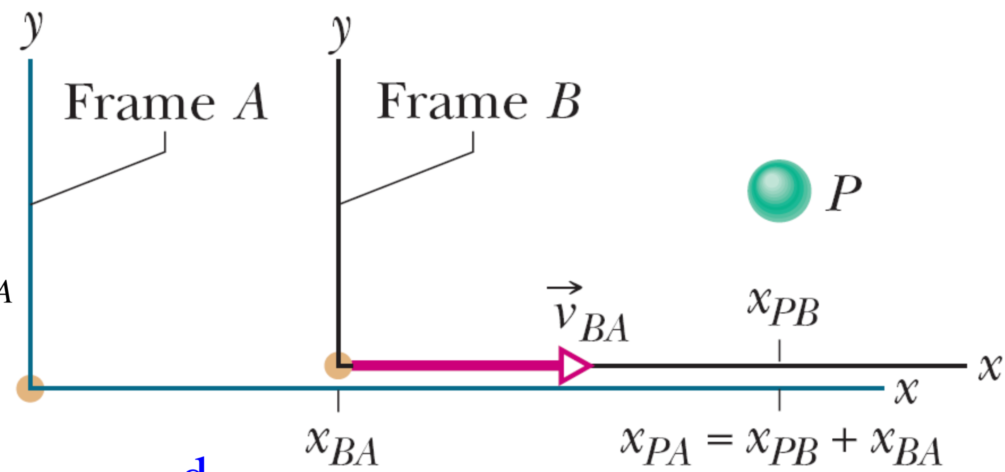
$$\Rightarrow \quad a = \sqrt{a_x^2 + a_y^2} = \frac{v^2}{r}\sqrt{\cos^2\theta + \sin^2\theta} = \frac{v^2}{r}, \qquad \tan\phi = \frac{a_y}{a_x} = \frac{-\left(v^2/r\right)\sin\theta}{-\left(v^2/r\right)\cos\theta} = \tan\theta$$

the acceleration is directed along the radius, toward the circle's center.

Problem 4.6

Relative Motion in One Dimension

- The velocity of a particle depends on the **reference frame** of whoever is observing or measuring the velocity.



- The coordinate x_{PA} of P as measured by A is *equal to* the coordinate x_{PB} of P as measured by B *plus* the coordinate x_{BA} of B as measured by A,

$$x_{PA} = x_{PB} + x_{BA}$$

- Taking the time derivative, $\frac{d}{dt}x_{PA} = \frac{d}{dt}x_{PB} + \frac{d}{dt}x_{BA}$ or $v_{PA} = v_{PB} + v_{BA}$

- The velocity v_{PA} of P as measured by A is *equal to* the velocity v_{PB} of P as measured by B *plus* the velocity v_{BA} of B as measured by A.

For v_{BA} is constant, $\frac{d}{dt}v_{PA} = \frac{d}{dt}v_{PB} + \frac{d}{dt}v_{BA}$ or $a_{PA} = a_{PB}$

- Observers on different frames of reference that move at constant velocity relative to each other will measure the same acceleration for a moving particle.

Problem 4.7

Relative Motion in Two Dimensions

- For the position vector,

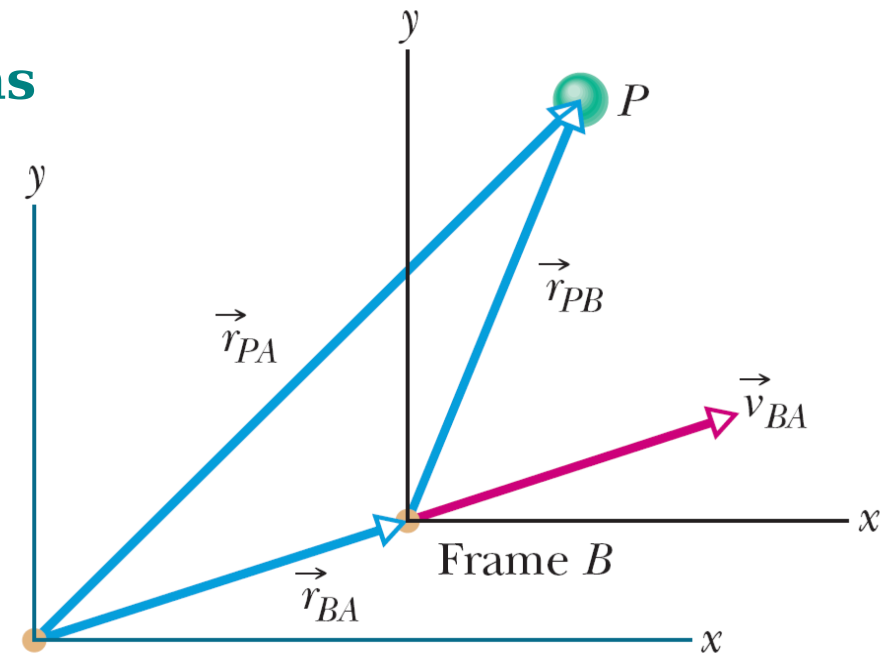
$$\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA}$$

- The velocity relation is

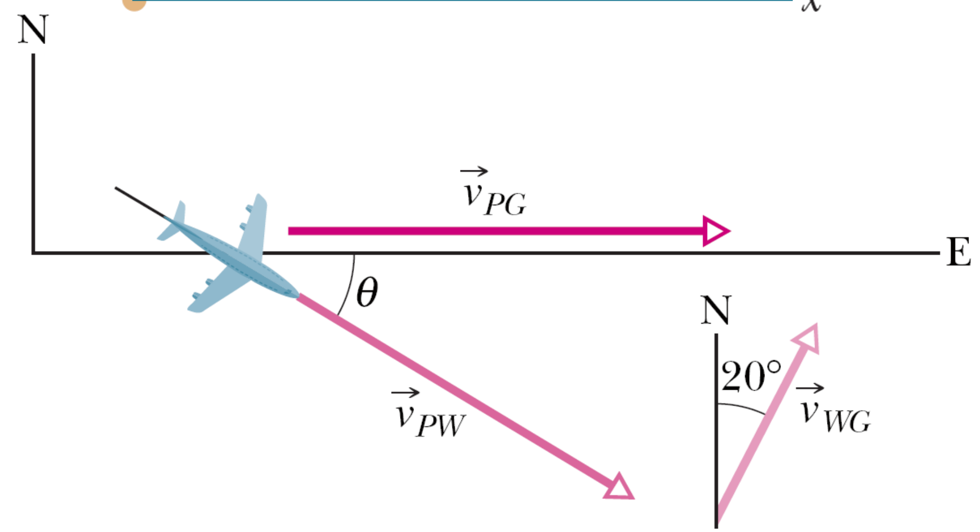
$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}$$

- For v_{BA} is constant, the the relation of the acceleration is

$$\vec{a}_{PA} = \vec{a}_{PB}$$



Problem 4.8



The chosen problems: 32, 48, 53.