Chapter 3 Vectors

Vectors and Scalars

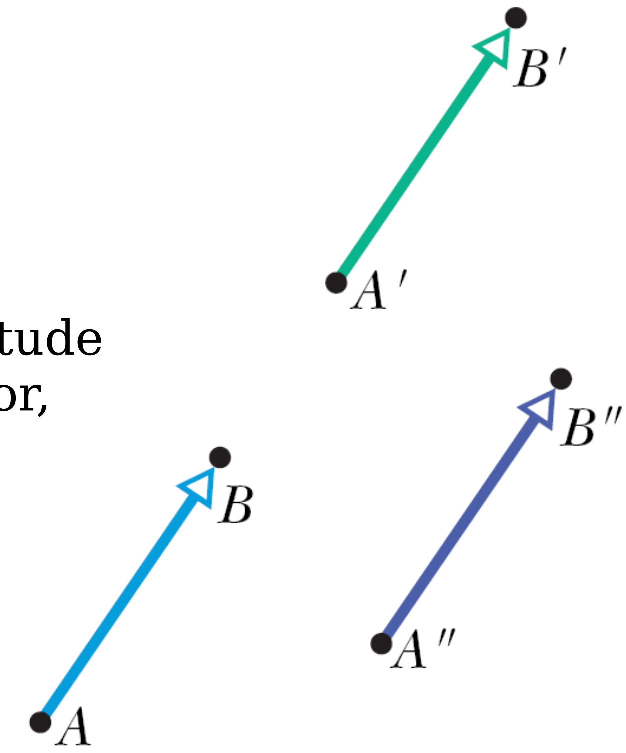
- A **vector** has magnitude as well as direction.

- A **vector quantity** is a quantity that has both a magnitude and a direction and thus can be represented with a vector, eg, displacement, velocity, acceleration.

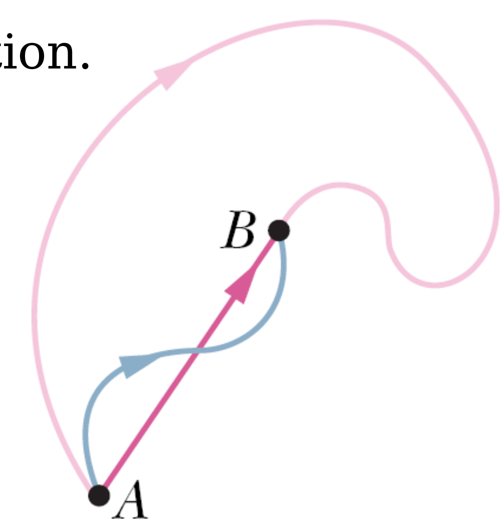
- A **scalar** has magnitude but no direction, eg, temperature, energy, mass, time.

- **Displacement vectors** AB, $A'B'$, $A''B''$ represent the same *change of position*.

- The displacement vector tell us nothing about the actual path that the particle take, but the overall effect of the motion.

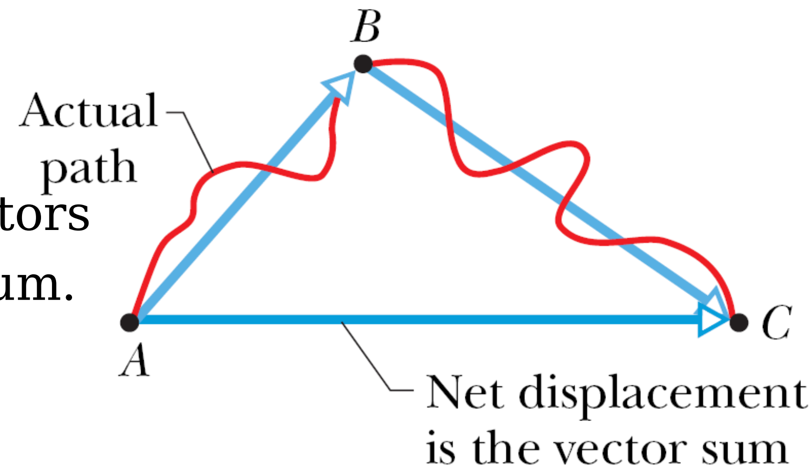


(a)



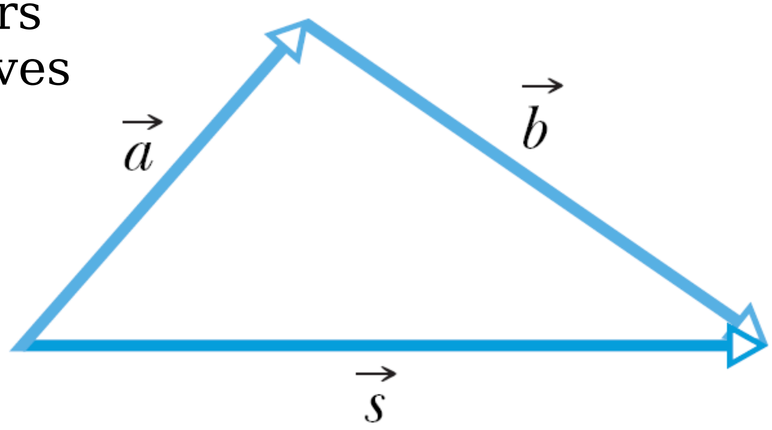
(b)

Adding Vectors Geometrically



- AC is the **vector sum** (or **resultant**) of the vectors AB and BC. This sum is not the usual algebraic sum.

- Vector equation: $\vec{s} = \vec{a} + \vec{b}$



- The symbol **+** has different meanings for vectors than it does in the usual algebra because it involves both magnitude *and* direction.

- 2 important properties:

(1) The order of addition does not matter,

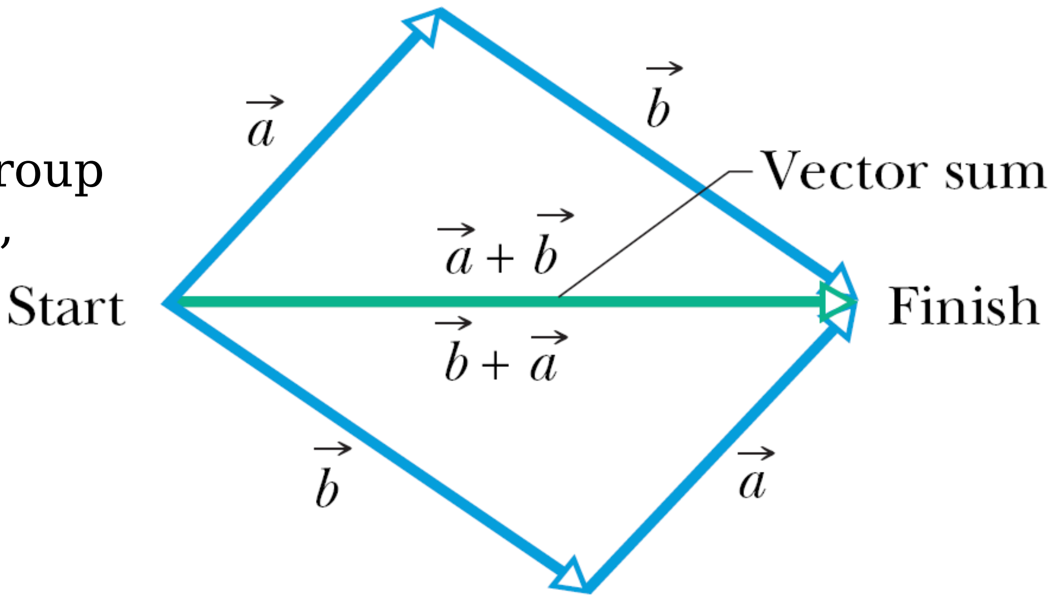
$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

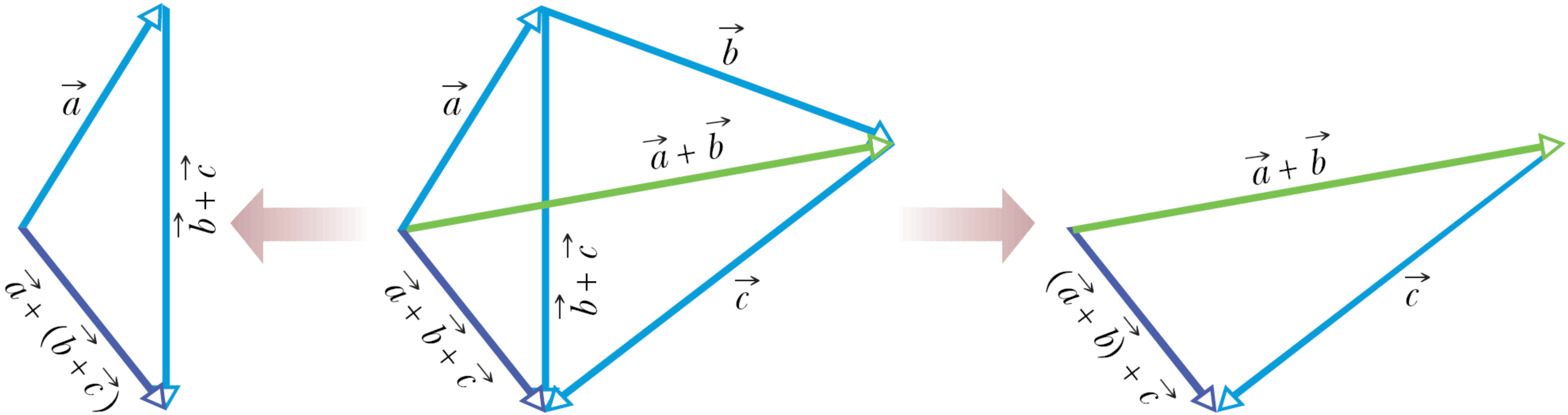
(commutative law)

(2) For more than 2 vectors, we can group them in any order as we add them,

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

(associative law)

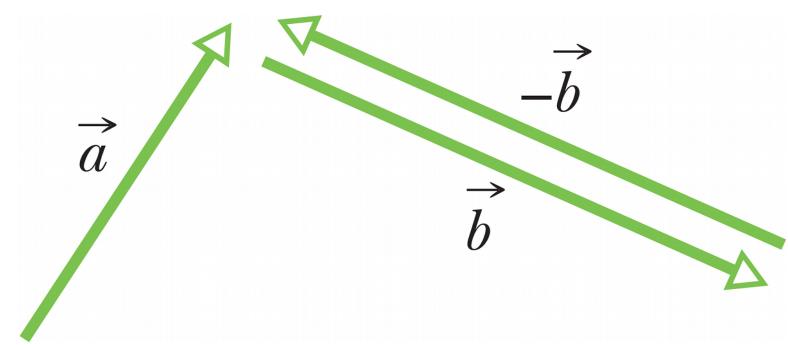


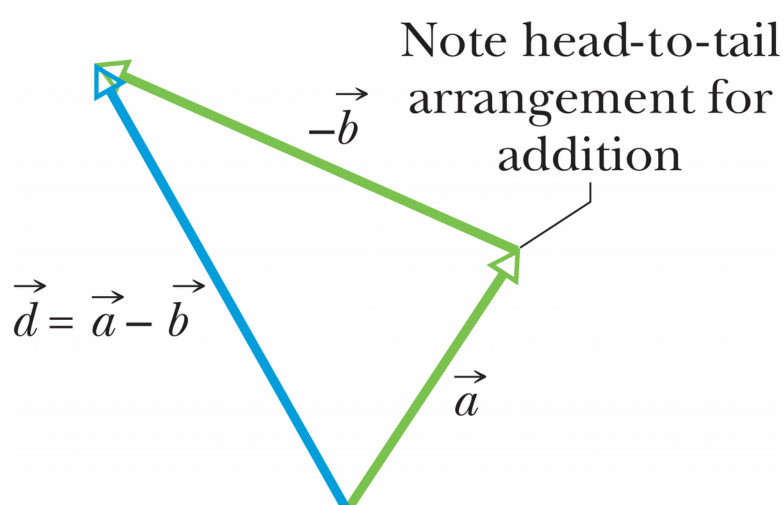


- $-\vec{b}$ is a vector with the same magnitude as $\vec{b}$ but the opposite direction.
- Adding $-\vec{b}$ has the effect of subtracting $\vec{b}$.

$$\vec{d}=\vec{a}-\vec{b}=\vec{a}+(-\vec{b}) \Rightarrow \vec{d}+\vec{b}=\vec{a}$$

(vector subtraction)





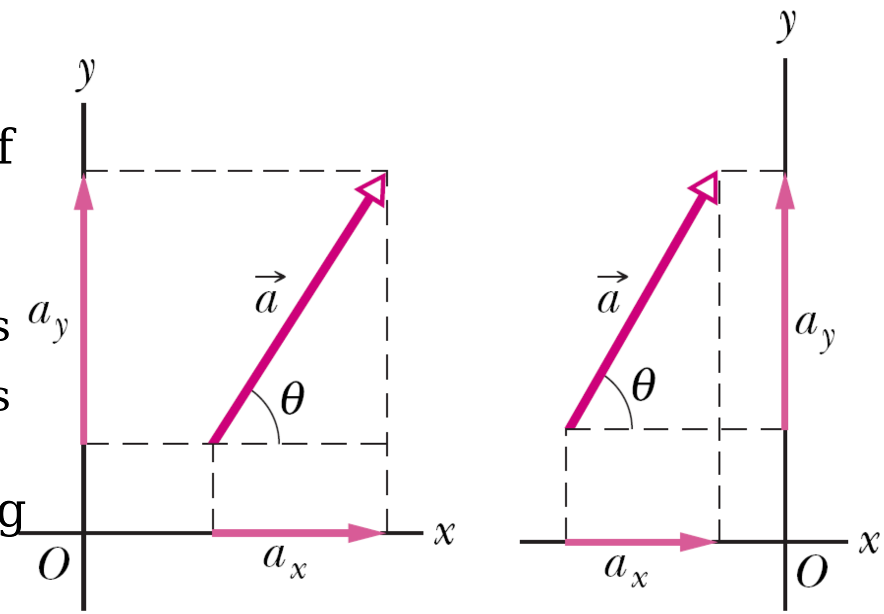
Components of Vectors

- A **component** of a vector is the projection of the vector on an axis.

x component: the projection of a vector on an x axis

y component: the projection of a vector on an y axis

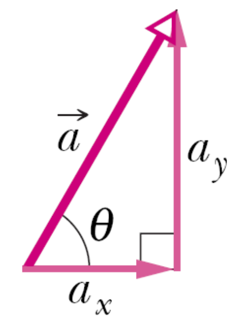
- **Resolving the vector**: the process of finding the components of a vector



(a) (b)

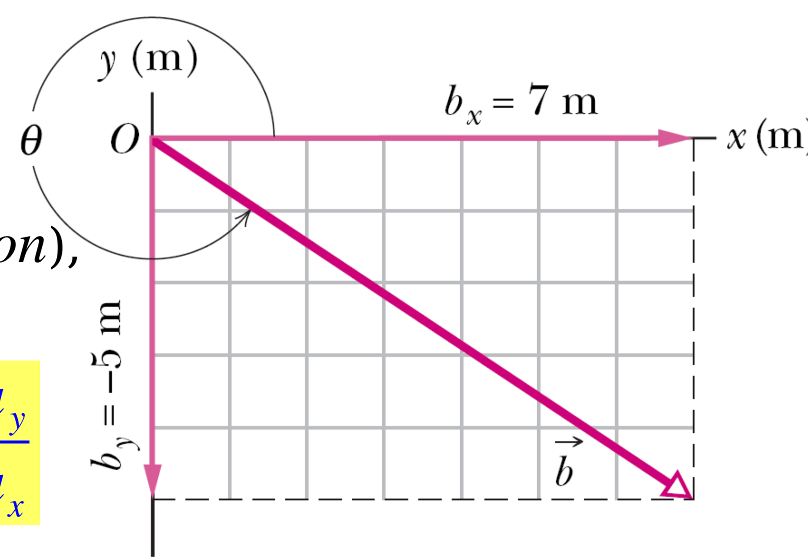
- A component of a vector has the same direction (along an axis) as the vector.

$$a_x = a\cos\theta\ ,\quad a_y = a\sin\theta$$



θ is the angle that the vector $\vec{a}$ makes with the positive direction of the x axis, $a = |\vec{a}|$.

- Reconstruct a vector from its component.



- $\vec{a}$ is determined by a_x and a_y (*component notation*), or by a and θ (*magnitude- angle notation*).

- For transformation $a = \sqrt{a_x^2 + a_y^2}$ and $\tan\theta = \dfrac{a_y}{a_x}$

- For 3D, we need a magnitude and 2 angles (a, θ, ϕ) or 3 components (a_x, a_y, a_z) to specify a vector.

Angles — Degrees and Radians

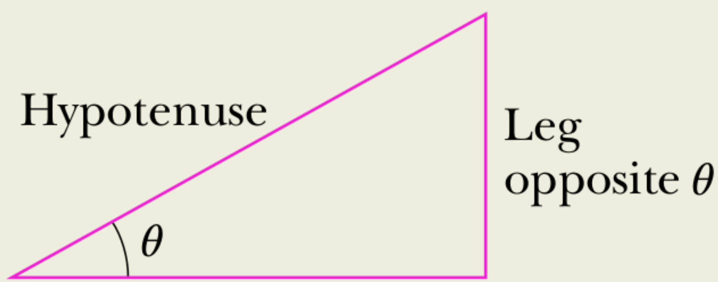
$$360^\circ\ (\text{degree}) = 2\pi\ (\text{radian})$$

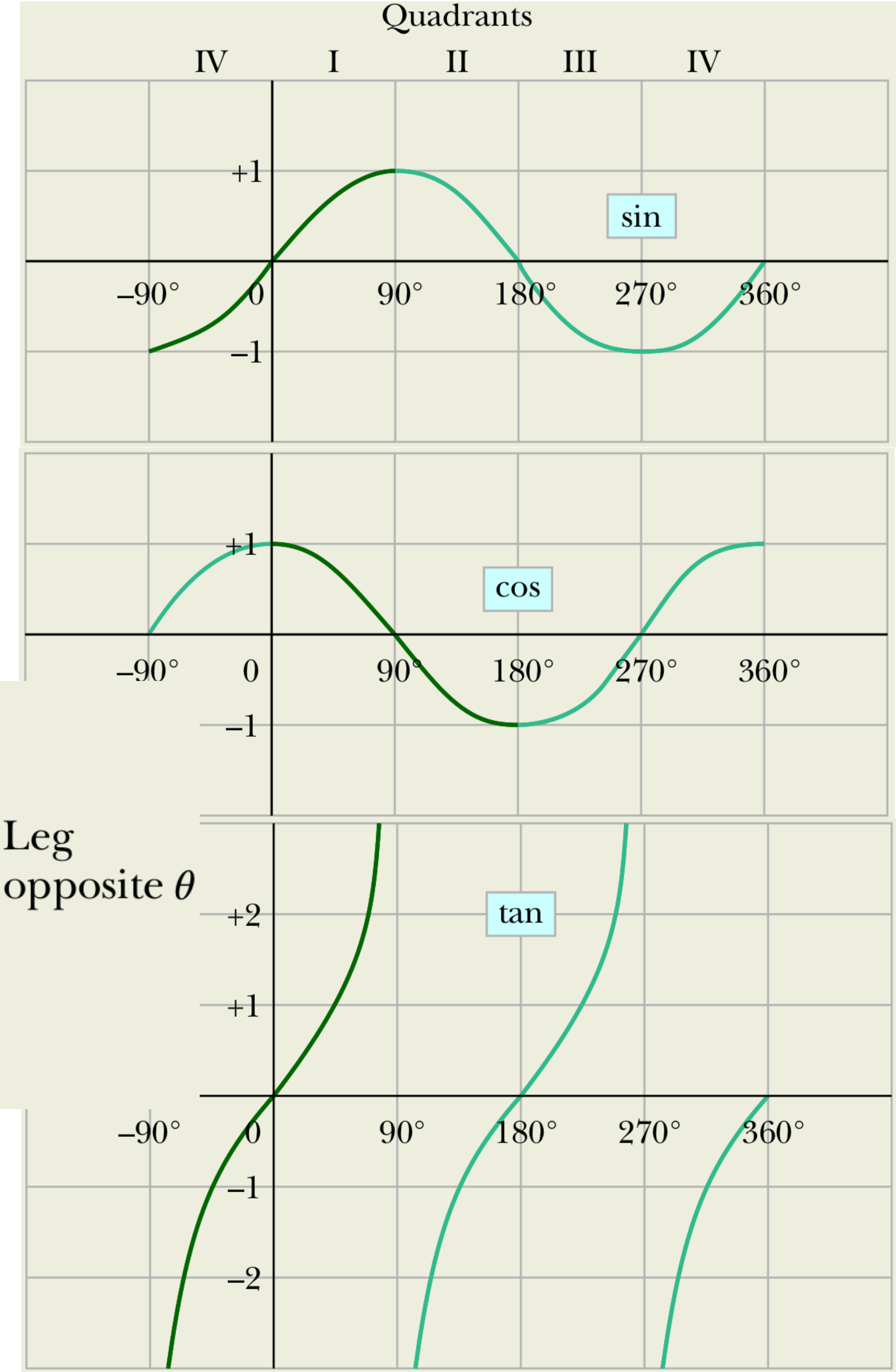
Ex: $40^\circ = 40^\circ \times \dfrac{2\pi}{360^\circ} = 0.70\ (\text{rad})$

$$\sin\theta = \frac{\text{leg opposite }\theta}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{leg adjacent to }\theta}{\text{hypotenuse}}$$

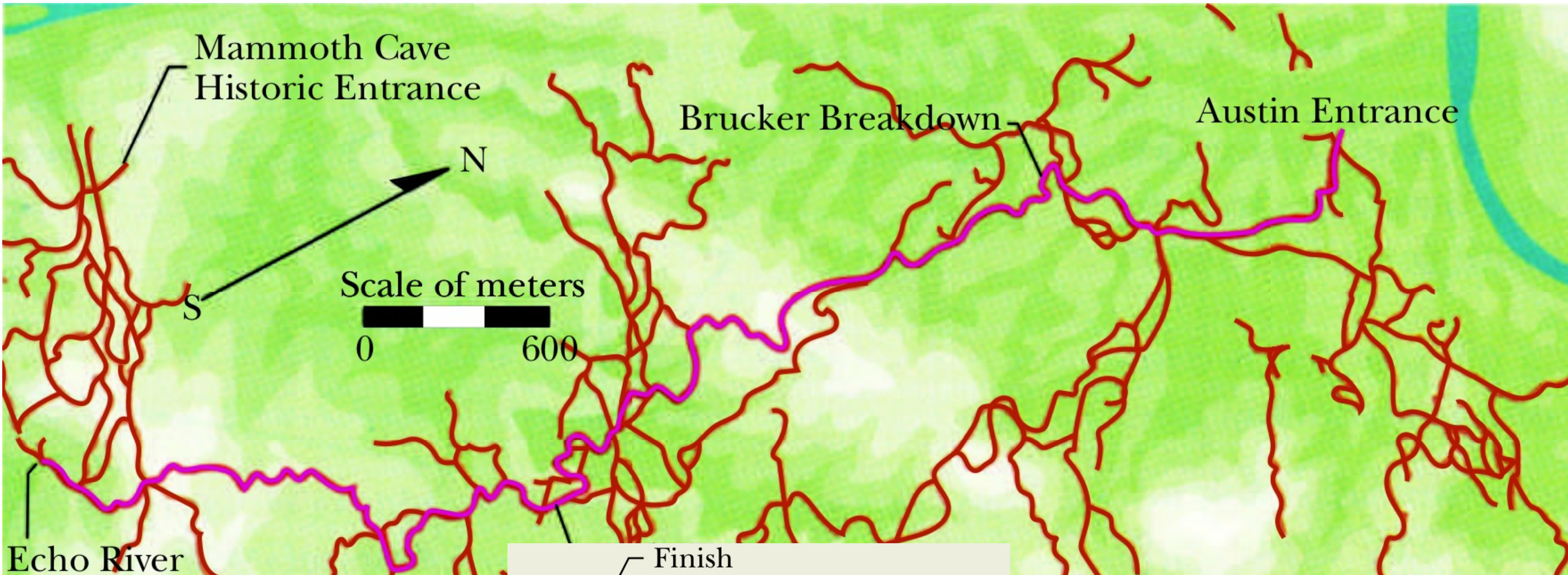
$$\tan\theta = \frac{\text{leg opposite }\theta}{\text{leg adjacent to }\theta}$$





<u>Sample 3.1.1</u>: **Spelunking**

A spelunking team found the connection from Austin Entrance in the Flint Ridge system to Echo River in Mammoth Cave, traveling a net 2.6 km westward, 3.9 km southward, and 25 m upward, the longest cave system in the world. What were the magnitude and angle of the team's displacement from start to finish?

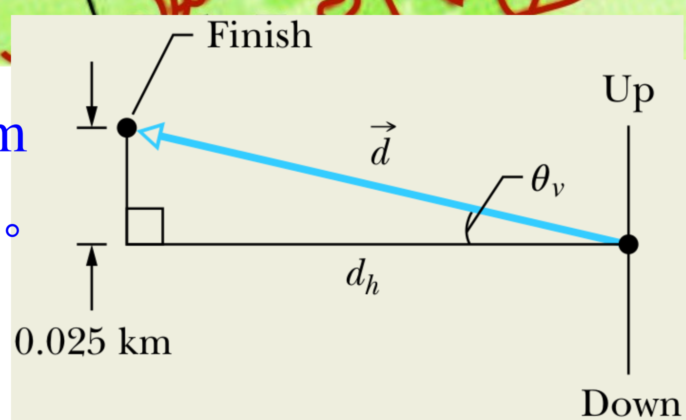


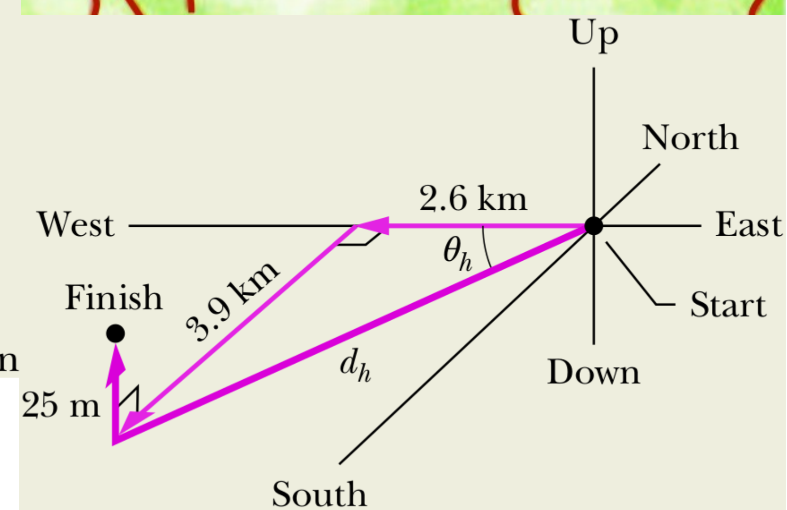
$$d_h = \sqrt{2.6^2 + 3.9^2} = 4.69\ \text{km}$$

$$\Rightarrow\ \tan\theta_h = \tan^{-1}\frac{3.9}{2.6} = 56^\circ$$

$$d = \sqrt{4.69^2 + 0.025^2}$$

$$\approx 4.7\ \text{km} \quad \Rightarrow \quad \theta_v = \tan^{-1}\frac{0.025}{4.69} = 0.3^\circ$$





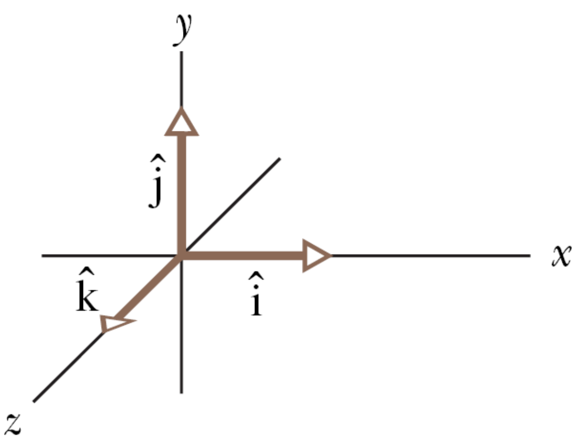
Unit Vectors

- A **unit vector** is a vector that has a magnitude of exactly 1 and points in a particular direction.

- Use **right-handed coordinate system**

Example of usage: $\vec{a} = a_x\,\hat{\mathrm{i}} + a_y\,\hat{\mathrm{j}}$

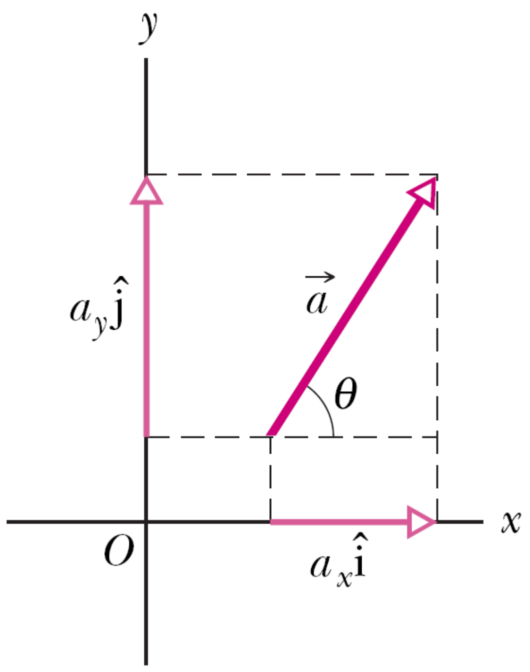
$\vec{b} = b_x\,\hat{\mathrm{i}} + b_y\,\hat{\mathrm{j}}$



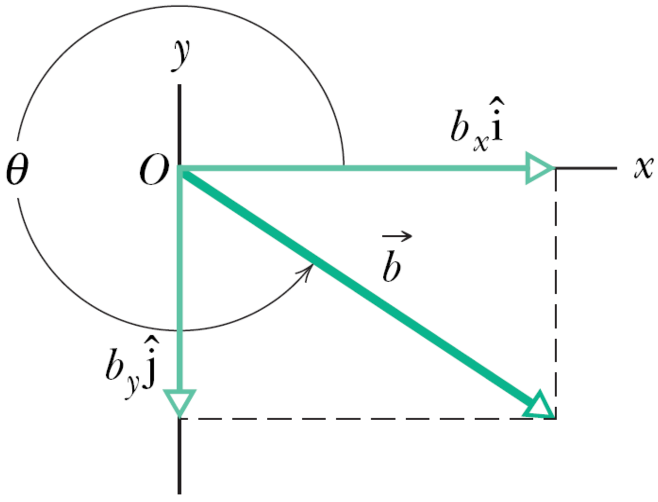
vector components: $a_x\,\hat{\mathrm{i}}\,,\quad a_y\,\hat{\mathrm{j}}$

scalar components: $a_x,\quad a_y.$

Example: $\vec{d} = (-2.6\text{ km})\,\hat{\mathrm{i}} + (0.025\text{ km})\,\hat{\mathrm{j}} + (3.9\text{ km})\,\hat{\mathrm{k}}$



(a)



(b)

Adding Vectors by Components

- Decompose $\vec{r} = \vec{a} + \vec{b}$ into
$$r_x = a_x + b_x$$
$$r_y = a_y + b_y$$
$$r_z = a_z + b_z$$

- 2 vectors must be equal if their corresponding components are equal.

- Add 2 vectors, we must
  (1) resolve the vectors into their scalar components;
  (2) combine these scalar components, axis by axis, to get the components of the sum;
  (3) combine the components to get the sum vector.

- This procedure also applies to vector subtractions:

$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + \left(-\vec{b}\right) = d_x\,\hat{\mathrm{i}} + d_y\,\hat{\mathrm{j}} + d_z\,\hat{\mathrm{k}} \Rightarrow \begin{aligned} d_x &= a_x - b_x \\ d_y &= a_y - b_y \\ d_z &= a_z - b_z \end{aligned}$$

Vectors and the Laws of Physics

- An infinite number of different pairs of components of a vector

- Which is the right pair of components?

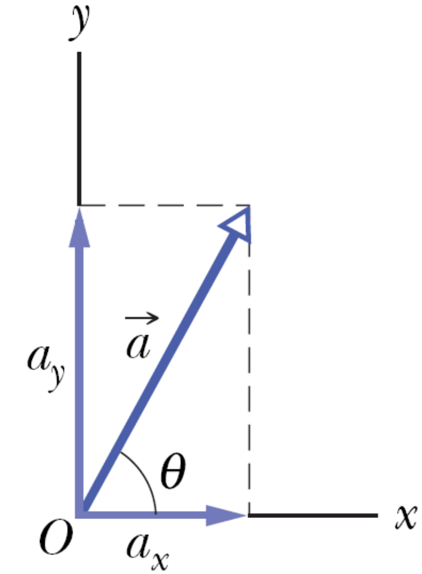
  Ans: they are all equally valid because each pair gives a different way of describing the same vector.

- The relation: $a = \sqrt{a_x^2 + a_y^2} = \sqrt{a_x'^2 + a_y'^2}$ and $\theta = \theta' + \phi$

- Freedom in choosing a coordinate system

- Also true of the relation of physics: independent of the choice of coordinate system

- Use the language of vector to present the laws of physics for its simplicity and richness.



(a)

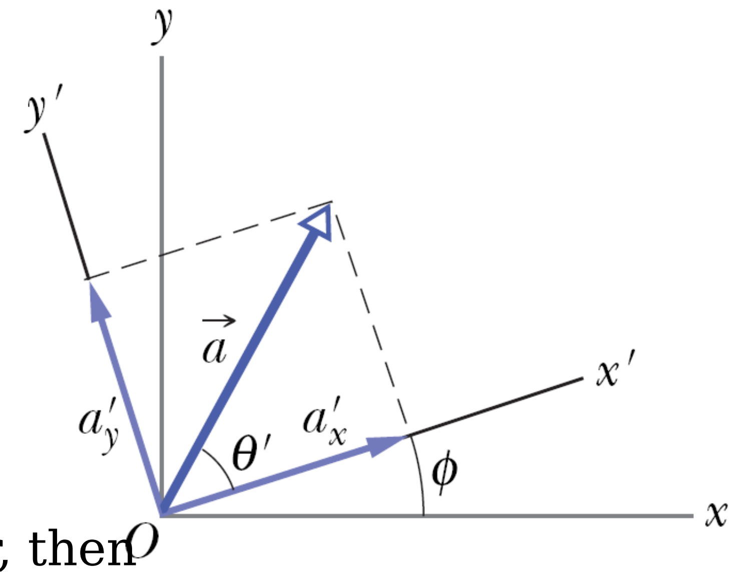
Multiplying Vectors

- 3 ways in which vectors can be multiplied:

  (1) **Multiplying a Vector by a Scalar**: s is a scalar, then

$$\vec{b} = (s)(\vec{a}) = s\,\vec{a}$$

$$\vec{c} = \frac{1}{s}(\vec{a}) = \frac{\vec{a}}{s}$$



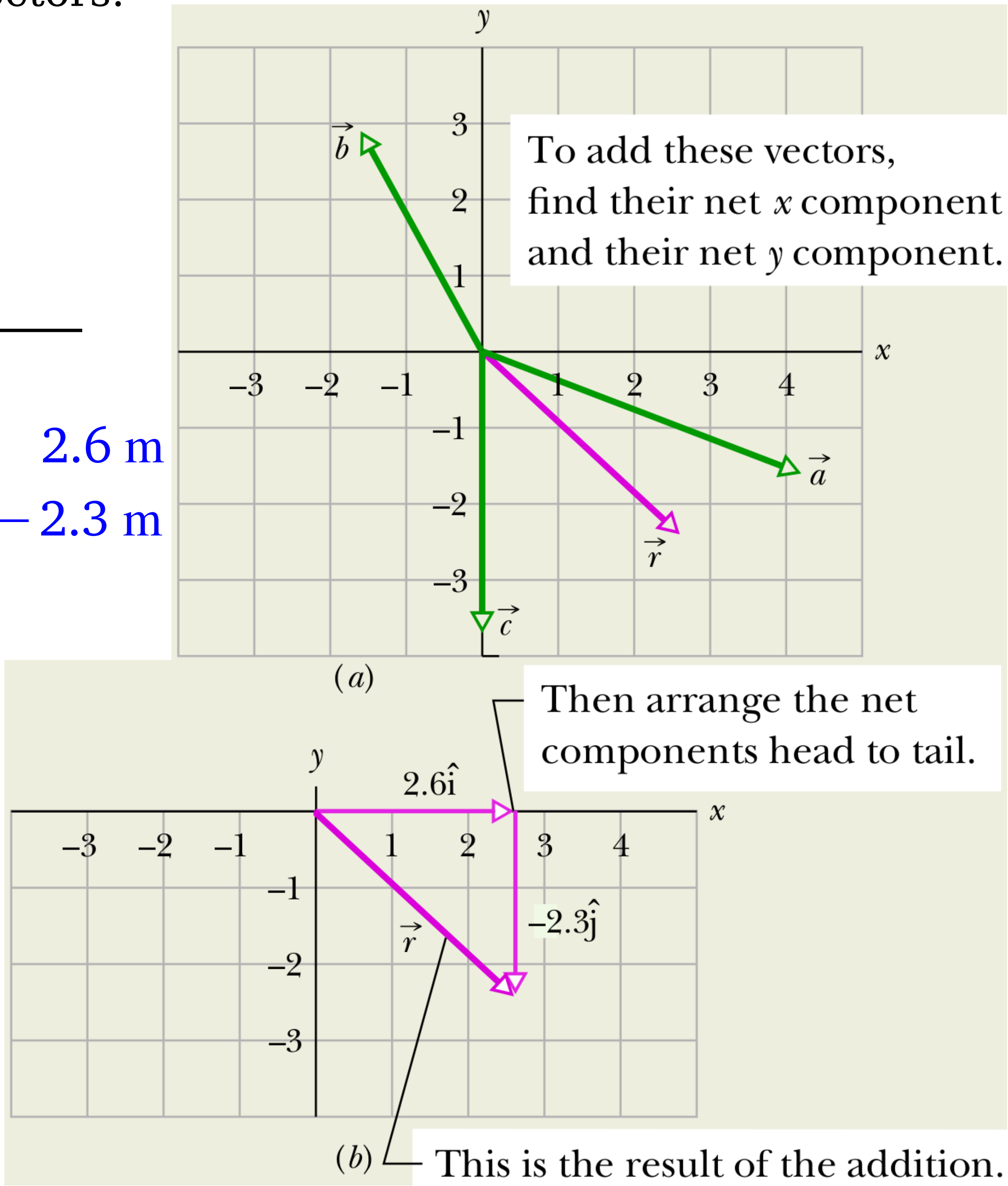
(b)

Sample 3.2.1: **Adding vectors, unit-vector components**

The figure shows the following 3 vectors:

$$\vec{a} = (4.2\text{ m})\,\hat{\text{i}} - (1.5\text{ m})\,\hat{\text{j}}$$
$$\vec{b} = (-1.6\text{ m})\,\hat{\text{i}} + (2.9\text{ m})\,\hat{\text{j}}$$
$$\vec{c} = (-3.7\text{ m})\,\hat{\text{j}}$$

What is their vector sum $\vec{r}$?



$$r_x = a_x + b_x + c_x = \quad 4.2 - 1.6 + \quad 0 \quad = \quad 2.6\text{ m}$$
$$r_y = a_y + b_y + c_y = -1.5 + 2.9 - 3.7 = -2.3\text{ m}$$

$$\Rightarrow \quad \vec{r} = 2.6\text{ m}\,\hat{\text{i}} - 2.3\text{ m}\,\hat{\text{j}}$$

$$\Rightarrow \quad r = \sqrt{2.6^2 + (-2.3)^2} = 3.5\text{ m}$$

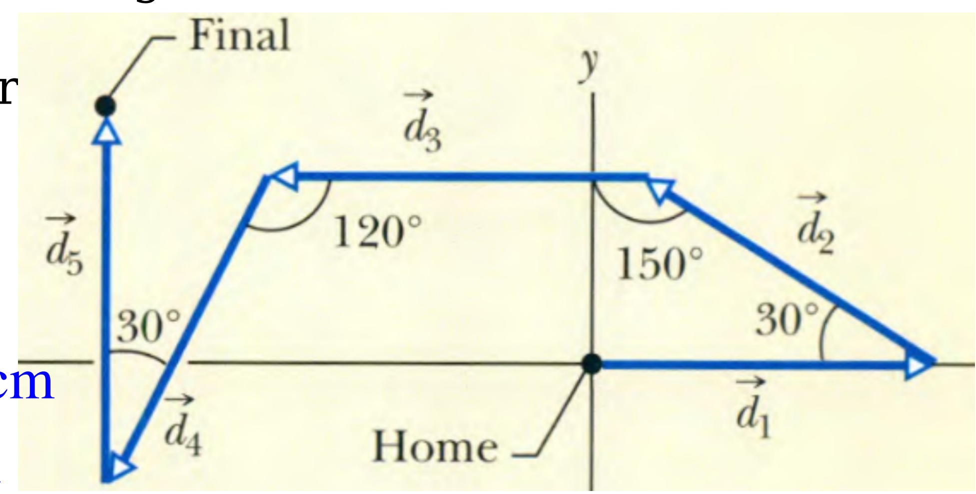
$$\theta = \tan^{-1}\frac{-2.3}{2.6} = -41^\circ$$

Sample 3.2.2: **Desert ant navigation**

The desert ant lives in the plains of the Sahara desert. The ant keeps track of its movements along a mental coordinate system. When it wants to return home, it effectively sums its displacements along the axes of the system to calculate a vector that points directly home.

Let's consider an ant making 5 runs of 6 cm each on an *xy* coordinate system, shown in the figure, starting from home. At the end of the 5th run, what are the magnitude and angle of the ant's displacement vector, and what are those of the homeward vector that extends from the ant's final position back to home?

Run	d_x (cm)	d_y (cm)
1	6	0
2	-5.2	3
3	-6	0
4	-3	-5.2
5	0	6
Net	-8.2	3.8



$$\vec{d}_{net} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3 + \vec{d}_4 + \vec{d}_5$$

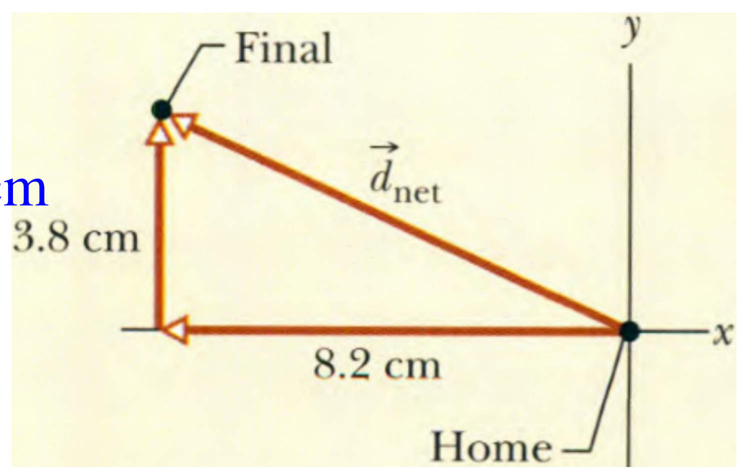
$$\Rightarrow \quad d_{net,x} = d_{1x} + d_{2x} + d_{3x} + d_{4x} + d_{5x} = -8.2 \text{ cm}$$

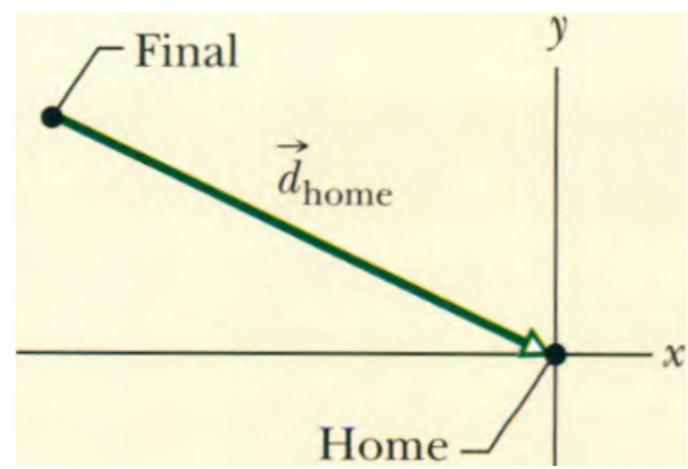
$$d_{net,y} = d_{1y} + d_{2y} + d_{3y} + d_{4y} + d_{5y} = 3.8 \text{ cm}$$

$$\Rightarrow \quad d_{net} = \sqrt{d_{net,x}^2 + d_{net,y}^2}$$

$$= \sqrt{(-8.2)^2 + 3.8^2} \approx 9 \text{ cm}$$

$$\theta = \tan^{-1} \frac{d_{net,y}}{d_{net,x}} = -24.86^\circ$$





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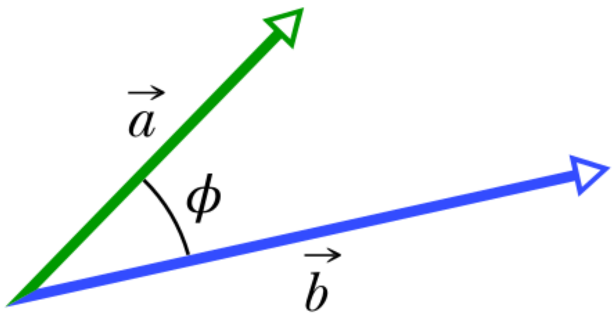
  (2) **Multiplying a Vector by a Vector**:

  (2a) **Scalar Product**: its result is a scalar;

  (2b) **Vector Product**: its result is a vector.

The Scalar Product (also **Dot Product**)



- The **scalar product** of 2 vectors is defined as

$$\vec{a}\cdot\vec{b} = a\,b\cos\phi$$

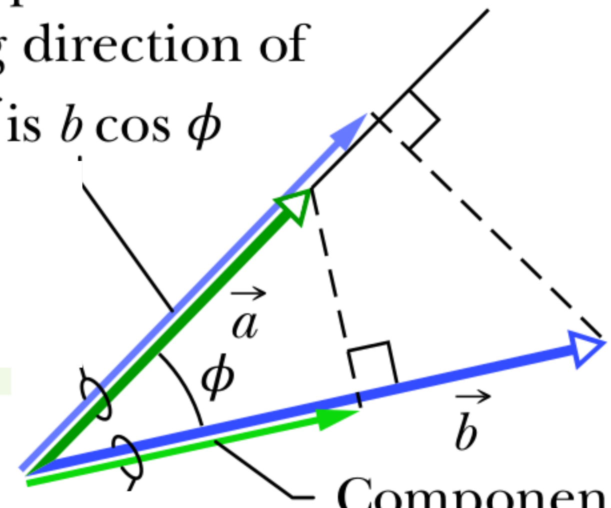
where a is the magnitude of $\vec{a}$, b is the magnitude of $\vec{b}$, and ϕ is the angle between $\vec{a}$ and $\vec{b}$.

- The angle ϕ and $360° - \phi$ is the same for the scalar product.

- $\phi = 0$ gives the scalar product its maximum;

  $\phi = 90°$ makes the scalar product be 0.

- A scalar product can be regarded as the product of 2 quantities:
  (1) the magnitude of one of the vectors,
  (2) the scalar component of the 2nd vector along the direction of the 1st vector.



- The commutative law applies to a scalar product:

$$\vec{a}\cdot\vec{b}=(a\cos\phi)(b)=(a)(b\cos\phi) \quad \Rightarrow \quad \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$$

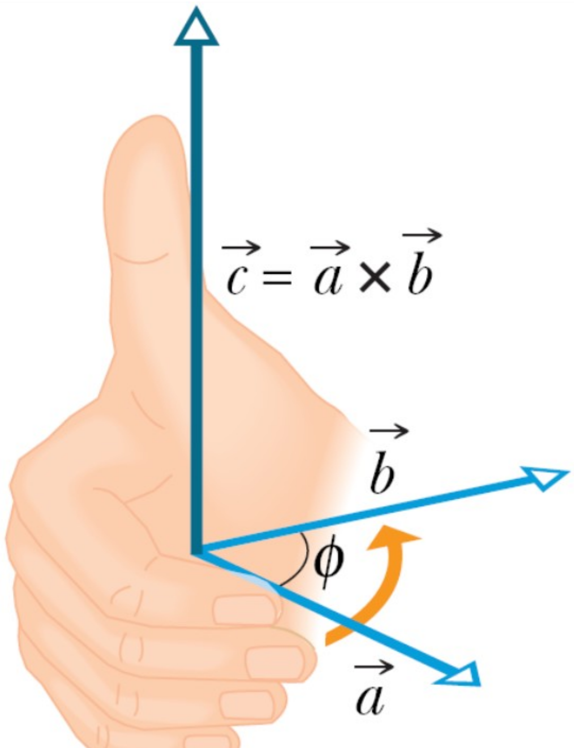
- Express the vectors into their components for the scalar product:

$$\vec{a}\cdot\vec{b}=(a_x\hat{\mathrm{i}}+a_y\hat{\mathrm{j}}+a_z\hat{\mathrm{k}})\cdot(b_x\hat{\mathrm{i}}+b_y\hat{\mathrm{j}}+b_z\hat{\mathrm{k}})$$

Use the distributive law to obtain the formula

$$\vec{a}\cdot\vec{b}=a_x b_x+a_y b_y+a_z b_z$$

The Vector Product (also Cross Product)



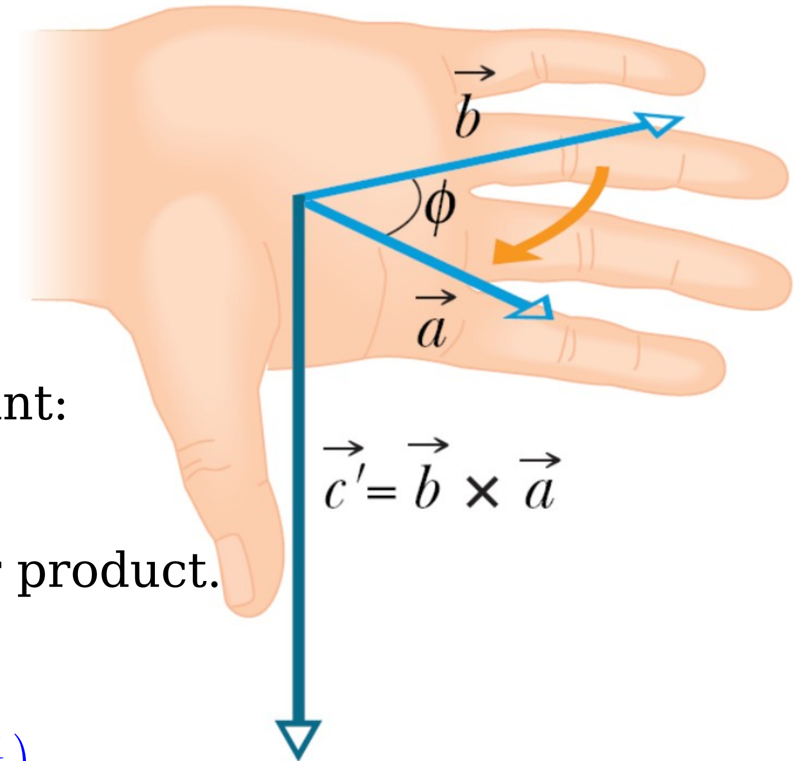
- The **vector product** of 2 vectors is defined as

$$\vec{c}=\vec{a}\times\vec{b} \quad \text{and} \quad c=a\,b\sin\phi$$

where ϕ is the *smaller* angle between $\vec{a}$ and $\vec{b}$.

- The direction of $\vec{c}$ is perpendicular to the plane that contains $\vec{a}$ and $\vec{b}$ with a **right-hand rule**.

- If 2 vectors are parallel ($\phi = 0$) or antiparallel ($\phi = 180°$), their cross product vanishes; if this 2 vectors are ⊥ each other, the magnitude of their cross product is maximum.



- The order of the vector multiplication is important:

$$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$$

- The commutative law does **not** apply to a vector product.

- In unit-vector notation

$$\vec{a} \times \vec{b} = (a_x \hat{\mathrm{i}} + a_y \hat{\mathrm{j}} + a_z \hat{\mathrm{k}}) \times (b_x \hat{\mathrm{i}} + b_y \hat{\mathrm{j}} + b_z \hat{\mathrm{k}})$$

- Since $a_x \hat{\mathrm{i}} \times b_x \hat{\mathrm{i}} = a_x b_x (\hat{\mathrm{i}} \times \hat{\mathrm{i}}) = 0$ and $a_x \hat{\mathrm{i}} \times b_y \hat{\mathrm{j}} = a_x b_y (\hat{\mathrm{i}} \times \hat{\mathrm{j}}) = a_x b_y \hat{\mathrm{k}}$;

it gives

$$\vec{a} \times \vec{b} = (a_y b_z - a_z b_y)\hat{\mathrm{i}} + (a_z b_x - a_x b_z)\hat{\mathrm{j}} + (a_x b_y - a_y b_x)\hat{\mathrm{k}}$$

$$= \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

The chosen problems: **5**, **12**, **25**

Sample 3.3.1: **Angle between 2 vectors using dot products**

What is the angle ϕ between $\vec{a} = 3\,\hat{\mathrm{i}} - 4\,\hat{\mathrm{j}}$ and $\vec{b} = -2\,\hat{\mathrm{i}} + 3\,\hat{\mathrm{k}}$?

$$a = \sqrt{3^2 + (-4)^2} = 5$$
$$b = \sqrt{(-2)^2 + 3^2} = 3.61$$

$$\vec{a} \cdot \vec{b} = (3\,\hat{\mathrm{i}} - 4\,\hat{\mathrm{j}} + 0\,\hat{\mathrm{k}}) \cdot (-2\,\hat{\mathrm{i}} + 0\,\hat{\mathrm{j}} + 3\,\hat{\mathrm{k}})$$
$$= 3 \cdot (-2) + (-4) \cdot 0 + 0 \cdot 3 = -6$$

$$\phi = \cos^{-1} \frac{\vec{a} \cdot \vec{b}}{a\,b} = 109^\circ \quad \Leftarrow \quad \vec{a} \cdot \vec{b} = a\,b\cos\phi$$

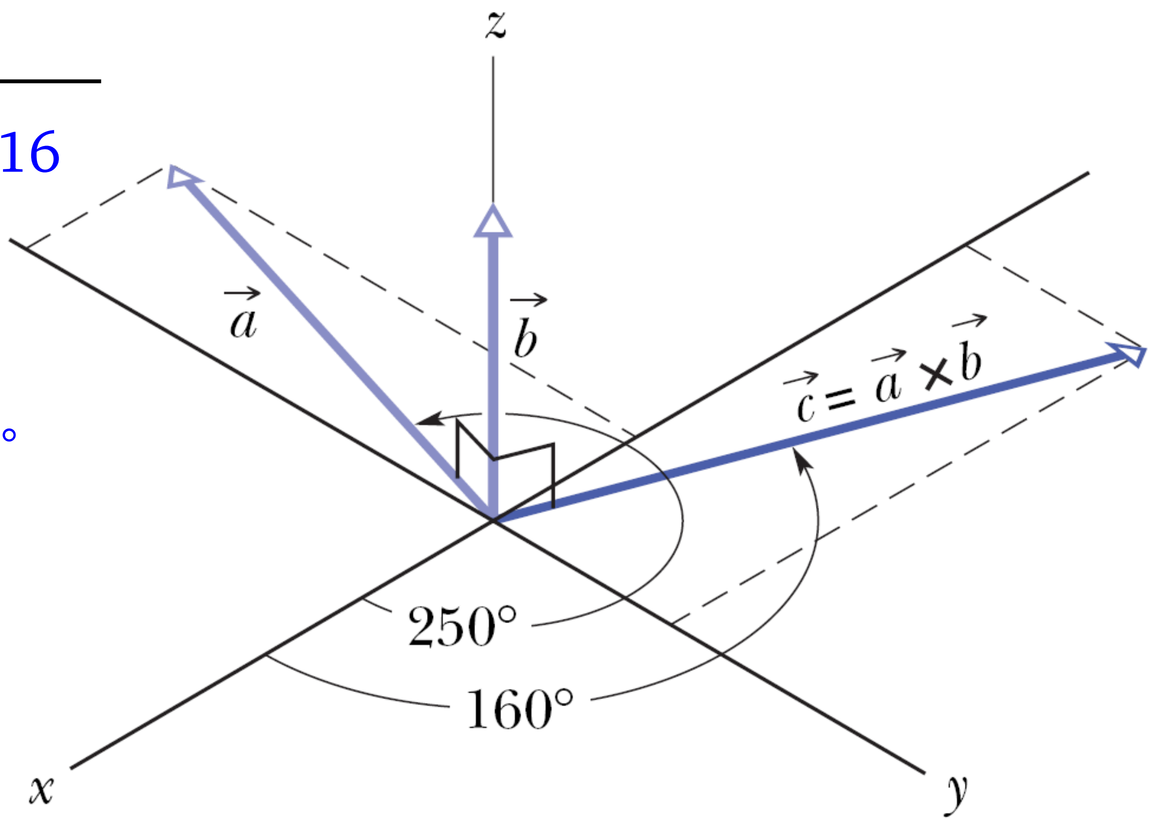
Sample 3.3.2: **Cross product, right-hand rule**

In the figure, vector $\vec{a}$ lies in the *xy* plane, has a magnitude of 18 units, and points in a direction 250° from the positive direction of the *x* axis. Also, vector $\vec{b}$ has a magnitude of 12 units and points in the positive direction of the *z* axis. What is the vector product $\vec{c} = \vec{a} \times \vec{b}$?

$$c = a\, b \sin\phi = 18 \cdot 12 \cdot \sin 90^\circ = 216$$

$$\vec{c} \perp \vec{b} \quad \Rightarrow \quad \vec{c} \quad \text{on } \ xy \ \text{ plane}$$

$$\vec{c} \perp \vec{a} \quad \Rightarrow \quad \theta = 250^\circ - 90^\circ = 160^\circ$$



Sample 3.3.3: **Cross product, unit-vector notation**

If $\vec{a} = 3\,\hat{\mathrm{i}} - 4\,\hat{\mathrm{j}}$ and $\vec{b} = -2\,\hat{\mathrm{i}} + 3\,\hat{\mathrm{k}}$, what is $\vec{c} = \vec{a} \times \vec{b}$?

$$\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ 3 & -4 & 0 \\ -2 & 0 & 3 \end{vmatrix} = -12\,\hat{\mathrm{i}} - 9\,\hat{\mathrm{j}} - 8\,\hat{\mathrm{k}}$$

Problem: Rock faults are ruptures along which opposite faces of rock have slid past each other. In the figure, points A and B coincided before the rock in the foreground slid down to the right. The net displacement AB is along the plane of the fault. The horizontal component of AB is the strike-slip AC. The component of AB that is directed down the plane of the fault is the dip-slip AD.

(a) What is the magnitude of the net displacement AB if the strike-slip is 22 m and the dip-slip is 17 m?

(b) If the plane of the fault is inclined at angle ϕ=52.0° to the horizontal, what is the vertical component of AB?

(a) $$\overline{AB} = \sqrt{\overline{AC}^2 + \overline{AD}^2} = \sqrt{22^2 + 17^2} = 27.8 \text{ m}$$

(b) $$\overline{AB}_\perp = \overline{AD}_\perp = -\overline{AD}\sin 52^\circ = -13.4 \text{ m}$$