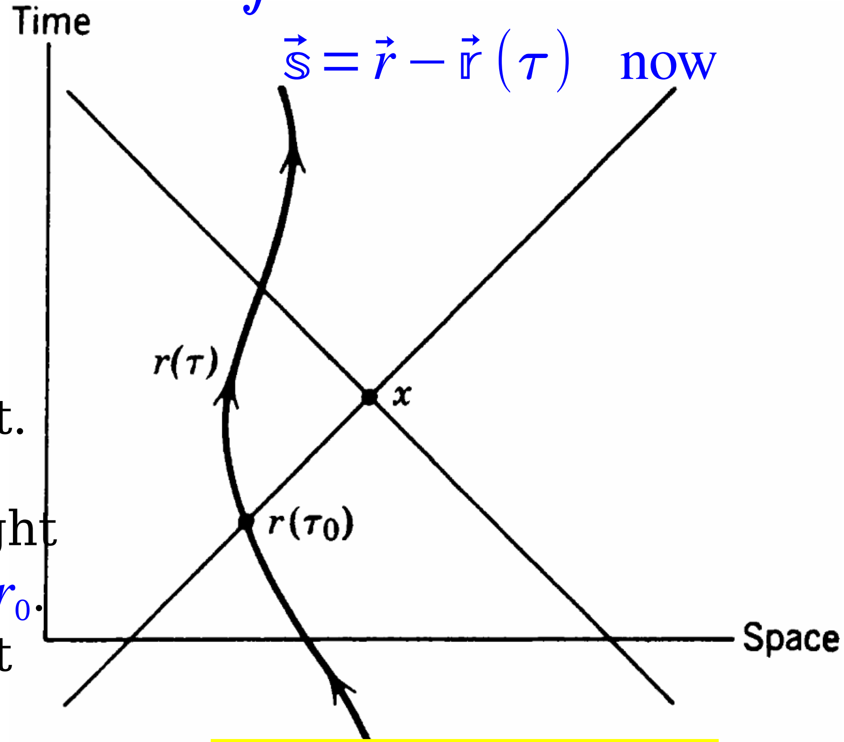
Chapter 14 Radiation by Moving Charges

Liénard-Wiechert Potentials and Fields for a Point Charge

- $\vec{A}(\vec{r}) = \frac{4\pi}{c} \int D_r(\vec{\mathbb{s}}) \vec{J}(\vec{r}') \, \mathrm{d}^4 x' \quad \Leftarrow \quad \vec{\mathbb{s}} \equiv \vec{r} - \vec{r}', \quad D_r(\vec{\mathbb{s}})$: retarded Green function

$$= 2e \int \vec{U}(\tau) \Theta(\mathbb{s}_0) \delta(\vec{\mathbb{s}}^2) \, \mathrm{d}\tau \quad \Leftarrow \quad \vec{J} = e c \int \vec{U} \delta[\vec{r}' - \vec{\mathbb{r}}(\tau)] \, \mathrm{d}\tau$$

$\vec{\mathbb{s}} = \vec{r} - \vec{\mathbb{r}}(\tau)$ now



- The integral gives a contribution only

(1) $\tau = \tau_0 \Leftarrow [\vec{r} - \vec{\mathbb{r}}(\tau_0)]^2 = 0$ for the light-cone

(2) $r_0 > \mathbb{r}_0(\tau_0)$ the retardation requirement

- The Green function is different from 0 only on the backward light cone of the observation point.

- The world line of the particle intersects the light cone at 2 points, one earlier and one later than r_0. The earlier point is the only part of the path that contributes to the fields at r^α.

- $\delta(\vec{\mathbb{s}}^2) = \frac{\delta(\vec{\mathbb{s}})}{2|\vec{\mathbb{s}} \cdot \vec{U}(\tau)|} \quad \Leftarrow \quad \delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{\left| \frac{\mathrm{d} f}{\mathrm{d} x} \right|_{x = x_i}} \quad \Rightarrow \quad \boxed{\vec{A}(\vec{r}) = \left. \frac{e \vec{U}(\tau)}{\vec{\mathbb{s}} \cdot \vec{U}} \right|_{\tau = \tau_0}}$

Lienard - Wiechert Potentials

- $\vec{U}\cdot\vec{\mathbb{s}}_{|\tau=\tau_0} = U_0\,\mathbb{s}_0 - \mathbf{U}\cdot\mathbf{s} = U_0\left[r_0 - \mathbb{r}_0(\tau_0)\right] - \mathbf{U}\cdot\left[\mathbf{r} - \mathbb{r}(\tau_0)\right]$

$$= \gamma\,c\,\mathbb{s} - \gamma\,\mathbf{v}\cdot\hat{\mathbb{s}}\,\mathbb{s} = \gamma\,c\,\mathbb{s}\left(1-\boldsymbol{\beta}\cdot\hat{\mathbb{s}}\right) \Leftarrow \quad \mathbb{s} \equiv r_0 - \mathbb{r}_0(\tau_0) = \left|\mathbf{r} - \mathbb{r}(\tau_0)\right|$$

$$\Rightarrow \quad \Phi(\mathbf{r},t) = \frac{e}{\mathbb{s}\left(1-\boldsymbol{\beta}\cdot\hat{\mathbb{s}}\right)_{|\mathrm{ret}}}\ , \quad \mathbf{A}(\mathbf{r},t) = \frac{e\,\boldsymbol{\beta}}{\mathbb{s}\left(1-\boldsymbol{\beta}\cdot\hat{\mathbb{s}}\right)_{|\mathrm{ret}}} \quad \Leftarrow \quad \boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c}$$

where $_{|\mathrm{ret}}$: evaluated at the retarded time τ_0 with $\mathbb{r}_0(\tau_0) = r_0 - \mathbb{s}$

$$\Rightarrow \quad \Phi(\mathbf{r},t) \to \frac{e}{\mathbb{s}},\quad \mathbf{A}(\mathbf{r},t) \to \frac{e}{\mathbb{s}}\,\boldsymbol{\beta} \quad \text{for } \beta \to 0 \quad \text{nonrelativistic motion}$$

- $\dfrac{\partial\,\delta\left(\vec{\mathbb{s}}^{\,2}\right)}{\partial x_\alpha} = -\dfrac{\mathbb{s}^\alpha}{\vec{U}\cdot\vec{\mathbb{s}}}\dfrac{\mathrm{d}}{\mathrm{d}\tau}\delta\left(\vec{\mathbb{s}}^{\,2}\right) \quad \Leftarrow \quad \dfrac{\partial\,\delta(f)}{\partial x_\alpha} = \dfrac{\partial f}{\partial x_\alpha}\dfrac{\mathrm{d}\tau}{\mathrm{d}f}\dfrac{\mathrm{d}\delta}{\mathrm{d}\tau}$

$$\Rightarrow \quad \frac{\partial A^\beta}{\partial x_\alpha} = 2\,e\int U^\beta\,\Theta(\mathbb{s}_0)\,\frac{\partial\,\delta\left(\vec{\mathbb{s}}^{\,2}\right)}{\partial x_\alpha}\,\mathrm{d}\tau = 2\,e\int \Theta(\mathbb{s}_0)\,\delta\left(\vec{\mathbb{s}}^{\,2}\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\mathbb{s}^\alpha\,U^\beta}{\vec{U}\cdot\vec{\mathbb{s}}}\,\mathrm{d}\tau$$

$$\Rightarrow \quad F^{\alpha\beta} = \frac{e}{\vec{U}\cdot\vec{\mathbb{s}}}\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\mathbb{s}^\alpha\,U^\beta - \mathbb{s}^\beta\,U^\alpha}{\vec{U}\cdot\vec{\mathbb{s}}}_{|\tau=\tau_0} \qquad \Uparrow \quad \frac{\partial\,\Theta}{\partial x_\alpha} \text{ has no contribution}$$

- $\vec{U} = \left(\gamma\,c\,,\,\gamma\,c\,\boldsymbol{\beta}\right) \quad \Rightarrow \quad \dfrac{\mathrm{d}\vec{U}}{\mathrm{d}\tau} = c\,\gamma^2\left[\gamma^2\,\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\,,\,\dot{\boldsymbol{\beta}} + \gamma^2\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)\boldsymbol{\beta}\right] \quad \Leftarrow \quad \dot{\boldsymbol{\beta}} = \dfrac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}t}$

$$\vec{\mathbb{s}} = \left(\mathbb{s}\,,\,\mathbb{s}\,\hat{\mathbb{s}}\right) \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\vec{U}\cdot\vec{\mathbb{s}}\right) = -c^2 + \vec{\mathbb{s}}\cdot\frac{\mathrm{d}\vec{U}}{\mathrm{d}\tau}$$

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = e\,\frac{\hat{\mathbb{s}}-\boldsymbol{\beta}}{\gamma^2\,\mathbb{s}^2\,(1-\boldsymbol{\beta}\cdot\hat{\mathbb{s}})^3}\Big|_{ret} + \frac{e}{c}\,\frac{\hat{\mathbb{s}}\times[(\hat{\mathbb{s}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{\mathbb{s}\,(1-\boldsymbol{\beta}\cdot\hat{\mathbb{s}})^3}\Big|_{ret}, \quad \mathbf{B}=\hat{\mathbb{s}}\times\mathbf{E}\big|_{ret} \qquad (0)$$

(velocity field) (acceleration field)

- The velocity fields are static fields falling off as $\dfrac{1}{\mathbb{s}^2}$, the acceleration fields are radiation fields, $\mathbf{E}$ & $\mathbf{B}$ being transverse to the radius vector and varying as $\dfrac{1}{\mathbb{s}}$.

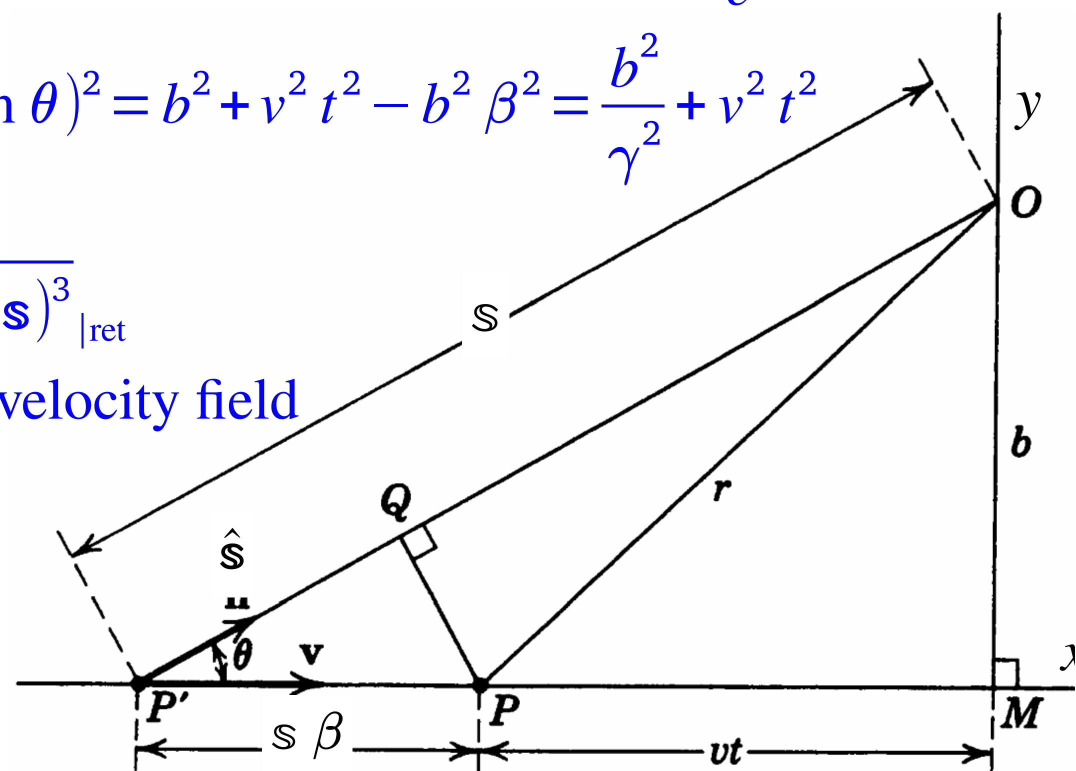
- $\vec{U} = \text{const} \quad \Rightarrow \quad F^{\alpha\beta} = e\,c^2\,\dfrac{\mathbb{s}^\alpha U^\beta - \mathbb{s}^\beta U^\alpha}{(\vec{U}\cdot\vec{\mathbb{s}})^3}\Big|_{\tau=\tau_0} \quad \Rightarrow$ Sec. 11.10 Problem 11.17

- $\overline{P'Q} = \mathbb{s}\,\beta\cos\theta = \boldsymbol{\beta}\cdot\mathbb{s}, \quad \overline{OQ} = \mathbb{s} - \boldsymbol{\beta}\cdot\mathbb{s}, \quad b = \mathbb{s}\sin\theta \quad \Leftarrow \quad \overline{P'P} = v\,\dfrac{\mathbb{s}}{c} = \mathbb{s}\,\beta$

$$\Rightarrow \quad (\mathbb{s}-\boldsymbol{\beta}\cdot\mathbb{s})^2 = r^2 - \overline{PQ}^2 = r^2 - (\mathbb{s}\,\beta\sin\theta)^2 = b^2 + v^2t^2 - b^2\beta^2 = \frac{b^2}{\gamma^2} + v^2t^2$$

$$\Rightarrow \quad E_y = \frac{e\,\gamma\,b}{(b^2+\gamma^2v^2t^2)^{3/2}} = \frac{e\,b}{\gamma^2(\mathbb{s}-\boldsymbol{\beta}\cdot\mathbb{s})^3}\Big|_{ret}$$

= the transverse component of the velocity field



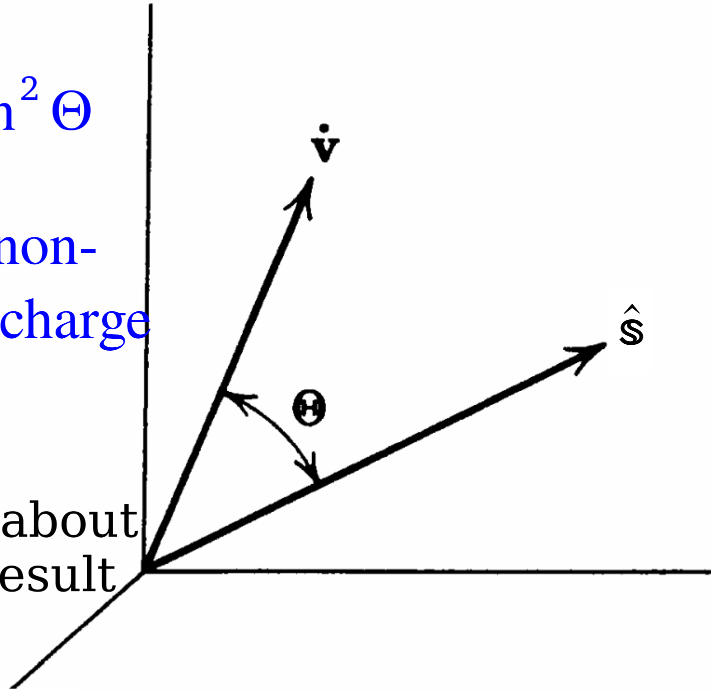
- The other components of $\mathbf{E}$ and $\mathbf{B}$ come out similarly.

Total Power Radiated by an Accelerated Charge: Larmor's Formula and Its Relativistic Generalization

- $\beta \ll 1 \;\Rightarrow\; \mathbf{E}_a \simeq \dfrac{e}{c}\,\dfrac{\hat{\mathbb{s}}\times\left(\hat{\mathbb{s}}\times\dot{\boldsymbol{\beta}}\right)}{\mathbb{s}}\Big|_{\text{ret}} \;\Rightarrow\; \mathbf{S} = \dfrac{c}{4\pi}\,\mathbf{E}\times\mathbf{B} \simeq \dfrac{c}{4\pi}\,E_a^2\,\hat{\mathbb{s}} \;\Leftarrow$ energy flux by Poynting vector

$$\Rightarrow \quad \frac{\mathrm{d}P}{\mathrm{d}\Omega} \simeq \frac{c\,\mathbb{s}^2 E_a^2}{4\pi} = \frac{e^2}{4\pi c}\left|\hat{\mathbb{s}}\times\left(\hat{\mathbb{s}}\times\dot{\boldsymbol{\beta}}\right)\right|^2 = \frac{e^2}{4\pi c}\,\dot{\beta}^2\sin^2\Theta$$

$$\Rightarrow \quad P = \frac{2}{3}\,\frac{e^2}{c}\,\dot{\beta}^2 = \frac{2}{3}\,\frac{e^2}{c^3}\,\dot{v}^2 \quad \Leftarrow$$ Larmor's formula for a non-relativistic, accelerated charge



the radiation is polarized in the plane of $\dot{\mathbf{v}}$ and $\hat{\mathbb{s}}$.

- Larmor's formula can be generalized by arguments about covariance under Lorentz transformations to yield a result that is valid for arbitrary velocities of the charge.

- Radiated EM energy behaves like the 0th component of a 4-vector, so the power is a Lorentz invariant.

- find a Lorentz invariant that involves only $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ and reduces to Larmor's formula for $\beta \ll 1$, then we have the desired generalization. The result is unique.

- $P = \dfrac{2}{3}\,\dfrac{e^2\,\dot{v}^2}{c^3} = \dfrac{2\,e^2}{3\,m^2 c^3}\,\dfrac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\cdot\dfrac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \;\Rightarrow\; P = -\dfrac{2}{3}\,\dfrac{e^2}{m^2 c^3}\,\dfrac{\mathrm{d}\vec{p}}{\mathrm{d}\tau}\cdot\dfrac{\mathrm{d}\vec{p}}{\mathrm{d}\tau} \quad (1) \;\Leftarrow$ generalization $\mathrm{d}t = \gamma\,\mathrm{d}\tau$

$$-\frac{d\vec{p}}{d\tau}\cdot\frac{d\vec{p}}{d\tau}=\left(\frac{d\mathbf{p}}{d\tau}\right)^2-\frac{1}{c^2}\left(\frac{d\mathcal{E}}{d\tau}\right)^2=\left(\frac{d\mathbf{p}}{d\tau}\right)^2-\beta^2\left(\frac{dp}{d\tau}\right)^2$$

where $\mathcal{E}=\gamma m c^2,\quad \mathbf{p}=\gamma m\mathbf{v}\quad\Rightarrow\quad d\mathcal{E}=m\gamma^3 v\,dv,\quad dp=m\gamma^3 dv$

$$\Rightarrow\quad P=\frac{2}{3}\frac{e^2}{c}\gamma^6[\dot{\boldsymbol{\beta}}^2-(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2]\quad \text{the Lienard result}$$

- The expression for radiated power can be used for charged-particle accelerators. Radiation losses are a limiting factor in the maximum practical energy attainable.

- For a given applied force, the radiated power (1) depends inversely on mass^2 of the particle. Consequently these radiative effects are largest for electrons.

- In a linear accelerator the motion is 1d $\frac{d\mathcal{E}}{dp}=\frac{dx}{dt}\quad\Rightarrow\quad P=\frac{2e^2}{3m^2c^3}\left(\frac{dp}{dt}\right)^2=\frac{2e^2}{3m^2c^3}\left(\frac{d\mathcal{E}}{dx}\right)^2$

$$\Rightarrow\quad P=\frac{2}{3}\frac{e^2}{m^2c^3}\left[\left(\frac{dp}{d\tau}\right)^2-\beta^2\left(\frac{dp}{d\tau}\right)^2\right]$$

for linear motion the power radiated depends only on the external forces that

Determine $\frac{d\mathcal{E}}{dx}$, not on the actual energy or momentum of the particle.

- $$\frac{\text{the radiated power}}{\text{power by external sources}}=\frac{P}{\frac{d\mathcal{E}}{dt}}=\frac{2e^2}{3m^2c^3}\frac{1}{v}\frac{d\mathcal{E}}{dx}\to\frac{2}{3}\frac{e^2/mc^2}{mc^2}\frac{d\mathcal{E}}{dx}\quad\text{for }\beta\to 1$$

- The radiation loss in an electron linear accelerator is unimportant unless the gain in energy is of the order of $\dfrac{m\,c^2}{e^2/m\,c^2} \sim 2\times 10^{14}\ \text{MeV/m}$. So radiation losses are negligible in linear accelerators, whether for electrons or heavier particles.

- In circular accelerators the momentum changes rapidly in direction as the particle rotates, but the change in energy per revolution is small

$$\Rightarrow \quad \left|\frac{\mathrm{d}\,\mathbf{p}}{\mathrm{d}\,\tau}\right| = \gamma\,\omega\,|\mathbf{p}| \gg \frac{1}{c}\,\frac{\mathrm{d}\,\mathscr{E}}{\mathrm{d}\,\tau} \quad \Rightarrow \quad P \approx \frac{2}{3}\,\frac{e^2}{m^2\,c^3}\,\gamma^2\,\omega^2\,|\mathbf{p}|^2 = \frac{2}{3}\,\frac{e^2\,c}{\rho^2}\,\gamma^4\,\beta^4 \quad \Leftarrow \quad \omega = \frac{c\,\beta}{\rho}$$

$$\Rightarrow \quad \frac{\text{radiative-energy loss}}{\text{revolution}} = \delta\,\mathscr{E} \approx \frac{2\,\pi\,\rho}{c\,\beta}\,P = \frac{4\,\pi}{3}\,\frac{e^2}{\rho}\,\gamma^4\,\beta^3 \to 0.1\,\frac{\mathscr{E}^4\,(\text{GeV})}{\rho\,(\text{meter})} \ \text{ for } \beta \to 1$$

$$\rho \simeq 1\ \text{meter}\,, \quad \mathscr{E}_{\text{max}} \simeq 0.3\ \text{GeV} \quad \Rightarrow \quad \delta\,\mathscr{E}_{\text{max}} = 1\ \text{keV}/\text{revolution}$$

This is less than, but not negligible to, the energy gain of a few KVs/turn.

- At higher energies the limitation on available radiofrequency power to overcome the radiation loss becomes a dominant consideration.

- The power radiated in circular electron accelerators can be expressed numerically as $P\,(\text{watts}) = 10^6\,\delta\,\mathscr{E}\,(\text{Mev})\,J\,(\text{amp})$

Angular Distribution of Radiation Emitted by an Accelerated Charge

- For an accelerated charge with $\beta \ll 1$, the radial component of Poynting's vector

$$[\mathbf{S}\cdot\hat{\mathbb{s}}]_{\rm ret} \simeq \frac{e^2}{4\pi c}\left|\frac{\hat{\mathbb{s}}\times[(\hat{\mathbb{s}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{\mathbb{s}\,(1-\hat{\mathbb{s}}\cdot\boldsymbol{\beta})^3}\right|^2_{\rm ret} \quad \Leftarrow \quad \mathbf{S} \simeq \frac{c}{4\pi}\mathbf{E}_a\times\mathbf{B}_a = \frac{c}{4\pi}|\mathbf{E}_a|^2\,\hat{\mathbb{s}}$$

energy/area/time at an observation point at t of radiation emitted at $t' = t - \dfrac{\mathbb{s}(t')}{c}$

- 2 types of relativistic effect:

(1) the effect of the spatial relationship between $\boldsymbol{\beta}$ & $\dot{\boldsymbol{\beta}}$, which determines the angular distribution.
(2) The relativistic effect from the transformation from the rest frame to the observer's frame and showing itself by the factors $1-\hat{\mathbb{s}}\cdot\boldsymbol{\beta}$ in the denominator.

- For ultrarelativistic particles effect (2) dominates the whole angular distribution.

- To calculate the energy radiated during a finite period $[T_1, T_2]$ of acceleration,

$$\mathcal{E} = \int_{T_1+\frac{\mathbb{s}(T_1)}{c}}^{T_2+\frac{\mathbb{s}(T_2)}{c}} [\mathbf{S}\cdot\hat{\mathbb{s}}]_{\rm ret}\,\mathrm{d}t = \int_{T_1}^{T_2} \mathbf{S}\cdot\hat{\mathbb{s}}\,\frac{\mathrm{d}t}{\mathrm{d}t'}\,\mathrm{d}t' \quad \Leftarrow \quad \mathbf{S}\cdot\hat{\mathbb{s}}\,\frac{\mathrm{d}t}{\mathrm{d}t'} : \ \text{(power radiated)/area in the charge's time}$$

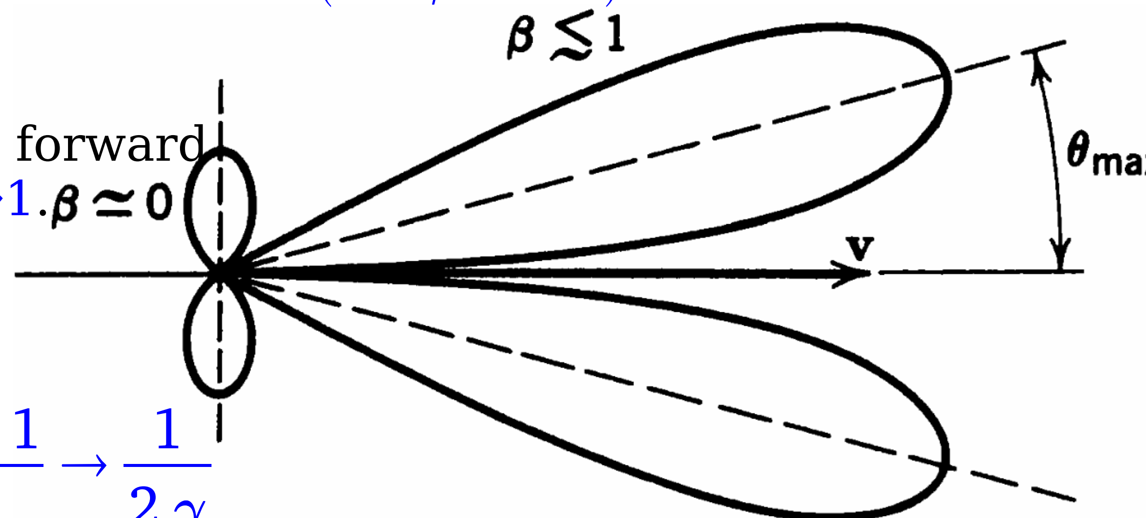
$$\Rightarrow \quad \frac{\text{power radiated}}{\text{solid angle}} = \frac{\mathrm{d}P}{\mathrm{d}\Omega}(t') = \mathbb{s}^2\,\mathbf{S}\cdot\hat{\mathbb{s}}\,\frac{\mathrm{d}t}{\mathrm{d}t'} = \mathbf{S}\cdot\mathbb{s}\,(\mathbb{s}-\boldsymbol{\beta}\cdot\mathbb{s}) = \frac{e^2\,|\hat{\mathbb{s}}\times[(\hat{\mathbb{s}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]|^2}{4\pi c\,(1-\hat{\mathbb{s}}\cdot\boldsymbol{\beta})^5}$$

- If the charge is accelerated only for a short time during which $\boldsymbol{\beta}$ & $\dot{\boldsymbol{\beta}}$ basically constant, and the observation point is far away away that $\hat{\mathbb{s}}$ & $\mathbb{s}$ change negligibly during the interval, then the power/(solid angle) is proportional to the angular distribution of the energy radiated.

- $\boldsymbol{\beta} \| \dot{\boldsymbol{\beta}}$ for a linear motion $\Rightarrow \quad \dfrac{dP}{d\Omega}(t') = \dfrac{e^2 \dot{\beta}^2}{4\pi c} \dfrac{\sin^2\theta}{(1-\beta\cos\theta)^5} \quad \Leftarrow \quad \cos\theta = \hat{\mathbb{s}} \cdot \hat{\boldsymbol{\beta}}$

$\Rightarrow$ Larmor's result for $\beta \ll 1$

- the angular distribution is tipped forward and increases in magnitude for $\beta \rightarrow 1$.



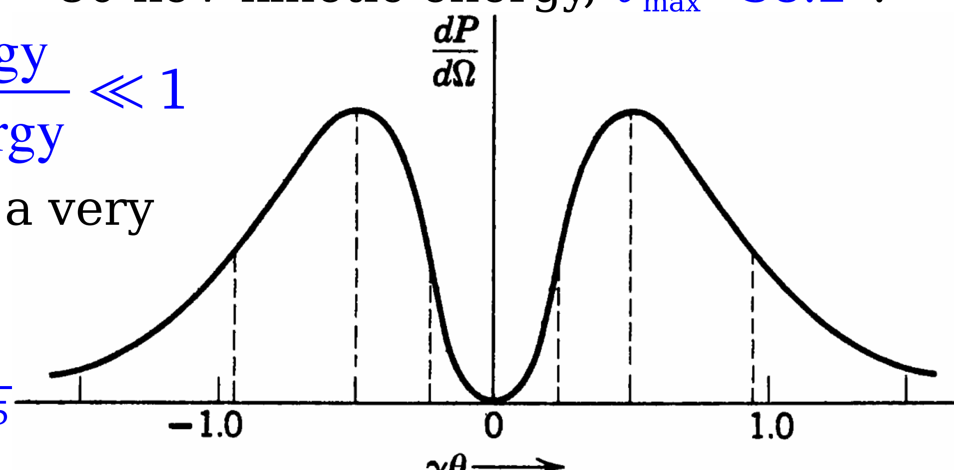
$$\beta \rightarrow 1 \quad \Rightarrow \quad \left.\frac{dP}{d\Omega}\right|_{max} \propto \gamma^8$$

$$\theta_{max} = \cos^{-1} \frac{\sqrt{15\beta^2+1}-1}{3\beta} \rightarrow \frac{1}{2\gamma}$$

- For β=0.5, corresponding to electrons of ~80 keV kinetic energy, θ_{max}=38.2°.

- For relativistic particles, $\theta_{max} \sim \dfrac{\text{rest energy}}{\text{total energy}} \ll 1$

and the angular distribution is confined to a very narrow cone in the direction of motion.



$$\theta \rightarrow 0 \quad \Rightarrow \quad \frac{dP}{d\Omega}(t') \simeq \frac{8 e^2 \dot{\beta}^2}{\pi c} \gamma^8 \frac{\gamma^2\theta^2}{(1+\gamma^2\theta^2)^5}$$

Find $\theta_{\max}$ at which the maximum radiation is emitted.

- $$\frac{d}{d\,\theta}\,\frac{d\,P}{d\,\Omega}=0 \quad\Rightarrow\quad 0=\frac{d}{d\,\theta}\,\frac{\sin\theta^2}{(1-\beta\cos\theta)^5}=\frac{\sin\theta\left(2\cos\theta-2\,\beta\cos^2\theta-5\,\beta\sin^2\theta\right)}{(1-\beta\cos\theta)^6}$$

$$\Rightarrow\quad 3\,\beta\cos^2\theta+2\cos\theta-5\,\beta=0 \quad\Rightarrow\quad \cos\theta=\frac{\pm\sqrt{15\,\beta^2+1}-1}{3\,\beta}$$ choose $+$ sign to fit $\beta\to 0$

$$\Rightarrow\quad \theta_{\max}=\cos^{-1}\frac{\sqrt{15\,\beta^2+1}-1}{3\,\beta}$$ and $\theta_{\min}=0,\pi$ for $\sin\theta_{\min}=0 \quad\Rightarrow\quad \frac{d\,P}{d\,\Omega}_{|\min}=0$

For ultra-relativistic speeds $\Rightarrow\quad \beta\to 1 \quad\Rightarrow\quad \beta=1-\delta\,,\quad \delta\ll 1 \quad\Rightarrow\quad \gamma=\frac{1}{\sqrt{1-\beta^2}}\simeq\frac{1}{\sqrt{2\,\delta}}$

$$\Rightarrow\quad \frac{\sqrt{15\,\beta^2+1}-1}{3\,\beta}=\frac{\sqrt{15\,(1-\delta)^2+1}-1}{3\,(1-\delta)}\simeq\frac{\sqrt{16-30\,\delta}-1}{3\,(1-\delta)}\simeq\frac{1+\delta}{3}\left(3-\frac{15}{4}\,\delta\right)$$

$$\simeq 1-\frac{\delta}{4} \quad\Rightarrow\quad \cos\theta_{\max}\simeq 1-\frac{\theta_{\max}^2}{2}\simeq 1-\frac{\delta}{4} \quad\Rightarrow\quad \theta_{\max}\simeq\sqrt{\frac{\delta}{2}}\simeq\sqrt{\frac{1-\beta}{2}}\simeq\frac{1}{2\,\gamma}$$

$$\Rightarrow\quad \frac{d\,P}{d\,\Omega}_{|\max}=\frac{\mu_0\,q^2}{16\,\pi^2\,\epsilon_0\,c}\,\frac{\dot\beta^2\sin\theta_{\max}^2}{(1-\beta\cos\theta_{\max})^5}\simeq\frac{\mu_0\,q^2\,\dot\beta^2}{16\,\pi^2\,\epsilon_0\,c}\,\frac{\theta_{\max}^2}{[1-(1-\delta)\,(1-\delta/4)]^5}$$

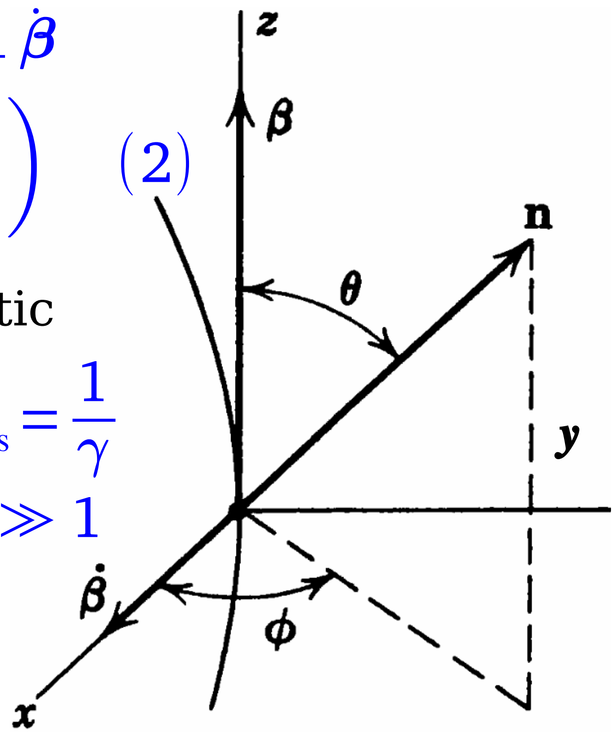
$$\simeq\frac{\mu_0\,q^2\,\dot\beta^2}{16\,\pi^2\,\epsilon_0\,c}\,\frac{\delta/2}{[1-(1-\delta)\,(1-\delta/4)]^5}\simeq\frac{\mu_0\,q^2\,\dot\beta^2}{16\,\pi^2\,\epsilon_0\,c}\,\frac{\delta/2}{(5\,\delta/4)^5}\simeq\frac{\mu_0\,q^2\,\dot\beta^2}{2\,\pi^2\,\epsilon_0\,c}\left(\frac{4}{5}\right)^5\gamma^8$$

- The peak occurs at $\gamma\,\theta = \pm\dfrac{1}{2}$, the half-power points at $\gamma\theta = \pm 0.23$ & $\gamma\theta = \pm 0.91$.
- The rms angle of radiation in the relativistic limit $\theta_{\text{rms}} \equiv \sqrt{\langle \theta^2 \rangle} = \dfrac{1}{\gamma} = \dfrac{m\,c^2}{\mathscr{E}}$

typical of the relativistic radiation patterns, regardless of the angle of $\boldsymbol{\beta}$ & $\dot{\boldsymbol{\beta}}$.

- The total power $P_{\text{linear}}(t') = \displaystyle\int \frac{\mathrm{d}\,P}{\mathrm{d}\,\Omega}(t')\,\mathrm{d}\,\Omega = \frac{2}{3}\frac{e^2}{c}\dot{\beta}^2\gamma^6 \quad \Rightarrow \quad$ the Lienard result
- For a charge in instantaneously circular motion $\Rightarrow \quad \boldsymbol{\beta} \perp \dot{\boldsymbol{\beta}}$

$$\Rightarrow \quad \frac{\mathrm{d}\,P}{\mathrm{d}\,\Omega}(t') = \frac{e^2}{4\,\pi\,c}\frac{\dot{\beta}^2}{(1-\beta\cos\theta)^3}\left(1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right) \qquad (2)$$



- In the relativistic limit, the same characteristic relativistic peaking at forward angles is present.

$$\frac{\mathrm{d}\,P}{\mathrm{d}\,\Omega}(t') = \frac{2\,e^2}{\pi\,c}\frac{\gamma^6\,\dot{\beta}^2}{(1+\gamma^2\theta^2)^3}\left(1 - \frac{4\,\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta^2)^2}\right) \quad \& \quad \theta_{\text{rms}} = \frac{1}{\gamma} \quad \text{for} \quad \gamma \gg 1$$

- The total power $P_{\text{circular}}(t') = \dfrac{2}{3}\dfrac{e^2}{c}\dot{\beta}^2\gamma^4$
- For circular motion $\dot{\mathbf{p}} = \gamma\,m\,\dot{\mathbf{v}} = \mathbf{F} \;\Rightarrow\; P_{\text{circular}}(t') = \dfrac{2\,e^2\gamma^2}{3\,m^2c^3}\left(\dfrac{\mathrm{d}\,\mathbf{p}}{\mathrm{d}\,t}\right)^2 = \gamma^2 P_{\text{linear}}$
- For a given magnitude of applied force the radiation emitted with a transverse acceleration is a factor of γ^2 larger than with a parallel acceleration.

Radiation Emitted by a Charge in Arbitrary, Extremely Relativistic Motion

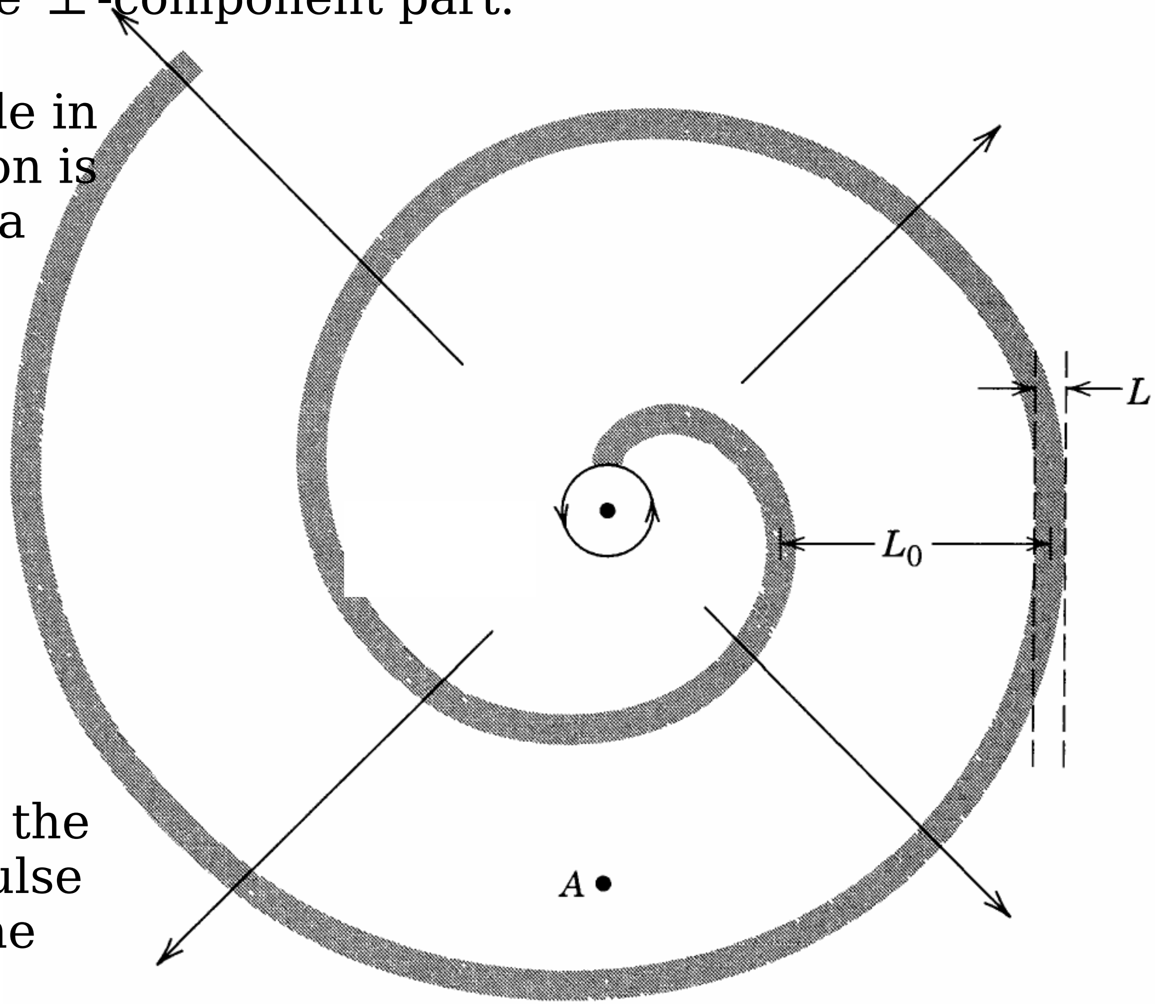
- In the case the radiation can be thought of as a coherent superposition of contributions coming from the components of acceleration ∥ & ⊥ to the velocity.

- Neglect the ∥-component part and approximate the radiation intensity with the ⊥-component part alone because the radiation from the ∥-component part is of order γ^{-2} compared to that from the ⊥-component part.

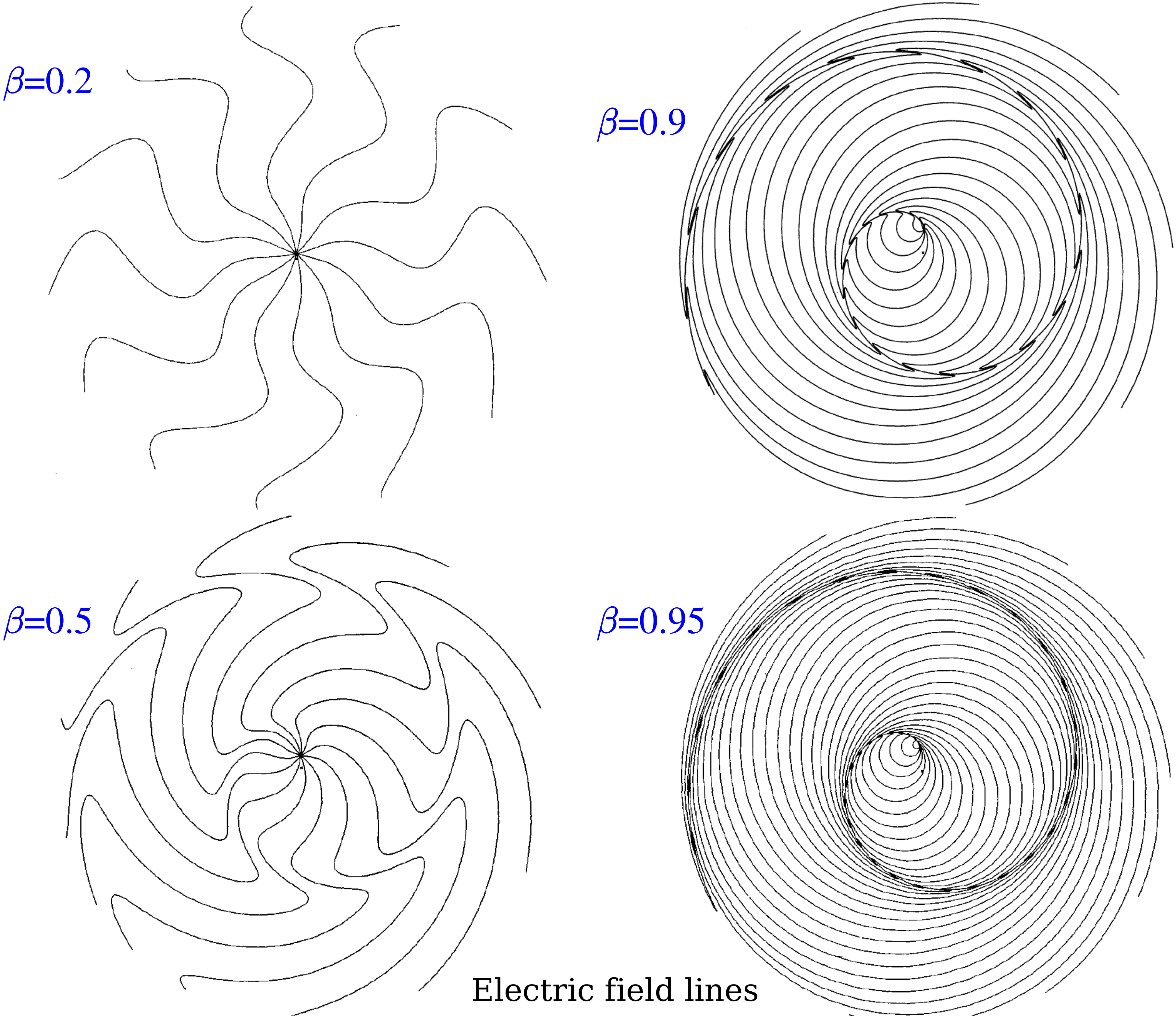
- the radiation by a charged particle in arbitrary, extreme relativistic motion is approximately the same as that by a particle moving instantaneously along the arc of a circular path of radius of curvature

$$\rho = \frac{v^2}{\dot{v}_\perp} \simeq \frac{c^2}{\dot{v}_\perp} \quad \Rightarrow \quad \frac{\mathrm{d}\,P}{\mathrm{d}\,\Omega}(t') = (2)$$

a narrow cone or searchlight beam of radiation directed along the velocity vector of the charge.

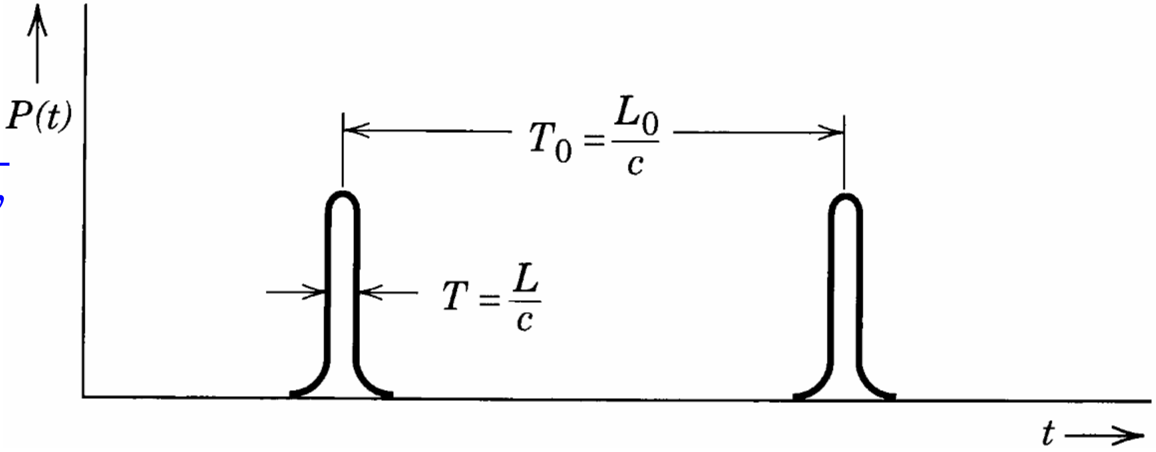


- For a particle in arbitrary motion the observer will detect a short-time pulse (or a succession of such bursts if the particle is in periodic motion).



Electric field lines

- $\Delta\theta \sim \dfrac{1}{\gamma} \;\Rightarrow\; d \sim \dfrac{\rho}{\gamma} \;\Rightarrow\; \Delta t \sim \dfrac{\rho}{\gamma v}$



$\Rightarrow\; D = c\,\Delta t \sim \dfrac{\rho}{\gamma\beta}$ pulse front's travelling distance

$\Rightarrow\; L = D - d = \dfrac{\rho}{\gamma\beta} - \dfrac{\rho}{\gamma} \simeq \dfrac{\rho}{2\gamma^3} \;\Leftarrow\;$ the length of the pulse $\;\Rightarrow\; T = \dfrac{L}{c}$

- By analyzing the wave trains it implies that the spectrum of the radiation will contain appreciable frequency components up to a critical frequency

$$\omega_c \sim \frac{c}{L} \sim \gamma^3 \omega_0 \quad (\ast) \quad \Leftarrow \quad \omega_0 = \frac{c}{\rho} \quad \text{the fundamental frequency}$$

- A relativistic particle emits a broad spectrum of frequencies, up to γ^3 times the fundamental frequency.

- 200 MeV synchrotron $\;\Rightarrow\; \gamma_{max} = 400\,,\quad \omega_0 \simeq 3\times 10^8/\mathrm{s}$

  $\Rightarrow\; \omega_c \sim 2\times 10^{16}/\mathrm{s}\,,\quad \lambda_c \sim 10^3\ \text{Å}$

- 10 GeV machine $\;\Rightarrow\; \gamma_{max} = 20000\,,\quad \omega_0 \simeq 3\times 10^6/\mathrm{s}$

  $\Rightarrow\; \omega_c \sim 2.4\times 10^{19}/\mathrm{s} \;\Rightarrow\;$ 16 keV x-ray

Distribution in Frequency and Angle of Energy Radiated by Accelerated Charges: Basic Results

- For relativistic motion the radiated energy is over a wide range of frequencies. The frequency spectrum can be analyzed precisely & quantitatively by the use of Parseval's theorem of Fourier analysis.

- **Parseval's theorem**: the sum/integral of the square of a function is equal to the sum/integral of the square of its transform, ie, the Fourier transform is unitary.

- $$\frac{\mathrm{d}\,P}{\mathrm{d}\,\Omega}(t)=|\mathbf{A}(t)|^2 \quad\Leftarrow\quad \mathbf{A}(t)=\sqrt{\frac{c}{4\,\pi}}\,[\mathbb{s}\,\mathbf{E}]_{\mathrm{ret}} \quad\Leftarrow\quad \text{in the observer's time}$$

$$A(\omega)=\frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{+\infty}\mathbf{A}(t)\,e^{i\,\omega\,t}\,\mathrm{d}\,t \quad\Leftrightarrow\quad A(t)=\frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{+\infty}\mathbf{A}(\omega)\,e^{-i\,\omega\,t}\,\mathrm{d}\,\omega$$

$$\Rightarrow\quad \frac{\mathrm{d}\,W}{\mathrm{d}\,\Omega}=\int\frac{\mathrm{d}\,P}{\mathrm{d}\,\Omega}(t)\,\mathrm{d}\,t=\int_{-\infty}^{+\infty}|\mathbf{A}(t)|^2\,\mathrm{d}\,t \quad\swarrow\quad \delta(\omega-\omega')=\frac{1}{2\,\pi}\int_{-\infty}^{+\infty}e^{i\,(\omega-\omega')\,t}\,\mathrm{d}\,t$$

$$=\frac{1}{2\,\pi}\int_{-\infty}^{+\infty}\mathbf{A}^*(\omega')\cdot\mathbf{A}(\omega')\,e^{i\,(\omega'-\omega)\,t}\,\mathrm{d}\,\omega'\,\mathrm{d}\,\omega\,\mathrm{d}\,t=\int_{-\infty}^{+\infty}|\mathbf{A}(\omega)|^2\,\mathrm{d}\,\omega$$

$$=\int_{-\infty}^{+\infty}\frac{\mathrm{d}^2\,I(\omega,\hat{\mathbb{s}})}{\mathrm{d}\,\omega\,\mathrm{d}\,\Omega}\,\mathrm{d}\,\omega \quad\Leftarrow\quad \frac{\mathrm{d}^2\,I(\omega,\hat{\mathbb{s}})}{\mathrm{d}\,\omega\,\mathrm{d}\,\Omega}=|\mathbf{A}(\omega)|^2+|\mathbf{A}(-\omega)|^2$$

$$\Rightarrow\quad \frac{\mathrm{d}^2\,I(\omega,\hat{\mathbb{s}})}{\mathrm{d}\,\omega\,\mathrm{d}\,\Omega}=2\,|\mathbf{A}(\omega)|^2 \quad\text{if}\quad \mathbf{A}(t)\in\mathbb{R} \quad\Leftarrow\quad \mathbf{A}(-\omega)=\mathbf{A}^*(\omega)$$

- $$\mathbf{A}(\omega)=\frac{e}{\sqrt{8\pi^2 c}}\int_{-\infty}^{+\infty}\left.\frac{\hat{\mathbb{s}}\times[(\hat{\mathbb{s}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\hat{\mathbb{s}}\cdot\boldsymbol{\beta})^3}\right|_{\text{ret}} e^{i\omega t}\,\mathrm{d}t \quad \text{for an accelerated charge} \Leftarrow \quad (0)$$

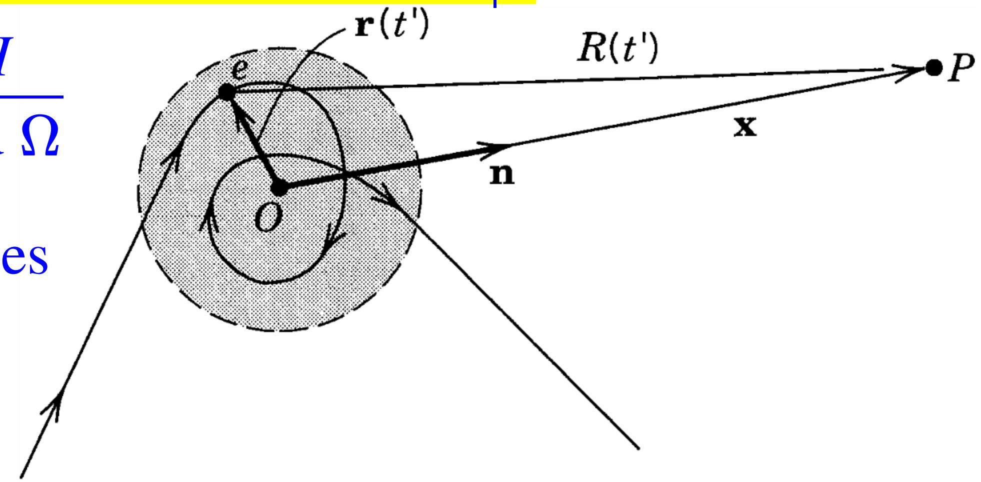
$$=\frac{e}{\sqrt{8\pi^2 c}}\int_{-\infty}^{+\infty}\frac{\hat{\mathbb{s}}\times[(\hat{\mathbb{s}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\hat{\mathbb{s}}\cdot\boldsymbol{\beta})^2}\, e^{i\omega\left(t'+\frac{\mathbb{s}(t')}{c}\right)}\,\mathrm{d}t' \quad \Leftarrow \quad t=t'+\frac{\mathbb{s}(t')}{c}$$

$$\approx e\,\frac{e^{i\omega\frac{r}{c}}}{\sqrt{8\pi^2 c}}\int_{-\infty}^{+\infty}\frac{\hat{\mathbb{s}}\times[(\hat{\mathbb{s}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\hat{\mathbb{s}}\cdot\boldsymbol{\beta})^2}\, e^{i\omega\left(t-\frac{\hat{\mathbf{r}}\cdot\mathbb{r}(t)}{c}\right)}\,\mathrm{d}t \quad \Leftarrow \quad \mathbb{s}(t')\approx r-\hat{\mathbf{r}}\cdot\mathbb{r}(t')$$

$$\Rightarrow \quad \boxed{\frac{\mathrm{d}^2 I}{\mathrm{d}\omega\,\mathrm{d}\Omega}=\frac{e^2}{4\pi^2 c}\left|\int_{-\infty}^{+\infty}\frac{\hat{\mathbf{n}}\times[(\hat{\mathbf{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^2}\, e^{i\omega\left(t-\frac{\hat{\mathbf{n}}\cdot\mathbf{r}(t)}{c}\right)}\,\mathrm{d}t\right|^2} \quad (3)$$

given $\mathbf{r}(t) \;\Rightarrow\; \boldsymbol{\beta}(t)\ \&\ \dot{\boldsymbol{\beta}}(t) \;\Rightarrow\; \dfrac{\mathrm{d}^2 I}{\mathrm{d}\omega\,\mathrm{d}\Omega}$

$\dfrac{\mathrm{d}^2 I}{\mathrm{d}\omega\,\mathrm{d}\Omega}=2\sum|\mathbf{A}_j(\omega)|^2$ for many particles



- $$\frac{\hat{\mathbf{n}}\times[(\hat{\mathbf{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^2}=\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\boldsymbol{\beta})}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}}$$

$$\Rightarrow \quad \boxed{\frac{\mathrm{d}^2 I}{\mathrm{d}\omega\,\mathrm{d}\Omega}=\frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{+\infty}\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\boldsymbol{\beta})\, e^{i\omega\left(t-\frac{\hat{\mathbf{n}}\cdot\mathbf{r}(t)}{c}\right)}\,\mathrm{d}t\right|^2} \quad (4)$$

- (4) is correct in all circumstances. For the acceleration being different from 0 for $T_1 \le t \le T_2$, by adding & subtracting the integrals over the times for v=const, (3) will give right answer.

- In processes like beta decay, involving the almost instantaneous halting or setting in motion of charges, extra care must be taken to specify each particle's velocity as a physically sensible function of time.

- The polarization of the radiation is given by the direction of the vector integral in each. The intensity of radiation of a fixed polarization can be obtained by the scalar product of the unit polarization vector with the vector integral.

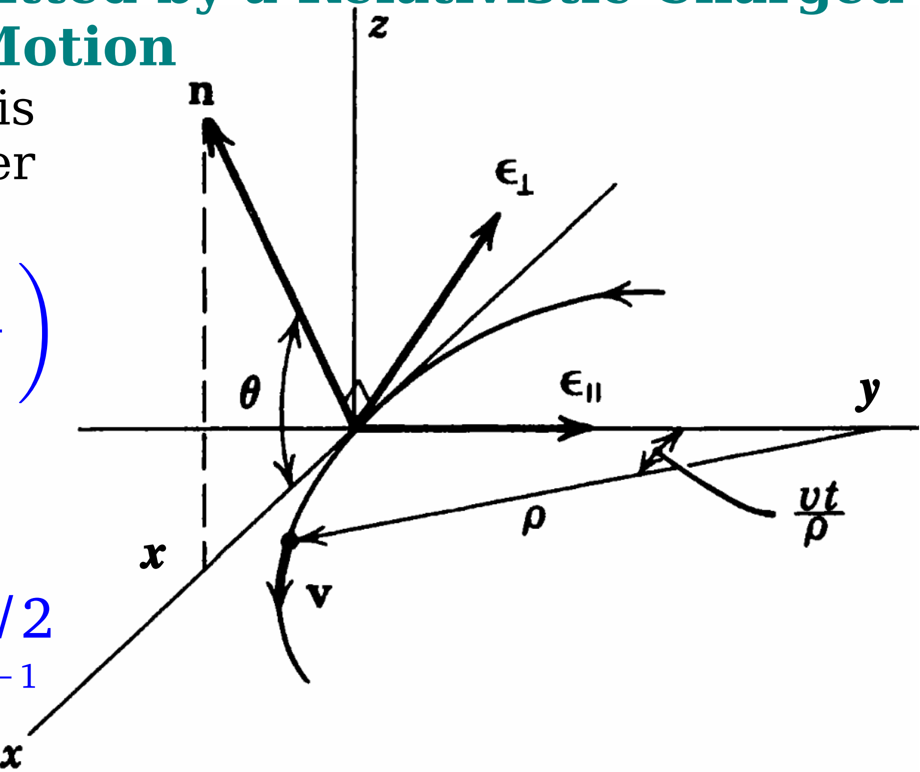
- For a number of charges

$$e\,\boldsymbol{\beta}\,e^{-i\frac{\omega}{c}\hat{\mathbf{n}}\cdot\mathbf{r}(t)} \to \sum_{j=1}^{N} e_j\,\boldsymbol{\beta}_j\,e^{-i\frac{\omega}{c}\hat{\mathbf{n}}\cdot\mathbf{r}_j(t)} \to \frac{1}{c}\int \boldsymbol{J}(\mathbf{r},t)\,e^{-i\frac{\omega}{c}\hat{\mathbf{n}}\cdot\mathbf{r}}\,\mathrm{d}^3x$$

$$\Rightarrow\quad \frac{\mathrm{d}^2 I}{\mathrm{d}\omega\,\mathrm{d}\Omega} = \frac{\omega^2}{4\pi^2 c^3}\left|\iint \hat{\mathbf{n}}\times[\hat{\mathbf{n}}\times\boldsymbol{J}(\mathbf{r},t)]\,e^{i\omega\left(t-\frac{\hat{\mathbf{n}}\cdot\mathbf{r}}{c}\right)}\,\mathrm{d}^3x\,\mathrm{d}t\right|^2$$

a result that can be obtained from the direct solution of the inhomogeneous wave equation for the vector potential.

Frequency Spectrum of Radiation Emitted by a Relativistic Charged Particle in Instantaneously Circular Motion



- If the duration of the pulse is very short, it is necessary to know the velocity & position over only a small arc of the trajectory.

- $$\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \boldsymbol{\beta}) = \beta \left(\hat{\mathbf{e}}_\perp \cos \frac{v\,t}{\rho} \sin\theta - \hat{\mathbf{e}}_\parallel \sin \frac{v\,t}{\rho} \right)$$

$$t - \frac{\hat{\mathbf{n}} \cdot \mathbf{r}(t)}{c} = t - \frac{\rho}{c} \sin \frac{v\,t}{\rho} \cos\theta$$

$$\simeq \frac{1+\gamma^2\theta^2}{2\gamma^2} t + \frac{c^2}{6\rho^2} t^3 \quad \Leftarrow \quad \begin{array}{l} \beta \to 1 - \gamma^{-2}/2 \\ \theta \leftarrow \theta_{\text{rms}} = \gamma^{-1} \end{array}$$

$$\Rightarrow \quad \frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2 \omega^2}{4\pi^2 c} \left| \hat{\mathbf{e}}_\perp A_\perp(\omega) - \hat{\mathbf{e}}_\parallel A_\parallel(\omega) \right|^2 \quad \Leftarrow \quad (3)$$

where

$$A_\parallel(\omega) \simeq \frac{c}{\rho} \int_{-\infty}^{+\infty} t\, e^{i\omega\left(\frac{1+\gamma^2\theta^2}{2\gamma^2} t + \frac{c^2 t^3}{6\rho^2}\right)} d\,t = \frac{\rho}{c} \frac{1+\gamma^2\theta^2}{\gamma^2} \int_{-\infty}^{+\infty} x\, e^{i\xi \frac{3x+x^3}{2}} d\,x$$

$$A_\perp(\omega) \simeq \theta \int_{-\infty}^{+\infty} e^{i\omega\left(\frac{1+\gamma^2\theta^2}{2\gamma^2} t + \frac{c^2 t^3}{6\rho^2}\right)} d\,t = \theta \frac{\rho}{c} \frac{\sqrt{1+\gamma^2\theta^2}}{\gamma} \int_{-\infty}^{+\infty} e^{i\xi \frac{3x+x^3}{2}} d\,x$$

where $\quad x = \dfrac{\gamma\, c\, t}{\rho\sqrt{1+\gamma^2\theta^2}}, \quad \xi = \dfrac{\omega\,\rho}{3\,c} \dfrac{\sqrt{(1+\gamma^2\theta^2)^3}}{\gamma^3}$

$$\int_0^\infty x \sin \frac{\xi\,(3x + x^3)}{2}\, d x = \frac{1}{\sqrt{3}} K_{2/3}(\xi), \qquad \int_0^\infty \cos \frac{\xi\,(3x + x^3)}{2}\, d x = \frac{1}{\sqrt{3}} K_{1/3}(\xi)$$

$$\Rightarrow \quad \frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2 \omega^2 \rho^2}{3\pi^2 c^3} \frac{(1+\gamma^2\theta^2)^2}{\gamma^4} \left(K^2_{2/3}(\xi) + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2} K^2_{1/3}(\xi) \right) \qquad (5)$$

radiation polarized $\|$ the plane of the orbit — radiation polarized $\perp$ the plane of the orbit

$$\Rightarrow \quad \frac{d I}{d\Omega} = \int_{00}^\infty \frac{d^2 I}{d\omega\, d\Omega}\, d\omega = \frac{7}{16} \frac{e^2}{\rho} \frac{\gamma^5}{(1+\gamma^2\theta^2)^{5/2}} \left(1 + \frac{5}{7} \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2} \right) \quad \Leftarrow \quad (2)$$

$\Rightarrow \quad I = I_{\|} + I_{\perp} \quad \Leftarrow \quad I_{\|} \approx 7\, I_{\perp} \quad \Rightarrow$ The radiation from a relativistically moving charge is very strongly polarized in the plane of motion.

- $I \to 0$ as $\xi \gg 1 \quad \Leftarrow \quad$ large θ :the radiation is largely confined to the plane of the motion, being more confined the higher the frequency relative to c/ρ.

- If ω gets too large, ξ will be large at *all* angles. Then there will be negligible total energy emitted at that frequency.

- Critical frequency: $\omega_c = \frac{3}{2}\gamma^3 \frac{c}{\rho} = \frac{3}{2}\left(\frac{E}{m c^2}\right)^3 \frac{c}{\rho} \simeq (\ast) \quad \Leftarrow \quad \xi(\omega_c, \theta = 0) = \frac{1}{2}$

$$\Rightarrow \quad \omega_c = n_c\, \omega_0 \qquad \Rightarrow \qquad n_c = \frac{3}{2}\gamma^3 \qquad \Leftarrow \qquad \omega_0 = \frac{c}{\rho}$$

critical harmonic frequency — harmonic number — fundamental harmonic frequency

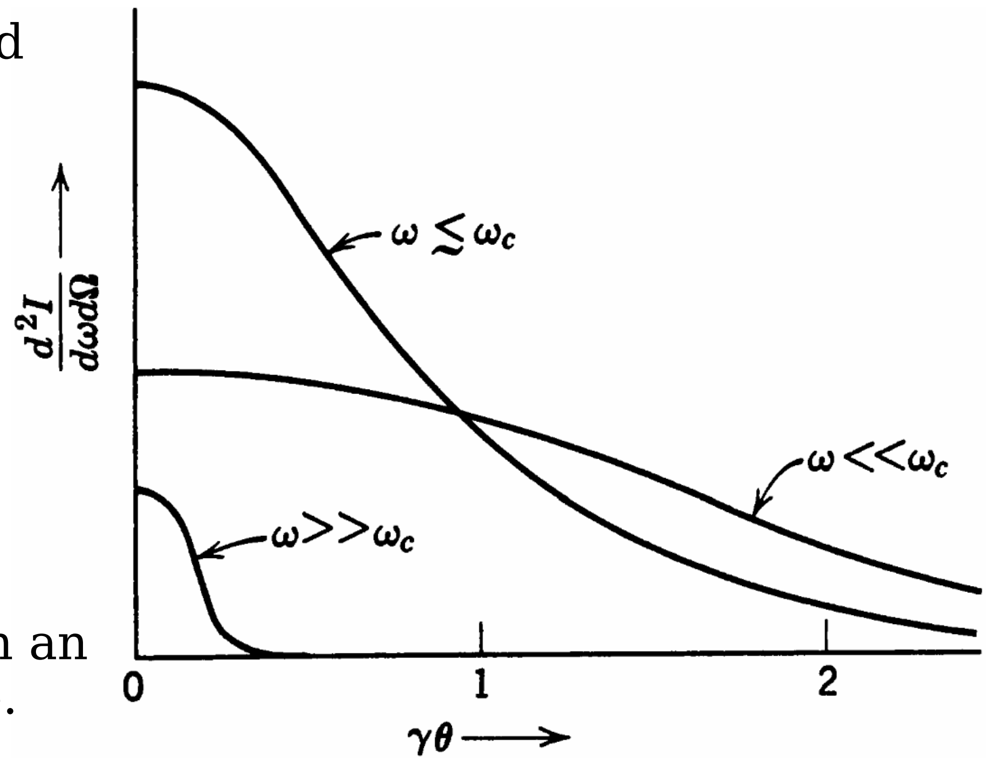
- $$\frac{d^2 I}{d\omega\, d\Omega}_{|\theta=0} = \begin{cases} \dfrac{e^2}{c}\left(\dfrac{\Gamma(3/2)}{\pi}\right)^2\left(\dfrac{3}{4}\right)^{1/3}\left(\dfrac{\omega\rho}{c}\right)^{2/3} & \text{for } \omega \ll \omega_c \\[2ex] \dfrac{3}{4\pi}\dfrac{e^2}{c}\gamma^2\dfrac{\omega}{\omega_c}e^{-\omega/\omega_c} & \text{for } \omega \gg \omega_c \end{cases}$$

the spectrum at $\theta=0$ increases with frequency as $\omega^{2/3}$ below the critical frequency, reaches a maximum near ω_c, drops exponentially to 0 above that frequency.

- Estimate the spread in angle at a fixed frequency by finding $\theta_c \Leftarrow \xi(\theta_c) \simeq \xi(0) + 1$

$$\omega \ll \omega_c \quad \Rightarrow \quad \xi(\theta_c) \simeq 1 \quad \Rightarrow \quad \theta_c \simeq \left(\frac{3c}{\omega\rho}\right)^{1/3} = \frac{1}{\gamma}\left(\frac{2\omega_c}{\omega}\right)^{1/3} = \left(\frac{2\omega_c}{\omega}\right)^{1/3}\theta_{\text{rms}} > \theta_{\text{rms}}$$

the low-frequency components are emitted at much wider angles than the average.



- $\omega > \omega_c \quad \Rightarrow \quad \xi(\theta_c) \gg 1$

$$\Rightarrow \quad \frac{d^2 I}{d\omega\, d\Omega} \simeq \frac{d^2 I}{d\omega\, d\Omega}_{|\theta=0} e^{-\frac{3\omega\gamma^2\theta^2}{2\omega_c}}$$

$$\Rightarrow \quad \theta_c \simeq \frac{1}{\gamma}\sqrt{\frac{2\omega_c}{3\omega}} \quad \Leftarrow \quad \frac{3\omega\gamma^2\theta_c^2}{2\omega_c} = 1$$

the high-frequency components are within an angular range much smaller than average.

- For the low-frequency range $\omega \ll \omega_c$

$$\frac{\mathrm{d}\,I}{\mathrm{d}\,\omega} = 2\,\pi \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{\mathrm{d}^2\,I}{\mathrm{d}\,\omega\,\mathrm{d}\,\Omega}\,\mathrm{d}\,\sin\theta$$

$$\simeq 2\,\pi \int_{-\infty}^{+\infty} \frac{\mathrm{d}^2\,I}{\mathrm{d}\,\omega\,\mathrm{d}\,\Omega}\,\mathrm{d}\,\theta \sim 2\,\pi\,\theta_c\,\frac{\mathrm{d}^2\,I}{\mathrm{d}\,\omega\,\mathrm{d}\,\Omega}_{|\theta=0} \sim \frac{e^2}{c}\left(\frac{\omega\,\rho}{c}\right)^{1/3}$$

the spectrum increases as $\omega^{1/3}$, and is very broad, flat at frequencies below ω_c.

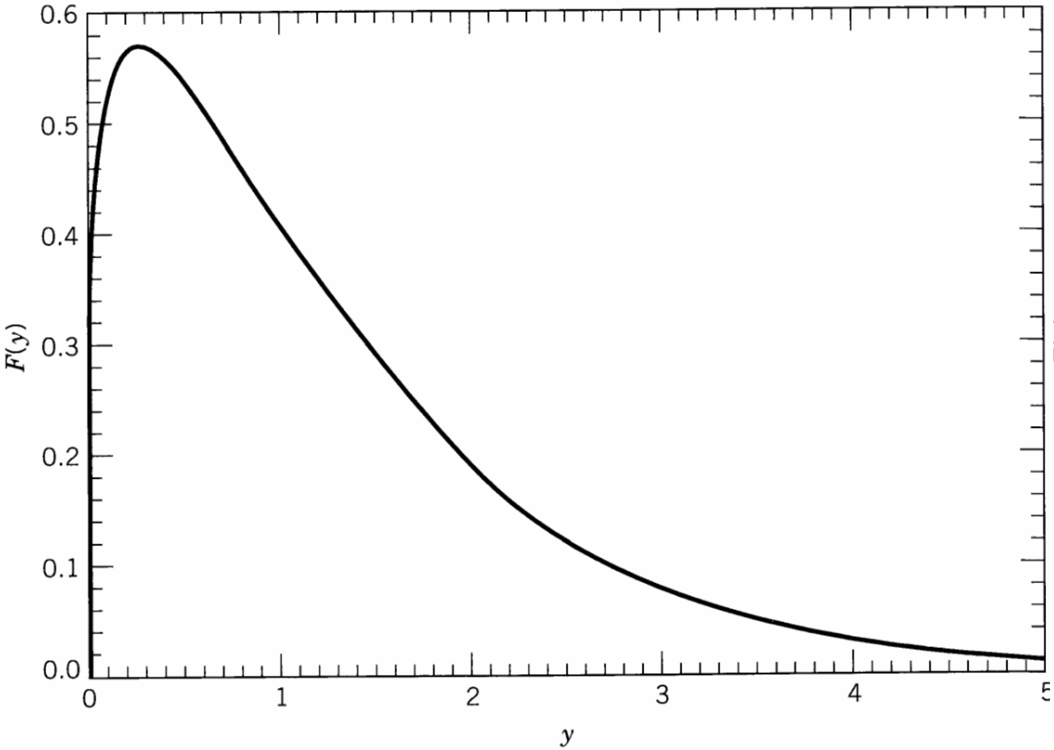
- For the high-frequency range $\omega \gg \omega_c \Rightarrow \dfrac{\mathrm{d}\,I}{\mathrm{d}\,\omega} \simeq \dfrac{e^2}{c}\,\gamma\,\sqrt{\dfrac{3\,\pi}{2}\,\dfrac{\omega}{\omega_c}}\,e^{-\frac{\omega}{\omega_c}}$

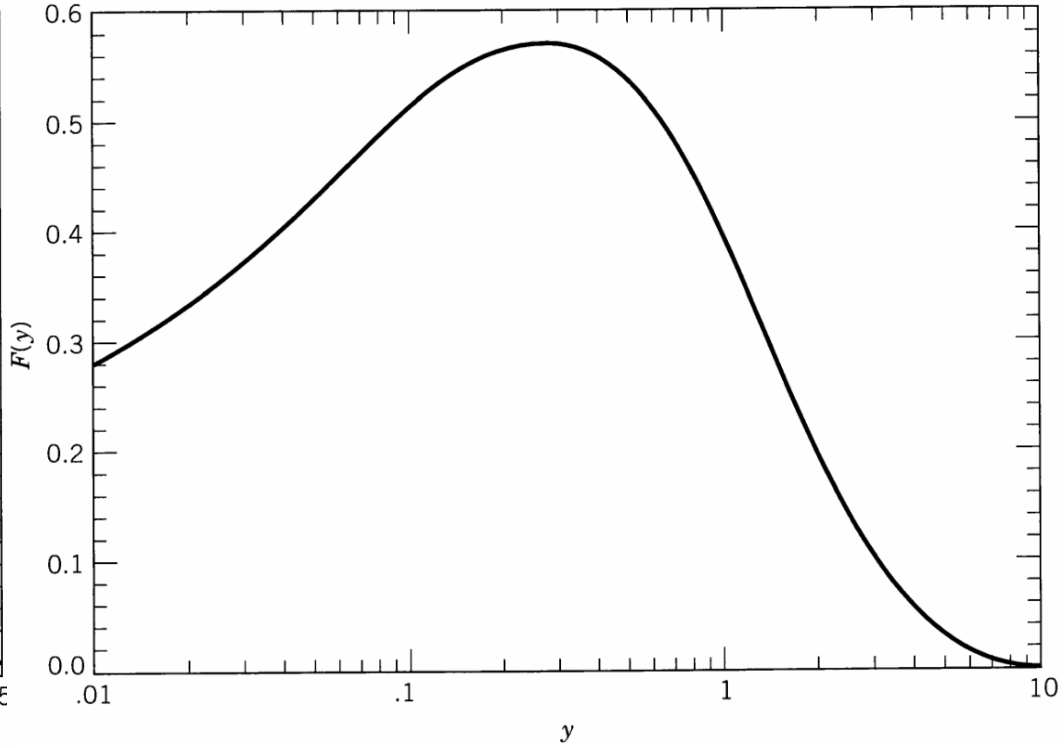
- A proper integration gives $\dfrac{\mathrm{d}\,I}{\mathrm{d}\,\omega} = \sqrt{3}\,\dfrac{e^2}{c}\,\gamma\,\dfrac{\omega}{\omega_c}\displaystyle\int_{\frac{\omega}{\omega_c}}^{\infty} K_{5/3}(x)\,\mathrm{d}\,x \qquad (6)$

- The radiation represented by (5) & (6) is called *synchrotron radiation*.

- For periodic circular motion the spectrum is actually discrete, being composed of frequencies that are integral multiples of the fundamental frequency $\omega_0 = c/\rho$.

- Since the charged particle repeats its motion at a rate of $c/2\pi\rho$ rev/sec, it is convenient to talk about the angular distribution of power radiated into the *n*th multiple of ω_0 instead of the energy radiated/frequency interval/particle.





$$\frac{\mathrm{d}\,P_n}{\mathrm{d}\,\Omega} = \frac{1}{2\,\pi}\,\frac{c^2}{\rho^2}\,\frac{\mathrm{d}^2\,I}{\mathrm{d}\,\omega\,\mathrm{d}\,\Omega}\bigg|_{\omega=n\,\omega_0}\,, \qquad P_n = \frac{1}{2\,\pi}\,\frac{c^2}{\rho^2}\,\frac{\mathrm{d}\,I}{\mathrm{d}\,\omega}\bigg|_{\omega=n\,\omega_0}$$

- Due to the broad frequency distribution covering the visible, UV, *x*-ray regions, synchrotron radiation is a useful tool for studies in condensed matter & biology.

- Electrons in the Crab nebula with energies ranging up to 10^{13} eV are emitting synchrotron radiation while moving in circular or helical orbits in a $\mathbf{B}\sim 10^{-4}$ gauss.

- The radio emission at $\sim 10^3$ MHz from Jupiter comes from energetic electrons trapped in Van Allen belts at distances up to 100 radii from Jupiter's surface.

- $B \sim 1$ gauss , $E_e \sim 5$ MeV $\Rightarrow$ $\rho \sim 100$ - 200 meters , $\omega_0 \sim 2 \times 10^6 /\mathrm{s}$

  $\Rightarrow$ 10^3 significant harmonics radiated

- (number of photons)/frequency is to divide the intensity distribution by $\hbar\omega$

  $$\frac{\mathrm{d}\, N}{\mathrm{d}\,(\omega/\omega_c)} = \frac{I}{\hbar\, \omega_c} \frac{9\sqrt{3}}{8\,\pi} \int_{\frac{\omega}{\omega_c}}^{\infty} K_{5/3}(x)\,\mathrm{d}\,x \quad \Leftarrow \quad I = \frac{4\,\pi\, e^2 \gamma^4}{3\,\rho}$$ total energy radiated per revolution

  $$\Rightarrow \quad \frac{\text{mean number of photons}}{(\text{revolution})(\text{particle})} = N = \frac{5\,\pi}{\sqrt{3}}\,\gamma\,\alpha$$

  $$\Rightarrow \quad \frac{\text{mean energy}}{\text{photon}} = \langle \hbar\,\omega \rangle = \frac{I}{N} = \frac{8\,\hbar\,\omega_c}{15\sqrt{3}}$$

  $$\Rightarrow \quad \omega_c \propto \gamma^3 , \quad \gamma\,(\mathrm{GeV}) = O\,(10^4) \quad \Rightarrow \quad \lambda_{\text{fundamental}} = 2\,\pi\,\rho \sim \text{hundred of meters}$$

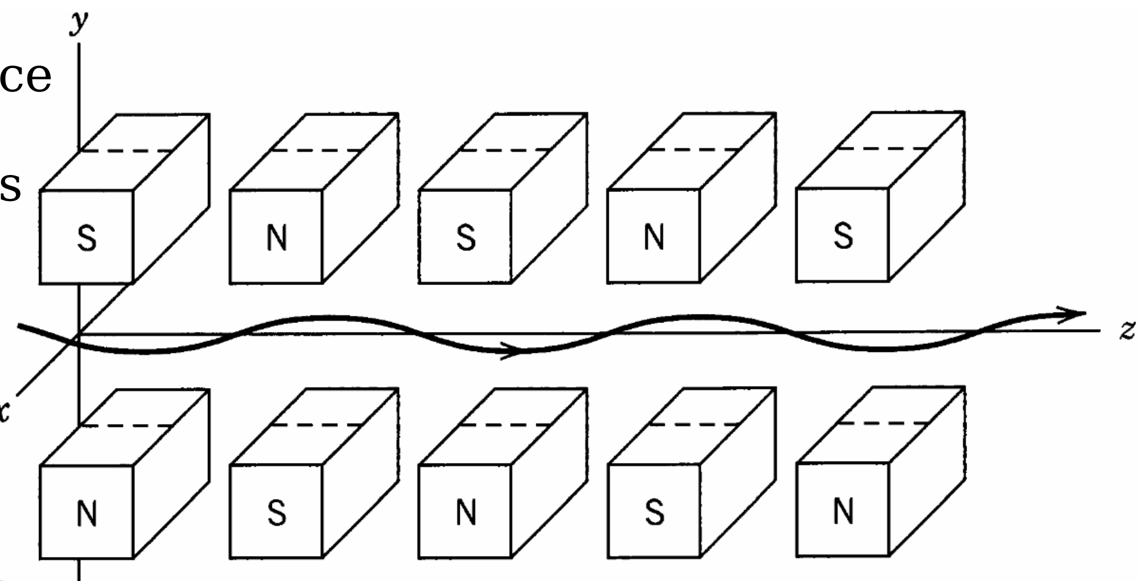
  $$\Rightarrow \quad \lambda_{\text{photon}} \sim 10^{-10}\ \text{meter} \Rightarrow \text{keV}\ \ x\text{-ray}$$

Undulators and Wigglers for Synchrotron Light Sources

- The magnetic properties of wigglers & undulators make the electrons undergo special motion that results in the concentration of the radiation into a much more monochromatic spectrum or series of separated peaks.

- The essential idea of undulators and wigglers is that a moving relativistically charged particle is caused to move transversely to its general forward motion by magnetic fields that alternate periodically.

- The external magnetic fields induce small transverse oscillations in the motion; the associated accelerations cause radiation to be emitted.



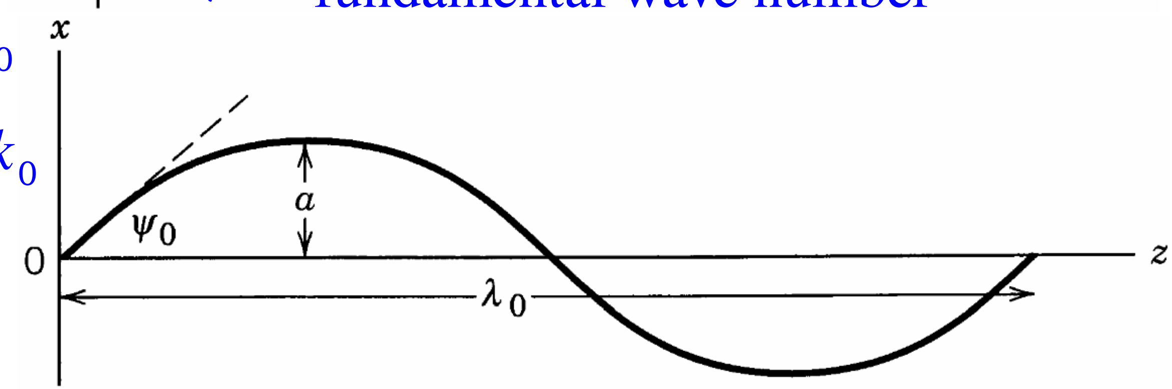
A. Qualitative Features

- $x \approx a\left(B_{\text{wiggler}},\, E_{\text{particle}}\right) \sin \dfrac{2\pi z}{\lambda_0}$

$$\Rightarrow \quad \psi_0 = \frac{\mathrm{d}\,x}{\mathrm{d}\,z}_{z=0} = k_0\, a \quad \Leftarrow \quad k_0 = \frac{2\pi}{\lambda_0}$$ $\Leftarrow$ fundamental wave number

$$\text{period} \quad T = \frac{\lambda_0}{\beta\, c} \quad \Rightarrow \quad k_{0,\,\text{radiation}} = \beta\, k_0$$

$$\text{for} \quad \gamma \gg 1 \quad \Rightarrow \quad k_0 \approx k_{0,\,\text{radiation}}$$



- For $\gamma \gg 1$, the radiation is confined to a width $\Delta\theta = O\left(\frac{1}{\gamma}\right)$ about the actual path.

- As the particle moves in its oscillatory path, the "searchlight" beam of radiation will flick back and forth about the forward direction.

(a) Wiggler ($\psi_0 \gg \Delta\theta$)

- $\nu_0 = \frac{\omega_0}{2\pi} = \frac{c\,k_0}{2\pi} = O(10\text{ GHz})$ for $\lambda_0 = O(\text{centimeters})$. The phenomenon is very much as in an ordinary synchrotron with bunches spaced a few cm's apart.

- The spectrum of radiation extends to frequencies about γ^3 times the basic freq.

$$\text{basic freq.}\quad \Omega = \frac{c}{R} \quad \Leftarrow \quad R : \begin{array}{l}\text{effective radius}\\ \text{of curvature}\end{array} \quad \Rightarrow \quad R_{\min} = \frac{1}{k_0^2\, a} = \frac{\lambda_0}{2\pi\,\psi_0}$$

- The wiggler radiation spectrum is very much like the synchrotron radiation spectrum, with a fundamental frequency Ω, $\Rightarrow \quad \omega_c = \gamma^3\,\Omega \quad \Leftarrow \quad \Omega = \frac{2\pi\,c\,\psi_0}{\lambda_0}$

- If the wiggler magnet structure has N periods, the intensity of radiation will be N times that for a single pass of a particle in the equivalent circular machine.

- $K \equiv \gamma\,\psi_0 \quad \Rightarrow \quad K \gg 1$ for wiggler $\Rightarrow \quad \omega_c = O\left(\gamma^2 K \frac{2\pi\,c}{\lambda_0}\right) \Leftarrow \lambda_c = O\left(\frac{\lambda_0}{\gamma^2 K}\right)$

(b) Undulators ($\psi_0 \ll \Delta\theta$ or $K \ll 1$)

- If $\psi_0 \ll \Delta\theta$, the searchlight beam of radiation moves negligibly compared to its own angular width.

- The radiation detected by an observer is an almost *coherent superposition* of the contributions from all the oscillations of the trajectory.

- For perfect coherence & an infinite number of magnet periods (and infinitesimal angular resolution of the detector), the radiation would be monochromatic.

- For finite N magnet periods the spread in frequency is $\dfrac{\Delta\omega}{\omega} = O\left(\dfrac{1}{N}\right)$; finite angular acceptance also causes a spread because of the Doppler shift.

- The frequency spectrum from an undulator is sharply peaked.

- The FitzGerald-Lorentz contraction means that in the particle's rest frame the magnet structure is rushing by the particle with a spatial period $\dfrac{\lambda_0}{\gamma}$

$$\omega_{\text{rest}} \approx \gamma \frac{2\pi c}{\lambda_0} \;\Rightarrow\; \omega_{\text{rest}} = \gamma\,\omega_{\text{lab}}(1 - \beta\cos\theta) \approx \omega_{\text{lab}} \frac{1+\gamma^2\theta^2}{2\gamma} \;\Rightarrow\; \omega_{\text{lab}} \approx \frac{2\gamma^2}{1+\gamma^2\theta^2}\frac{2\pi c}{\lambda_0}$$

For $\gamma\theta \ll 1 \;\Rightarrow\; \omega_{\text{lab}} = O(\gamma^2)$ has the same γ dependence as ω_c with fixed K

B. Some Details of the Kinematics and Particle Dynamics

- to consider the particle in its average rest frame, in which it oscillates both transversely and longitudinally.

- Its initial γ and β remain unchanged because $\mathbf{B}$ does no work on the particle.

- Due to the transverse motion, the particle's average speed in the z-direction, $c\,\bar{\beta} < c\,\beta$, and its associated $\bar{\gamma} < \gamma$. The average rest frame moves with speed $c\,\bar{\beta}$.

$$s = \int_0^{\lambda_0} \sqrt{1 + \left(\frac{\mathrm{d}\,x}{\mathrm{d}\,z}\right)^2}\,\mathrm{d}\,z = \int_0^{\lambda_0} \left[1 + \frac{1}{2}\left(\frac{\mathrm{d}\,x}{\mathrm{d}\,z}\right)^2 + \cdots\right]\mathrm{d}\,z \quad \Leftarrow \quad \text{length per cycle}$$

$$\approx \lambda_0 \left(1 + \frac{\psi_0^2}{4}\right) \text{ for } \psi_0 \ll 1 \;\Rightarrow\; \bar{\beta} = \frac{\beta}{1 + \psi_0^2/4} \approx \beta\left(1 - \frac{1}{4}\psi_0^2\right) \approx 1 \text{ for } \beta \approx 1$$

$$\Rightarrow \quad \frac{1}{\bar{\gamma}^2} = 1 - \bar{\beta}^2 \approx 1 - \beta^2\left(1 - \frac{\psi_0^2}{2}\right) \approx \frac{1}{\gamma^2} + \frac{\psi_0^2}{2} = \frac{2 + K^2}{2\,\gamma^2} \quad \Rightarrow \quad \bar{\gamma} = \gamma\sqrt{\frac{2}{2 + K^2}}$$

since $K \gg 1$, $\bar{\gamma}$ can differ significantly from γ even if $\psi_0 \ll 1$.

- Lorentz force equation $\quad \dfrac{\mathrm{d}\,\mathbf{p}}{\mathrm{d}\,\tau} = e\,\gamma\left(\cancel{\mathbf{E}} + \boldsymbol{\beta} \times \mathbf{B}\right) \;\Rightarrow\; \ddot{x} = -\dfrac{e\,B_y\,\beta_z}{\gamma\,m} \;\Leftarrow\; \begin{array}{l}\beta\,,\;\; \gamma = \text{const} \\ \beta_y\,B_z \to \;\; 0\end{array}$

$$\begin{array}{l} z \simeq c\,t \\ \beta_z \simeq 1 \end{array} \;\Rightarrow\; B_y(z) = -\frac{\gamma\,m\,c^2}{e}\frac{\mathrm{d}^2\,x}{\mathrm{d}\,z^2} = B_0 \sin k_0\,z \;\Leftarrow\; B_0 = \gamma\,m\,c^2\frac{k_0^2\,a}{e}$$

the requisite magnetic structure to have a sinusoidal transverse motion

- $K \equiv \gamma\,\psi_0 = \gamma\,k_0\,a = \dfrac{e\,B_0}{k_0\,m\,c^2} = \dfrac{e\,B_0\,\lambda_0}{2\,\pi\,m\,c^2}$
- An actual magnet structure will be periodic, but not sinusoidal.

- We can make a Fourier decomposition of the actual B_y in multiples of k_0. Each component will contribute to the motion. The fundamental will dominate. For simplicity, we keep only that contribution.

- The longitudinal oscillations can be found from the constancy of β

$$\beta_z(t) \approx \beta - \frac{\beta_x^2}{2\,\beta} \approx \beta - \frac{\beta_x^2}{2} \quad \Leftarrow \quad \beta_z^2 = \beta^2 - \beta_x^2\,, \quad |\beta_x| \ll \beta$$

$$\approx \beta - \frac{1}{2}\,k_0^2\,a^2\cos^2(k_0\,c\,t) \quad \Leftarrow \quad \beta_x \approx k_0\,a\cos(k_0\,c\,t) \quad \Leftarrow \quad \begin{aligned} x &= a\sin k_0\,z \\ &\approx a\sin(k_0\,c\,t) \end{aligned}$$

$$= \beta - \frac{1}{4}\,k_0^2\,a^2\left[1+\cos(2\,k_0\,c\,t)\right] = \overline{\beta} - \frac{K^2}{4\,\gamma^2}\cos(2\,k_0\,c\,t)$$

- $$\begin{aligned} z(t) &= \int c\,\beta_z(t)\,\mathrm{d}\,t = c\,\overline{\beta}\,t - \frac{\lambda_0\,K^2}{16\,\pi\,\gamma^2}\sin(2\,k_0\,c\,t) && \text{longitudinal} \\ x(t) &= \int c\,\beta_x(t)\,\mathrm{d}\,t = \frac{\lambda_0\,K}{2\,\pi\,\gamma}\sin(k_0\,c\,t) && \text{transverse} \end{aligned} \qquad (7)$$

C Particle Motion in the Average Rest Frame

- Lorentz transforma

$$x' = x$$
$$z' = \bar{\gamma}\,(z - c\,\bar{\beta}\,t) \;\Rightarrow\; c\,t' = \bar{\gamma}\left(c\,t\,(1-\bar{\beta}^2) + \frac{\bar{\beta}\,K^2 \sin 2\theta}{8\,k_0\,\gamma^2} \right) \;\Leftarrow\; (7),\quad \theta = k_0\,c\,t$$
$$c\,t' = \bar{\gamma}\,(c\,t - \bar{\beta}\,z)$$

$$\Rightarrow\quad t = \bar{\gamma}\,t' - \frac{1}{4\,k_0\,c}\,\frac{K^2}{2+K^2}\,\sin\left(2\,\bar{\gamma}\,k_0\,c\,t'\right) \;\Leftarrow\; t \approx \bar{\gamma}\,t' \text{ to the 1st approximation}$$

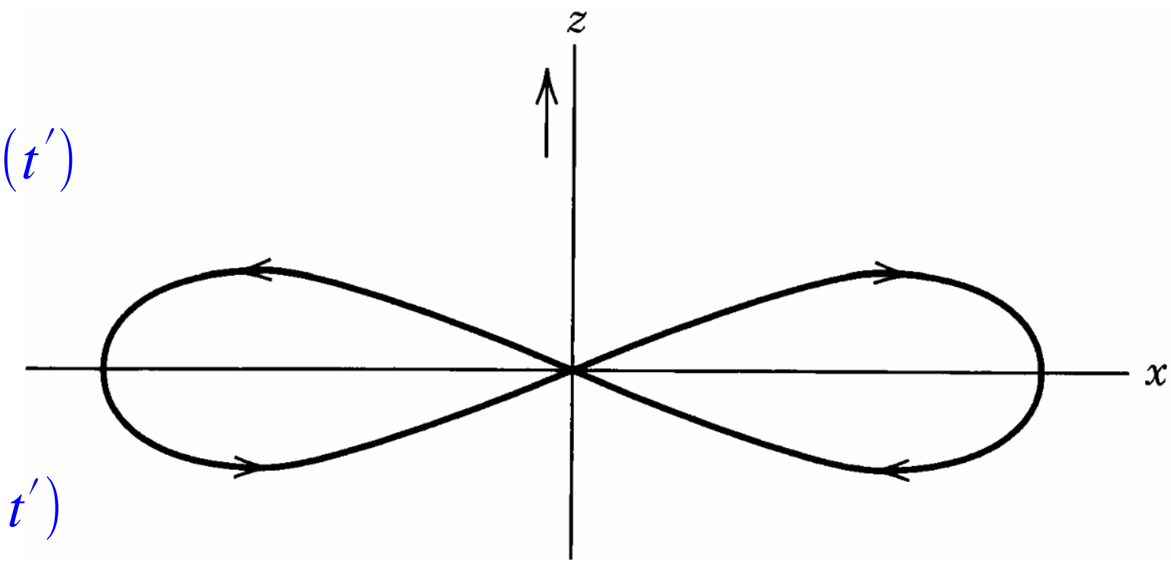
$$\Rightarrow\quad \theta = \bar{\gamma}\,k_0\,c\,t' - \frac{1}{4}\,\frac{K^2}{2+K^2}\,\sin\left(2\,\bar{\gamma}\,k_0\,c\,t'\right) \;\Leftarrow$$

usually using the 1st term is good enough, the 2nd term is used in differentiation

$$x'(t') = \frac{K}{\gamma\,k_0}\,\sin\theta\,(t') = a\,\sin\theta\,(t')$$

$$\Rightarrow\quad z'(t') = -\frac{\bar{\gamma}\,K^2}{8\,\gamma^2\,k_0}\,\sin 2\,\theta\,(t')$$

$$= -\frac{K\,a}{8}\sqrt{\frac{2}{2+K^2}}\,\sin 2\,\theta\,(t')$$



$$\Rightarrow\quad z' = \mp 2\,z'_{\max}\,\frac{x'}{a}\sqrt{1-\frac{x'^2}{a^2}} \;\Leftarrow\; z'_{\max} = \frac{K\,a}{8}\sqrt{\frac{2}{2+K^2}}$$

$$\Rightarrow\; \begin{array}{l} K \gg 1 \;\Rightarrow\; \infty\text{-pattern} \\ K \ll 1 \;\Rightarrow\; \text{1d SHM in } x \end{array}$$

- $$\left(\begin{array}{l}\text{particle's speed in}\\ \text{the moving frame}\end{array}\right)^2 = \beta'^2 = \frac{1}{c^2}\left[\left(\frac{\mathrm{d}\,x'}{\mathrm{d}\,t'}\right)^2 + \left(\frac{\mathrm{d}\,z'}{\mathrm{d}\,t'}\right)^2\right] \quad ⇱ \quad \frac{\mathrm{d}}{\mathrm{d}\,t'}\theta\left(t'\right)$$

$$\Rightarrow \quad \beta'^2 = \left(\frac{2\,K^2}{2+K^2}\cos^2\theta + \frac{K^4\cos^2 2\,\theta}{4\left(2+K^2\right)^2}\right)\left(1 - \frac{K^2\cos 2\,\theta}{2\left(2+K^2\right)}\right)^2 \quad \Leftarrow \quad \theta = \bar{\gamma}\,k_0\,c\,t' \text{ now}$$

$$\beta' \approx K\cos\theta \qquad \text{for } K \ll 1 \quad \Rightarrow \quad \text{nonrelativistic SHM} \quad \Rightarrow \quad \text{undulator}$$

$$\Rightarrow \quad \beta' \approx 1 - \frac{\left(2\cos^2\theta - 1\right)^2}{4} \quad \text{for } K \to \infty \quad \Rightarrow \quad \frac{3}{4} < \beta' < 1 \quad \text{relativistic} \quad \Rightarrow \quad \text{wiggler}$$

- The radiation in the *moving* frame consists of many harmonics of the basic frequency, with an angular distribution that is far from a simple dipole pattern.

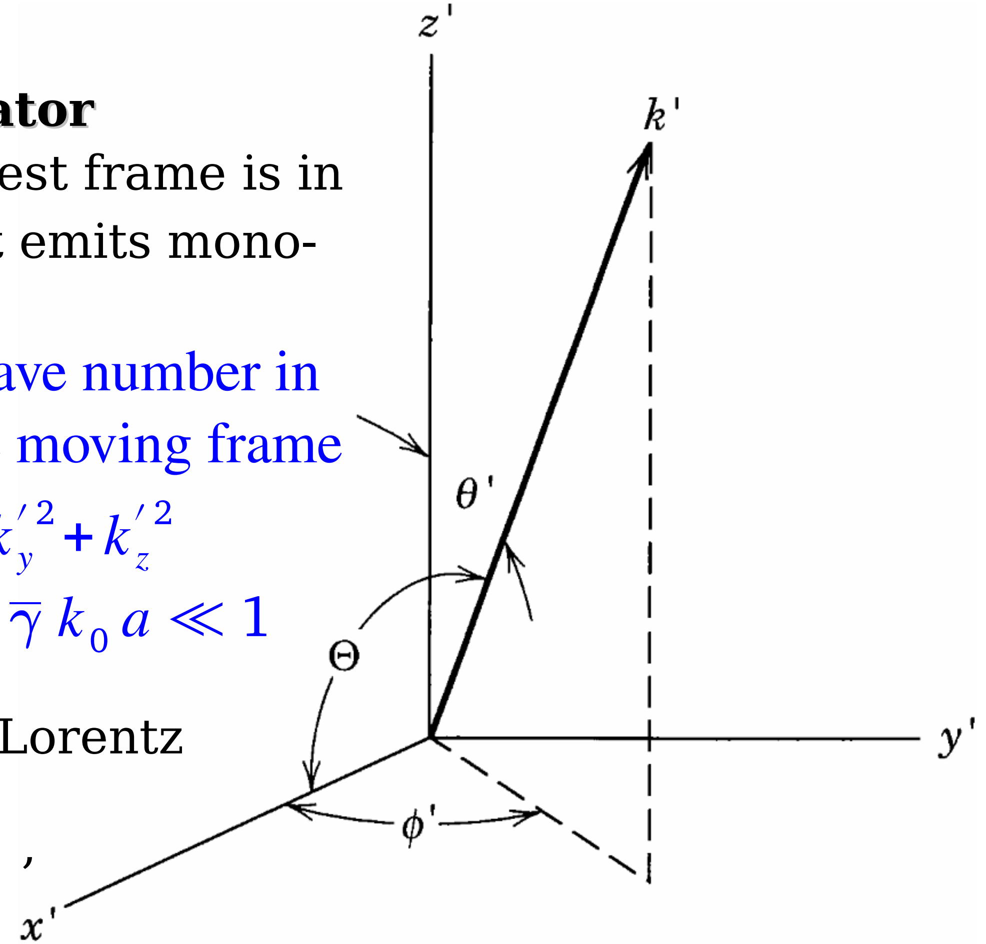
- The laboratory radiation pattern from a strong wiggler is better described by the contributions in the direction of observation.

D. Radiation Spectrum from an Undulator

- When $K \ll 1$, the motion in the average rest frame is in nonrelativistic SHM along the x axis and it emits monochromatic dipole radiation

$$\frac{\mathrm{d}P'}{\mathrm{d}\Omega'} = \frac{e^2 c}{8\pi} k'^4 a^2 \sin^2\Theta \quad \Leftarrow \quad k' = \bar{\gamma} k_0$$

wave number in the moving frame



$$= \frac{e^2 c}{8\pi} K^2 \left(k_y'^2 + k_z'^2\right) \quad \Leftarrow \quad \begin{aligned} & k'^2 \sin^2\Theta = k_y'^2 + k_z'^2 \\ & K = \gamma k_0 a \approx \bar{\gamma} k_0 a \ll 1 \end{aligned}$$

- Since the phase-space density $\dfrac{\mathrm{d}^3 k}{\omega}$ is a Lorentz invariant, it is useful to consider $\omega' \dfrac{\mathrm{d}^3 P'}{\mathrm{d}^3 k'}$,

$$\mathrm{d}^3 P' \equiv \mathrm{d}P' \, \mathrm{d}k' = \frac{\mathrm{d}P'}{\mathrm{d}\Omega'} \frac{c}{k'} \frac{\mathrm{d}^3 k'}{\omega'}$$

$$= \left(\frac{e^2 c^2}{8\pi} K^2 \left(k_y'^2 + k_z'^2\right) \frac{\delta\left(k' - \bar{\gamma} k_0\right)}{\bar{\gamma} k_0} \right) \frac{\mathrm{d}^3 k'}{\omega'} \quad \Leftarrow \quad \mathrm{d}^3 k' = k'^2 \, \mathrm{d}k' \, \mathrm{d}\Omega'$$

Inserting $\delta\left(k' - \bar{\gamma} k_0\right)$ to assure the monochromatic nature

$$\Delta t' = \frac{\lambda_0}{\bar{\gamma} \bar{\beta} c} \approx \frac{\lambda_0}{\bar{\gamma} c} \quad \Leftarrow$$

time for passing one period of the magnet structure in the moving frame

$$\Rightarrow \quad \begin{matrix}\text{\# of photon} \\ \text{emitted}\end{matrix} = \frac{\Delta t'}{\hbar \omega'} \frac{\mathrm{d}^3 P'}{\mathrm{d}^3 k' / \omega'} \Leftarrow N' = N \Rightarrow \frac{\Delta t}{\hbar \omega} \frac{\mathrm{d}^3 P}{\mathrm{d}^3 k / \omega} \quad \Leftarrow \quad \text{invariant}$$

$$\Rightarrow \quad \frac{d^3 P}{d k \, d \Omega} = \frac{k^2 \Delta t'}{\omega' \Delta t} \frac{d^3 P'}{d^3 k' / \omega'} \qquad \Leftarrow \quad \frac{\Delta t}{\Delta t'} = \bar{\gamma}$$

$$= \frac{c e^2 K^2}{8 \pi \bar{\gamma}^3} \frac{k^2}{k_0^2} \left(k_y'^2 + k_z'^2 \right) \delta \left(k' - \bar{\gamma} k_0 \right) \qquad \frac{d^3 k}{\omega} = \frac{k \, d k \, d \Omega}{c}$$

$$\phi' = \phi$$

$$k_y' = k_y = k \sin\theta \sin\phi$$

$$k_z' = \bar{\gamma} k \left(\cos\theta - \bar{\beta} \right)$$

$$k' = \bar{\gamma} k \left(1 - \bar{\beta} \cos\theta \right)$$

in the lab variables

$+$

$$k = \frac{k_0}{1 - \bar{\beta} \cos\theta} \quad \Leftarrow \quad k' = \bar{\gamma} k_0$$

$$\bar{\gamma} \gg 1 \quad \Rightarrow \quad \theta \ll 1 , \quad \bar{\beta} \approx 1 - \frac{1}{2 \bar{\gamma}^2}$$

$$\Rightarrow \quad \frac{d^3 P}{d \eta \, d k \, d \phi} = \frac{c e^2 \bar{\gamma}^2 K^2 k_0^2}{2 \pi} \frac{(1-\eta)^2 + 4 \eta \sin^2 \phi}{(1+\eta)^4} \delta \left[k (1+\eta) - 2 \bar{\gamma}^2 k_0 \right] \Leftarrow \eta = \left(\bar{\gamma} \theta \right)^2$$

- Because of the delta function, the frequency and angular distributions are not independent.

(a) Angular Distribution

- $$\frac{d^2 P}{d \eta \, d \phi} = \int \frac{d^3 P}{d \eta \, d k \, d \phi} d k = \frac{c e^2 \bar{\gamma}^2 K^2 k_0^2}{2 \pi} \frac{(1-\eta)^2 + 4 \eta \sin^2 \phi}{(1+\eta)^5}$$

$$\Rightarrow \quad \text{total radiated power} \quad P = \frac{c e^2 \bar{\gamma}^2 K^2 k_0^2}{3} \quad \Leftarrow \quad \frac{d P}{d \eta} = \int \frac{d^2 P}{d \eta \, d \phi} d \phi = 3 P \frac{1 + \eta^2}{(1-\eta)^5}$$

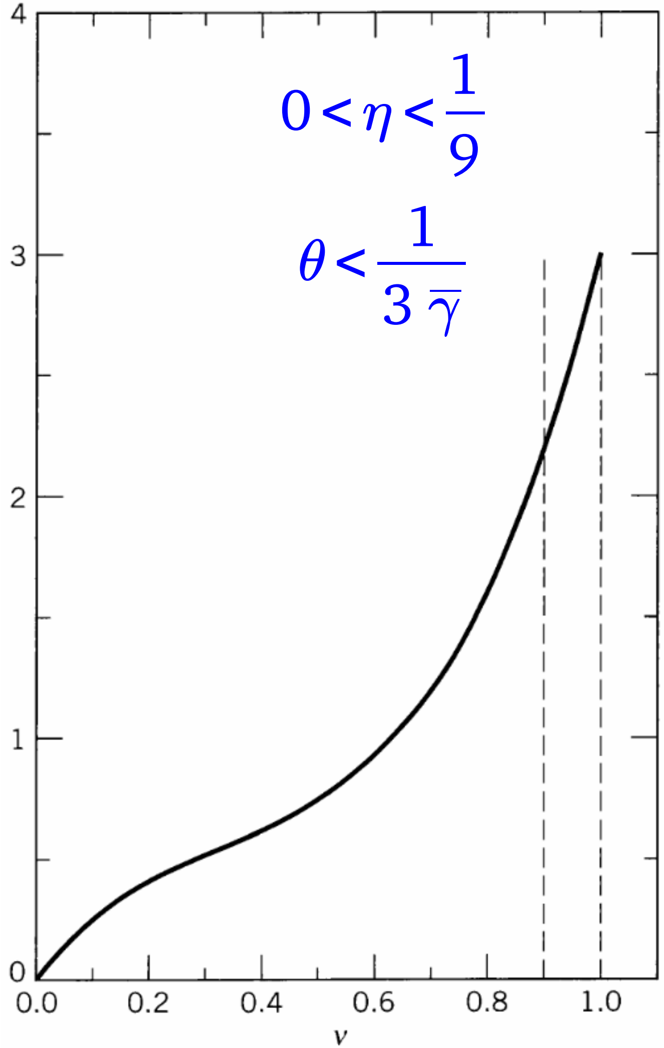
- $\langle \eta \rangle = 1 \quad \Rightarrow \quad \langle \theta \rangle = \dfrac{1}{\bar{\gamma}}$

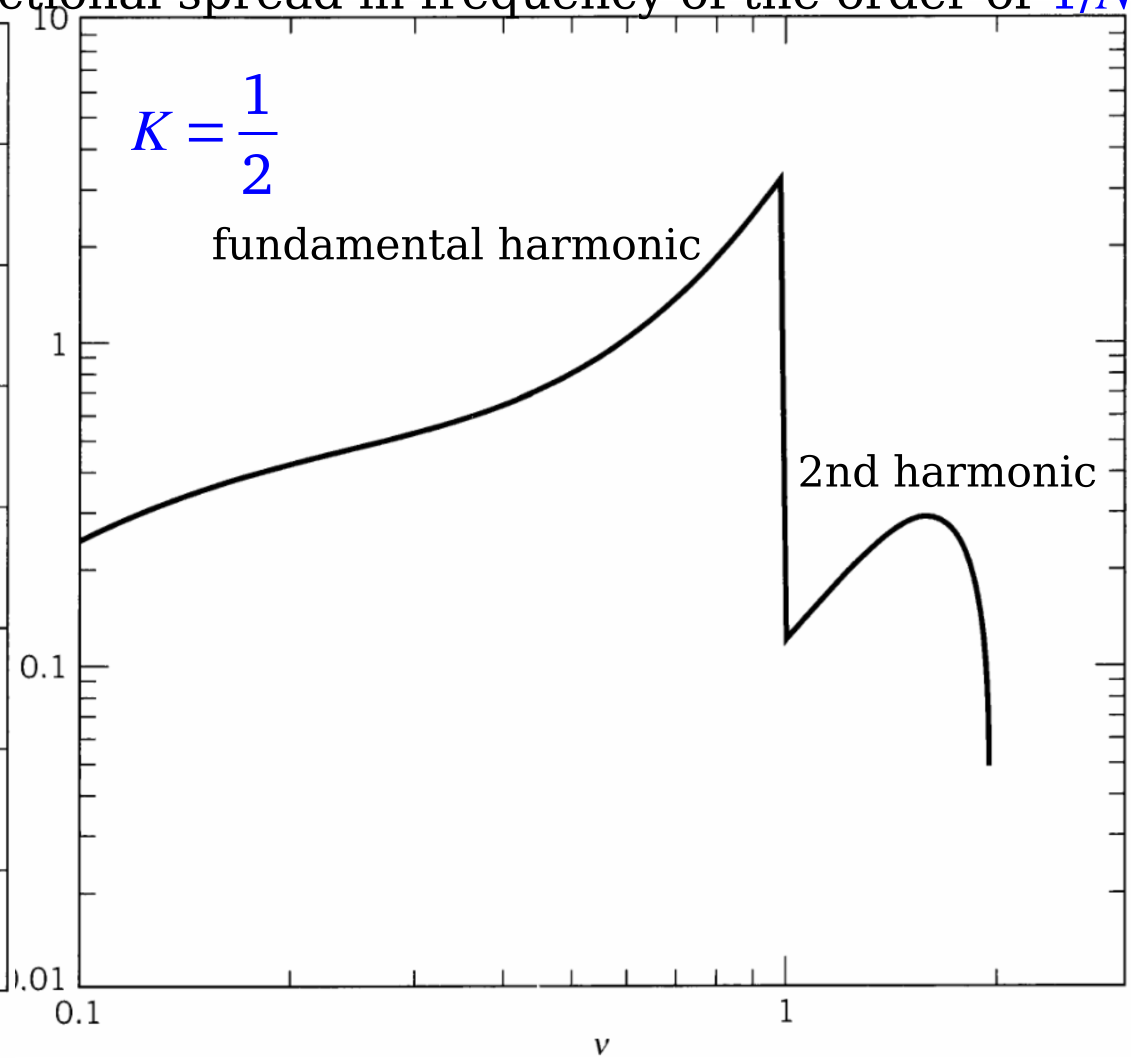
$$\Downarrow \quad \nu \equiv \frac{k}{2\,\bar{\gamma}^2 k_0}$$

(b) Frequency Distribution

- $$\frac{dP}{d\nu} = \int_{\eta_1}^{\eta_2} \frac{d^3 P}{d\eta\, d\nu\, d\phi}\, d\eta\, d\phi = 3P\nu\left(1 - 2\nu + 2\nu^2\right) \quad \Leftarrow \quad \frac{1}{1+\eta_2} < \nu < \frac{1}{1+\eta_1} \qquad (8)$$
- This spectrum is for perfectly sinusoidal motion of the particle at all times.

- If N of magnet periods is finite, the duration of the oscillatory motion is finite; the wave train will have a fractional spread in frequency of the order of $1/N$.





- For large N the spread is small compared to the spread from finite acceptance.

- For small K, there are higher harmonics, coming from higher multipoles caused by the ∞-pattern motion.

- The 2nd harmonic comes from a coherent superposition of the fields of a dipole in the z-direction [$z' \propto \sin 2\theta(t')$] and a quadrupole caused by the x'-motion.

(c) Energy of Photons and Number Emitted per Magnet Period

- $\hbar\,\omega_{\max} = 2\,\hbar\,\bar{\gamma}^2\,k_0\,c$ at $\begin{array}{c} \nu = 1 \\ \eta = 0 \end{array}$ + $\begin{array}{c} \text{energy radiated} \\ \text{per magnet period} \end{array}$ $\Delta E = P\,\Delta t \Leftarrow \Delta t = \dfrac{\lambda_0}{c}$

$$\Rightarrow \begin{array}{l} \text{\# of photon per} \\ \text{magnet period} \end{array} N_\gamma \geq \frac{P\,\Delta t}{\hbar\,\omega_{\max}} = O\left(\alpha\,K^2\right) \Rightarrow N_\gamma = \frac{2}{3}\,\pi\,\alpha\,K^2 \Leftarrow (8)$$

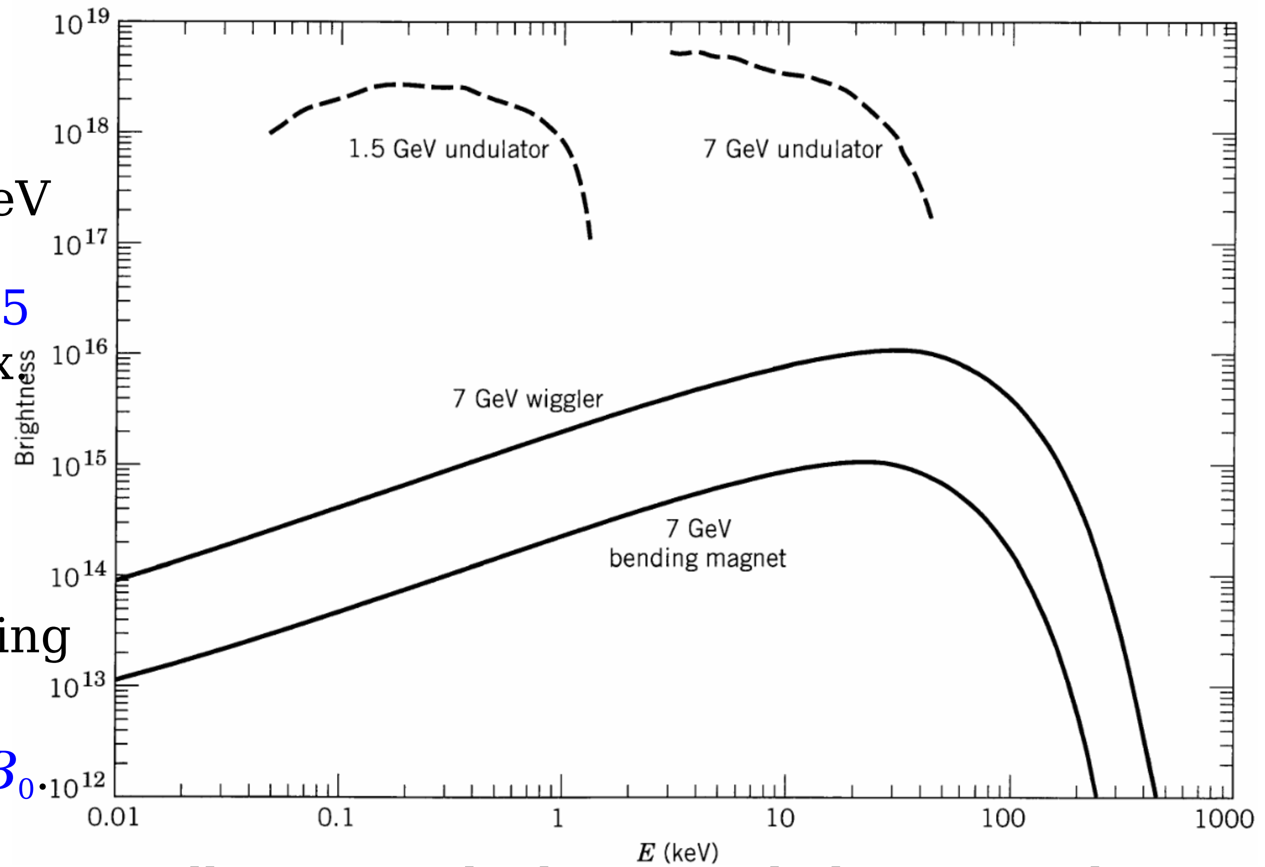
E. Numerical Values and Representative Spectra and Facilities

- The parameters K and $\hbar\,\omega_{\max}$ are given for electrons

$$K = \frac{e\,B_0}{k_0\,m\,c^2} = \frac{e\,B_0\,\lambda_0}{2\,\pi\,m\,c^2} = 93.4\,B_0(T)\,\lambda_0(\mathrm{m}), \quad \hbar\,\omega_{\max}(\mathrm{eV}) = \frac{9.496\,[E(\mathrm{GeV})]^2}{(1+K^2/2)\,\lambda_0(\mathrm{m})}$$

$$\Rightarrow \begin{array}{l} \text{Typical undulator: } B_0 \sim 0.5\,T,\ \lambda_0 \sim 4\text{ cm},\ E \sim 1\text{ - }7\text{ GeV} \Rightarrow \begin{array}{l} K \sim 2 \\ \hbar\,\omega_{\max} \sim 80\text{ eV- }4\text{ keV} \end{array} \\ \text{Typical wiggler: } B_0 \sim 1\,T,\ \lambda_0 \sim 20\text{ cm} \Rightarrow K \sim 20 \end{array}$$

- The lower energy facilities provide photons in the tens of eV to several KeV range; the high-energy facilities extend to 10-75 keV, and higher at reduced flux.



F. Additional Comments

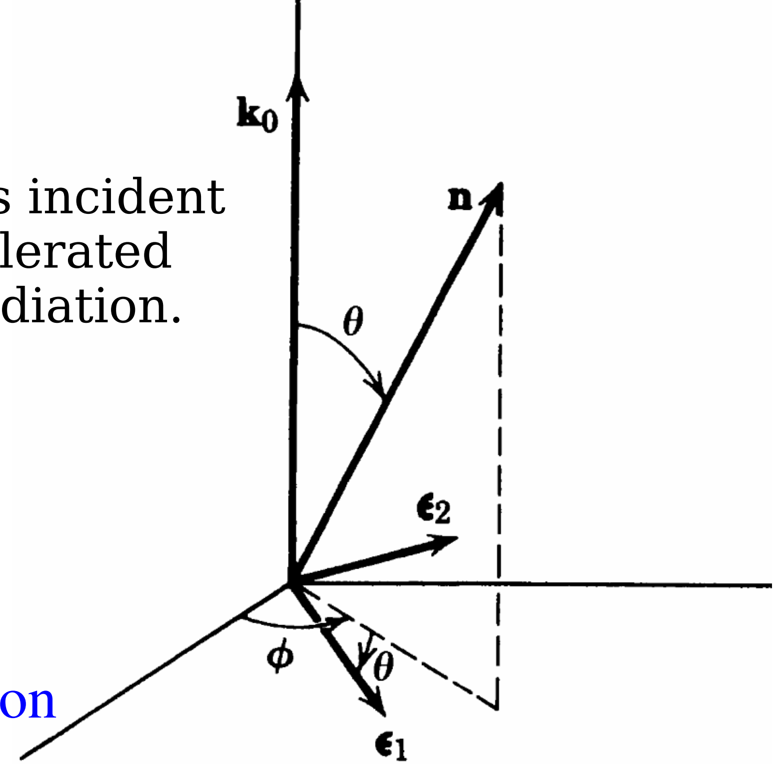
- An undulator's fundamental freq. ω_{max} can be tuned by varying K by changing the gap in the magnet structure & changing B_0.

- The simple undulator with beam oscillations in the horizontal plane provides linearly polarized light. Circular polarization can be provided by use of a designed helical undulator. Or, 2 undulators at right angles with an adjustable longitudinal spacing between them can be used to produce circular polarization or any other state.

- Free electron lasers are related to wigglers and undulators. An undulator can be thought of as radiating in the forward direction at freq. ω_{max} by spontaneous emission. Addition of a co-traveling EM wave of the same frequency provides the possibility of interaction and stimulated emission and growth of the wave.

Thomson Scattering of Radiation

- If a plane wave of monochromatic EM radiation is incident on a free charged particle, the particle will be accelerated and so emit radiation—scattering of the incident radiation.



- $\mathbf{E}(\mathbf{r}, t) = \hat{\mathbf{e}}_0 E_0 e^{i(\boldsymbol{k}_0 \cdot \mathbf{r} - \omega t)}$

$$\Rightarrow \quad \dot{\mathbf{v}}(t) = \hat{\mathbf{e}}_0 \frac{e}{m} E_0 e^{i(\boldsymbol{k}_0 \cdot \mathbf{r} - \omega t)} \quad \Leftarrow \quad \hat{\mathbf{e}}_0 : \text{polarization of EM wave}$$

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3} |\hat{\mathbf{e}}^* \cdot \dot{\mathbf{v}}|^2 \quad \Leftarrow \quad \hat{\mathbf{e}} : \text{polarization of radiation}$$

$$\Rightarrow \quad \left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c}{8\pi} \frac{e^4}{m^2 c^4} |E_0|^2 |\hat{\mathbf{e}}^* \cdot \hat{\mathbf{e}}_0|^2 \quad \Leftarrow \quad \langle |\dot{\mathbf{v}}|^2 \rangle = \frac{1}{2} \Re(\dot{\mathbf{v}} \cdot \dot{\mathbf{v}}^*)$$

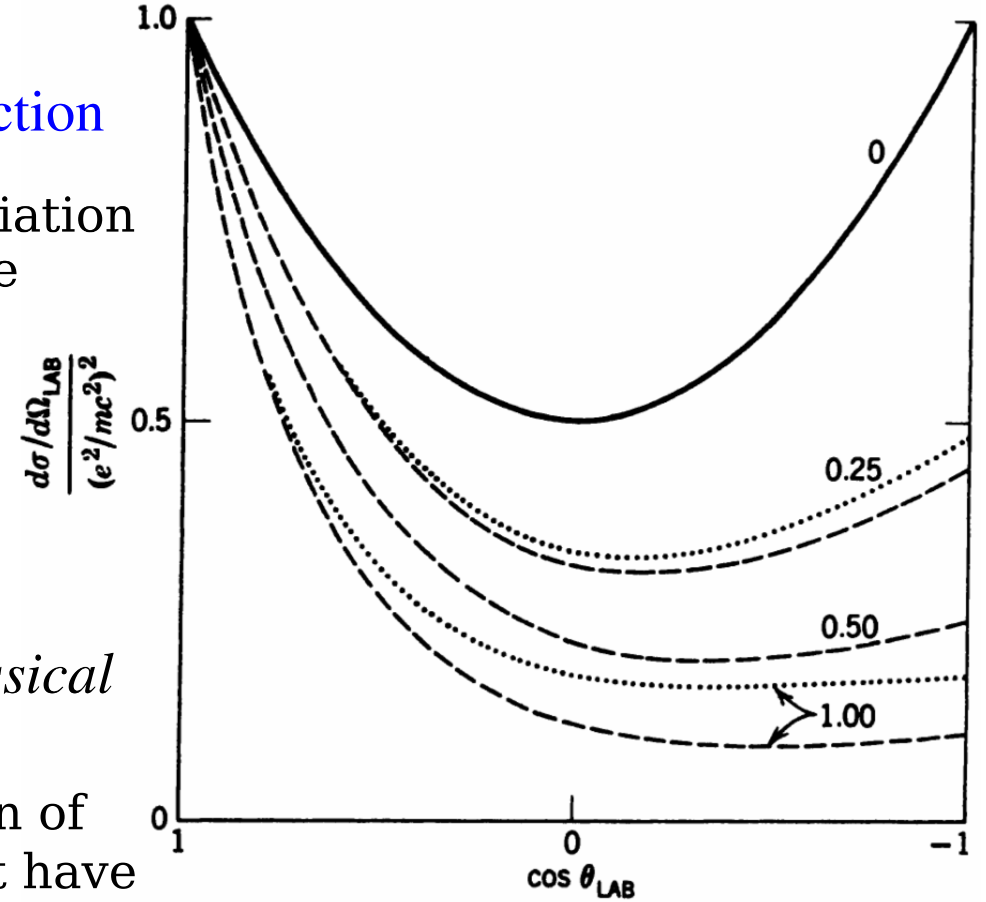
$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{\text{Energy radiated/time/solid angle}}{\text{Incident energy flux in energy/area/time}} = \frac{dP/d\Omega}{c|E_0|^2/8\pi} = \frac{e^4}{m^2 c^4} |\hat{\mathbf{e}}^* \cdot \hat{\mathbf{e}}_0|^2$$

$$\hat{\mathbf{e}}_1 = \cos\theta(\hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi) - \hat{\mathbf{z}} \sin\theta, \quad \hat{\mathbf{e}}_2 = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{e^4}{m^2 c^4} \cdot \begin{cases} (\cos^2\theta \cos^2\phi + \sin^2\phi) & \text{linear polarization } \parallel x\text{-axis} \\ (\cos^2\theta \sin^2\phi + \cos^2\phi) & \text{linear polarization } \parallel y\text{-axis} \end{cases}$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{e^4}{m^2 c^4} \frac{1 + \cos^2\theta}{2} \quad \text{unpolarized} \quad \Leftarrow \quad \text{Thomson formula}$$

$$\Rightarrow \quad \sigma_T = \frac{8\pi}{3}\frac{e^4}{m^2 c^4} \quad \Leftarrow \quad \text{Thomson cross section}$$



- Thomson formula is for scattering of radiation by a free charge, and is appropriate for the scattering of x-rays by electrons or γ-rays by protons.

- The Thomson cross section is equal to 0.665×10^{-24} cm^2 for electrons.

- $\dfrac{e^2}{mc^2} = 2.82\times10^{-13}\ \text{cm}$ is called the *classical electron radius* since a classical distribution of charge totaling the electronic charge must have a radius of this order if its electrostatic self-energy is to equal the electron mass.

- The classical Thomson formula is valid only at low frequencies where the momentum of the incident photon can be ignored.

- When the photon's momentum $\dfrac{\hbar\omega}{c}$ becomes comparable to or larger than mc, modifications occur—quantum-mechanical effects.

- The energy or momentum of the scattered photon is less than the incident energy because the charged particle recoils during the collision.

- $\dfrac{k'}{k}=\dfrac{mc^2}{mc^2+\hbar\omega(1-\cos\theta)}$ Compton formula $\Leftarrow$ θ : scattering angle in the lab

$$\Rightarrow \quad \frac{d\sigma}{d\Omega}=\frac{e^4}{m^2c^4}\frac{k'^2}{k^2}\left|\boldsymbol{\epsilon}^*\cdot\boldsymbol{\epsilon}_0\right|^2 \quad \Leftarrow \quad \text{spinless particle}$$

- $\left(\dfrac{k'}{k}\right)^2$ comes entirely from the phase space. Its presence causes the differential cross section to decrease relative to the Thomson result at large angles.

$$\Rightarrow \quad \frac{\sigma}{\sigma_T}=\begin{cases} 1-2\dfrac{\hbar\omega}{mc^2}+\cdots & \text{for} \quad \hbar\omega\ll mc^2 \\ \begin{cases} \dfrac{3}{4}\dfrac{mc^2}{\hbar\omega} & \text{spinless} \\ \dfrac{3}{4}\dfrac{mc^2}{\hbar\omega}\left(\dfrac{1}{4}+\dfrac{1}{2}\ln\dfrac{2\hbar\omega}{mc^2}\right) & \text{electron} \end{cases} & \text{for} \quad \hbar\omega\gg mc^2 \end{cases}$$

- For protons the departures from the Thomson formula occur at $\hbar\omega>100$ MeV. This is far below the critical energy $\hbar\omega\sim Mc^2\sim 1\text{GeV}$.

- The reason is that a proton is not a point particle but having a spread-out charge distribution by the strong interactions.