Chapter 12 Dynamics of Relativistic Particles & EM Fields

Lagrangian & Hamiltonian for a Relativistic Charged Particle in External EM Fields

- $$\frac{d\mathbf{p}}{dt} = e\left(\mathbf{E} + \frac{\mathbf{u}}{c}\times\mathbf{B}\right) \quad \Rightarrow \quad \frac{dU^{\alpha}}{d\tau} = \frac{e}{mc}\mathbb{F}^{\alpha\beta}U_{\beta} \quad (\&) \quad \Leftarrow \quad \vec{U} = (\gamma c, \gamma\mathbf{u}) = \frac{\vec{p}}{m}$$
  $$\frac{d\mathscr{E}}{dt} = e\,\mathbf{u}\cdot\mathbf{E} \qquad \frac{d\vec{U}}{d\tau} = \frac{e}{mc}\mathbb{F}\cdot\vec{U}$$

- It is useful to consider the formulation of the dynamics from the viewpoint of Lagrangian and Hamiltonian mechanics.

- *The principle of least action*: the motion of a mechanical system is such that in going from one configuration at one time to another configuration at another time, the action is an extremum.

$$\mathcal{A} = \int_{t_1}^{t_2} L\left[q_i(t), \dot{q}_i(t); t\right] dt \quad \Rightarrow \quad \delta\mathcal{A} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \qquad \text{Euler-Lagrange equation of motion}$$

- Wish to extend the formalism to relativistic motion in a manner consistent with special relativity and leading for charged particles in external fields to the right equations.

A. Elementary Approach to a Relativistic Lagrangian

- From the 1st postulate of SR the action integral must be a Lorentz scalar because the equations of motion are determined by the extremum condition.

- $\mathcal{A} = \int_{t_1}^{t_2} L \,\mathrm{d}t = \int_{\tau_1}^{\tau_2} \gamma L \,\mathrm{d}\tau \;\Rightarrow\; \gamma L$ is *Lorentz invariant* $\;\Leftarrow\; \tau, \mathcal{A}$ invariant

- The Lagrangian for a free particle can be a function of its velocity & mass, but not of its position. The only Lorentz invariant of the velocity available is $U_\alpha U^\alpha = c^2$

$$\Rightarrow \quad \gamma L_{\text{free}} \propto c^2 \quad \Rightarrow \quad L_{\text{free}} = -\frac{m c^2}{\gamma} = -m c^2 \sqrt{1 - \frac{u^2}{c^2}} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m \mathbf{u}) = 0$$

- If a particle stays at rest initially and after in the frame, the integral over proper time will be larger than if it moves with a nonzero velocity along its path. So a straight world line gives the maximum integral over proper time.

- This motion at constant velocity is the solution of the free-particle equation of motion.

- For a relativistic charged particle in external EM fields

Interaction part $L_{\text{int}} \rightarrow L_{\text{int}}^{\text{nonrel}} = -e\Phi \quad \Rightarrow \quad L_{\text{int}} = -\frac{e}{\gamma c} \vec{U} \cdot \vec{A} = -e\left(\Phi - \frac{\mathbf{u}}{c} \cdot \mathbf{A}\right)$

$$\Rightarrow \quad L = -m c^2 \sqrt{1 - \frac{u^2}{c^2}} - e\Phi + e\frac{\mathbf{u}}{c} \cdot \mathbf{A} \quad \Rightarrow \text{Lorentz force law} \Leftarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

canonical momentum $P_j \equiv \dfrac{\partial L}{\partial u^j} = \gamma m u_j + \dfrac{e}{c} A_j \;\Rightarrow\; \mathbf{P} = \mathbf{p} + \dfrac{e}{c}\mathbf{A} \;\Leftarrow\; \mathbf{p} = \gamma m \mathbf{u}$

$\Rightarrow\; H \equiv \mathbf{P}\cdot\mathbf{u} - L = \sqrt{(c\mathbf{P} - e\mathbf{A})^2 + m^2 c^4} + e\Phi \;\Leftarrow\; \dfrac{\mathbf{u}}{c} = \dfrac{c\mathbf{P} - e\mathbf{A}}{\sqrt{(c\mathbf{P} - e\mathbf{A})^2 + m^2 c^4}}$

Hamiltonian

Total energy $W = H \;\Rightarrow\; (W - e\Phi)^2 - (c\mathbf{P} - e\mathbf{A})^2 = m^2 c^4$

$$\Rightarrow\; \vec{p}^{\,2} = m^2 c^2 \;\Leftarrow\; \vec{p} = \left(\frac{E}{c}, \mathbf{p}\right) = \left(\frac{W - e\Phi}{c}, \mathbf{P} - \frac{e}{c}\mathbf{A}\right)$$

- The total energy $\dfrac{W}{c}$ acts as the time component P^0 of a canonically conjugate 4-momentum $\vec{P}$ of which $\mathbf{P}$ is the space part.

- The equations of motion are invariant under a gauge transformation of the potentials.

- Since the Lagrangian involves the potentials explicitly, it is not invariant. But the change in the Lagrangian is of such a form (a total time derivative) that it does not alter the action integral or the equations of motion. [Problem 12.2]

B. Manifestly Covariant Treatment of the Relativistic Lagrangian

- $L_{\text{free}} = -\dfrac{m\,c}{\gamma}\sqrt{U_\alpha U^\alpha} \;\Rightarrow\; \mathcal{A} = -m\,c\displaystyle\int_{\tau_1}^{\tau_2}\sqrt{U_\alpha U^\alpha}\,\mathrm{d}\tau$

$$= -m\,c\int_{\tau_1}^{\tau_2}\sqrt{\vec{U}^2}\,\mathrm{d}\tau \;\Rightarrow\; \frac{\mathrm{d}\,U^\alpha}{\mathrm{d}\,\tau} = 0$$

- The equation of constraint: $\vec{U}^2 = U_\alpha U^\alpha = c^2 \;\Leftrightarrow\; \vec{U}\cdot\dfrac{\mathrm{d}\,\vec{U}}{\mathrm{d}\,\tau} = 0$

can be incorporated by the Lagrange multiplier method, but we try another way.

- $\sqrt{\vec{U}^2}\,\mathrm{d}\,\tau = \sqrt{\dfrac{\mathrm{d}\,x_\alpha}{\mathrm{d}\,\tau}\dfrac{\mathrm{d}\,x^\alpha}{\mathrm{d}\,\tau}}\,\mathrm{d}\,\tau \;\Rightarrow\; \mathcal{A} = -m\,c\displaystyle\int_{s_1}^{s_2}\sqrt{g_{\alpha\beta}\dfrac{\mathrm{d}\,x^\alpha}{\mathrm{d}\,s}\dfrac{\mathrm{d}\,x^\beta}{\mathrm{d}\,s}}\,\mathrm{d}\,s \quad (\ast)$

$$= \sqrt{g_{\alpha\beta}\,\mathrm{d}\,x^\alpha\,\mathrm{d}\,x^\beta}$$

s : affine parameter

$$\Rightarrow \sqrt{g_{\alpha\beta}\frac{\mathrm{d}\,x^\alpha}{\mathrm{d}\,s}\frac{\mathrm{d}\,x^\beta}{\mathrm{d}\,s}}\,\mathrm{d}\,s = c\,\mathrm{d}\,\tau \;\Rightarrow\; \vec{U}^2 = c^2 \;\Rightarrow\; m\,c\,\frac{\mathrm{d}}{\mathrm{d}\,s}\frac{\dfrac{\mathrm{d}\,x^\alpha}{\mathrm{d}\,s}}{\sqrt{\dfrac{\mathrm{d}\,x_\beta}{\mathrm{d}\,s}\dfrac{\mathrm{d}\,x^\beta}{\mathrm{d}\,s}}} = 0 \;\Leftarrow\; (\ast)$$

$$\Rightarrow m\,\frac{\mathrm{d}^2\,x^\alpha}{\mathrm{d}\,\tau^2} = 0 \quad \text{free particle motion}$$

- For a charged particle in an external field

$$\mathcal{A}=\int_{s_1}^{s_2}\mathcal{L}\,\mathrm{d}s=-\int_{s_1}^{s_2}\left(m\,c\sqrt{g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s}}+\frac{e}{c}A_\alpha(x)\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\right)\mathrm{d}s$$

$$\Rightarrow\quad \frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial\mathcal{L}}{\partial\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}}-\frac{\partial\mathcal{L}}{\partial x^\alpha}=0\quad\Leftarrow\quad\mathcal{L}=-\left(m\,c\sqrt{g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s}}+\frac{e}{c}A_\alpha\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\right)$$

$$\Rightarrow\quad m\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2}+\frac{e}{c}\frac{\mathrm{d}A^\alpha}{\mathrm{d}\tau}-\frac{e}{c}\frac{\partial A_\beta}{\partial x_\alpha}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}=0\qquad ⇦\qquad\frac{\mathrm{d}A^\alpha}{\mathrm{d}\tau}=\frac{\mathrm{d}x_\beta}{\mathrm{d}\tau}\frac{\partial A^\alpha}{\partial x_\beta}$$

$$\Rightarrow\quad m\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2}=\frac{e}{c}\left(\frac{\partial A^\beta}{\partial x_\alpha}-\frac{\partial A^\alpha}{\partial x_\beta}\right)\frac{\mathrm{d}x_\beta}{\mathrm{d}\tau}=\frac{e}{c}\mathbb{F}^{\alpha\beta}U_\beta=m\times(\&)$$

$$P^\alpha\equiv-\frac{\partial\mathcal{L}}{\partial\frac{\mathrm{d}x_\alpha}{\mathrm{d}s}}=m\,U^\alpha+\frac{e}{c}A^\alpha\quad\Rightarrow\quad\mathcal{H}=P_\alpha U^\alpha+\mathcal{L}=\frac{(c\,\vec{P}-e\,\vec{A})^2}{m\,c^2}-\sqrt{(c\,\vec{P}-e\,\vec{A})^2}$$

$$\Rightarrow\quad\begin{aligned}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}&=+\frac{\partial\mathcal{H}}{\partial P_\alpha}=\frac{c\,P^\alpha-e\,A^\alpha}{m\,c}\\ \frac{\mathrm{d}P^\alpha}{\mathrm{d}\tau}&=-\frac{\partial\mathcal{H}}{\partial x_\alpha}=e\,\frac{c\,P_\beta-e\,A_\beta}{m\,c^2}\frac{\partial A^\beta}{\partial x_\beta}\end{aligned}\quad\Leftarrow\quad(c\,\vec{P}-e\,\vec{A})^2=m^2c^4$$

$$\Rightarrow\quad\mathcal{H}\simeq0\neq W$$

Motion in a Uniform, Static Magnetic Field

- $\mathbf{E} = 0$, $\mathbf{B} = \text{const}$ $\Rightarrow$ $\dfrac{d\,\mathbf{p}}{d\,t} = e\,\dfrac{\mathbf{v}}{c} \times \mathbf{B}$, $\dfrac{d\,\mathscr{E}}{d\,t} = 0$ $\Rightarrow$ $v = \text{const}$, $\gamma = \text{const}$ $\Rightarrow$ $\dfrac{d\,\mathbf{v}}{d\,t} = \mathbf{v} \times \boldsymbol{\omega}_B$ $\Leftarrow$ $\boldsymbol{\omega}_B = \dfrac{e\,\mathbf{B}}{\gamma\,m\,c} = \dfrac{c\,e}{\mathscr{E}}\,\mathbf{B}$ gyration precession frequency

A circular motion $\perp$ $\mathbf{B}$ & a uniform translation $\parallel$ $\mathbf{B}$

$$\Rightarrow \quad \mathbf{v}(t) = a\,\omega_B\,e^{-i\,\omega_B\,t}\left(\hat{\mathbf{x}} - i\,\hat{\mathbf{y}}\right) + v_\parallel\,\hat{\mathbf{z}} \quad (0) \quad \Leftarrow$$

counterclock rotation (for positive charge), a : gyration radius

$$\Rightarrow \quad \mathbf{r}(t) = \mathbf{r}_0 + a\,e^{-i\,\omega_B\,t}\left(i\,\hat{\mathbf{x}} + \hat{\mathbf{y}}\right) + v_\parallel\,t\,\hat{\mathbf{z}} \quad (\%) \quad \Rightarrow$$

helix radius $= a$, pitch angle : $\tan\alpha = \dfrac{v_\parallel}{a\,\omega_B}$

- This form is convenient for the determination of particle momenta.

$$v_\perp = a\,\omega_B \quad \Rightarrow \quad c\,p_\perp = a\,e\,B$$

- For particles with charge the same in magnitude as the electronic charge

$$p_\perp\left(\text{MeV}/c\right) = 3.00 \times 10^{-4}\,B\,a\left(\text{gauss-cm}\right) = 300\,B\,a\left(\text{Tesla-m}\right)$$

Motion in Combined, Uniform, Static E & B Fields

- $\mathbf{E}' = \gamma(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}) - (\gamma-1)(\hat{\boldsymbol{\beta}}\cdot\mathbf{E})\,\hat{\boldsymbol{\beta}}$
  $\mathbf{B}' = \gamma(\mathbf{B} - \boldsymbol{\beta}\times\mathbf{E}) - (\gamma-1)(\hat{\boldsymbol{\beta}}\cdot\mathbf{B})\,\hat{\boldsymbol{\beta}}$ $\Rightarrow$ Let $\mathbf{E}\perp\mathbf{B}$, choose $\boldsymbol{\beta} = \dfrac{\mathbf{u}}{c} = \dfrac{\mathbf{E}\times\mathbf{B}}{B^2}$

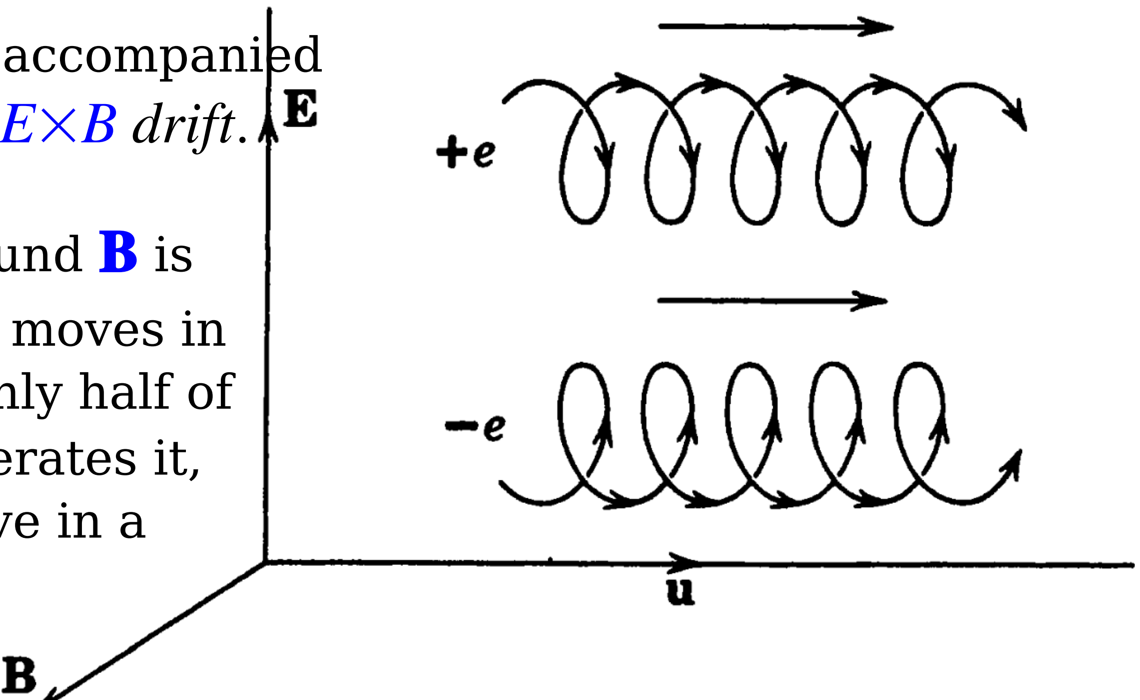
$$\Rightarrow \quad \mathbf{E}' = 0\,, \quad \mathbf{B}'_{\parallel} = 0\,, \quad \mathbf{B}'_{\perp} = \frac{\mathbf{B}}{\gamma_u} = \sqrt{B^2 - E^2}\,\hat{\mathbf{B}} \quad \Leftarrow \quad E < B\,, \quad \gamma_u = \frac{B}{\sqrt{B^2 - E^2}}$$

$$\Rightarrow \quad \text{Lorentz force equation} \quad \frac{d\,\mathbf{p}'}{d\,t'} = e\left(\mathbf{E}' + \frac{\mathbf{v}'}{c}\times\mathbf{B}'\right) = \frac{e}{\gamma_u}\,\frac{\mathbf{v}'}{c}\times\mathbf{B} \quad \text{in } K'$$

- In K' the only field acting is a static $\mathbf{B}'$ pointing in the same direction as $\mathbf{B}$, but weaker by a factor $\dfrac{1}{\gamma}$.

- As viewed from K, the gyration is accompanied by a uniform drift $\mathbf{u}\perp\mathbf{E}$ & $\mathbf{B}$ ← the *$E\times B$ drift*.

- A particle that starts gyrating around $\mathbf{B}$ is accelerated by $\mathbf{E}$, gains energy, and moves in a path with a larger radius for roughly half of its cycle. On the other half, $\mathbf{E}$ decelerates it, causing it to lose energy and so move in a tighter arc.



- The combination of arcs produces a translation $\perp$ $\mathbf{E}$ & $\mathbf{B}$. The direction of drift is independent of the sign of the charge.

- The drift velocity has physical meaning only if it is less than c, ie, if $E < B$.

- If $E > B$, $\mathbf{E}$ is so strong that the particle is continually accelerated in the direction of $\mathbf{E}$ and its average energy continues to increase with time

- For $E > B$ choose $\dfrac{\mathbf{u}'}{c} = \dfrac{\mathbf{E}\times\mathbf{B}}{E^2} \;\Rightarrow\; \mathbf{E}''_{\parallel} = 0\,,\quad \mathbf{E}''_{\perp} = \dfrac{\mathbf{E}}{\gamma_{u'}} = \sqrt{E^2 - B^2}\,\hat{\mathbf{E}}\,,\quad \mathbf{B}'' = 0$

$$\Rightarrow \quad \frac{d\,\mathbf{p}''}{d\,t''} = e\left(\mathbf{E}'' + \frac{\mathbf{v}''}{c}\times\mathbf{B}''\right) = \frac{e}{\gamma_{u'}}\mathbf{E} \;\text{ in K}'' \quad \Leftarrow \quad \gamma_{u'} = \frac{E}{\sqrt{E^2 - B^2}}$$

- In K'' the particle is acted on by a purely $\mathbf{E}''$ which causes hyperbolic motion with ever-increasing velocity.

- If a beam of particles having a spread in velocities is normally incident on a region containing uniform crossed $\mathbf{E}$ & $\mathbf{B}$, only those particles with velocities equal to $\dfrac{c\,E}{B}$ will travel without deflection.

- Suitable entrance and exit slits will allow only a very narrow band of velocities around $\dfrac{c\,E}{B}$ to be transmitted.

- Combined with momentum selectors, like a deflecting magnet, the $\mathbf{E}\times\mathbf{B}$ velocity selectors can extract a very pure & monoenergetic beam of particles of a definite mass from a mixed beam with different masses and momenta — commonly used in high-energy accelerators.

- If $\mathbf{E}$ has a component parallel to $\mathbf{B}$, the behavior of the particle cannot be understood in such simple term

$$\mathbf{E}\cdot\mathbf{B} \text{ and } E^2 - B^2 \text{ are the only 2 Lorentz invariants} \quad \Leftarrow \quad \text{[Problem 11.4]}$$

$$\mathbf{E}\perp\mathbf{B} \quad \Rightarrow \quad \mathbf{E}\cdot\mathbf{B}=0 \quad \Rightarrow \quad \exists \text{ a Lorentz frame where } \begin{array}{l} \mathbf{E}=0 \text{ if } B>E \\ \mathbf{B}=0 \text{ if } E>B \end{array}$$

- If $\mathbf{E}\cdot\mathbf{B}\neq 0$, $\mathbf{E}$ & $\mathbf{B}$ will exist simultaneously in all Lorentz frames. Consequently motion in combined fields must be considered. [Problem 12.6]

Particle Drifts in Nonuniform, Static B Fields

- Often the variations of the magnetic field are gentle enough that a perturbation solution to the motion is an adequate approximation.

- Consider a gradient ⊥ the direction of $\mathbf{B}$ $\quad \Rightarrow \quad B_\perp \equiv \hat{\mathbf{n}} \cdot \mathbf{B} = 0$

$$\Rightarrow \quad \boldsymbol{\omega}_B(\mathbf{r}) = \frac{e}{\gamma m c} \mathbf{B}(\mathbf{r}) \simeq \boldsymbol{\omega}_0 \left(1 + \frac{\partial B}{\partial \xi}_{|0} \frac{\hat{\mathbf{n}} \cdot \mathbf{r}}{B_0} \right) \Leftarrow$$

expansion about the origin of coordinates where $\boldsymbol{\omega}_B = \boldsymbol{\omega}_0$

ξ : coordinate in the direction $\hat{\mathbf{n}}$

- Since the direction of $\mathbf{B}$ is unchanged, the motion ∥ $\mathbf{B}$ remains a uniform translation. We then consider only modifications in the transverse motion.

$$\mathbf{v}_\perp = \mathbf{v}_0 + \mathbf{v}_1$$

$$\frac{d\,\mathbf{v}_\perp}{d\,t} = \mathbf{v}_\perp \times \boldsymbol{\omega}_B(\mathbf{r}) \quad \Rightarrow \quad \frac{d\,\mathbf{v}_1}{d\,t} \simeq [\mathbf{v}_1 + (\hat{\mathbf{n}} \cdot \mathbf{r}_0)\, \partial_\xi \ln B_{|0}\, \mathbf{v}_0] \times \boldsymbol{\omega}_0 \quad \Leftarrow \quad \frac{d\,\mathbf{v}_0}{d\,t} = \mathbf{v}_0 \times \boldsymbol{\omega}_0$$

$$(0) \atop (\%) \quad \Rightarrow \quad \mathbf{v}_0 = -\boldsymbol{\omega}_0 \times (\mathbf{r}_0 - \mathbb{r}_0) \quad \Rightarrow \quad (\mathbf{r}_0 - \mathbb{r}_0) = \frac{\boldsymbol{\omega}_0 \times \mathbf{v}_0}{\omega_0^2} \quad \Leftarrow$$

$\mathbb{r}_0$: center of gyration

$(\mathbb{r}_0 = 0 \ \text{here})$

$$\Rightarrow \quad \frac{d\,\mathbf{v}_1}{d\,t} \simeq [\mathbf{v}_1 - (\hat{\mathbf{n}} \cdot \mathbf{r}_0)\, \partial_\xi \ln B_{|0}\, \boldsymbol{\omega}_0 \times \mathbf{r}_0] \times \boldsymbol{\omega}_0 \quad \Rightarrow \quad \begin{aligned} \mathbf{r}_{0\perp} &\equiv \mathbf{r}_0 - \hat{\boldsymbol{\omega}}_0 (\hat{\boldsymbol{\omega}}_0 \cdot \mathbf{r}_0) \\ &= \hat{\boldsymbol{\omega}}_0 \times (\mathbf{r}_0 \times \hat{\boldsymbol{\omega}}_0) \end{aligned}$$

$$\Rightarrow \quad \mathbf{v}_G \equiv \langle \mathbf{v}_1 \rangle = \partial_\xi \ln B_{|0}\, \boldsymbol{\omega}_0 \times \langle (\hat{\mathbf{n}} \cdot \mathbf{r}_0)\, \mathbf{r}_{0\perp} \rangle$$

gradient drift velocity

- The rectangular components of $\mathbf{r}_{0\perp}$ oscillate sinusoidally with peak amplitude a and a phase difference of 90°, so only the component of $\mathbf{r}_{0\perp} \parallel \hat{\mathbf{n}}$ contributes

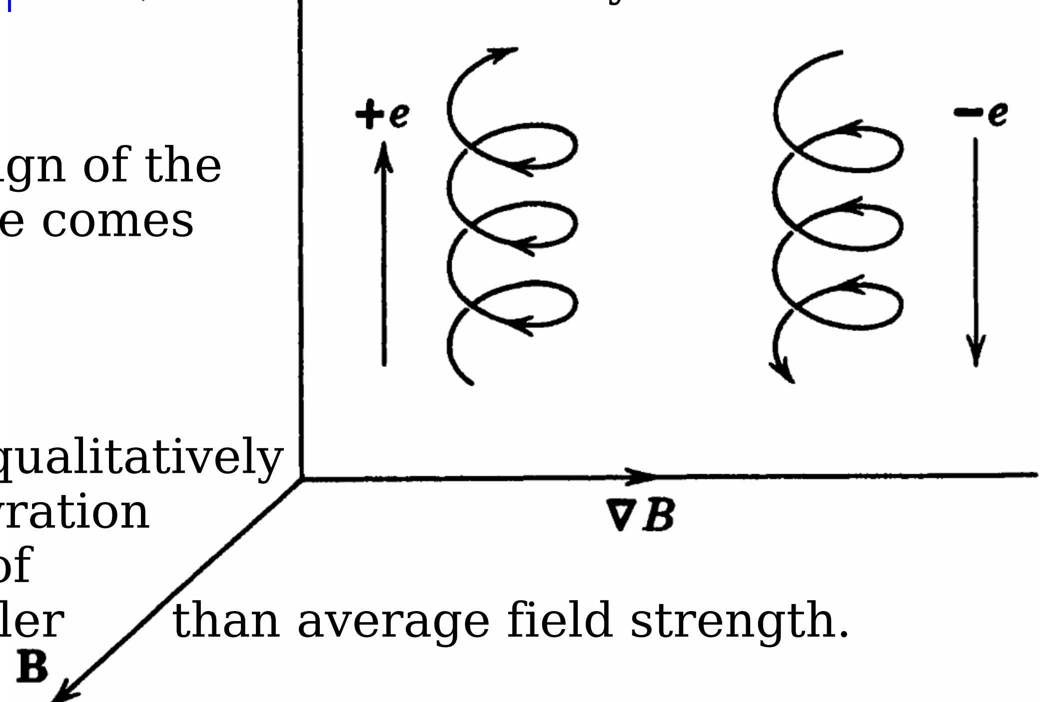
$$\Rightarrow \quad \langle \mathbf{r}_{0\perp} (\hat{\mathbf{n}} \cdot \mathbf{r}_0) \rangle = \frac{a^2}{2} \hat{\mathbf{n}} \quad \Rightarrow \quad \mathbf{v}_G = \frac{a^2}{2} \partial_\xi \ln B_0 \, \boldsymbol{\omega}_0 \times \hat{\mathbf{n}} \quad \Rightarrow \quad \frac{\mathbf{v}_G}{\omega_B a} = \frac{a}{2} \hat{\mathbf{B}} \times \nabla_\perp \ln B$$

coordinate independance

- If the gradient of the field is $a |\nabla \ln B| \ll 1$, the drift velocity is small compared to the orbital velocity $\omega_B a$.

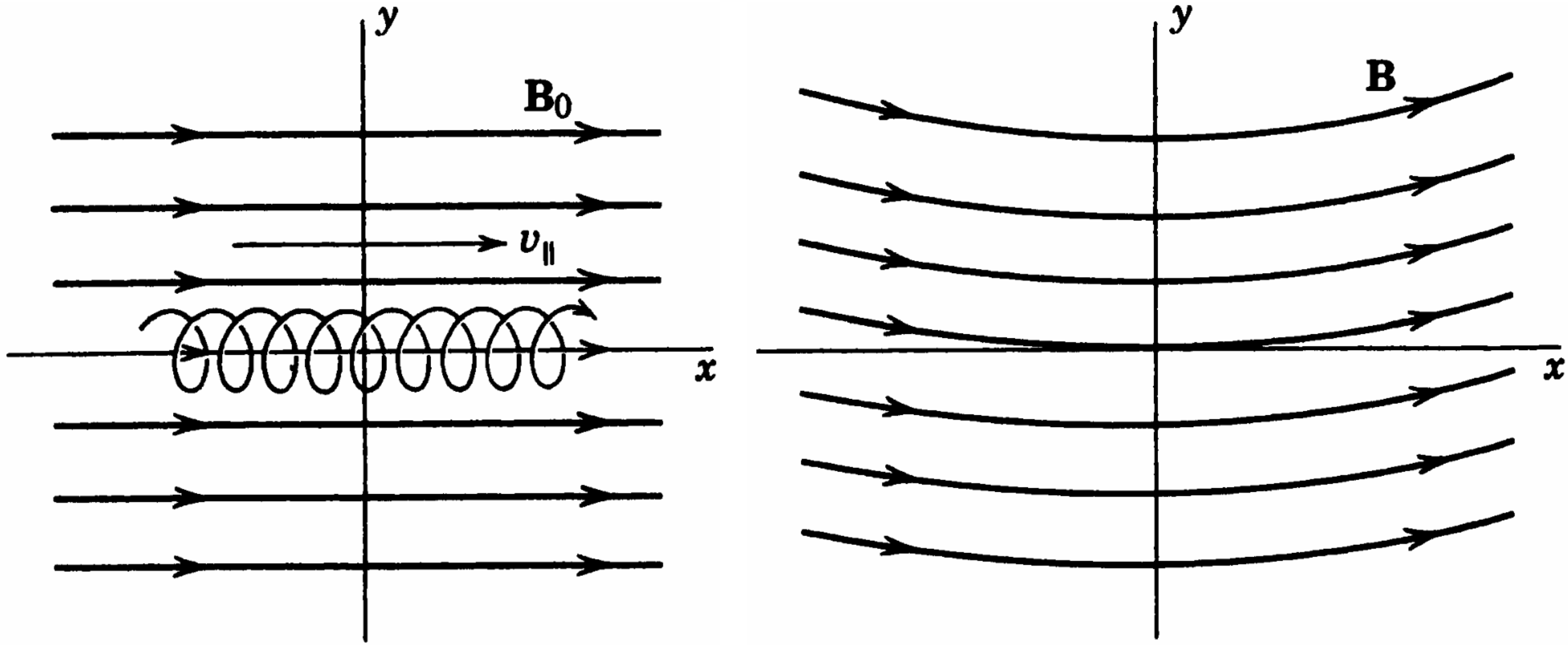
- For negatively charged particles the sign of the drift velocity is opposite; the sign change comes from the definition of ω_B: $\boldsymbol{\omega}_B = \dfrac{e \mathbf{B}}{\gamma m c}$

- The gradient drift can be understood qualitatively from consideration of the variation of gyration radius as the particle moves in and out of regions of larger than average and smaller than average field strength.



- Another type of field variation that causes a drifting of the particle's guiding center is curvature of the lines of force.

- The particle tends to spiral around a field line, but the field line curves off to the side. This is equivalent to a centrifugal acceleration of magnitude $\dfrac{v_\parallel^2}{R}$.



- The acceleration can be viewed as arising from an effective electric field

$$\mathbf{E}_{\text{eff}} = \frac{\gamma\, m}{e}\frac{v_\parallel^2}{R}\hat{\mathbf{R}} \quad \Rightarrow \quad \begin{matrix}\text{curvature}\\ \text{drift velocity}\end{matrix} \quad \mathbf{v}_C \simeq c\,\frac{\gamma\, m}{e\, B_0}\frac{v_\parallel^2}{R}\hat{\mathbf{R}}\times\hat{\mathbf{B}}_0 = \frac{v_\parallel^2}{\omega_B\, R}\hat{\mathbf{R}}\times\hat{\mathbf{B}}_0 \quad (\$)$$

- The sign in the equation is appropriate for positive charges and is independent of the sign of $v_\parallel$. For negative particles the opposite sign arises from ω_B.

- A straightforward derivation comes from solving the Lorentz force eqn directly.

- With origin at the center of curvature, $\mathbf{B}$ has only a ϕ-component, $B_\phi = B_0\,\dfrac{R}{\rho}$.

Lorentz force equation $\Rightarrow \frac{d\mathbf{v}}{dt} = \mathbf{v} \times \boldsymbol{\omega}_B$ using the cylindrical coordinates

$$\ddot{z} = \omega_B \frac{R}{\rho}\dot{\rho} \quad \Rightarrow \quad \dot{z} = \omega_B R \ln\frac{\rho}{R} + v_0 \approx \omega_B x + v_0 \quad \Leftarrow \quad \begin{matrix} \rho = R + x \\ \ln(1+y) \simeq y \end{matrix}$$

$$\Rightarrow \quad \rho\ddot{\phi} + 2\dot{\rho}\dot{\phi} = 0 \quad \Rightarrow \quad \rho^2\dot{\phi} = R v_\parallel = \text{const} \quad \Leftarrow \quad \text{angular momentum}$$

$$\ddot{\rho} - \rho\dot{\phi}^2 = -\omega_B \frac{R}{\rho}\dot{z} \quad \Rightarrow \quad \ddot{x} + \left(\omega_B^2 + 3\frac{v_\parallel^2}{R^2} - \frac{\omega_B}{R}v_0\right)x \approx \frac{v_\parallel^2}{R} - \omega_B v_0$$

$$\Rightarrow \quad \langle x \rangle \approx \frac{v_\parallel^2}{\omega_B^2 R} - \frac{v_0}{\omega_B} \quad \Leftarrow \quad v_\parallel, v_0 \ll \omega_B R \quad \Rightarrow \quad \langle \dot{z} \rangle \approx v_0 + \omega_B \langle x \rangle \approx \frac{v_\parallel^2}{\omega_B R} \Rightarrow (\$)$$

- If $\boldsymbol{J} = 0$ $\Rightarrow \nabla \times \mathbf{B} = 0$ $\Rightarrow \quad \nabla_\perp \ln B = \frac{\nabla_\perp B}{B} = -\frac{\hat{\mathbf{R}}}{R} \quad \Leftarrow$ Let $\mathbf{B} = B(\rho)\hat{\boldsymbol{\phi}} \quad \Rightarrow \quad \nabla \times \mathbf{B} = 0$ $\Rightarrow \quad \partial_\rho(\rho B) = 0 \quad \Rightarrow \quad \frac{dB}{B} = -\frac{d\rho}{\rho}$

transverse velocity of gyration $v_\perp = \omega_B a \quad \Rightarrow \quad \mathbf{v}_G = \frac{\omega_B}{2}a^2\hat{\mathbf{B}} \times \nabla_\perp \ln B = \frac{v_\perp^2}{2\omega_B R}\hat{\mathbf{R}} \times \hat{\mathbf{B}}$

$$\Rightarrow \quad \mathbf{v}_D = \mathbf{v}_G + \mathbf{v}_C = \frac{2v_\parallel^2 + v_\perp^2}{2\omega_B R}\hat{\mathbf{R}} \times \hat{\mathbf{B}} \quad \Rightarrow \quad v_D(\text{cm/s}) = \frac{172\,T(\text{K})}{R(\text{m})\,B(\text{gauss})} \quad \text{for } v_D \ll c$$

- With typical parameters of R=1 m, $B=10^3$ gauss, particles in a 1eV plasma ($T \approx 10^4$ K) will have drift velocities $v_D \sim 18$ m/s.

- This meaps that they will drift out to the walls in a small fraction of a second. Hotter the plasmas, greater the drift rate.

- To prevent this 1st-order drift in toroidal tube is to twist the torus into a 8.

- The particles make many circuits around the closed path, so they feel no net curvature or gradient of the field, and no net drift, at least to 1st order in $\dfrac{1}{R}$.

Adiabatic Invariance of Flux Through Orbit of Particle

- We now consider motion parallel to the lines of force.

- For slowly varying fields a powerful tool is the concept of adiabatic invariants.

- If q_i and p_i are the generalized canonical coordinates & momenta, and for each coordinate which is periodic, the action integral is defined by

$$J_i \equiv \oint p_i \, \mathrm{d}\, q_i \quad \Leftarrow \quad \text{over a complete cycle of } \; q_i$$

- For a given mechanical system the action integrals are constants.

- If a change is slow compared to the periods of motion and is not related to the periods (ie, *adiabatic change*), the action integrals are invariant.

- One system can be changed into another system with an adiabatic change, but the values of the action integrals have the same values in both systems.

- For a charged particle in a uniform, static $\mathbf{B}$, the transverse motion is periodic.

$$J = \oint \mathbf{P}_\perp \cdot \mathrm{d}\,\boldsymbol{\ell} = \oint \gamma\, m\, \mathbf{v}_\perp \cdot \mathrm{d}\,\boldsymbol{\ell} + \frac{e}{c} \oint \mathbf{A} \cdot \mathrm{d}\,\boldsymbol{\ell} \quad \Leftarrow \quad \mathbf{P} = \mathbf{p} + \frac{e}{c}\mathbf{A}$$

$$= \oint \gamma\, m\, \omega_B\, a^2 \, \mathrm{d}\,\theta + \frac{e}{c} \oint \mathbf{A} \cdot \mathrm{d}\,\boldsymbol{\ell} = 2\,\pi\,\gamma\, m\, \omega_B\, a^2 + \frac{e}{c} \int_S \mathbf{B} \cdot \mathrm{d}\,\boldsymbol{a} \quad \Leftarrow \quad \mathbf{v}_\perp \| \mathrm{d}\,\boldsymbol{\ell}$$

$\Rightarrow \quad J = \pi \gamma m \omega_B a^2 = \frac{e}{c} B \pi a^2 \quad \Leftarrow \quad \mathrm{d}\boldsymbol{a}$ is antiparallel to $\mathbf{B} \quad \Leftarrow$ Lorentz force law

$B \pi a^2$ is the flux through the particle's orbit.

- If the particle moves through regions where $\mathbf{B}$ varies slowly, the adiabatic invariance of J means that the flux linked by the particle's orbit remains constant.

- If $\mathbf{B}$ increases, a will decrease so that $B \pi a^2$ remains unchanged.

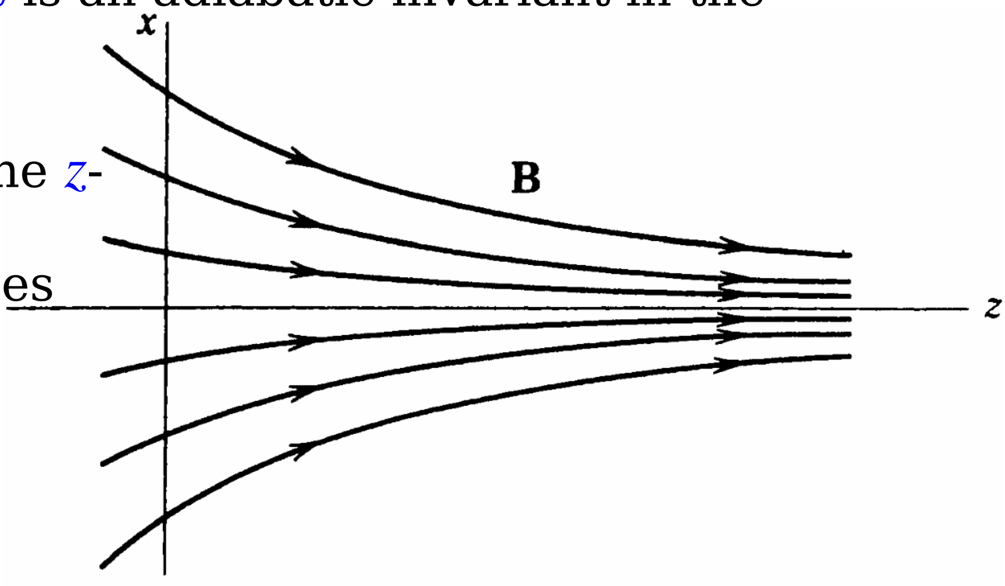
$$\left.\begin{matrix} B a^2 \\ \frac{p_\perp^2}{B} \\ \gamma \mu \end{matrix}\right] \text{ adiabatic invariants} \quad \Leftarrow \quad \mu = \frac{e \omega_B a^2}{2c} \text{ magnetic moment of the current loop}$$

- If $B = \text{const} \Rightarrow v = \text{const} \Rightarrow \text{energy=const} \Rightarrow \mu$ is itself an adiabatic invariant

- In time-varying fields or with static $\mathbf{E}$, μ is an adiabatic invariant in the nonrelativistic limit.

- Assume cylindrical symmetry. Besides the z-component of field there is a small radial component due to the curvature of the lines of force.

- A particle spirals around the z axis.

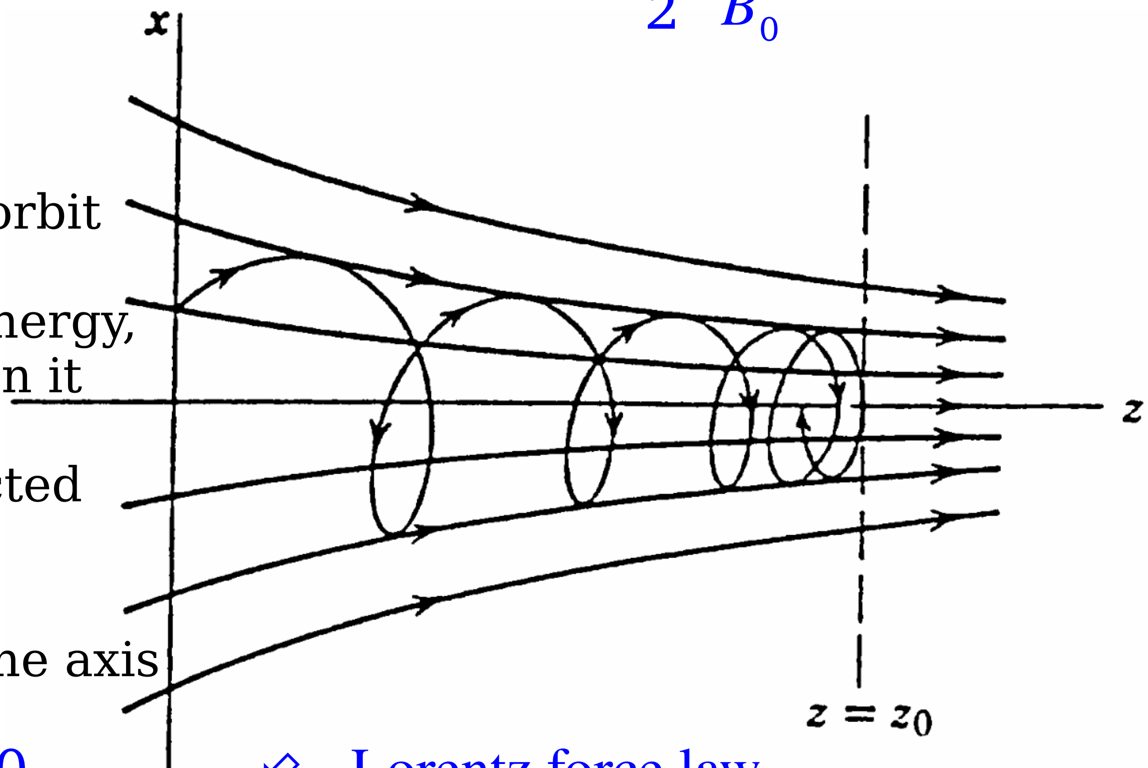


$$\mathbf{v} = v_\perp \hat{\boldsymbol{\rho}} + v_\parallel \hat{\mathbf{z}}$$
$$\mathbf{v}_0 = \mathbf{v}(z=0) = v_{\perp 0} \hat{\boldsymbol{\rho}} + v_{\parallel 0} \hat{\mathbf{z}} \quad \Rightarrow \quad v_\parallel^2 + v_\perp^2 = v_0^2 = v_{\parallel 0}^2 + v_{\perp 0}^2 = \text{const} \quad \Leftarrow \quad \text{no energy loss in a } B \text{ field}$$

$$\frac{v_\perp^2}{B} = \frac{v_{\perp 0}^2}{B_0} \quad \text{invariant} \quad \Rightarrow \quad v_\parallel^2 = v_0^2 - v_{\perp 0}^2 \frac{B(z)}{B_0} \qquad (1) \quad \Leftarrow \quad \begin{array}{c} B = B_z = \mathbf{B}\cdot\hat{\mathbf{z}} \\ \text{axial magnetic induction} \end{array}$$

$$\Rightarrow \quad \frac{1}{2} m v_\parallel^2 = \frac{1}{2} m v_0^2 - \frac{1}{2} m v_{\perp 0}^2 \frac{B(z)}{B_0} = \mathscr{E} - V(z) \quad \Leftarrow \quad V(z) = \frac{m}{2} \frac{v_{\perp 0}^2}{B_0} B(z)$$

$$\Rightarrow \quad \mathscr{E} = V(z_0) \quad \Rightarrow \quad v_\parallel(z_0) = 0$$



- The particle spirals in an tighter orbit along the lines of force, converting translational energy into rotation energy, until its axial velocity vanishes. Then it turns around, still spiraling in the same sense, and moves back, reflected by the magnetic field.

- The radial component of **B** near the axis

$$B_\rho(\rho, z) \simeq -\frac{\rho}{2} \frac{\partial B}{\partial z} \quad \Leftarrow \quad \nabla \cdot \mathbf{B} = 0 \qquad ⬃ \quad \text{Lorentz force law}$$

$$\Rightarrow \quad \ddot{z} = \frac{e(-\rho \dot{\phi} B_\rho)}{\gamma m c} \simeq \frac{e \rho^2 \dot{\phi}}{2 \gamma m c} \frac{\partial B}{\partial z} \simeq -\frac{v_{\perp 0}^2}{2 B_0} \frac{\partial B}{\partial z} \quad \Leftarrow \quad \rho^2 \dot{\phi} \simeq -(a^2 \omega_B)_0 = -\frac{v_{\perp 0}^2}{\omega_{B0}}$$

$$\Rightarrow \quad \frac{d\, v_\parallel^2}{2} \Leftarrow V_\parallel \, d\, v_\parallel \Leftarrow d\, z \frac{d}{d\, t} v_\parallel \Leftarrow \ddot{z}\, d\, z \simeq -\frac{v_{\perp 0}^2}{2\, B_0} d\, B \quad \Rightarrow \quad (1) \quad \Leftarrow \quad v_\parallel = \frac{d\, z}{d\, t}$$

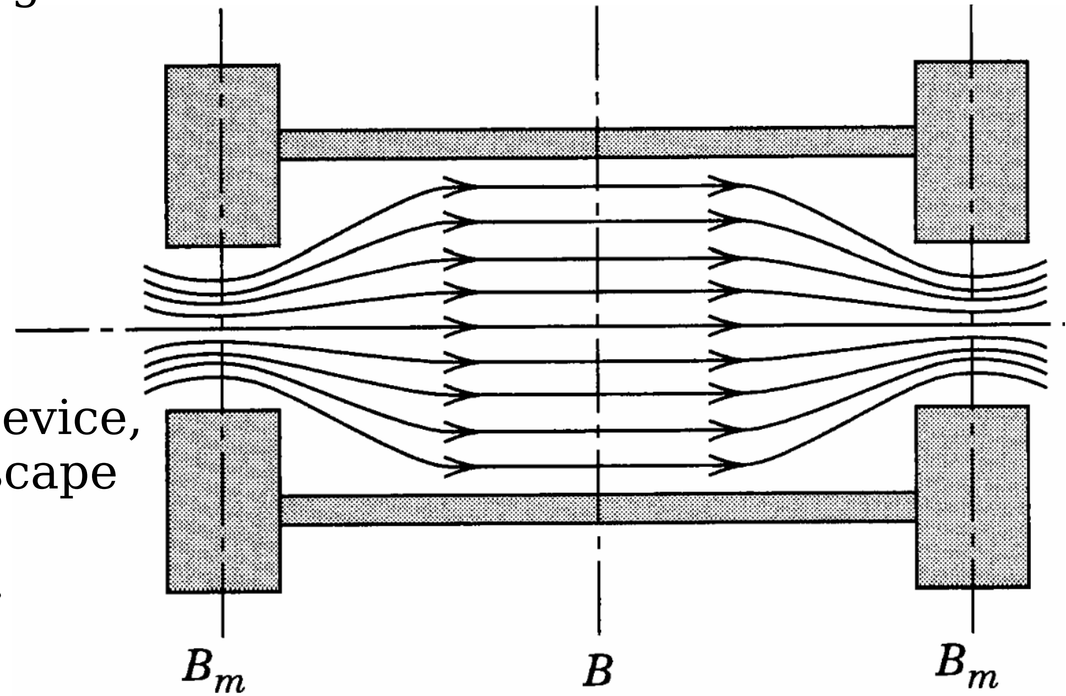
- To 1st order in small quantities the constancy of flux linking the orbit follows directly from the equations of motion.

- **Magnetic mirror**: charged particles are reflected by regions of strong $\mathbf{B}$.

- Particles created or injected into the field in the central region will spiral along the axis, but will be reflected by the magnetic mirrors at each end.

- The criterion for trapping

$$\left|\frac{v_{\parallel 0}}{v_{\perp 0}}\right| < \sqrt{\frac{B_m}{B} - 1} \quad \Leftarrow \quad v_\parallel^2 = v_0^2 - v_\perp^2 \frac{B(z)}{B_0}$$



- If the particles are injected into the device, it is easy to satisfy the criterion. The escape of particles depend on the rate of their scattering with residual gas atoms, etc.

- The adiabatic invariance of the flux linking an orbit is useful in particle motions in all types of spatially varying magnetic fields.

Lowest Order Relativistic Corrections to the Lagrangian for Interacting Charge Particles: The Darwin Lagrangian

- When the finite velocity of propagation of EM fields is taken into account, this isn't possible that the Lagrangian is a function of the instantaneous velocities and coordinates of all the particles, since the values of the potentials at one particle due to the other particles depend on their state of motion at "retarded" times.

- Consider a conventional Lagrangian of the interaction of two or more charged particles with each other, and it is possible only at nonrelativistic velocities.

- $L_{\text{int}}^{\text{NR}} = -\dfrac{q_1 q_2}{r} = -q_1 \Phi_{12} \;\Leftarrow\; r = |\mathbf{r}| = |\mathbf{r}_1 - \mathbf{r}_2|, \quad \boldsymbol{\beta}_i \equiv \dfrac{\mathbf{v}_i}{c}, \quad \boldsymbol{\beta} \equiv \dfrac{\mathbf{v}}{c}, \quad \begin{matrix} \beta_i = |\boldsymbol{\beta}_i| \\ \beta = |\boldsymbol{\beta}| \end{matrix}$

- To generalize beyond the static limit, we must determine both Φ_{12} and $\mathbf{A}_{12}$.

- In general there are relativistic corrections to Φ_{12} & $\mathbf{A}_{12}$. But in the *Coulomb gauge*, the scalar potential is given correctly to all orders in β by the Coulomb potential. Thus, all that needs to be considered is the vector potential $\mathbf{A}_{12}$.

- If only to the lowest order relativistic corrections, retardation effects can be neglected in computing $\mathbf{A}_{12}$ because $\mathbf{A}_{12}$ enters the Lagrangian in $q_1 \boldsymbol{\beta}_1 \cdot \mathbf{A}_{12}$. Since $\mathbf{A}_{12}$ is of the order of β_2, greater accuracy in computing $\mathbf{A}_{12}$ is unnecessary.

- The magnetostatic expression $\mathbf{A}_{12} \simeq \dfrac{1}{c} \displaystyle\int \dfrac{\boldsymbol{J}_t(\mathbf{r}')\, \mathrm{d}^3 x'}{|\mathbf{r}_1 - \mathbf{r}'|} \;\Leftarrow\; \boldsymbol{J}_t$ the transverse part of the current from q_2

$$\boldsymbol{J}_t(\mathbf{r}') = q_2 \mathbf{v}_2 \delta(\mathbf{r}' - \mathbf{r}_2) - \frac{q_2}{4\pi} \nabla' \frac{\mathbf{v}_2 \cdot (\mathbf{r}' - \mathbf{r}_2)}{|\mathbf{r}' - \mathbf{r}_2|^3} \quad \Leftarrow \boldsymbol{J}_t(\mathbf{r}) = \nabla \times \nabla \times \int \frac{\boldsymbol{J}(\mathbf{r}')\, d^3x'}{4\pi |\mathbf{r} - \mathbf{r}'|}$$

$$\boldsymbol{J}(\mathbf{r}) = q_2 \mathbf{v}_2 \delta(\mathbf{r} - \mathbf{r}_2)$$

$$\Rightarrow \quad \mathbf{A}_{12} \simeq \frac{q_2 \boldsymbol{\beta}_2}{r} - \frac{q_2}{4\pi} \int \frac{d^3x'}{|\mathbf{r}' - \mathbf{r}_1|} \nabla' \frac{\boldsymbol{\beta}_2 \cdot (\mathbf{r}' - \mathbf{r}_2)}{|\mathbf{r}' - \mathbf{r}_2|^3} \qquad ⬃ \quad \mathbf{s} = \mathbf{r}' - \mathbf{r}_2, \quad s = |\mathbf{s}|$$

$$= \frac{q_2 \boldsymbol{\beta}_2}{r} - \frac{q_2}{4\pi} \nabla_r \int \frac{\boldsymbol{\beta}_2 \cdot \mathbf{s}}{s^3} \frac{d^3 s}{|\mathbf{s} - \mathbf{r}|} = q_2 \left(\frac{\boldsymbol{\beta}_2}{r} - \nabla_r \frac{\boldsymbol{\beta}_2 \cdot \hat{\mathbf{r}}}{2} \right) = q_2 \frac{\boldsymbol{\beta}_2 + (\boldsymbol{\beta}_2 \cdot \hat{\mathbf{r}})\, \hat{\mathbf{r}}}{2r}$$

$$\Rightarrow \quad L_{\text{int}} = -q_1 \Phi_{12} + \frac{q_1}{c} \mathbf{v}_1 \cdot \mathbf{A}_{12} \simeq \frac{q_1 q_2}{r} \left(-1 + \frac{\boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2 + (\boldsymbol{\beta}_1 \cdot \hat{\mathbf{r}})(\boldsymbol{\beta}_2 \cdot \hat{\mathbf{r}})}{2} \right)$$

Darwin 1920

- Important in a quantum mechanics of relativistic corrections in 2-electron atom

- Breit interaction: replace the velocity vectors with their corresponding quantum-mechanical operators.

- For a system of particles, correct to order $\frac{1}{c^2}$ inclusive,

$$L_{\text{Darwin}} = \sum \frac{m_i v_i^2}{2} \left(1 + \frac{\beta_i^2}{4} \right) - \sum_{i \neq j} \frac{q_1 q_2}{2 r_{ij}} \left(1 - \frac{\boldsymbol{\beta}_i \cdot \boldsymbol{\beta}_j + (\boldsymbol{\beta}_i \cdot \hat{\mathbf{r}}_{ij})(\boldsymbol{\beta}_j \cdot \hat{\mathbf{r}}_{ij})}{2} \right)$$

⇩ no self-energy

- The Darwin Lagrangian has uses usually in the purely classical domain.

Due to the spherical symmetry, let $\mathbf{r} = r\,\hat{\mathbf{z}}$

$$\mathbf{s} = s\,\left[\sin\theta\,\left(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}\right) + \cos\theta\,\hat{\mathbf{z}}\right]$$

$$\boldsymbol{\beta}_2 = \beta_2\left[\sin\gamma\,\left(\cos\psi\,\hat{\mathbf{x}} + \sin\psi\,\hat{\mathbf{y}}\right) + \cos\gamma\,\hat{\mathbf{z}}\right]$$

$$\int \frac{\boldsymbol{\beta}_2\cdot\mathbf{s}}{s^3}\,\frac{\mathrm{d}^3 s}{|\mathbf{s}-\mathbf{r}|} = \int \frac{\beta_2\left[\cos\gamma\cos\theta + \sin\gamma\sin\theta\cos(\phi-\psi)\right]}{\sqrt{s^2+r^2-2\,r\,s\cos\theta}}\,\sin\theta\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= 2\pi\,\beta_2\cos\gamma\,\int_0^\infty \mathrm{d}s\int_{-1}^{+1}\frac{-t\,\mathrm{d}t}{\sqrt{s^2+r^2+2\,s\,r\,t}} \quad \Leftarrow \quad t \equiv -\cos\theta$$

$$= 2\pi\,\beta_2\cos\gamma\left(\int_0^r \frac{2\,s}{3\,r^2}\,\mathrm{d}s + \int_r^\infty \frac{2\,r}{3\,s^2}\,\mathrm{d}s\right)$$

$$= 2\pi\,\beta_2\cos\gamma = 2\pi\,\boldsymbol{\beta}_2\cdot\hat{\mathbf{r}}$$

$$\nabla_r\left(\boldsymbol{\beta}_2\cdot\hat{\mathbf{r}}\right) = \nabla\frac{\boldsymbol{\beta}_2\cdot\mathbf{r}}{r} = \nabla\frac{\beta_x\,x+\beta_y\,y+\beta_z\,z}{\sqrt{x^2+y^2+z^2}} = \frac{\boldsymbol{\beta}_2}{r} - \frac{\boldsymbol{\beta}_2\cdot\hat{\mathbf{r}}}{r}\,\hat{\mathbf{r}}$$

$$L_{\text{free}} = -m\,c^2\sqrt{1-\frac{u^2}{c^2}} = -m\,c^2 + \frac{m}{2}\,u^2 + \frac{m}{8\,c^2}\,u^4 + \cdots$$

Lagrangian for the EM Field

- The Lagrangian approach to continuous fields closely parallels the techniques used for discrete point particles. The finite number of coordinates are replaced by an infinite number of degrees of freedom. Each point in space-time corresponds to a finite number of values of the discrete index. The generalized coordinate is replaced by a continuous field. The generalized velocity is replaced by the 4-vector gradient

$$i \to x^\alpha, k \qquad q_i \to \phi_k(x) \qquad \dot{q}_i \to \partial^\alpha \phi_k(x) \quad \Rightarrow \quad L = \sum L_i(q_i, \dot{q}_i) \to \int \mathscr{L}(\phi_k, \partial^\alpha \phi_k)\, d^3x$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \to \partial^\alpha \frac{\partial \mathscr{L}}{\partial \partial^\alpha \phi_k} = \frac{\partial \mathscr{L}}{\partial \phi_k} \qquad \Leftarrow \quad \mathscr{L}: \text{Lagrangian } \textit{density}$$

$$\Rightarrow \quad \text{Action } \mathcal{A} = \iint \mathscr{L}\, d^3x\, dt = \frac{1}{c}\int \mathscr{L}\, d^4x$$

- The Lorentz-invariant nature of the action is preserved provided the Lagrangian density is a *Lorentz scalar* since the 4d volume element is invariant.

- Expect the free-field Lagrangian to be quadratic in the velocity, a scalar under proper Lorentz transformations, and the interaction involves the source densities

$$\Rightarrow \quad \mathscr{L} = -\frac{1}{16\pi}\mathbb{F}_{\alpha\beta}\mathbb{F}^{\alpha\beta} - \frac{J_\alpha A^\alpha}{c} = -\frac{1}{16\pi}\mathbb{F}^2 - \frac{1}{c}\vec{J}\cdot\vec{A}$$

$$= -\frac{g_{\mu\lambda} g_{\nu\sigma}}{16\pi}\left(\partial^\lambda A^\sigma - \partial^\sigma A^\lambda\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - \frac{J_\alpha A^\alpha}{c}$$

$$\Rightarrow \frac{\partial \mathscr{L}}{\partial \partial^{\beta} A^{\alpha}} = -\frac{g_{\mu\lambda} g_{\nu\sigma}}{16\pi} [(\delta_{\beta}{}^{\lambda} \delta_{\alpha}{}^{\sigma} - \delta_{\beta}{}^{\sigma} \delta_{\alpha}{}^{\lambda}) \mathbb{F}^{\mu\nu} + (\delta_{\beta}{}^{\mu} \delta_{\alpha}{}^{\nu} - \delta_{\beta}{}^{\nu} \delta_{\alpha}{}^{\mu}) \mathbb{F}^{\lambda\sigma}] = \frac{\mathbb{F}_{\alpha\beta}}{4\pi}$$

$$\frac{\partial \mathscr{L}}{\partial A^{\alpha}} = -\frac{1}{c} J_{\alpha}$$

$$\partial^{\beta} \frac{\partial \mathscr{L}}{\partial \partial^{\beta} A^{\alpha}} = \frac{\partial \mathscr{L}}{\partial A^{\alpha}} \quad \Rightarrow \quad \partial^{\beta} \mathbb{F}_{\beta\alpha} = \frac{4\pi}{c} J_{\alpha} \quad (\#)$$

$$\boldsymbol{\partial} \cdot \boldsymbol{\mathbb{F}} = \frac{4\pi}{c} \vec{J}$$

inhomogeneous Maxwell equations

- The definition of the field strength tensor $\boldsymbol{\mathbb{F}}$ in terms of the 4-vector potential A^{λ} was chosen so that the homogeneous equations were satisfied automatically.

$$\partial_{\alpha} \mathscr{F}^{\alpha\beta} = \frac{1}{2} \partial_{\alpha} (\epsilon^{\alpha\beta\mu\nu} \mathbb{F}_{\mu\nu}) = \partial_{\alpha} (\epsilon^{\alpha\beta\mu\nu} \partial_{\mu} A_{\nu}) = \epsilon^{\alpha\beta\mu\nu} \partial_{\alpha} \partial_{\mu} A_{\nu} = 0 \quad \Rightarrow \quad \boldsymbol{\partial} \cdot \boldsymbol{\mathscr{F}} = 0$$

- The conservation of the source current density can be obtained from (#)

$$0 \Leftarrow \partial^{\alpha} \partial^{\beta} \mathbb{F}_{\beta\alpha} = \frac{4\pi}{c} \partial^{\alpha} J_{\alpha} \quad \Rightarrow \quad \partial^{\alpha} J_{\alpha} = 0$$

$$0 \Leftarrow \boldsymbol{\partial} \cdot (\boldsymbol{\partial} \cdot \boldsymbol{\mathbb{F}}) = \frac{4\pi}{c} \boldsymbol{\partial} \cdot \vec{J} \quad \Rightarrow \quad \boldsymbol{\partial} \cdot \vec{J} = 0$$

Proca Lagrangian; Photon Mass Effects

- The conventional Maxwell equations and the Lagrangian are based on the hypothesis that the photon has zero mass.

- The *Proca Lagrangian*: add a "mass" term into the Lagrangian

$$\mathscr{L}_{\text{Proca}} = -\frac{1}{16\pi}\mathbb{F}^2 + \frac{\mu^2}{8\pi}\vec{A}^2 - \frac{1}{c}\vec{J}\cdot\vec{A} \quad \Leftarrow \quad \mu = \frac{m_\gamma c}{\hbar} \text{ reciprocal Compton wavelength of photon}$$

$$\Rightarrow \quad \partial\cdot\mathbb{F} + \mu^2\vec{A} = \frac{4\pi}{c}\vec{J}, \quad \partial\cdot\mathscr{F} = 0$$

- In the Proca equations the potentials as well as the fields enter, thus the potentials acquire real physical significance through the mass term.

- Lorenz gauge $\partial\cdot\vec{A} = 0 \quad \Rightarrow \quad \partial\cdot\mathbb{F} + \mu^2\vec{A} = \Box\vec{A} + \mu^2\vec{A} = \frac{4\pi}{c}\vec{J} \quad \Rightarrow \quad \partial\cdot\vec{J} = 0$

$$\Rightarrow \quad \nabla^2\vec{A} - \mu^2\vec{A} = -\frac{4\pi}{c}\vec{J} \quad \text{static limit} \quad \Rightarrow \quad A_\alpha = \delta_\alpha^0\,\Phi(x) \text{ for a rest point charge}$$

$$\Rightarrow \quad \Phi(x) = q\frac{e^{-\mu r}}{r} \quad \Leftarrow \quad \text{spherically symmetric Yukawa form}$$

- The exponential factor alters the character of the earth's $\mathbf{B}$ sufficiently to permit us to set quite stringent limits on the photon mass. [Chap. 0, Prob. 12.15]

- absence of sources $\Rightarrow$ $\Box \vec{A} + \mu^2 \vec{A} = 0$ $\Rightarrow$ $\omega^2 = c^2 k^2 + \mu^2 c^2$ $\Leftarrow$ $\left(\frac{\mathscr{E}}{\hbar}\right)^2$
- Consider some resonant system (cavity/lumped circuit) $\omega^2 = \omega_0^2 + \mu^2 c^2$ $\Leftarrow \omega_0 = c\, k$
- Measure the difference between ω and ω_0 in a circuit for a given photon mass.
- However, lumped circuits are incapable of setting any limit on the photon mass.
- For a solid conducting sphere of radius a at the center of a hollow conducting shell of inner radius b held at zero potential, the capacitance is increased by

$$\Delta C = \frac{\mu^2 a^2 b}{3} \text{ for } \mu b \ll 1 \quad \Rightarrow \quad \frac{\Delta \omega}{\omega_0} \simeq \frac{\mu^2 c^2}{2\omega_0^2} \quad \Rightarrow \quad \frac{\Delta \omega}{\omega_0} = O\left(\mu^2 d^2\right) \quad \Leftarrow \quad \omega_0 = \frac{1}{\sqrt{L\, C}}$$

very insensitive in practice to a possible photon mass.

- For μ=0 the TEM modes of a transmission line are degenerate modes, with the phase velocity=c. The situation does not alter if $\mu \neq 0$, but now the transverse behavior of the fields is governed by $(\nabla_t^2 - \mu^2)\psi = 0$ instead of the Laplace eqn.
- C & L of the transmission line are altered by the order $\mu^2 d^2$, but nothing else.
- TM modes propagate with the naive eqn, but TE modes propagate differently.
- For Schumann resonances the 1[st] order of $\frac{h}{R}$ is modified with $\mu \neq 0$.

Effective "Photon" Mass in Superconductivity; London Penetration Depth

- **Meissner effect**: the expulsion of $\mathbf{B}$ from the interior of a superconductor as it transits from the normal state ($T > T_c$) to the superconducting state ($T < T_c$).

- If $\mathbf{B}$ is applied after the material is superconducting, it penetrates a very small distance called the London penetration depth λ_L (~10 nm).

- Being a perfect conductor, a superconductor is perfectly diamagnetic.

- $$\boldsymbol{J} = q\, n\, \mathbf{v} = q\, n \frac{\mathbf{P}}{m} - \frac{n\, q^2}{m\, c}\mathbf{A} \quad \Leftarrow \quad \mathbf{P} = m\,\mathbf{v} + \frac{q}{c}\mathbf{A} \qquad \Leftarrow \quad \mathscr{L}_{\text{eff}} = -\frac{\mathbb{F}^2}{16\,\pi} + \frac{n\, q^2}{2\, m\, c^2}\vec{A}^2$$

The superconducting state is a coherent state of the charge carriers with $\mathbf{P} = 0$

$$\Rightarrow \quad \boldsymbol{J} = -\frac{n\, q^2}{m\, c}\mathbf{A}$$

$$\Box\, \vec{A} = \frac{4\,\pi}{c}\vec{J} \quad \Rightarrow \quad \nabla^2 \mathbf{A} - \partial_0^2 \mathbf{A} - \mu^2 \mathbf{A} = 0 \quad \Leftarrow \quad \mu^2 = 8\,\pi \frac{n\, q^2}{2\, m\, c^2} \quad \Leftarrow \quad \partial_0 = \frac{1}{c}\frac{\partial}{\partial\, t}$$

- No current flows across the interface between normal and superconducting media, so the normal component of $\mathbf{A}$, ie, $A_\perp$, vanishes (exponentially).

- For $\partial_t\, \mathbf{A} = 0$ + planar symmetry $\Rightarrow \quad A_\perp \propto e^{\pm \mu\, x} \quad \Rightarrow \quad \lambda_L = \frac{1}{\mu} = \frac{c}{|q|}\sqrt{\frac{m}{4\,\pi\, n}}$

$\Rightarrow$ the effective photon mass $m_{\gamma,\,\text{eff}} = \dfrac{\hbar}{\lambda_L c}$ + Bohr radius $a_0 = \dfrac{4\pi\epsilon_0 \hbar^2}{m_e e^2}$ $\Leftarrow$ $\epsilon_0 = \dfrac{1}{4\pi}$ Gaussian unit

$$\Rightarrow \quad m_{\gamma,\,\text{eff}}\, c^2 = \frac{|q|}{e}\sqrt{\frac{4\pi n a_0^3 m_e}{m}}\,\frac{e^2}{a_0} \sim \text{few eVs} \quad \Leftarrow \quad \text{the order of the Rydberg energy}$$

- The charge carriers in low-temperature superconductors are (the Cooper) pairs of electrons loosely bound by a 2nd-order interaction through lattice phonons

$q = -2e$, $m = 2m_e$, $n = \dfrac{n_{\text{eff}}}{2} = O\left(10^{22}\,\text{cm}^3\right) \Rightarrow \lambda_L = O\left(4\times 10^{-6}\,\text{cm}\right)$ $\Leftarrow$ $\mu^2 = 8\pi r_0 n$, $r_0 =$ electron radius

- BCS theory $\Rightarrow$ $n(T=0) = \dfrac{n_{\text{eff}}}{2} = \dfrac{2}{3}\mathscr{E}_F N(0)$ $\Leftarrow$ $\mathscr{E}_F$: the Fermi energy of the valence band; $N(0)$: density of state

- In high-temperature superconductors penetration depths are found to be an order of magnitude smaller than in conventional superconductors.

- Measurements of $\lambda_L(T)$ can be done by incorporating the superconductor into a resonant circuit and studying the shift in resonant frequency with change in temperature. $Z_s \approx -i\,\dfrac{8\pi^2}{c}\dfrac{\lambda_L}{\lambda}$ (Gaussian units) $= -i\,\dfrac{2\pi\lambda_L}{\lambda} Z_0$ (SI units) problem 12.20

- The impedance is inductive, corresponding to an inductance per unit area, $L = \mu_0 \lambda_L$.

Canonical and Symmetric Stress Tensors; Conservation Laws

A. Generalization of the Hamiltonian: Canonical Stress Tensor

- $p_i = \dfrac{\partial L}{\partial \dot{q}^i} \quad \Rightarrow \quad H = p_i \dot{q}^i - L \quad \Rightarrow \quad \dfrac{\mathrm{d} H}{\mathrm{d} t} = 0 \quad \text{if} \quad \dfrac{\partial L}{\partial t} = 0$

- Hamiltonian density: $H = \int \mathscr{H} \, \mathrm{d}^3 x$

- Since the energy of a particle is the time component of a 4-vector, H should transform in the same way. Since the invariant 4-volume element is $\mathrm{d}^4 x = \mathrm{d}^3 x \, \mathrm{d} x^0$, $\mathscr{H}$ transform as the time-time component of a 2nd-rank tensor.

$$\Rightarrow \quad \mathscr{H} = \frac{\partial \mathscr{L}}{\partial \partial_t \phi_k} \partial_t \phi_k - \mathscr{L} \quad \text{vs} \quad H = p_i \dot{q}^i - L$$

- The inferred Lorentz transformation properties of $\mathscr{H}$ suggest that the *covariant generalization of the Hamiltonian density* is the *canonical stress tensor*:

$$\mathbb{T}^{\alpha\beta} \equiv \frac{\partial \mathscr{L}}{\partial \partial_\alpha \phi_k} \partial^\beta \phi_k - g^{\alpha\beta} \mathscr{L}$$

- For the free EM field Lagrangian $\mathscr{L}_{\mathrm{em}} = -\dfrac{\mathbb{F}_{\mu\nu} \mathbb{F}^{\mu\nu}}{16\pi} = -\dfrac{1}{16\pi} \mathbb{F}^2 = \dfrac{E^2 - B^2}{8\pi}$

$$\Rightarrow \quad \mathbb{T}^{\alpha\beta} \equiv \frac{\partial \mathscr{L}_{\mathrm{em}}}{\partial \partial_\alpha A^\lambda} \partial^\beta A^\lambda - g^{\alpha\beta} \mathscr{L}_{\mathrm{em}} = -\frac{1}{4\pi} \mathbb{F}^\alpha{}_\lambda \partial^\beta A^\lambda - g^{\alpha\beta} \mathscr{L}_{\mathrm{em}}$$

$$8\pi\,\mathbb{T}^{00} = \left(E^2 + B^2\right) + 2\,\nabla\cdot\left(\Phi\,\mathbf{E}\right)$$

$$\Rightarrow\quad 4\pi\,\mathbb{T}^{0\,i} = \left(\mathbf{E}\times\mathbf{B}\right)^i + \nabla\cdot\left(A^i\,\mathbf{E}\right) \quad\Leftarrow\quad \nabla\cdot\mathbf{E} = 0 \quad \& \quad \nabla\times\mathbf{B} = \partial_0\,\mathbf{E}$$

$$4\pi\,\mathbb{T}^{i\,0} = \left(\mathbf{E}\times\mathbf{B} + \nabla\times\Phi\,\mathbf{B}\right)^i - \partial_0\left(\Phi\,E^i\right)$$

- Suppose that the fields are localized in some finite region of space

$$\int \mathbb{T}^{00}\,\mathrm{d}^3x = \frac{1}{8\pi}\int\left(E^2 + B^2\right)\mathrm{d}^3x = \mathscr{E}_{\text{field}} \quad\Leftarrow\quad \text{total energy of the fields}$$

$$\int \mathbb{T}^{0\,i}\,\mathrm{d}^3x = \frac{1}{4\pi}\int\left(\mathbf{E}\times\mathbf{B}\right)^i\mathrm{d}^3x = c\,P^i_{\text{field}} \quad\Leftarrow\quad \text{linear momentum of the fields}$$

- The differential conservation statement: $\partial_\alpha\,\mathbb{T}^{\alpha\,\beta} = 0 \quad\Rightarrow\quad \boldsymbol{\partial}\cdot\mathbb{T} = 0$

$$\partial_\alpha\,\mathbb{T}^{\alpha\,\beta} = \partial_\alpha\left(\frac{\partial\,\mathscr{L}}{\partial\,\partial_\alpha\,\phi_k}\,\partial^\beta\,\phi_k\right) - \partial^\beta\,\mathscr{L} = \partial^\beta\,\phi_k\,\partial_\alpha\,\frac{\partial\,\mathscr{L}}{\partial\,\partial_\alpha\,\phi_k} + \frac{\partial\,\mathscr{L}}{\partial\,\partial_\alpha\,\phi_k}\,\partial_\alpha\,\partial^\beta\,\phi_k - \partial^\beta\,\mathscr{L}$$

$$= \frac{\partial\,\mathscr{L}}{\partial\,\phi_k}\,\partial^\beta\,\phi_k + \frac{\partial\,\mathscr{L}}{\partial\,\partial_\alpha\,\phi_k}\,\partial^\beta\,\partial_\alpha\,\phi_k - \partial^\beta\,\mathscr{L} = \partial^\beta\,\mathscr{L}\left(\phi_k\,,\,\partial^\alpha\,\phi_k\right) - \partial^\beta\,\mathscr{L} = 0$$

- The conservation law or continuity equation yields the conservation of total energy and momentum upon integration over all of 3-space at fixed time

$$0 = \int \partial_\alpha\,\mathbb{T}^{\alpha\,\beta}\,\mathrm{d}^3x = \frac{1}{c}\,\frac{\mathrm{d}}{\mathrm{d}\,t}\int \mathbb{T}^{0\,\beta}\,\mathrm{d}^3x + \int \partial_i\,\cancel{\mathbb{T}^{i\,\beta}}\,\mathrm{d}^3x \quad\Rightarrow\quad \frac{\mathrm{d}}{\mathrm{d}\,t}\,\mathscr{E}_{\text{field}} = 0\,,\quad \frac{\mathrm{d}}{\mathrm{d}\,t}\,\mathbf{P}_{\text{field}} = 0$$

- One expects that a covariant integral statement is also possible, ie, the derivation of conservation integrals is valid in all frames.

B. Symmetric Stress Tensor

- Deficiencies (1) $\mathbb{T}^{00}$ & $\mathbb{T}^{0i}$ differ from the usual expressions for $\mathscr{E}$ & P densities.

  (2) lack of symmetry

  (3) it involves the potentials explicitly, and so is not gauge invariant

  (4) its trace is not zero, as required for zero-mass photons.

- The angular momentum of the field $\mathbf{L}_{\text{field}} = \dfrac{1}{4\pi c}\displaystyle\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})\,\mathrm{d}^3 x$

- Its covariant generalization—3rd-rank tensor $\mathbb{M}^{\alpha\beta\gamma} = \mathbb{T}^{\alpha\beta} x^{\gamma} - \mathbb{T}^{\alpha\gamma} x^{\beta}$

$$\partial_\alpha \mathbb{M}^{\alpha\beta\gamma} = 0 \quad\Rightarrow\quad \boldsymbol{\partial}\cdot\mathbb{M} = 0 \quad\Rightarrow\quad \text{conservation of the total angular momentum of the field}$$

$$\Rightarrow\quad 0 = x^{\gamma}\partial_\alpha \mathbb{T}^{\alpha\beta} + \mathbb{T}^{\gamma\beta} - x^{\beta}\partial_\alpha \mathbb{T}^{\alpha\gamma} - \mathbb{T}^{\beta\gamma} = \mathbb{T}^{\gamma\beta} - \mathbb{T}^{\beta\gamma}$$

$$\Rightarrow\quad \text{conservation of angular momentum requires } \mathbb{T}^{\alpha\beta} = \mathbb{T}^{(\alpha\beta)}$$

- $\mathbb{T}^{\alpha\beta} = -\dfrac{\mathbb{F}^{\alpha}{}_{\lambda}}{4\pi}\partial^{\beta}A^{\lambda} - g^{\alpha\beta}\mathscr{L}_{\text{em}} = \dfrac{1}{4\pi}\left(\mathbb{F}^{\alpha}{}_{\lambda}\mathbb{F}^{\lambda\beta} + \dfrac{1}{4}g^{\alpha\beta}\mathbb{F}^2\right) - \dfrac{\mathbb{F}^{\alpha}{}_{\lambda}}{4\pi}\partial^{\lambda}A^{\beta}$

$$\Rightarrow\quad \mathbb{T}_D^{\alpha\beta} \equiv -\frac{\mathbb{F}^{\alpha}{}_{\lambda}}{4\pi}\partial^{\lambda}A^{\beta} = \frac{\mathbb{F}^{\lambda\alpha}}{4\pi}\partial_{\lambda}A^{\beta} = \frac{\mathbb{F}^{\lambda\alpha}\partial_\lambda A^{\beta} + A^{\beta}\partial_\lambda \mathbb{F}^{\lambda\alpha}}{4\pi} \quad\Leftarrow\quad \boldsymbol{\partial}\cdot\mathbb{F} = 0$$

$$= \frac{1}{4\pi}\partial_\lambda\left(\mathbb{F}^{\lambda\alpha}A^{\beta}\right) \quad\Rightarrow\quad \boldsymbol{\partial}\cdot\mathbb{T}_D = 0\,, \quad \int \mathbb{T}_D^{0\beta}\,\mathrm{d}^3 x = 0$$

$\Rightarrow$ symmetric stress tensor

$$\Theta^{\alpha\beta} = \mathbb{T}^{\alpha\beta} - \mathbb{T}_D^{\alpha\beta} = \frac{1}{4\pi}\left(\mathbb{F}^{\alpha}{}_{\lambda}\,\mathbb{F}^{\lambda\beta} + \frac{1}{4}\,\mathbb{g}^{\alpha\beta}\,\mathbb{F}_{\mu\nu}\,\mathbb{F}^{\mu\nu} \right)$$

$$\boldsymbol{\Theta} = \mathbf{T} - \mathbf{T}_D = \frac{1}{4\pi}\left(\mathbb{F}\cdot\mathbb{F} + \frac{1}{4}\,\mathbb{g}\,\mathbb{F}^2 \right)$$

$\Theta^{00} = \dfrac{1}{8\pi}\left(E^2 + B^2\right) \Rightarrow u$: energy density

$\Rightarrow$ $\Theta^{0i} = \dfrac{1}{4\pi}\left(\mathbf{E}\times\mathbf{B}\right)^i \Rightarrow c\,g^i$: momentum density

$\Theta^{ij} = -\dfrac{1}{4\pi}\left(E^i E^j + B^i B^j - \dfrac{1}{2}\,\delta^{ij}\left(E^2 + B^2\right) \right) = -\mathbb{t}_{ij}$: Maxwell stress tnesor

$\Rightarrow$ $\Theta^{\alpha\beta} = \begin{bmatrix} u & c\,\mathbf{g} \\ c\,\mathbf{g} & -\mathbb{t} \end{bmatrix}$, $\Theta_{\alpha\beta} = \begin{bmatrix} u & -c\,\mathbf{g} \\ -c\,\mathbf{g} & -\mathbb{t} \end{bmatrix}$, $\Theta^{\alpha}{}_{\beta} = \begin{bmatrix} u & -c\,\mathbf{g} \\ c\,\mathbf{g} & \mathbb{t} \end{bmatrix}$, $\Theta_{\alpha}{}^{\beta} = \begin{bmatrix} u & c\,\mathbf{g} \\ -c\,\mathbf{g} & \mathbb{t} \end{bmatrix}$

$\Rightarrow$ $\partial\cdot\boldsymbol{\Theta} = 0 \Rightarrow$ $0 = \partial_\alpha\,\Theta^{\alpha 0} = \dfrac{1}{c}\left(\partial_t\,u + \nabla\cdot\mathbf{S}\right) \Leftarrow \mathbf{S} = c^2\,\mathbf{g}$: Poynting vector

$0 = \partial_\alpha\,\Theta^{\alpha i} = \partial_t\,g^i - \partial^j\,\mathbb{t}_{ij} \Leftarrow (6.121)$ $\Leftarrow$ both for source-free

$\Rightarrow$ $\mathbb{M}^{\alpha\beta\gamma} = \Theta^{\alpha\beta}\,x^\gamma - \Theta^{\alpha\gamma}\,x^\beta \Rightarrow \partial\cdot\mathbb{M} = 0 \Leftarrow$ angular momentum conservation

$\Rightarrow$ The conservation of $\mathbb{M}^{0\beta\gamma}$ is a statement on the center of mass motion

C. Conservation Laws for EM Fields Interacting with Charged Particles

- In the presence of external sources $\mathscr{L} = -\frac{1}{16\pi}\mathbb{F}^2 - \frac{1}{c}\vec{J}\cdot\vec{A}$

- The symmetric stress tensor for the EM field retains its form, but the coupling to the source current makes its divergence nonvanishing.

$$4\pi\,\partial_\alpha\Theta^{\alpha\beta} = \partial^\mu\left(\mathbb{F}_{\mu\nu}\mathbb{F}^{\nu\beta}\right) + \frac{1}{4}\partial^\beta\mathbb{F}^2 = \mathbb{F}^{\nu\beta}\partial^\mu\mathbb{F}_{\mu\nu} + \mathbb{F}_{\mu\nu}\partial^\mu\mathbb{F}^{\nu\beta} + \frac{\mathbb{F}_{\mu\nu}}{2}\partial^\beta\mathbb{F}^{\mu\nu}$$

$$\Rightarrow \quad \partial_\alpha\Theta^{\alpha\beta} + \frac{1}{c}\mathbb{F}^{\beta\nu}J_\nu = \frac{1}{8\pi}\mathbb{F}_{\mu\nu}\left(\partial^\mu\mathbb{F}^{\nu\beta} + \partial^\mu\mathbb{F}^{\nu\beta} + \partial^\beta\mathbb{F}^{\mu\nu}\right) \quad \Leftarrow \quad \frac{1}{4\pi}\partial\cdot\mathbb{F} = \frac{1}{c}\vec{J}$$

$$= \frac{1}{8\pi}\mathbb{F}_{\mu\nu}\left(\partial^\mu\mathbb{F}^{\nu\beta} + \partial^\nu\mathbb{F}^{\mu\beta}\right) = 0 \quad \Leftarrow \quad \partial^\mu\mathbb{F}^{\nu\beta} + \partial^\beta\mathbb{F}^{\mu\nu} + \partial^\nu\mathbb{F}^{\beta\mu} = 0 \quad \Leftarrow \quad \partial\cdot\mathscr{F} = 0$$

$$\Rightarrow \quad \partial\cdot\boldsymbol{\Theta} = -\frac{1}{c}\mathbb{F}\cdot\vec{J} \quad \Rightarrow \quad \begin{aligned} &\partial_t u + \nabla\cdot\mathbf{S} = -\boldsymbol{J}\cdot\mathbf{E} \\ &\partial_t\mathbf{g} - \nabla\cdot\mathbb{t} = -\rho\,\mathbf{E} - \frac{\boldsymbol{J}\times\mathbf{B}}{c} \end{aligned} \quad \Leftarrow$$

conservation of energy & momentum for EM fields with sources

$$\vec{J} = (c\,\rho, \boldsymbol{J})$$

$$\Rightarrow \quad \vec{f} \equiv \frac{1}{c}\mathbb{F}\cdot\vec{J} = \frac{1}{c}\left(\boldsymbol{J}\cdot\mathbf{E},\, c\,\rho\,\mathbf{E} + \boldsymbol{J}\times\mathbf{B}\right)$$ *Lorentz force density*

For the sources are discrete $\int \vec{f}\,\mathrm{d}^3x = \frac{\mathrm{d}}{\mathrm{d}t}\vec{P}_{\text{particle}} \quad \Leftarrow \quad \frac{\mathrm{d}\,\mathbf{p}}{\mathrm{d}\,t} = e\left(\mathbf{E} + \frac{\mathbf{u}}{c}\times\mathbf{B}\right), \quad \frac{\mathrm{d}\,\mathscr{E}}{\mathrm{d}\,t} = e\,\mathbf{u}\cdot\mathbf{E}$

$$\Rightarrow \int \left(\partial \cdot \boldsymbol{\Theta} + \vec{f}\right) \mathrm{d}^3 x = \frac{\mathrm{d}}{\mathrm{d}\, t}\left(\vec{P}_{\text{field}} + \vec{P}_{\text{particle}}\right) = 0 \quad \Leftarrow$$ conservation of 4-momentum for the system of particles and fields

- A more equitable treatment of a combined system of particles and fields

$$\mathscr{L} = \mathscr{L}_{\text{free-field}} + \mathscr{L}_{\text{free-particle}} + \mathscr{L}_{\text{interaction}}$$

Selected Problems: 5, 9, 14

Solution of the Wave Equation in Covariant Form; Invariant Green Functions

- $\partial \cdot \mathbb{F} = \frac{4\pi}{c} \vec{J} \quad \Rightarrow \quad \Box \vec{A} - \partial (\partial \cdot \vec{A}) = \Box \vec{A} = \frac{4\pi}{c} \vec{J}(\vec{r}) \quad \Leftarrow \quad \partial \cdot \vec{A} = 0$
- The solution can be accomplished by finding a Green function for the equation

$$\Box D(\vec{r}, \vec{r}') = \delta(\vec{r} - \vec{r}') = \delta(t - t')\,\delta(\mathbf{r} - \mathbf{r}')$$

- In the absence of boundary surfaces, the Green function can depend only on the 4-vector difference

$$\vec{\mathbb{s}} = \vec{r} - \vec{r}' \quad \Rightarrow \quad D(\vec{r}, \vec{r}') = D(\vec{r} - \vec{r}') = D(\vec{\mathbb{s}}) \quad \Rightarrow \quad \Box_{\mathbb{s}} D(\vec{\mathbb{s}}) = \delta(\vec{\mathbb{s}})$$

$$\Rightarrow \quad D(\vec{\mathbb{s}}) = \frac{1}{(2\pi)^4} \int \widetilde{D}(\vec{k})\, e^{-i\vec{k}\cdot\vec{\mathbb{s}}}\, \mathrm{d}^4 k \quad \Leftarrow \quad \vec{k}\cdot\vec{\mathbb{s}} = k_0\, \mathbb{s}_0 - \boldsymbol{k}\cdot\mathbb{s}$$

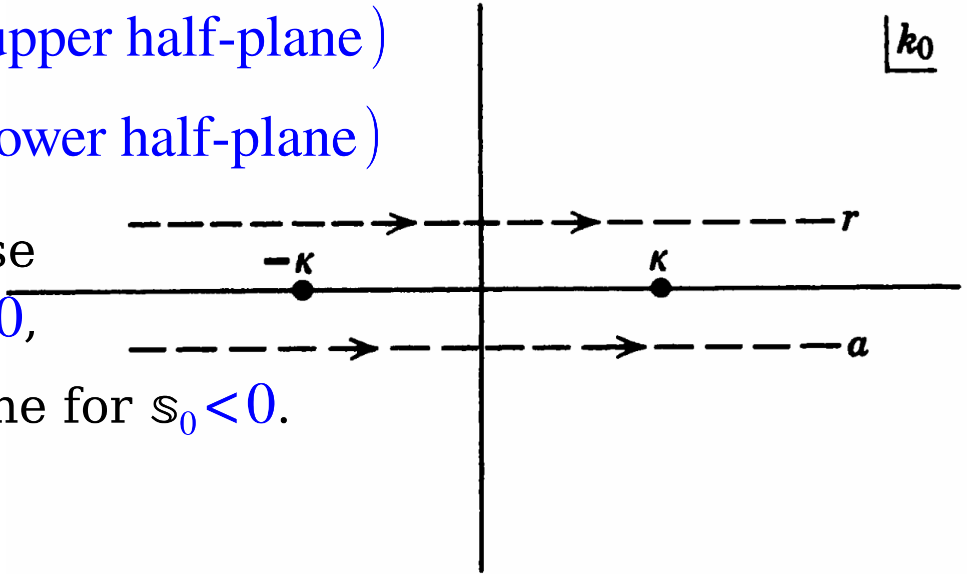
$$\Rightarrow \quad \widetilde{D}(\vec{k}) = -\frac{1}{\vec{k}\cdot\vec{k}} \quad \Leftarrow \quad \delta(\vec{\mathbb{s}}) = \frac{1}{(2\pi)^4} \int e^{-i\vec{k}\cdot\vec{\mathbb{s}}}\, \mathrm{d}^4 k \qquad \Downarrow \quad \kappa = |\boldsymbol{k}|$$

$$\Rightarrow \quad D(\vec{\mathbb{s}}) = -\frac{1}{(2\pi)^4} \int \frac{e^{-i\vec{k}\cdot\vec{\mathbb{s}}}}{\vec{k}\cdot\vec{k}}\, \mathrm{d}^4 k = -\frac{1}{(2\pi)^4} \int e^{i\boldsymbol{k}\cdot\mathbb{s}}\, \mathrm{d}^3 k \int_{-\infty}^{+\infty} \frac{e^{-i k_0 \mathbb{s}_0}}{k_0^2 - \kappa^2}\, \mathrm{d} k_0$$

- The k_0 integrand has 2 simple poles at $k_0 = \pm\kappa$.

- Green functions that differ in their behavior are obtained by choosing different contours of integration relative to the poles.

- $\mathbb{s}_0 > 0 \;\Rightarrow\; e^{-i k_0 \mathbb{s}_0} \to \infty$ for $\Im[k_0] > 0$ (upper half-plane)
  $\mathbb{s}_0 < 0 \;\Rightarrow\; e^{-i k_0 \mathbb{s}_0} \to \infty$ for $\Im[k_0] < 0$ (lower half-plane)



- To use the residue theorem, we must close the contour in the lower half-plane for $\mathbb{s}_0 > 0$, and close the contour in the upper half-plane for $\mathbb{s}_0 < 0$.

- Consider the contour r

$$\mathbb{s}_0 < 0 \;\Rightarrow\; \oint_r \frac{e^{-i k_0 \mathbb{s}_0}}{k_0^2 - \kappa^2} \, \mathrm{d} k_0 = 0 \;\Leftarrow$$ the contour is closed in the upper half-plane and encircles no singularities

$$\mathbb{s}_0 > 0 \;\Rightarrow\; \oint_r \frac{e^{-i k_0 \mathbb{s}_0}}{k_0^2 - \kappa^2} \, \mathrm{d} k_0 = -2 \pi i \, \Re\left[\frac{e^{-i k_0 \mathbb{s}_0}}{k_0^2 - \kappa^2}\right] = -\frac{2\pi}{\kappa} \sin \kappa \, \mathbb{s}_0 \quad ⬃ \quad \mathbb{s} = |\boldsymbol{\mathbb{s}}|$$

$$\Rightarrow\; D_r(\vec{\mathbb{s}}) = \frac{\Theta(\mathbb{s}_0)}{(2\pi)^3} \int \frac{\sin \kappa \, \mathbb{s}_0}{\kappa} e^{i \boldsymbol{k} \cdot \boldsymbol{\mathbb{s}}} \, \mathrm{d}^3 k = \frac{\Theta(\mathbb{s}_0)}{2 \pi^2 \mathbb{s}} \int_0^\infty \sin \kappa \, \mathbb{s}_0 \sin \kappa \, \mathbb{s} \, \mathrm{d} \kappa$$

$$\int \frac{\sin \kappa \, \mathbb{s}_0}{\kappa} e^{i \boldsymbol{k} \cdot \boldsymbol{\mathbb{s}}} \, \mathrm{d}^3 k = 2 \pi \int e^{i \kappa \mathbb{s} \cos \theta} \kappa \sin \kappa \, \mathbb{s}_0 \sin \theta \, \mathrm{d} \theta \, \mathrm{d} \kappa \quad ⬃ \quad u = -\cos \theta$$

$$= 2 \pi \int \sin \kappa \, \mathbb{s}_0 \, \mathrm{d} \kappa \int_{-1}^{+1} e^{-i \kappa \mathbb{s} u} \kappa \, \mathrm{d} u = \frac{4 \pi}{\mathbb{s}} \int_0^\infty \sin \kappa \, \mathbb{s}_0 \sin \kappa \, \mathbb{s} \, \mathrm{d} \kappa$$

$$\Rightarrow \quad D_r(\vec{\mathbb{s}}) = \frac{\Theta(\mathbb{s}_0)}{8\pi^2 \mathbb{s}} \int_{-\infty}^{+\infty} \left(e^{i\kappa(\mathbb{s}_0-\mathbb{s})} - e^{i\kappa(\mathbb{s}_0+\mathbb{s})}\right) \mathrm{d}\kappa = \frac{\Theta(\mathbb{s}_0)}{4\pi \mathbb{s}} \left[\delta(\mathbb{s}_0-\mathbb{s}) - \delta(\mathbb{s}_0+\mathbb{s})\right]$$

$$\Rightarrow \quad D_r(\vec{\mathbb{s}}) = \frac{\Theta(\mathbb{s}_0)}{4\pi \mathbb{s}} \delta(\mathbb{s}_0-\mathbb{s})$$

retarded (causal) Green function :the source time t' is always earlier than the observation time t

- $\widetilde{D}(\omega) = \dfrac{e^{i\frac{\omega}{c}\mathbb{s}}}{4\pi \mathbb{s}} \Leftarrow$ Fourier transform of $D_r(\vec{\mathbb{s}})$ with respect to $r_0 = c\,t$ vs chapter 6

- $D_a(\vec{\mathbb{s}}) = \dfrac{\Theta(-\mathbb{s}_0)}{4\pi \mathbb{s}} \delta(\mathbb{s}_0+\mathbb{s}) \Leftarrow$ *advanced Green function* choosing the contour a

- These Green functions can be put in covariant form

$$\delta(\vec{\mathbb{s}}^2) = \delta(\mathbb{s}_0^2 - \mathbb{s}^2) = \delta((\mathbb{s}_0-\mathbb{s})(\mathbb{s}_0+\mathbb{s})) = \frac{\delta(\mathbb{s}_0-\mathbb{s}) + \delta(\mathbb{s}_0+\mathbb{s})}{2\mathbb{s}} \quad \Leftarrow \quad \vec{\mathbb{s}} = \vec{r} - \vec{r}\,'$$

$$\Rightarrow \quad D_{r/a}(\vec{\mathbb{s}}) = \frac{\Theta(\pm \mathbb{s}_0)}{2\pi} \delta(\vec{\mathbb{s}}^2) \quad \Leftarrow \quad \text{explicitly invariant expression}$$

- The Θ functions, apparently noninvariant, are actually invariant under proper Lorentz transformations when constrained by the delta functions.

- The retarded (advanced) Green function is different from 0 only on the forward (backward) light cone of the source point.

- Solution of the wave equation

$$\vec{A}(\vec{r})=\vec{A}_{\text{in}}(\vec{r})+\frac{4\pi}{c}\int D_r(\vec{\mathbb{s}})\ \vec{J}(\vec{r}')\,\mathrm{d}^4x' \quad (\ast 1)$$

$$\vec{A}(\vec{r})=\vec{A}_{\text{out}}(\vec{r})+\frac{4\pi}{c}\int D_a(\vec{\mathbb{s}})\,\vec{J}(\vec{r}')\,\mathrm{d}^4x'$$

- In the limit $r_0=c\,t\rightarrow-\infty$, the integral in (✶1) vanishes, assuming the sources are localized in space and time, the retarded nature of the Green function.

- $\vec{A}_{\text{in}}$: the incident or incoming potential specified at $t\rightarrow-\infty$
  $\vec{A}_{\text{out}}$: the asymptotic outgoing potential specified at $t\rightarrow+\infty$

- The *radiation* fields: difference between the outgoing and the incoming fields.

$$\vec{A}_{\text{rad}}(\vec{r})=\vec{A}_{\text{out}}(\vec{r})-\vec{A}_{\text{in}}(\vec{r})=\frac{4\pi}{c}\int D(\vec{\mathbb{s}})\,\vec{J}(\vec{r}')\,\mathrm{d}^4x' \quad\Leftarrow\quad D(\vec{\mathbb{s}})=D_r(\vec{\mathbb{s}})-D_a(\vec{\mathbb{s}})$$

- For a charged particle

$$\rho(\mathbf{r},t)=e\,\delta(\mathbf{r}-\mathbb{r}(t)) \qquad \boldsymbol{J}(\mathbf{r},t)=e\,\mathbf{v}(t)\,\delta(\mathbf{r}-\mathbb{r}(t)) \quad\Leftarrow\quad \mathbb{r}\ \text{is the position in K},\quad \mathbf{v}=\dot{\mathbb{r}}\ \text{is the velocity}$$

can be written as a 4-vector current in manifestly covariant form

$$\vec{J}(\vec{r})=e\,c\int \vec{U}(\tau)\,\delta(\vec{r}-\vec{\mathbb{r}}(\tau))\,\mathrm{d}\tau \quad\Leftarrow\quad \vec{\mathbb{r}}=(c\,t,\mathbb{r}(t)),\quad \vec{U}=(\gamma\,c,\gamma\,\mathbf{v})\ \text{in the inertial frame K}$$