Chapter 8 Waveguides, Resonant Cavities, Optical Fibers

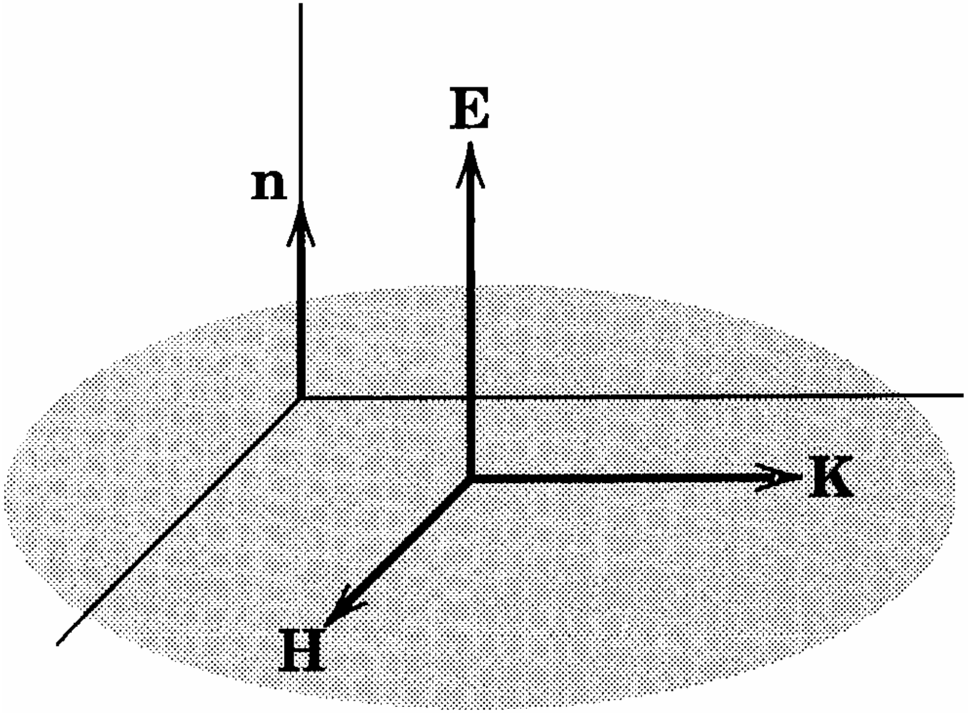
- At high frequencies with wavelengths~meters or less, generating & transmitting EM radiation involves metallic structures with dimensions~the wavelengths.

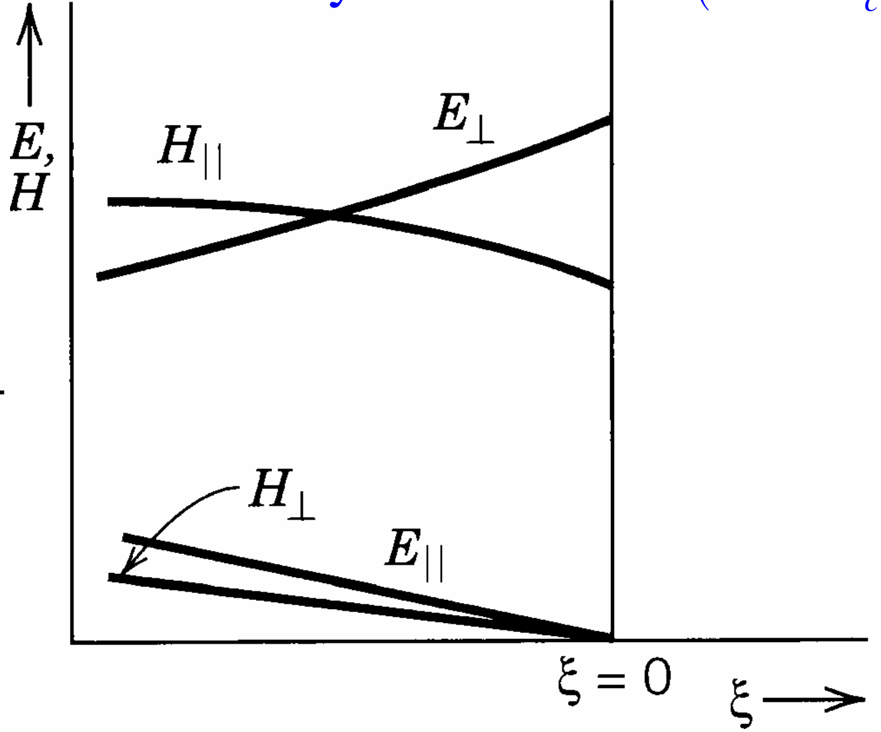
- At much higher (infrared) frequencies, dielectric optical fibers are exploited in the telecommunications industry.

Fields at the Surface of and Within a Conductor

- Consider a surface with unit normal $\hat{\mathbf{n}}$ outward from a *perfect* conductor into a nonconducting medium

$$\begin{array}{ll} \mathbf{E}=0 & \text{in the} \\ \mathbf{B}=0 & \text{conductor} \end{array} \Rightarrow \begin{array}{ll} \hat{\mathbf{n}}\cdot\mathbf{D}=\Sigma & \text{surface charge density} \\ \hat{\mathbf{n}}\times\mathbf{H}=\boldsymbol{K} & \text{surface current density} \end{array} \quad (1) \quad + \quad \begin{array}{l} \hat{\mathbf{n}}\cdot(\mathbf{B}-\mathbf{B}_c)=0 \\ \hat{\mathbf{n}}\times(\mathbf{E}-\mathbf{E}_c)=0 \end{array}$$





- Outside the surface of a perfect conductor only *normal* $E_\perp$ and *tangential* $\mathbf{H}_\parallel$ fields exist, and that the fields drop abruptly to 0 inside the perfect conductor.

- Inside a good (not perfect) conductor the fields are attenuated exponentially in the skin depth δ. For moderate frequencies, $\delta < 1\text{cm}$. So the boundary condition (1) are approximately true, aside from a thin transitional layer at the surface.

- $\boldsymbol{J} = \sigma \mathbf{E}$ with a finite conductivity, there can't be a surface layer of current but

$$\hat{\mathbf{n}} \times (\mathbf{H} - \mathbf{H}_c) = 0 \quad \Rightarrow \quad \mathbf{H}_\parallel = \mathbf{H}_{\parallel c} \neq 0$$

- First assume that just outside the conductor there exists only $E_\perp$ & $\mathbf{H}_\parallel$, as for a perfect conductor.

- Then use the boundary conditions & Maxwell's eqns in the conductor to find the fields within the transition layer and small corrections to the fields outside.

- The spatial variation of the fields ⊥ the surface is much more rapid than the variations ∥ the surface in solving the Maxwell equations within the conductor ⇒ neglect all derivatives with respect to coordinates ∥ the surface.

- $$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \boldsymbol{J} \qquad \mathbf{E}_c \simeq \frac{1}{\sigma} \nabla \times \mathbf{H}_c \quad \Leftarrow$$

neglect the displacement current in the conductor

$$\Rightarrow$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \mathbf{H}_c = -\frac{i}{\mu_c \omega} \nabla \times \mathbf{E}_c \quad \Leftarrow \quad \mathbf{B}_c = \mu_c \mathbf{H}_c \propto e^{-i\omega t}$$

$$\nabla \simeq -\hat{\mathbf{n}}\,\frac{\partial}{\partial \xi} \quad \Leftarrow$$

$\hat{\mathbf{n}}$: unit normal outward

ξ : normal coordinate inward

$$\Rightarrow \quad \begin{bmatrix} \mathbf{E}_c \\ \mathbf{H}_c \end{bmatrix} \simeq \begin{bmatrix} -\dfrac{1}{\sigma} \\ \dfrac{i}{\mu_c\,\omega} \end{bmatrix} \hat{\mathbf{n}} \times \frac{\partial}{\partial \xi} \begin{bmatrix} \mathbf{H}_c \\ \mathbf{E}_c \end{bmatrix}$$

$$\Rightarrow \quad \frac{\partial^2}{\partial \xi^2}\left(\hat{\mathbf{n}} \times \mathbf{H}_c\right) + \frac{2\,i}{\delta^2}\,\hat{\mathbf{n}} \times \mathbf{H}_c \simeq 0 \quad \Leftarrow \quad \delta = \sqrt{\frac{2}{\mu_c\,\omega\,\sigma}} \quad \Rightarrow \quad \boxed{\begin{aligned} \mathbf{H}_c &\simeq \mathbf{H}_{\parallel}\, e^{\frac{\xi\,(i-1)}{\delta}} \\ \mathbf{E}_c &\simeq \frac{1-i}{\delta\,\sigma}\,\hat{\mathbf{n}} \times \mathbf{H}_c \end{aligned}} \qquad (0)$$

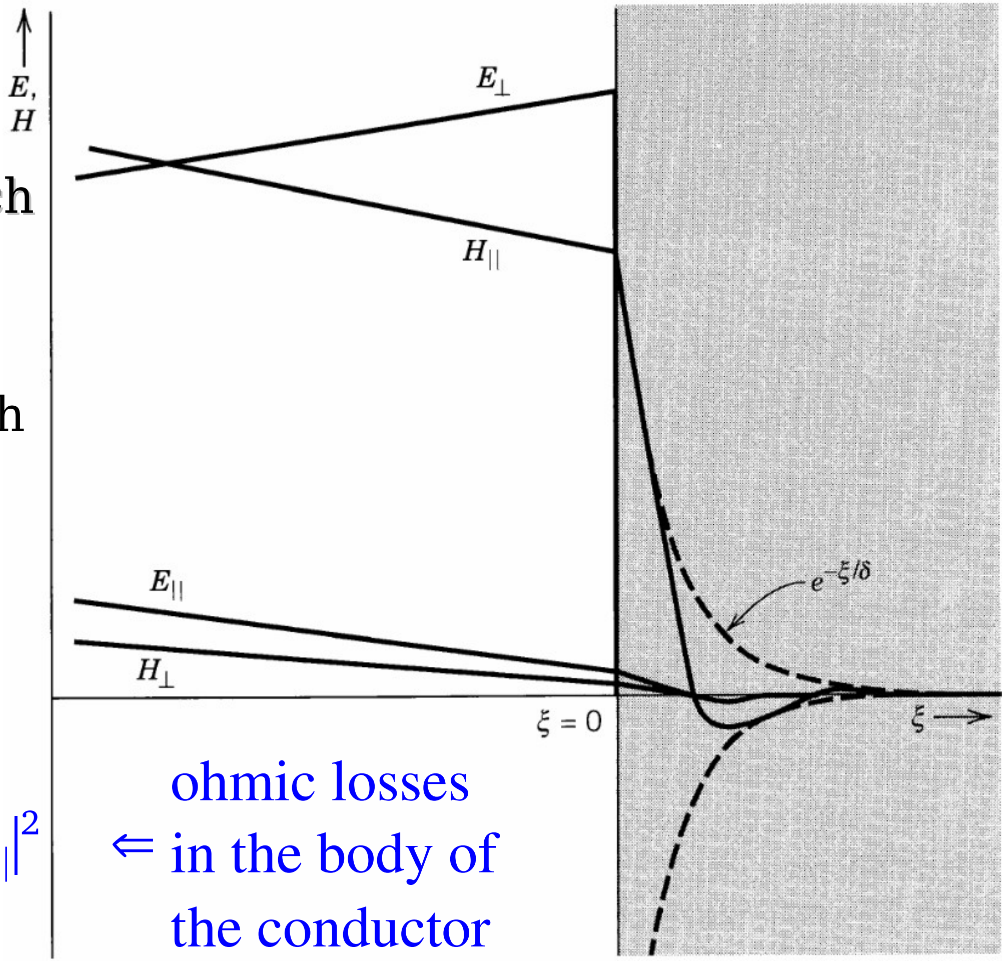
$$\hat{\mathbf{n}} \cdot \mathbf{H}_c \simeq 0$$

- **H** & **E** inside the conductor exhibit the properties of rapid exponential decay, phase difference, and magnetic field much larger than the electric field.

- For a good conductor, the fields inside are $\parallel$ the surface and propagate $\perp$ it, with magnitude that depend only on the tangential magnetic field $\mathbf{H}_{\parallel}$ just outside.

- $$\mathbf{E}_{\parallel} = \mathbf{E}_c\,(\xi = 0) = \frac{1-i}{\delta\,\sigma}\,\hat{\mathbf{n}} \times \mathbf{H}_{\parallel} \qquad (13)$$

$$\Rightarrow \quad \frac{\mathrm{d}\,P_{\mathrm{loss}}}{\mathrm{d}\,a} = -\frac{\Re\left[\hat{\mathbf{n}} \cdot \mathbf{E} \times \mathbf{H}^{*}\right]}{2} = \frac{\mu_c\,\omega\,\delta}{4}\left|\mathbf{H}_{\parallel}\right|^2$$



$\Leftarrow$ ohmic losses in the body of the conductor

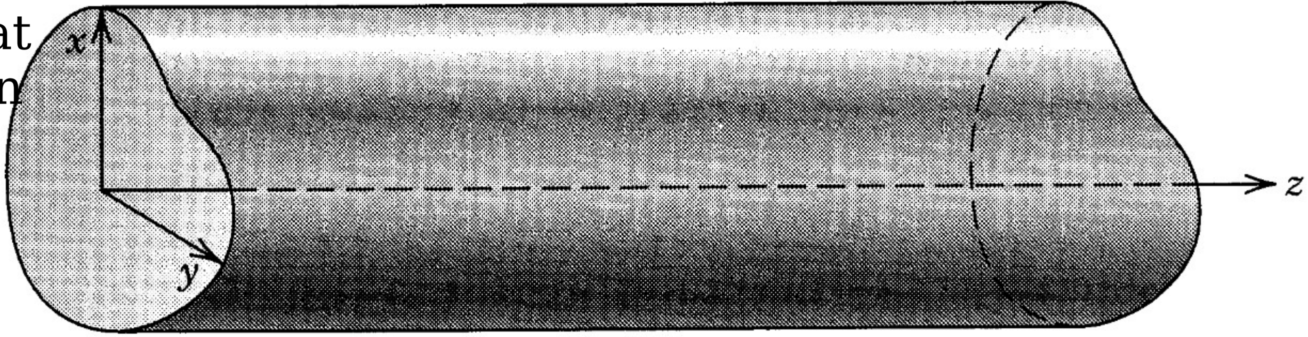
- $\boldsymbol{J} = \sigma\,\mathbf{E}_c = \frac{1-i}{\delta}\,\hat{\mathbf{n}} \times \mathbf{H}_{\parallel}\, e^{\frac{\xi(i-1)}{\delta}}$

$$\Rightarrow \quad \frac{\mathrm{d}\,P_{\text{loss}}}{\mathrm{d}\,a} = \int \frac{\mathrm{d}\,P_{\text{loss}}}{\mathrm{d}\,v}\,\mathrm{d}\,\xi = \int \frac{1}{2}\,\boldsymbol{J} \cdot \mathbf{E}^{*}\,\mathrm{d}\,\xi = \int \frac{1}{2\,\sigma}\left|\boldsymbol{J}\right|^2 \mathrm{d}\,\xi = \frac{\mu_c\,\omega\,\delta}{4}\left|\mathbf{H}_{\parallel}\right|^2$$

- $\mathbf{K}_{\text{eff}} = \int_0^{\infty} \boldsymbol{J}\,\mathrm{d}\,\xi = \hat{\mathbf{n}} \times \mathbf{H}_{\parallel} \quad \Rightarrow \quad \frac{\mathrm{d}\,P_{\text{loss}}}{\mathrm{d}\,a} = \frac{\left|\mathbf{K}_{\text{eff}}\right|^2}{2\,\sigma\,\delta} \quad (15) \quad \Leftarrow \quad \frac{1}{\sigma\,\delta}$: surface resistance of the conductor

- A good conductor behaves effectively like a perfect conductor, with the idealized surface current replaced by an effective surface current, which is distributed throughout a very small, but finite, thickness at the surface.

Cylindrical Cavities and Waveguides



- A practical situation of great importance is the propagation or excitation of EM waves in hollow metallic cylinders.

- If the cylinder has end surfaces, it is called a cavity; otherwise, a waveguide.

- The boundary surfaces are assumed to be perfect conductors.

- $$\begin{matrix} \nabla\times\mathbf{E}=i\,\omega\,\mathbf{B}\,, & \nabla\cdot\mathbf{B}=0 \\ \nabla\times\mathbf{B}=-i\,\mu\,\epsilon\,\omega\,\mathbf{E}\,, & \nabla\cdot\mathbf{E}=0 \end{matrix} \Leftarrow \begin{matrix}\text{Maxwell's eqns} \\ \text{inside the cylinder}\end{matrix} \Rightarrow \left(\nabla^2+\mu\,\epsilon\,\omega^2\right)\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix}=0$$

$$\Rightarrow \begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix}(x\,,y\,,z\,,t)=\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix}(x\,,y)\,e^{i(\pm k\,z-\omega\,t)} \Rightarrow \left(\nabla_t^2+\mu\,\epsilon\,\omega^2-k^2\right)\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix}=0$$

$$\text{where } \underset{\text{transverse part}}{\nabla_t^2=\nabla^2-\partial_z^2} \Rightarrow \begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix}=\begin{bmatrix}\mathbf{E}_z\\ \mathbf{B}_z\end{bmatrix}+\begin{bmatrix}\mathbf{E}_t\\ \mathbf{B}_t\end{bmatrix} \Leftarrow \begin{bmatrix}\mathbf{E}_z\\ \mathbf{B}_z\end{bmatrix}=\begin{bmatrix}E_z\\ B_z\end{bmatrix}\hat{\mathbf{z}}\,,\quad \begin{bmatrix}\mathbf{E}_t\\ \mathbf{B}_t\end{bmatrix}=\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix}-\begin{bmatrix}\mathbf{E}_z\\ \mathbf{B}_z\end{bmatrix}$$

$$\Rightarrow \begin{matrix} \dfrac{\partial\,\mathbf{E}_t}{\partial\,z}+i\,\omega\quad\hat{\mathbf{z}}\times\mathbf{B}_t=\nabla_t\,E_z\,,\ \left(\nabla_t\times\mathbf{E}_t\right)_z=\ i\quad\omega\,B_z\,,\ \nabla_t\cdot\mathbf{E}_t=-\dfrac{\partial\,E_z}{\partial\,z} \\ \dfrac{\partial\,\mathbf{B}_t}{\partial\,z}-i\,\mu\,\epsilon\,\omega\,\hat{\mathbf{z}}\times\mathbf{E}_t=\nabla_t\,B_z\,,\ \left(\nabla_t\times\mathbf{B}_t\right)_z=-i\,\mu\,\epsilon\,\omega\,E_z\,,\ \nabla_t\cdot\mathbf{B}_t=-\dfrac{\partial\,B_z}{\partial\,z} \end{matrix} \qquad (6)$$

$$\begin{aligned} \partial_z \mathbf{E}_t + i \quad \omega\, \hat{\mathbf{z}} \times \mathbf{B}_t &= \nabla_t E_z \\ \partial_z \mathbf{B}_t - i\, \mu\, \epsilon\, \omega\, \hat{\mathbf{z}} \times \mathbf{E}_t &= \nabla_t B_z \end{aligned} \quad + \quad \begin{bmatrix} \mathbf{E} \\ \mathbf{B} \end{bmatrix} (x, y, z, t) = \begin{bmatrix} \mathbf{E} \\ \mathbf{B} \end{bmatrix} (x, y)\, e^{i(\pm k z - \omega t)}$$

$$\Rightarrow \begin{aligned} i(\pm k\, \mathbf{E}_t + \quad \omega\, \hat{\mathbf{z}} \times \mathbf{B}_t) &= \nabla_t E_z \qquad \times \quad \pm k \\ i(\pm k\, \mathbf{B}_t - \mu\, \epsilon\, \omega\, \hat{\mathbf{z}} \times \mathbf{E}_t) &= \nabla_t B_z \qquad \times \quad \omega\, \hat{\mathbf{z}} \end{aligned}$$

$$\Rightarrow \begin{aligned} k^2 \mathbf{E}_t \pm \quad \omega\, k\, \hat{\mathbf{z}} \times \mathbf{B}_t &= \mp i\, k\, \nabla_t E_z \\ \mp \omega\, k\, \hat{\mathbf{z}} \times \mathbf{B}_t - \mu\, \epsilon\, \omega^2 \mathbf{E}_t &= i\, \omega\, \hat{\mathbf{z}} \times \nabla_t B_z \quad \Leftarrow \quad (\hat{\mathbf{z}} \times \mathbf{E}_t) \times \hat{\mathbf{z}} = (\hat{\mathbf{z}} \cdot \hat{\mathbf{z}})\, \mathbf{E}_t - \hat{\mathbf{z}}\, (\cancel{\hat{\mathbf{z}} \cdot \mathbf{E}_t}) \end{aligned}$$

$$\Rightarrow \quad (k^2 - \mu\, \omega\, \omega^2)\, \mathbf{E}_t = i\, (\mp k\, \nabla_t E_z + \omega\, \hat{\mathbf{z}} \times \nabla_t B_z)$$

$$\Rightarrow \quad \mathbf{E}_t = \frac{i}{\mu\, \epsilon\, \omega^2 - k^2} (\pm k\, \nabla_t E_z - \omega\, \hat{\mathbf{z}} \times \nabla_t B_z)$$

Similarly

$$\Rightarrow \quad \mathbf{B}_t = \frac{i}{\mu\, \epsilon\, \omega^2 - k^2} (\pm k\, \nabla_t B_z + \mu\, \epsilon\, \omega\, \hat{\mathbf{z}} \times \nabla_t E_z)$$

$\left.\begin{matrix} \mathbf{E}_{\parallel} \\ \mathbf{B}_{\parallel} \end{matrix}\right] \perp \hat{\mathbf{n}}$ normal to the interface , $\left.\begin{matrix} \mathbf{E}_t \\ \mathbf{B}_t \end{matrix}\right] \perp \hat{\mathbf{z}}$ propagating direction

- If E_z and B_z are known, the transverse components of $\mathbf{E}$ and $\mathbf{B}$ are determined

$$\left.\begin{matrix}\mathbf{E}_t\\ \mathbf{B}_t\end{matrix}\right] = \frac{i}{\mu\,\epsilon\,\omega^2 - k^2}\left(\pm k\,\nabla_t\left[\begin{matrix}E_z\\ B_z\end{matrix}\right] + \left[\begin{matrix}(-1)\\ \mu\,\epsilon\end{matrix}\right]\omega\,\hat{\mathbf{z}}\times\nabla_t\left[\begin{matrix}B_z\\ E_z\end{matrix}\right]\right) \quad (2) \quad \Leftarrow \quad \begin{matrix}\text{for } E_z \neq 0\\ \text{and/or } B_z \neq 0\end{matrix}$$

- The *transverse electromagnetic* (TEM) wave: only field components transverse to the direction of propagation ⇐ a degenerate solution;

$$\left.\begin{matrix}E_z\\ B_z\end{matrix}\right] = 0 \quad \Rightarrow \quad \left.\begin{matrix}\mathbf{E}_{\text{TEM}}\\ \mathbf{B}_{\text{TEM}}\end{matrix}\right] = \left[\begin{matrix}\mathbf{E}_t\\ \mathbf{B}_t\end{matrix}\right. \quad \Rightarrow \quad \nabla_t \times \left[\begin{matrix}\mathbf{E}_{\text{TEM}}\\ \mathbf{B}_{\text{TEM}}\end{matrix}\right] = 0\,, \quad \nabla_t \cdot \left[\begin{matrix}\mathbf{E}_{\text{TEM}}\\ \mathbf{B}_{\text{TEM}}\end{matrix}\right] = 0$$

- $\mathbf{E}_{\text{TEM}}$ is a solution of an *electrostatic* problem in 2d: $\Leftarrow \quad \nabla_t^2\,\mathbf{E}_{\text{TEM}} = 0$

$$(1) \quad k = k_0 = \omega\sqrt{\mu\,\epsilon} \quad \Leftarrow \quad \left(\nabla_t^2 + \mu\,\epsilon\,\omega^2 - k^2\right)\mathbf{E}_{\text{TEM}} = 0$$

$$\Rightarrow \quad (2) \quad \mathbf{B}_{\text{TEM}} = \pm\sqrt{\mu\,\epsilon}\;\hat{\mathbf{z}}\times\mathbf{E}_{\text{TEM}} \quad \text{for waves propagating as } e^{i(\pm k\,z - \omega\,t)}$$

$$(3) \quad \text{TEM mode can't exist inside a single, hollow, cylindrical perfect conductor}$$

$$\Leftarrow \quad \mathbf{E}_{\text{inside}} = 0 \quad \Leftarrow \quad \text{The surface is an equipotential}$$

- It is necessary to have 2 or more cylindrical surfaces to support the TEM mode, e.g., The coaxial cable and the parallel-wire transmission line.

- The TEM mode is without a cutoff frequency. The wave number is real for all ω. This is not true for the other modes.

- It turns out that TEM waves cannot occur in a hollow wave guide.

Proof:

$$E_z = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 \quad \Leftarrow \quad \nabla \cdot \mathbf{E} = 0$$

$$B_z = 0 \quad \Rightarrow \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \quad \Leftarrow \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\Rightarrow \quad \nabla_t \equiv \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y}$$

$$\nabla_t \cdot \mathbf{E}_t = \nabla_t \times \mathbf{E}_t = 0$$

$$\Rightarrow \quad \mathbf{E}_t = -\nabla_t \Phi$$

No charge inside

$$\Rightarrow \quad \nabla_t^2 \Phi = 0, \quad \oint_S \mathbf{E} \cdot \mathrm{d}\boldsymbol{a}_\perp = 0 \quad \Rightarrow \quad \Phi = \text{constant} \quad \Rightarrow \quad \mathbf{E}_t = 0$$

- This argument applies only to a completely *empty* pipe—if you run a separate conductor down the middle, the potential at *its* surface need not be the same as on the outer wall, eg, the coaxial cable and the parallel-wire transmission line, and hence a nontrivial potential is possible.

- For a perfectly conducting cylinder ($\mathbf{E}_c = \mathbf{B}_c = 0$) the boundary conditions are

$$\begin{array}{l} \mathbf{E}_\parallel = 0 \\ B_\perp = 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} E_{z|S} = E_{\ell|S} = 0 \\ B_{\perp|S} = 0 \end{array} \quad \Rightarrow \quad \frac{\partial B_z}{\partial n}_{|S} = 0 \quad \Leftarrow \quad \hat{\mathbf{n}} \cdot \left(\frac{\partial \mathbf{B}_t}{\partial z} - i \mu \epsilon \omega \hat{\mathbf{z}} \times \mathbf{E}_t = \nabla_t B_z \right)$$

use $\hat{\mathbf{n}}$, $\hat{\boldsymbol{\ell}}$, $\hat{\mathbf{z}}$ to verify

- The 2d wave equations for E_z and B_z, together with the boundary conditions, specify eigenvalue problems of the usual sort.

- For a given frequency ω, only certain values of wave number k can occur (waveguide). For a given k, only certain ω values are allowed (resonant cavity).

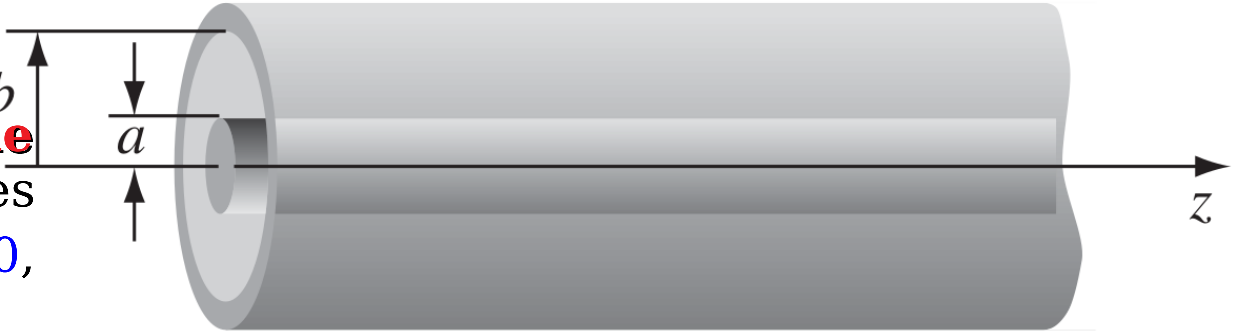
- *Since the boundary conditions on E_z and B_z are different, the eigenvalues will in general be different.*

- Transverse Magnetic (TM) Waves: $\begin{cases} B_z = 0 \ \text{everywhere;} \\ \text{boundary condition} \ \ E_{z|S} = 0 \end{cases}$

  Transverse Electric (TE) Waves: $\begin{cases} E_z = 0 \ \text{everywhere;} \\ \text{boundary condition} \ \ \dfrac{\partial B_z}{\partial n}_{|S} = 0 \end{cases}$

- The various TM and TE waves, plus the TEM wave if it can exist, constitute a complete set to describe an arbitrary EM disturbance in a waveguide or cavity.

The Coaxial Transmission Line



- A coaxial transmission line does admit modes with $E_z=0$ and $B_z=0$, ie, the TEM waves.

- In this case Maxwell's equations yield $k=\frac{\omega}{c} \quad \Rightarrow \quad v=c$, and are nondispersive

$$\Rightarrow \quad c\,B_y = E_x\,, \quad c\,B_x = -E_y\,, \quad \mathbf{E}\perp\mathbf{B}\,, \quad \nabla\cdot\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix}=0$$

$$\Rightarrow \quad \frac{\partial}{\partial x}\begin{bmatrix}E_x\\ B_x\end{bmatrix}+\frac{\partial}{\partial y}\begin{bmatrix}E_y\\ B_y\end{bmatrix}=0\,, \quad \frac{\partial}{\partial x}\begin{bmatrix}E_y\\ B_y\end{bmatrix}-\frac{\partial}{\partial y}\begin{bmatrix}E_x\\ B_x\end{bmatrix}=0$$

- These are precisely the equations of *electrostatics* and *magnetostatics*, for empty space, in 2D; the solution with cylindrical symmetry can be borrowed from the case of an infinite line charge and an infinite straight current

$$\mathbf{E}_t(s\,,\phi)=\frac{A}{s}\,\hat{\mathbf{s}}\,, \quad \mathbf{B}_t(s\,,\phi)=\frac{A}{c\,s}\,\hat{\boldsymbol{\phi}} \quad \Leftarrow \quad A:\ \text{some constant}$$

$$\Rightarrow \quad \text{the real part} \quad \begin{aligned}\mathbf{E}(s\,,\phi\,,z\,,t)&=\frac{A}{s}\cos(k\,z-\omega\,t)\,\hat{\mathbf{s}}\\ \mathbf{B}(s\,,\phi\,,z\,,t)&=\frac{A}{c\,s}\cos(k\,z-\omega\,t)\,\hat{\boldsymbol{\phi}}\end{aligned}$$

Waveguides

- For the propagation of waves inside a hollow waveguide of uniform cross section, it is found from (2)

$$\begin{aligned}\mathbf{H}_t &= \pm\frac{\hat{\mathbf{z}}\times\mathbf{E}_t}{Z} \\ \mathbf{E}_t &= \mp\left(\hat{\mathbf{z}}\times\mathbf{H}_t\right)\end{aligned} \quad (3) \quad \Leftarrow \quad Z = \left[\begin{array}{ll} \dfrac{k}{\epsilon\,\omega} = \dfrac{k}{k_0}\sqrt{\dfrac{\mu}{\epsilon}} & \text{TM} \\ \dfrac{\mu\,\omega}{k} = \dfrac{k_0}{k}\sqrt{\dfrac{\mu}{\epsilon}} & \text{TE} \end{array}\right. \quad \begin{matrix}\text{wave}\\ \text{impedance}\end{matrix} \quad \Leftarrow \quad k_0 = \omega\sqrt{\mu\,\epsilon}$$

$$\Rightarrow \quad \begin{matrix}\text{TM}\\ \text{TE}\end{matrix}\ \text{Waves}\ \begin{bmatrix}\mathbf{E}_t\\ \mathbf{H}_t\end{bmatrix} = \pm\frac{i\,k}{\gamma^2}\nabla_t\,\psi \quad \Leftarrow \quad \begin{bmatrix}E_z\\ H_z\end{bmatrix} = \psi\, e^{\pm i\,k\,z} \quad (4) \quad \Leftarrow \quad \left[\begin{array}{ll} \psi_{|S} = 0 & \text{TM} \\ \dfrac{\partial\,\psi}{\partial\, n}_{|S} = 0 & \text{TE} \end{array}\right.$$

$$\text{where} \quad \left(\nabla_t^2 + \gamma^2\right)\psi = 0\,, \quad \gamma^2 = \mu\,\epsilon\,\omega^2 - k^2$$

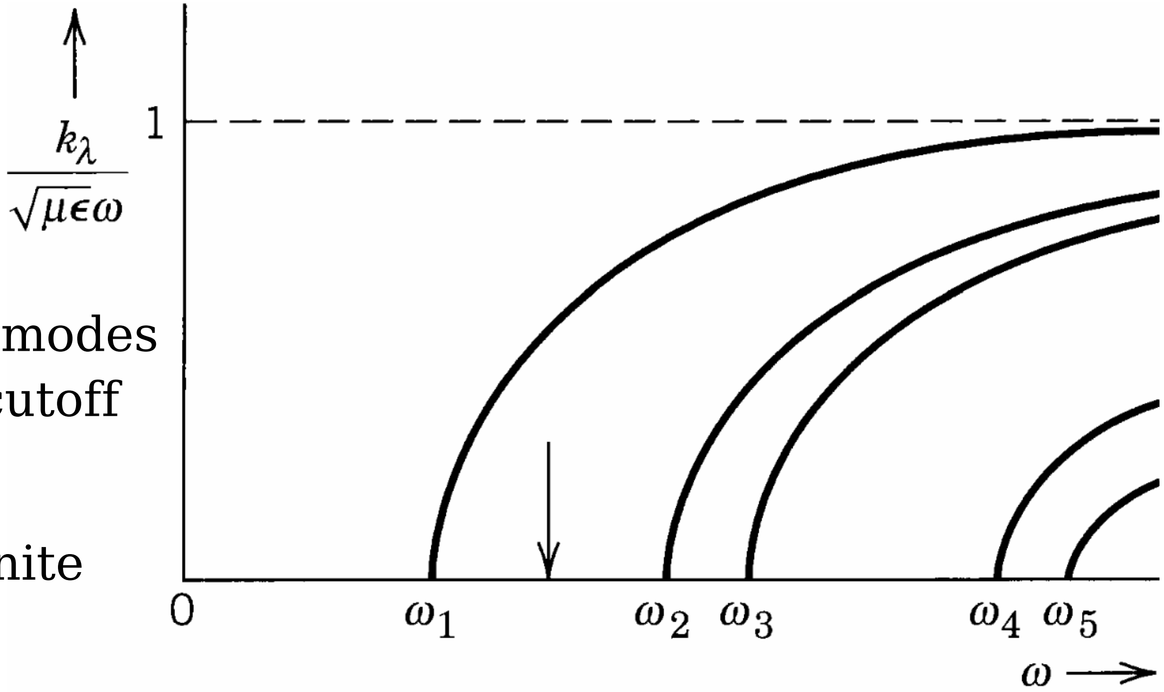
- The elliptic eqn of ψ with boundary conditions specifies an eigenvalue problem.

- γ^2 must be nonnegative because ψ must be oscillatory to satisfy the boundary condition on opposite sides of the cylinder.

- There will be a spectrum of eigenvalues $\gamma_\lambda{}^2$ and corresponding solutions ψ_λ which form an orthogonal set — *modes of the guide*.

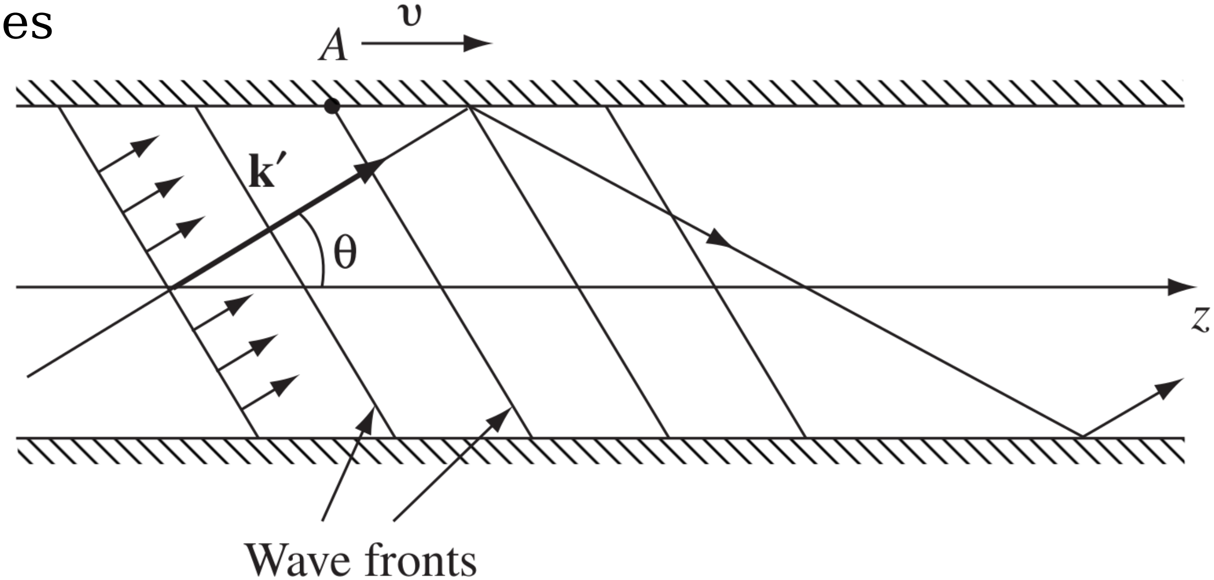
$$\text{Given}\ \ \omega\,, \quad k_\lambda^2 = \mu\,\epsilon\,\omega^2 - \gamma_\lambda^2 \quad \Rightarrow \quad \begin{matrix}\text{cutoff}\\ \text{frequency}\end{matrix} \quad \omega_\lambda = \frac{\gamma_\lambda}{\sqrt{\mu\,\epsilon}} \quad \Rightarrow \quad k_\lambda = \sqrt{\mu\,\epsilon}\,\sqrt{\omega^2 - \omega_\lambda^2}$$

- For $\omega > \omega_\lambda$, then k_λ is real; waves of the λ mode can propagate in the guide.
- For $\omega < \omega_\lambda$, k_λ is imaginary; such modes cannot propagate and are called cutoff modes or evanescent modes.
- At any given frequency only a finite number of modes can propagate.

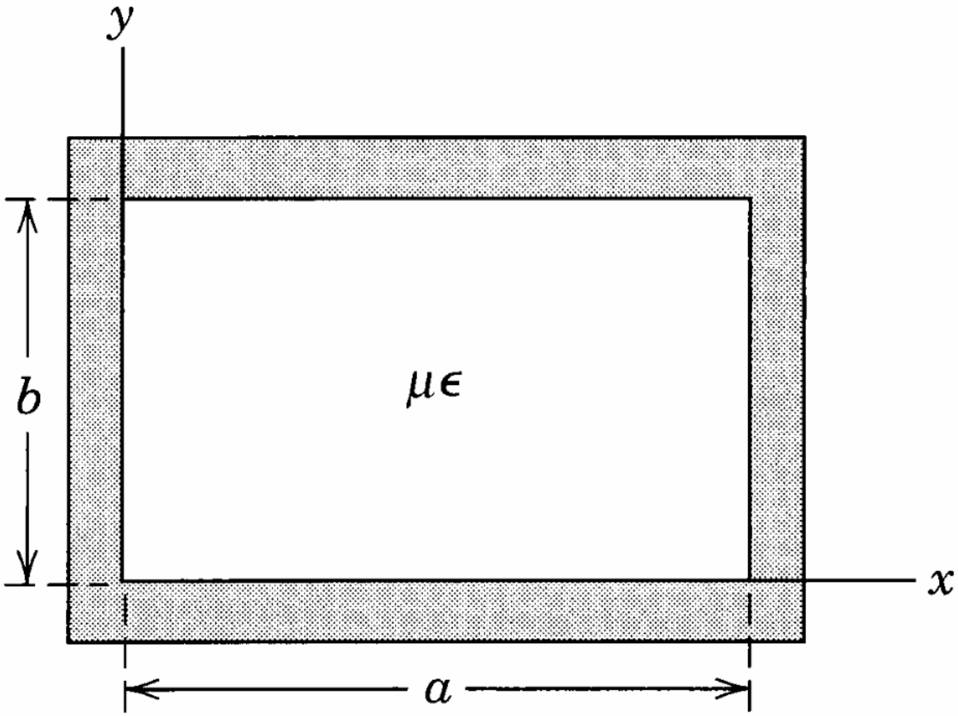


- It is often convenient to choose the dimensions of the guide so that at the operating frequency only the lowest mode can occur.
- $k_\lambda < k_0 = \sqrt{\mu\,\epsilon}\,\omega \quad \Rightarrow \quad$ phase velocity $\quad v_p = \frac{\omega}{k_\lambda} = \frac{\omega}{\sqrt{\mu\,\epsilon\,(\omega^2 - \omega_\lambda^2)}} > \frac{1}{\sqrt{\mu\,\epsilon}}$

The phase velocity becomes infinite exactly at cutoff.



Modes in a Rectangular Waveguide



- TE wave: $(\partial_x^2+\partial_y^2+\gamma^2)\psi=0 \quad \Leftarrow \quad \psi=\mathcal{H}_z$

Condition: $\mathcal{H}_\perp=0 \quad \Rightarrow \quad \frac{\partial\psi}{\partial n}_{|S}=\frac{\partial\mathcal{H}_z}{\partial n}_{|S}=0$

- Do it by separation of variables

$$\mathcal{H}_z=X(x)\,Y(y)$$

$$\Rightarrow \quad Y\frac{\mathrm{d}^2X}{\mathrm{d}x^2}+X\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}+\gamma^2XY=0$$

$$\Rightarrow \quad \frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2}=-k_x^2,\quad \frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}=-k_y^2 \quad \Leftarrow \quad k_x^2+k_y^2=\gamma^2=\mu\epsilon\omega^2-k^2=\mu\epsilon\omega^2-k_z^2$$

$$\Rightarrow \quad X(x)=A\sin k_x x+B\cos k_x x$$

$$\mathcal{H}_{x|S}=0 \quad \Rightarrow \quad \frac{\mathrm{d}X}{\mathrm{d}x}(0)=\frac{\mathrm{d}X}{\mathrm{d}x}(a)=0 \quad \Rightarrow A=0,\ k_x=\frac{m\pi}{a},\ m=0,1,2,\cdots$$

similarly $k_y=\frac{n\pi}{b},\quad n=0,1,2,\cdots \quad \Rightarrow \quad k_z^2=\mu\epsilon\omega^2-\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$

$$\Rightarrow \quad \psi_{mn}(x,y)=\mathcal{H}_{zmn}=H_{0mn}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b},\qquad \gamma_{mn}^2=\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

$$\Rightarrow \quad \mathcal{H}_z = \sum_{m,n} H_{0\,m\,n} \cos\frac{m\,\pi\,x}{a} \cos\frac{n\,\pi\,y}{b}, \qquad \omega^2_{m\,n} = \frac{\gamma^2_{m\,n}}{\mu\,\epsilon}$$

$$\Rightarrow \quad \left.\begin{matrix}\mathcal{H}_x \\ \\ \mathcal{H}_y\end{matrix}\right] = \sum_{m,n} \frac{-i\,k_z}{\gamma^2_{m\,n}} H_{0\,m\,n} \left[\begin{matrix} \dfrac{m\,\pi}{a} \sin\dfrac{m\,\pi\,x}{a} \cos\dfrac{n\,\pi\,y}{b} \\ \dfrac{n\,\pi}{b} \cos\dfrac{m\,\pi\,x}{a} \sin\dfrac{n\,\pi\,y}{b} \end{matrix}\right. , \qquad \mathbf{E} = -\frac{\mu\,\omega}{k_z}\,\hat{\mathbf{z}} \times \mathbf{H}$$

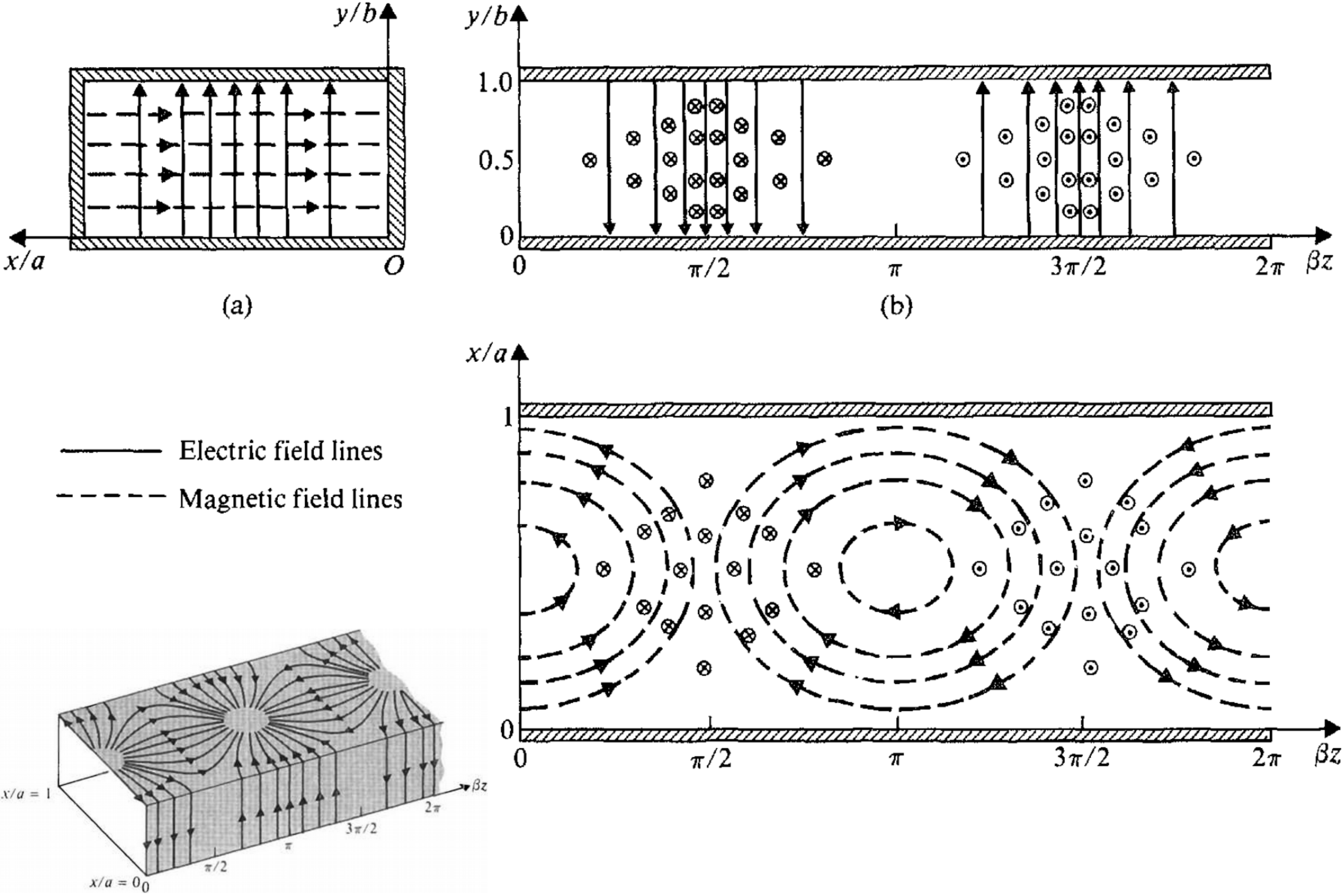
$$\Rightarrow \quad \begin{matrix}\text{cutoff} \\ \text{freqency}\end{matrix} \quad \omega_{m\,n} = \frac{\gamma_{m\,n}}{\sqrt{\mu\,\epsilon}} \quad \Rightarrow \quad \omega_{10} = \frac{\pi}{a\,\sqrt{\mu\,\epsilon}} \quad \begin{matrix}\text{lowest cutoff} \\ \text{frequency}\end{matrix} \quad \text{for } a > b$$

$$\Rightarrow \quad H_z = H_0 \cos\frac{\pi\,x}{a}\, e^{i(k_z z - \omega\,t)}, \quad \left.\begin{matrix} H_x \\ E_y \end{matrix}\right] = i \begin{bmatrix} (-k_z) \\ \mu\,\omega \end{bmatrix} \frac{a}{\pi} H_0 \sin\frac{\pi\,x}{a}\, e^{i(k_z z - \omega\,t)} \quad \Leftarrow \quad \begin{matrix} \mathrm{TE}_{10} \\ k_z = k_{10} \end{matrix}$$

- The presence of a factor i in $H_x\,(E_y)$ means that there is a spatial (or temporal) phase difference of 90° between $H_x\,(E_y)$ and H_z in the propagation direction.

- The TE_{10} mode has the lowest cutoff frequency of both TE and TM modes, and so is the one used in most practical situations.

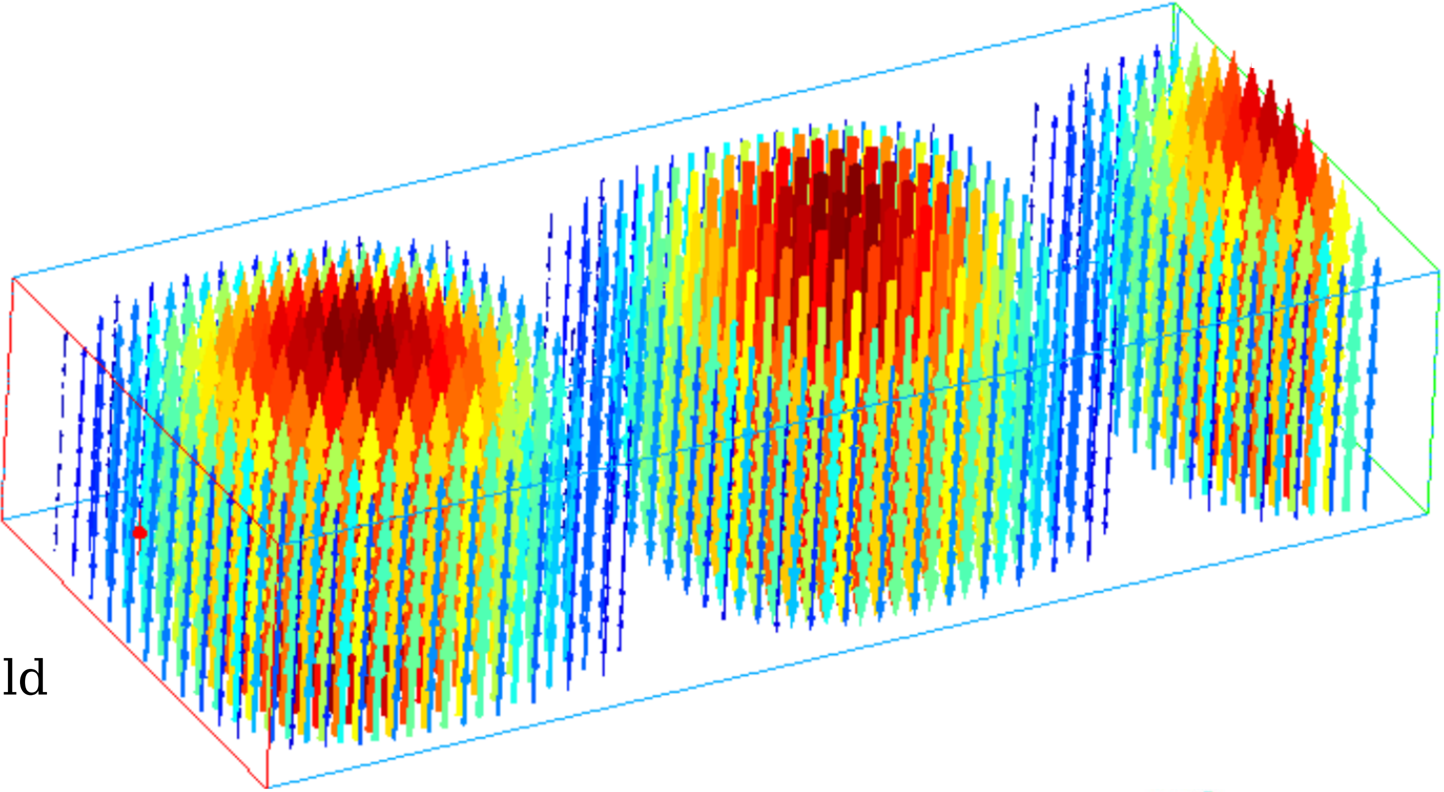
- There is a frequency range from cutoff to twice cutoff or to $\dfrac{a}{b}$ times cutoff where the TE_{10} mode is the only propagating mode for $a = 2\,b$.


(a)

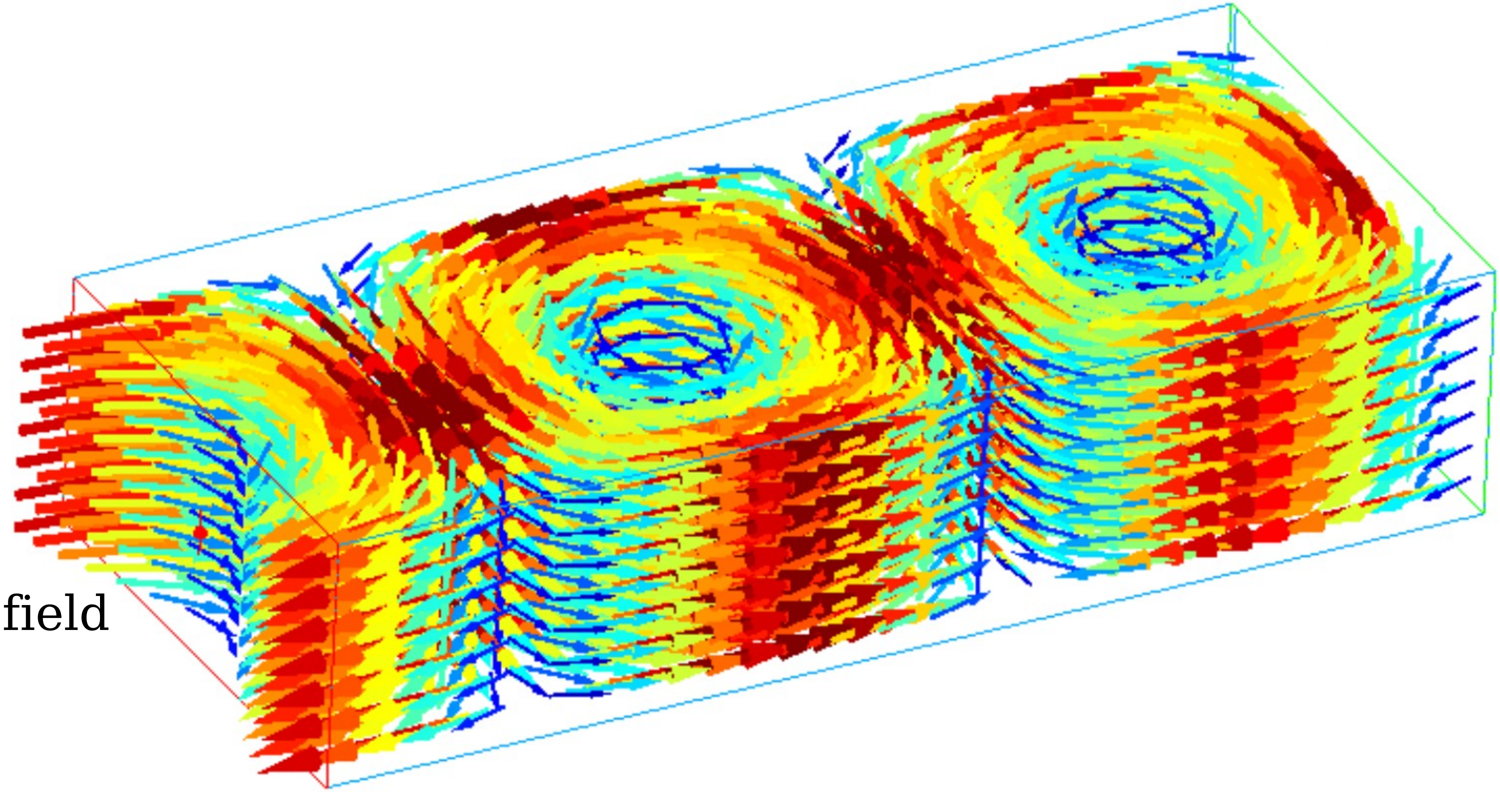
(b)

y/b
x/a
O
y/b
1.0
0.5
0
0
π/2
π
3π/2
2π
βz
x/a
1
0
0
π/2
π
3π/2
2π
βz
Electric field lines
Magnetic field lines
x/a = 1
x/a = 0
0
π/2
π
3π/2
2π
βz

TE$_{10}$ mode



Electric field



Magnetic field

ω_{mn}

$m \backslash n$	0	1	2	3
0		2.00	4.00	6.00
1	1.00	2.24	4.13	
2	2.00	2.84	4.48	
3	3.00	3.61	5.00	
4	4.00	4.48	5.66	
5	5.00	5.39		
6	6.00			

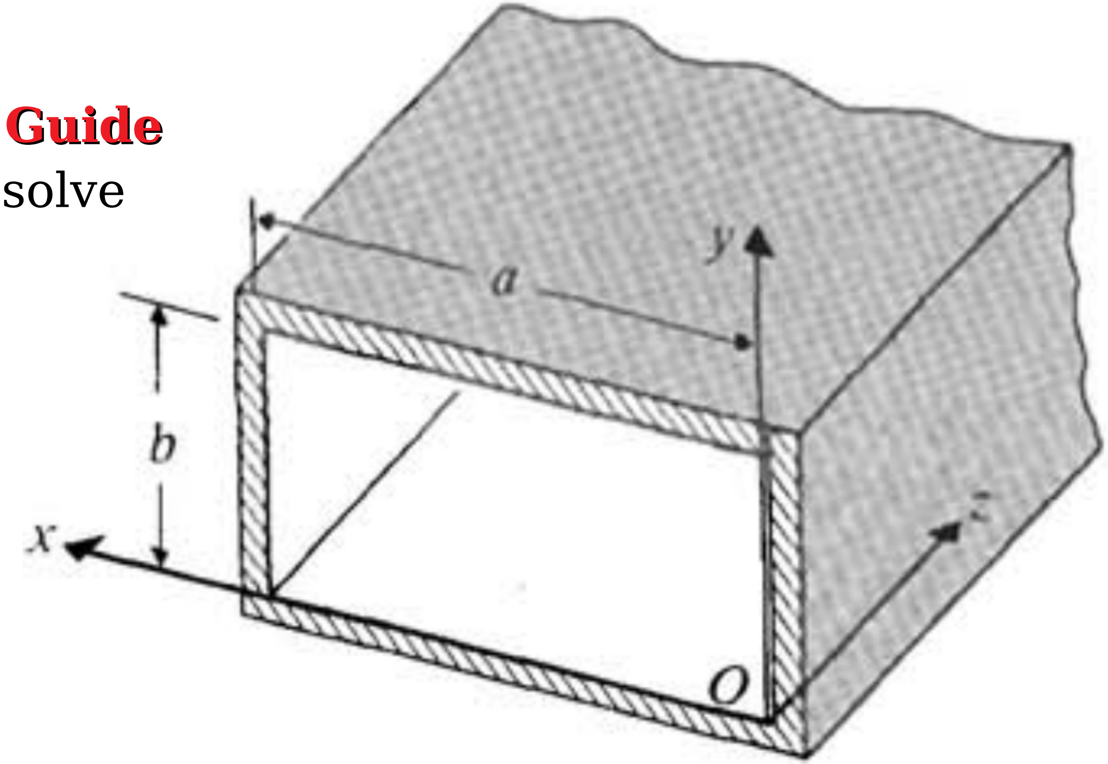
for $a = 2b$

<u>Example</u>: Standard air-filled waveguides have been designed for the radar bands (300MHz–300 GHz). One type, designated WG-16, is suitable for X-band (8GHz-12.4GHz) applications. Its dimensions are: a=2.29cm and b=1.02cm. If it is desired that a WG-16 waveguide operate only in the dominant TE$_{10}$ mode and that the operating frequency be at least 25% above the cutoff frequency of the TE$_{10}$ mode but no higher than 95% of the next higher cutoff frequency, what is the allowable operating-frequency range?

- $$f_{mn} = \frac{\omega_{mn}}{2\pi} = \frac{c}{2}\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}} \quad \Rightarrow \quad \begin{aligned} & f_{10} = \frac{c}{2a} = 6.55 \times 10^9 \text{ Hz}, \; f_{20} = \frac{c}{a} = 13.1 \times 10^9 \text{ Hz} \\ & f_{11} = \frac{c}{2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}} = 16.1 \times 10^9 \text{ Hz} \end{aligned}$$
- Thus the allowable operating-frequency range under the specified conditions is

$$1.25\, f_{10} < f < 0.95\, f_{20} \quad \Rightarrow \quad 8.19 \text{ GHz} < f < 12.45 \text{ GHz}$$

TM Waves in a Rectangular Wave Guide



- The TM wave ($H_z=0$) problem is to solve

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\gamma^2\right)\mathcal{E}_z=0 \quad \text{using} \quad \mathbf{E}_\parallel=0$$

- Do it by separation of variables

$$\mathcal{E}_z=X(x)\,Y(y)$$

$$\Rightarrow \quad Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}+\gamma^2XY=0$$

$$\Rightarrow \quad \frac{1}{X}\frac{d^2X}{dx^2}=-k_x^2, \quad \frac{1}{Y}\frac{d^2Y}{dy^2}=-k_y^2 \quad \Leftarrow \quad k_x^2+k_y^2=\gamma^2=\mu\epsilon\omega^2-k^2=\mu\epsilon\omega^2-k_z^2$$

$$\Rightarrow \quad X(x)=A\sin k_x x+B\cos k_x x$$

$$\mathcal{E}_{z|S}=0 \quad \Rightarrow \quad X(0)=X(a)=0 \quad \Rightarrow \quad B=0, \quad k_x=\frac{m\pi}{a}, \quad m=0,\ 1,\ 2,\cdots$$

$$\text{Similarly} \quad k_y=\frac{n\pi}{b}, \quad n=0,\ 1,\ 2,\cdots \quad \Rightarrow \quad k_z^2=\mu\epsilon\omega^2-\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

$$\Rightarrow \quad \mathcal{E}_{zmn}=E_{0mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad \Rightarrow \quad \mathcal{E}_z=\sum_{m,n}E_{0mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

- The other field components are

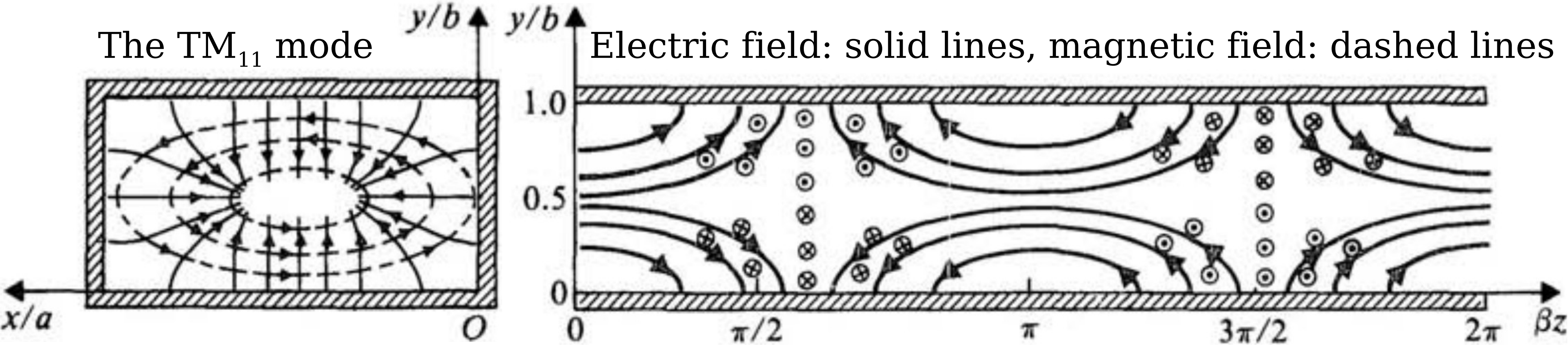
$$\left.\begin{matrix}\mathcal{E}_x\\ \mathcal{E}_y\end{matrix}\right] = \sum_{m,n} \frac{i\,k_z}{\gamma_{m\,n}^2} E_{0\,m\,n} \left[\begin{matrix}\frac{m\,\pi}{a}\cos\frac{m\,\pi\,x}{a}\sin\frac{n\,\pi\,y}{b}\\ \frac{n\,\pi}{b}\sin\frac{m\,\pi\,x}{a}\cos\frac{n\,\pi\,y}{b}\end{matrix}\right. , \quad \mathbf{H} = \frac{\epsilon\,\omega}{k_z}\hat{\mathbf{z}}\times\mathbf{E}$$

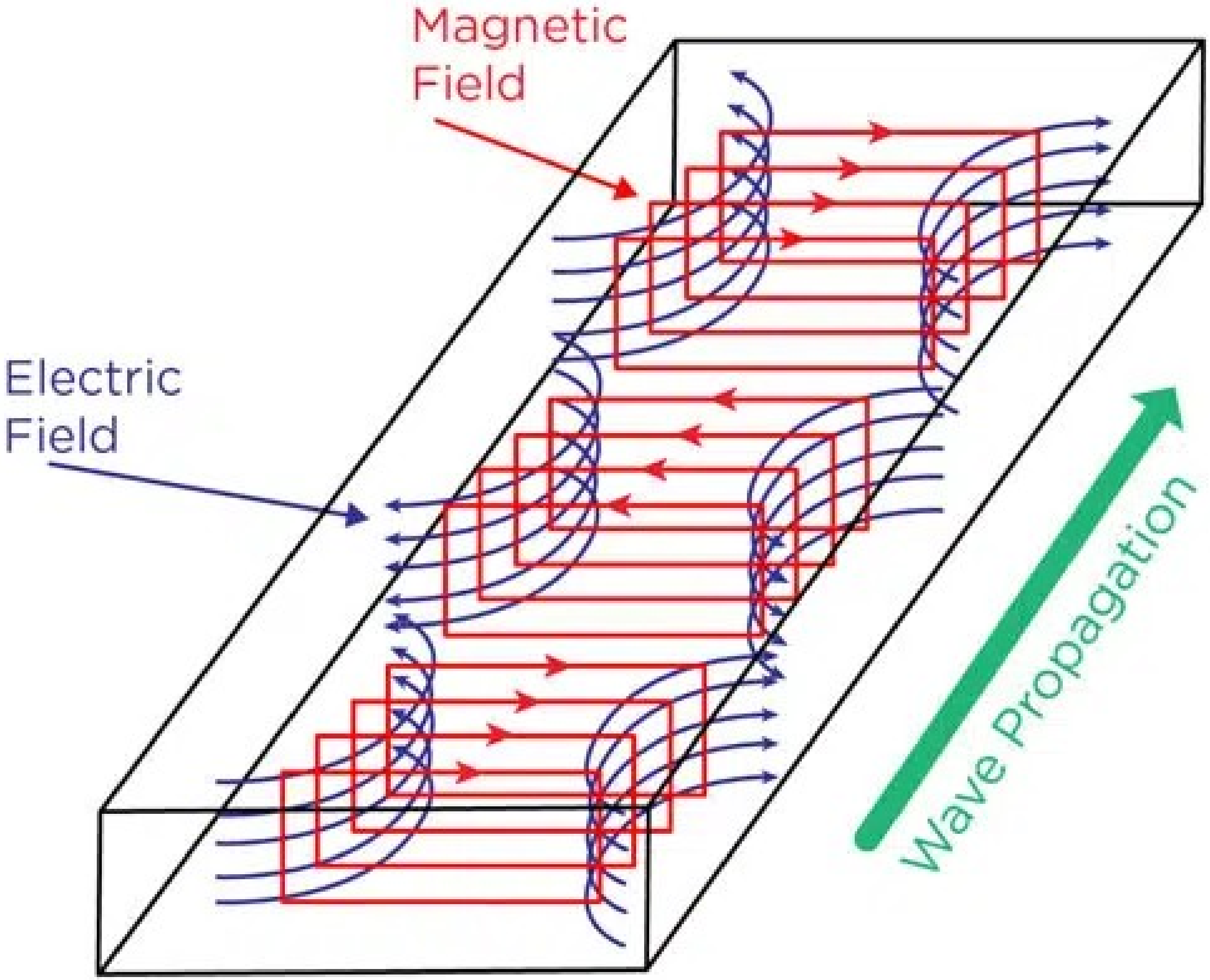
- The solution is called the TM_{mn} mode, assuming $a \geq b$. Neither one of the indices can be 0.
- If $\omega < c\,\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}} \equiv \omega_{m\,n}$, the wave number k_z is imaginary, and instead of a traveling wave we have exponentially attenuated fields. For this reason, $\omega_{m\,n}$ is called the **cutoff frequency** for the mode in question.
- The **cutoff wavelength**: $\lambda_{m\,n} = \frac{2\,\pi\,c}{\omega_{m\,n}} = \frac{2\,a\,b}{\sqrt{b^2\,m^2 + a^2\,n^2}}$
- The *lowest* cutoff frequency for a given wave guide occurs for the mode TM_{11}: $\omega_{11} = c\,\pi\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}$. Frequencies less than this will not propagate at all.
- The wave number can be written in terms of the cutoff frequency:

$$k_z = \frac{\sqrt{\omega^2 - \omega_{m\,n}^2}}{c} \quad \Rightarrow \quad \text{phase velocity} \quad v_p = \frac{\omega}{k_z} = \frac{\omega\,c}{\sqrt{\omega^2 - \omega_{m\,n}^2}} > c$$

The TM$_{11}$ mode

Electric field: solid lines, magnetic field: dashed lines





Energy Flow and Attenuation in Waveguides

- $$\mathbf{S} = \frac{\mathbf{E}\times\mathbf{H}^*}{2} = \frac{\omega\, k}{2\,\gamma^4}\begin{bmatrix}\epsilon\\ \mu\end{bmatrix}\left(\hat{\mathbf{z}}\,|\nabla_t\,\psi|^2 + i\,\frac{\gamma^2}{k}\begin{bmatrix}\psi\\ (-\psi^*)\end{bmatrix}\nabla_t\begin{bmatrix}\psi^*\\ \psi\end{bmatrix}\right) \quad \begin{matrix}\text{TM}\\ \text{TE}\end{matrix} \quad \Leftarrow \quad (3)\ \&\ (4)$$

$\Rightarrow$ $\mathbf{S}_t$: reactive energy flow
$\mathbf{S}_z$: time-averaged flow of energy $\quad \Leftarrow \quad \psi\in\mathbb{R} \quad \Downarrow \quad (\nabla_t^2+\gamma^2)\,\psi=0$

$$\Rightarrow \quad P = \int_{\mathcal{A}} \mathbf{S}\cdot\mathrm{d}\,\boldsymbol{a}_z = \frac{\omega\, k}{2\,\gamma^4}\begin{bmatrix}\epsilon\\ \mu\end{bmatrix}\int_{\mathcal{A}} \nabla_t\,\psi^*\cdot\nabla_t\,\psi\,\mathrm{d}\,a \qquad \Leftarrow \quad \psi_{|S}=0 \ \ \text{TM}$$

$$= \frac{\omega\, k}{2\,\gamma^4}\begin{bmatrix}\epsilon\\ \mu\end{bmatrix}\left(\oint_{\mathcal{C}} \psi^*\,\cancel{\frac{\partial\,\psi}{\partial\, n}}\,\mathrm{d}\,\ell - \int_{\mathcal{A}} \psi^*\,\nabla_t^2\,\psi\,\mathrm{d}\,a\right) \quad \Leftarrow \quad \frac{\partial\,\psi}{\partial\, n}_{|S} = 0 \ \ \text{TE}$$

$$\Rightarrow \quad P = \frac{\omega}{2\sqrt{\mu\,\epsilon}}\,\frac{\sqrt{\omega^2-\omega_\lambda^2}}{\omega_\lambda^2}\begin{bmatrix}\epsilon\\ \mu\end{bmatrix}\int_{\mathcal{A}} |\psi|^2\,\mathrm{d}\,a\,, \quad \text{similarly} \quad U = \frac{1}{2}\,\frac{\omega^2}{\omega_\lambda^2}\begin{bmatrix}\epsilon\\ \mu\end{bmatrix}\int_{\mathcal{A}} |\psi|^2\,\mathrm{d}\,a$$

$$v_g < v_p\,, \quad v_g(\omega_\lambda) = 0$$

$$\Rightarrow \quad \frac{P}{U} = \frac{k}{\omega}\,\frac{1}{\mu\,\epsilon} = \frac{1}{\sqrt{\mu\,\epsilon}}\,\frac{\sqrt{\omega^2-\omega_\lambda^2}}{\omega} = v_g = \frac{\mathrm{d}\,\omega}{\mathrm{d}\,k} \quad \begin{matrix}\text{group}\\ \text{velocity}\end{matrix} \quad \Rightarrow \quad v_g\,v_p = \frac{1}{\mu\,\epsilon} = c^2$$

$$\Rightarrow \quad \omega\,\mathrm{d}\,\omega \propto k\,\mathrm{d}\,k$$

- If the walls have a finite conductivity, there is ohmic losses and the power flow along the guide will be attenuated.

α_λ : unimportant except near cutoff when $k_\lambda^{(0)} \to 0$

$$k_\lambda \simeq k_\lambda^{(0)} + \alpha_\lambda + i\,\beta_\lambda \quad \Leftarrow \quad \beta_\lambda = -\frac{1}{2\,P}\,\frac{\mathrm{d}\,P}{\mathrm{d}\,z} \quad \Leftarrow \quad P = P_0\,e^{-2\,\beta_\lambda\, z}$$

For the TM modes: $H_z = 0\,, \quad \mathbf{E}_t = \dfrac{i\,k}{\gamma^2}\nabla_t E_z\,, \quad \mathbf{H}_t = \dfrac{\hat{\mathbf{z}}\times\mathbf{E}}{Z}\,, \quad Z = \dfrac{k}{\epsilon\,\omega}$

$$\Rightarrow \quad \mathbf{S} = \frac{\mathbf{E}\times\mathbf{H}^*}{2} = \frac{\mathbf{E}\times\mathbf{H}_t^*}{2} = \frac{(\mathbf{E}_t+\mathbf{E}_z)\times(\hat{\mathbf{z}}\times\mathbf{E}_t^*)}{2\,Z} = \frac{|\mathbf{E}_t|^2\,\hat{\mathbf{z}} - E_z\,\mathbf{E}_t^*}{2\,Z}$$

$$= \frac{\epsilon\,\omega}{2\,k}\left(\frac{k^2}{\gamma^4}|\nabla_t E_z|^2\,\hat{\mathbf{z}} + i\,\frac{k}{\gamma^2}E_z\nabla_t E_z^*\right)$$

$$= \frac{\epsilon\,\omega\,k}{2\,\gamma^4}\left(|\nabla_t E_z|^2\,\hat{\mathbf{z}} + i\,\frac{\gamma^2}{k}E_z\nabla_t E_z^*\right)$$

For the TE modes: $E_z = 0\,, \quad \mathbf{H}_t = \dfrac{i\,k}{\gamma^2}\nabla_t H_z\,, \quad \mathbf{E}_t = Z\,(\mathbf{H}\times\hat{\mathbf{z}})\,, \quad Z = \dfrac{\mu\,\omega}{k}$

$$\Rightarrow \quad \mathbf{S} = \frac{\mathbf{E}\times\mathbf{H}^*}{2} = \frac{\mathbf{E}_t\times\mathbf{H}^*}{2} = Z\,\frac{(\mathbf{H}_t\times\hat{\mathbf{z}})\times(\mathbf{H}_t^*+\mathbf{H}_z^*)}{2} = Z\,\frac{|\mathbf{H}_t|^2\,\hat{\mathbf{z}} - H_z^*\,\mathbf{H}_t}{2}$$

$$= \frac{\mu\,\omega}{2\,k}\left(\frac{k^2}{\gamma^4}|\nabla_t H_z|^2\,\hat{\mathbf{z}} - i\,\frac{k}{\gamma^2}H_z^*\nabla_t H_z\right)$$

$$= \frac{\mu\,\omega\,k}{2\,\gamma^4}\left(|\nabla_t H_z|^2\,\hat{\mathbf{z}} - i\,\frac{\gamma^2}{k}H_z^*\nabla_t H_z\right)$$

$$-\frac{d\,P}{d\,z}=\frac{1}{2\,\sigma\,\delta}\oint_{C}|\hat{\mathbf{n}}\times\mathbf{H}|^{2}\,d\,\ell \underset{(11)}{\overset{(15)\ \Downarrow}{=}} \frac{1}{2\,\sigma\,\delta}\overset{\Downarrow(3)\ \&\ (4)}{\left[\begin{array}{lr}\displaystyle\oint_{C}\frac{\omega^{2}}{\mu^{2}\,\omega_{\lambda}^{4}}\left|\frac{\partial\,\psi}{\partial\,n}\right|^{2}d\,\ell & \text{TM}\\ \displaystyle\oint_{C}\left(\frac{\omega^{2}-\omega_{\lambda}^{2}}{\mu\,\epsilon\,\omega_{\lambda}^{4}}|\hat{\mathbf{n}}\times\nabla_{t}\,\psi|^{2}+|\psi|^{2}\right)d\,\ell & \text{TE}\end{array}\right.}$$

$$\left\langle\left|\frac{\partial\,\psi}{\partial\,n}\right|^{2}\right\rangle\sim\langle|\hat{\mathbf{n}}\times\nabla_{t}\,\psi|^{2}\rangle\sim\mu\,\epsilon\,\omega_{\lambda}^{2}\,\langle|\psi|^{2}\rangle \quad \Leftarrow \quad (\nabla_{t}^{2}+\mu\,\epsilon\,\omega_{\lambda}^{2})\,\psi=0$$

$$\Rightarrow \quad \oint_{C}\left|\frac{\partial\,\psi}{\partial\,n}\right|^{2}\frac{d\,\ell}{\omega_{\lambda}^{2}}=\xi_{\lambda}\,\mu\,\epsilon\,\frac{C}{A}\int_{A}|\psi|^{2}\,d\,a\,, \qquad \delta_{\lambda}\equiv\sqrt{\frac{2}{\mu\,\sigma\,\omega_{\lambda}}}=\delta\,\sqrt{\frac{\omega_{\lambda}}{\omega}}$$

$$\Rightarrow \quad \beta_{\lambda}=\sqrt{\frac{\epsilon}{\mu}}\,\frac{1}{\sigma\,\delta_{\lambda}}\,\frac{C}{2\,A}\,\frac{\xi_{\lambda}\,\omega^{2}+\eta_{\lambda}\,\omega_{\lambda}^{2}}{\sqrt{\omega\,\omega_{\lambda}\,(\omega^{2}-\omega_{\lambda}^{2})}} \qquad (5) \qquad \Leftarrow \quad \eta_{\lambda}=0 \quad \text{for TM}$$

- For the TE modes in a rectangular guide, $\xi_{m\,0}=\frac{a}{a+b}\,,\quad \eta_{m\,0}=\frac{2\,b}{a+b} \quad \Leftarrow$ order of unity
- For TM modes the minimum always occurs at $\omega_{\min}=\sqrt{3}\,\omega_{\lambda} \quad \Leftarrow \quad \frac{d\,\beta_{\lambda}}{d\,\omega}=0$
- At high frequencies the attenuation increases as $\sqrt{\omega}$.

For the TM modes
consider $E_z = \psi_\lambda$: $H_z = 0\,, \quad \mathbf{E}_t = \dfrac{i\,k_\lambda}{\gamma_\lambda^2}\,\nabla_t\,E_z\,, \quad \mathbf{H}_t = \dfrac{\hat{\mathbf{z}}\times\mathbf{E}_t}{Z}\,, \quad Z = \dfrac{k_\lambda}{\epsilon\,\omega}$

$$\Rightarrow \quad |\hat{\mathbf{n}}\times\mathbf{H}|^2 = (\hat{\mathbf{n}}\times\mathbf{H}_t)\cdot(\hat{\mathbf{n}}\times\mathbf{H}_t^\star) = \frac{[\hat{\mathbf{n}}\times(\hat{\mathbf{z}}\times\mathbf{E}_t)]\cdot[\hat{\mathbf{n}}\times(\hat{\mathbf{z}}\times\mathbf{E}_t^\star)]}{|Z|^2}$$

$$= \frac{|\hat{\mathbf{n}}\cdot\mathbf{E}_t|^2}{|Z|^2} = \frac{\epsilon^2\,\omega^2}{\gamma_\lambda^4}\,|\hat{\mathbf{n}}\cdot\nabla_t\,E_z|^2 = \frac{\omega^2}{\mu^2\,\omega_\lambda^4}\left|\frac{\partial\,E_z}{\partial\,n}\right|^2 \quad \Leftarrow \quad \omega_\lambda^2 = \frac{\gamma_\lambda^2}{\mu\,\epsilon}$$

For the TE modes
consider $H_z = \psi_\lambda$: $E_z = 0\,, \quad \mathbf{H}_t = \dfrac{i\,k}{\gamma^2}\,\nabla_t\,H_z$

$$\Rightarrow \quad |\hat{\mathbf{n}}\times\mathbf{H}|^2 = [\hat{\mathbf{n}}\times(\mathbf{H}_t + \mathbf{H}_z)]\cdot[\hat{\mathbf{n}}\times(\mathbf{H}_t^\star + \mathbf{H}_z^\star)]$$

$$= |\hat{\mathbf{n}}\times\mathbf{H}_t|^2 + (\hat{\mathbf{n}}\times\mathbf{H}_t)\cdot(\hat{\mathbf{n}}\times\mathbf{H}_z^\star) + (\hat{\mathbf{n}}\times\mathbf{H}_z)\cdot(\hat{\mathbf{n}}\times\mathbf{H}_t^\star) + |\hat{\mathbf{n}}\times\mathbf{H}_z|^2$$

$$= \frac{k_\lambda^2}{\gamma_\lambda^4}\,|\hat{\mathbf{n}}\times\nabla_t\,H_z|^2 + |H_z|^2 \quad \Leftarrow \quad \begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = {} & (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) \\ & -(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})\end{aligned}$$

$$= \frac{\omega^2 - \omega_\lambda^2}{\mu\,\epsilon\,\omega_\lambda^4}\,|\hat{\mathbf{n}}\times\nabla_t\,H_z|^2 + |H_z|^2 \quad \Leftarrow \quad k_\lambda^2 = \mu\,\epsilon\,(\omega^2 - \omega_\lambda^2)$$

For the TM modes $\left(H_z = 0\right)$: Consider $E_z = \psi_\lambda$

$$-\frac{\mathrm{d}\,P}{\mathrm{d}\,z} = \frac{1}{2\,\sigma\,\delta} \oint_{\mathcal{C}} \frac{\omega^2}{\mu^2\,\omega_\lambda^4} \left|\frac{\partial\,E_z}{\partial\,n}\right|^2 \mathrm{d}\,\ell = \frac{\xi_\lambda}{2\,\sigma\,\delta_\lambda} \sqrt{\frac{\omega}{\omega_\lambda}}\,\frac{\epsilon\,\omega^2}{\mu\,\omega_\lambda^2}\,\frac{C}{A} \int_{\mathcal{A}} |E_z|^2\,\mathrm{d}\,a$$

$$P = \frac{\omega}{2}\sqrt{\frac{\epsilon}{\mu}}\,\frac{\sqrt{\omega^2 - \omega_\lambda^2}}{\omega_\lambda^2} \int_{\mathcal{A}} |E_z|^2\,\mathrm{d}\,a$$

$$\Rightarrow \quad \beta_\lambda = -\frac{1}{2\,P}\,\frac{\mathrm{d}\,P}{\mathrm{d}\,z} = \sqrt{\frac{\epsilon}{\mu}}\,\frac{C}{2\,A\,\sigma\,\delta_\lambda}\,\frac{\xi_\lambda\,\omega^2}{\sqrt{\omega\,\omega_\lambda\left(\omega^2 - \omega_\lambda^2\right)}}$$

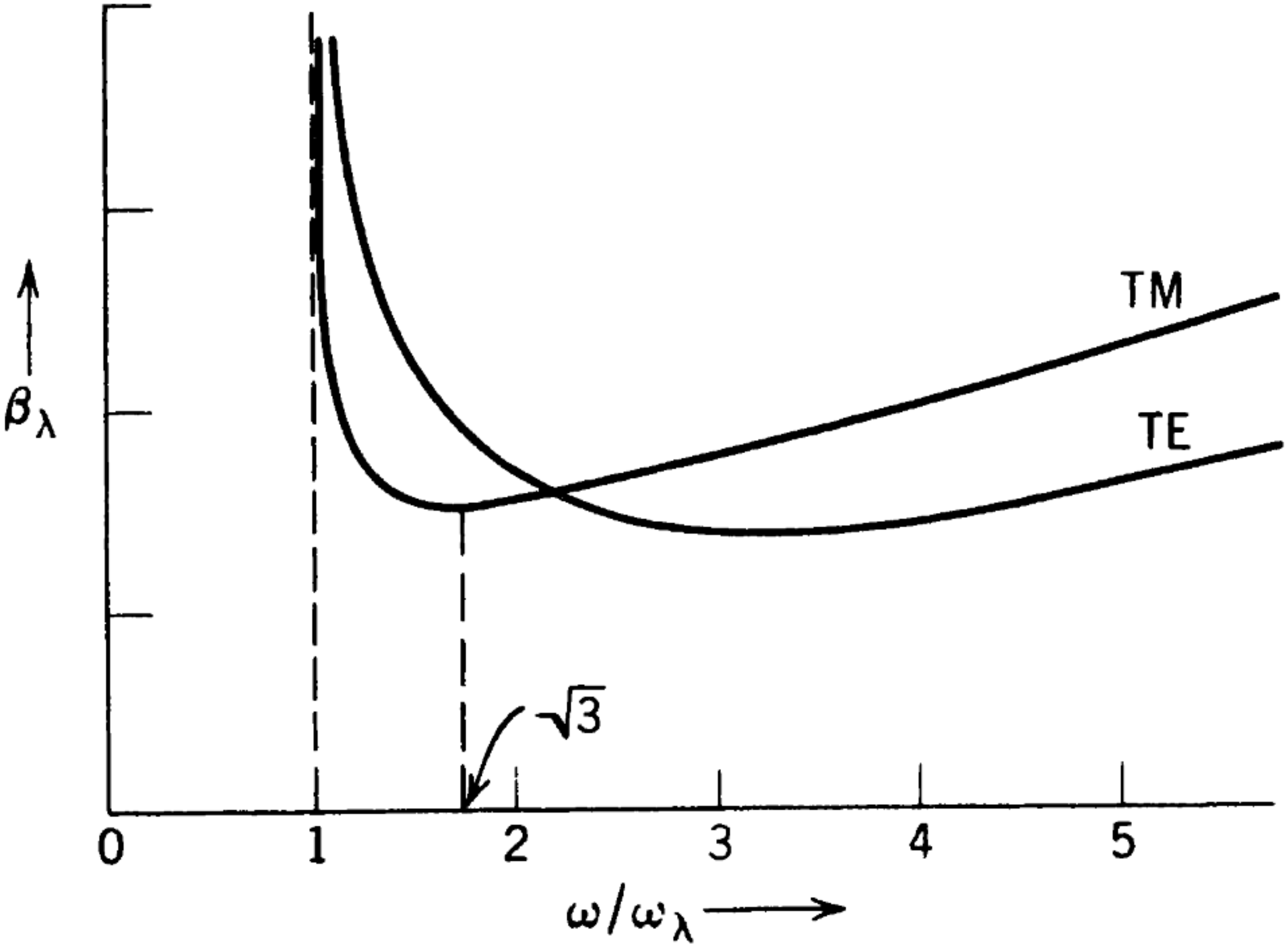
For the TE modes $\left(E_z = 0\right)$: Consider $H_z = \psi_\lambda$

$$-\frac{\mathrm{d}\,P}{\mathrm{d}\,z} = \frac{1}{2\,\sigma\,\delta} \oint_{\mathcal{C}} \left(\frac{\omega^2 - \omega_\lambda^2}{\mu\,\epsilon\,\omega_\lambda^4}\,|\hat{\mathbf{n}} \times \nabla_t\,H_z|^2 + |H_z|^2\right) \mathrm{d}\,\ell$$

$$= \frac{1}{2\,\sigma\,\delta_\lambda}\sqrt{\frac{\omega}{\omega_\lambda}}\,\frac{C}{A}\left(\xi_\lambda\,\frac{\omega^2 - \omega_\lambda^2}{\omega_\lambda^2} + \eta_\lambda'\right) \int_{\mathcal{A}} |H_z|^2\,\mathrm{d}\,a$$

$$= \frac{1}{2\,\sigma\,\delta_\lambda}\sqrt{\frac{\omega}{\omega_\lambda}}\,\frac{C}{A}\,\frac{\xi_\lambda\,\omega^2 + \eta_\lambda\,\omega_\lambda^2}{\omega_\lambda^2} \int_{\mathcal{A}} |H_z|^2\,\mathrm{d}\,a \quad \Leftarrow \quad \eta_\lambda = \eta_\lambda' - \xi_\lambda$$

$$\Rightarrow \quad \beta_\lambda = -\frac{1}{2\,P}\,\frac{\mathrm{d}\,P}{\mathrm{d}\,z} = \sqrt{\frac{\epsilon}{\mu}}\,\frac{C}{2\,A\,\sigma\,\delta_\lambda}\,\frac{\xi_\lambda\,\omega^2 + \eta_\lambda\,\omega_\lambda^2}{\sqrt{\omega\,\omega_\lambda\left(\omega^2 - \omega_\lambda^2\right)}}$$



- In the microwave region typical attenuation constants for copper guides are of the order $\beta_\lambda \sim 10^{-4}\dfrac{\omega_\lambda}{c}$, giving $\dfrac{1}{e}$ distances of 200-400 meters.
- (5) break down close to cutoff for $\beta_\lambda \to \infty \quad \text{at} \quad \omega = \omega_\lambda$

Perturbation of Boundary Conditions

- The use of energy conservation can determine the attenuation constant β_λ, but gives physically meaningless results at cutoff and fails to yield a value for α_λ.

—can be remedied by use of the *perturbation of boundary conditions*.

- Consider a single TM mode with no other mode degenerate with it

$$(\nabla_t^2+\gamma_0^2)\,\psi_0=0\,,\quad \psi_{0|S}=0\,,\quad \gamma_0^2\in\mathbb{R}\quad\Leftarrow\quad E_z=\psi_0\quad \text{unperturbed}$$

$$(6)\quad\Rightarrow\quad \partial_z\,\mathbf{E}_t+i\,\mu\,\omega\,\hat{\mathbf{z}}\times\mathbf{H}_t=\nabla_t\,E_z$$

$$(4)\quad\Rightarrow\quad \mathbf{E}_t=\frac{i\,k_0}{\gamma_0^2}\nabla_t\,E_z\quad\Rightarrow\quad\left(\frac{\gamma_0^2+k_0^2}{\gamma_0^2}\nabla_t\,E_z=i\,\mu\,\omega\,\hat{\mathbf{z}}\times\mathbf{H}_t\right)\cdot\hat{\mathbf{n}}$$

$$\Rightarrow\quad\frac{\gamma_0^2+k_0^2}{\gamma_0^2}\,\frac{\partial E_z}{\partial n}_{|S}=-i\,\mu\,\omega\,H_\ell=\frac{\delta\,\sigma\,\mu\,\omega}{1+i}E_z\quad\Leftarrow\quad\mathbf{E}_c\simeq\frac{1-i}{\delta\,\sigma}\hat{\mathbf{n}}\times\mathbf{H}_c\quad\Leftarrow\quad(0)$$

$$\Rightarrow\quad E_z=\frac{1+i}{\delta\,\sigma\,\mu\,\omega}\,\frac{\mu\,\epsilon\,\omega^2}{\mu\,\epsilon\,\omega_0^2}\,\frac{\partial E_z}{\partial n}_{|S}=\frac{1+i}{2}\,\frac{\mu_c}{\mu}\,\frac{\omega^2}{\omega_0^2}\,\delta\,\frac{\partial E_z}{\partial n}_{|S}\quad\Leftarrow\begin{matrix}\mu\,\epsilon\,\omega^2-k_0^2=\gamma_0^2=\mu\,\epsilon\,\omega_0^2\\ \mu_c\,\sigma\,\omega\,\delta^2=2\end{matrix}$$

$$\Rightarrow\quad\psi_{|S}\simeq f\,\frac{\partial\psi_0}{\partial n}_{|S}\quad\Leftarrow\quad f=\frac{1+i}{2}\,\frac{\mu_c}{\mu}\,\frac{\omega^2}{\omega_0^2}\,\delta\quad\Leftarrow\quad\omega_0:$$

cutoff frequency of
the unperturbed mode

the perturbed problem $(\nabla_t^2+\gamma^2)\,\psi=0\,,\quad\psi_{|S}\simeq f\,\dfrac{\partial\psi_0}{\partial n}_{|S}\qquad(7)$

$$\int_{\mathcal{A}} \left(\phi\, \nabla_t^2 \psi - \psi\, \nabla_t^2 \phi\right) \mathrm{d}\, a = \oint_{\mathcal{C}} \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n}\right) \mathrm{d}\, \ell$$

2d Green's theorem

$+ \quad \phi = \psi_0^\star$

$$\Rightarrow \quad \left(\gamma_0^2 - \gamma^2\right) \int_{\mathcal{A}} \psi_0^\star \psi \, \mathrm{d}\, a = f \oint_{\mathcal{C}} \left|\frac{\partial \psi_0}{\partial n}\right|^2 \mathrm{d}\, \ell = \frac{1+i}{2} \frac{\mu_c}{\mu} \frac{\omega^2}{\omega_0^2} \delta \oint_{\mathcal{C}} \left|\frac{\partial \psi_0}{\partial n}\right|^2 \mathrm{d}\, \ell$$

$$\Rightarrow \quad \gamma_0^2 - \gamma^2 = k^2 - k^{(0)\,2} \simeq f \frac{\displaystyle\oint_{\mathcal{C}} \left|\frac{\partial \psi_0}{\partial n}\right|^2 \mathrm{d}\, \ell}{\displaystyle\int_{\mathcal{S}} \left|\psi_0\right|^2 \mathrm{d}\, a} \simeq 2\, k^{(0)} \left(\alpha^{(0)} + i\, \beta_{\mathrm{TM}}^{(0)}\right) \quad \Leftarrow \quad \begin{aligned} &k - k^{(0)} \\ &= \alpha^{(0)} + i\, \beta^{(0)} \end{aligned}$$

$$\Rightarrow \quad k^{(0)} \beta_{\mathrm{TM}}^{(0)} \int_{\mathcal{A}} \left|\psi_0\right|^2 \mathrm{d}\, a = \frac{\delta}{4} \frac{\mu_c}{\mu} \frac{\omega^2}{\omega_0^2} \oint_{\mathcal{C}} \left|\frac{\partial \psi_0}{\partial n}\right|^2 \mathrm{d}\, \ell \quad \Rightarrow \quad \beta_{\mathrm{TM}}^{(0)} = (5)$$

$$\Rightarrow \quad \alpha^{(0)} = \beta^{(0)} \quad \Leftarrow \quad \alpha^{(0)},\, \beta^{(0)} \ll k^{(0)} \quad \Rightarrow \quad k^2 \simeq k^{(0)\,2} + 2\left(1+i\right) k^{(0)} \beta^{(0)} \qquad (8)$$

for both the TM/TE mode

- At cutoff and below where the earlier results failed, (8) yields sensible results because $k^{(0)}\beta^{(0)}$ is finite and well behaved in the neighborhood of $k^{(0)}=0$.

- The transition from a propagating mode to a cutoff mode is not a sharp one if the walls are less than perfect conductors, but the attenuation is sufficiently large immediately above & below the cutoff frequency that little error is made in assuming a sharp cutoff.

- If a TM and a TE mode are degenerate, then any perturbation can cause sizable mixing of the 2 modes. The methods used so far fail in such circumstances.

- The breakdown of this method occurs in the perturbed boundary condition (7), now involving the tangential derivative of H_z and the normal derivative of E_z.

- The perturbed modes are orthogonal linear combinations of the unperturbed

TM and TE modes, $\beta = \dfrac{\beta_{\mathrm{TM}} + \beta_{\mathrm{TE}} \pm \sqrt{(\beta_{\mathrm{TM}} - \beta_{\mathrm{TE}})^2 + 4|K|^2}}{2}$ ⇐ K : coupling parameter

Resonant Cavities

- An important class of cavities is produced by placing end faces on a length of cylindrical waveguide—the end surfaces are plane ⊥ the axis of the cylinder.

- Because of reflections at the end surfaces, the z dependence of the fields is that appropriate to standing waves: $A \sin k z + B \cos k z$

- $z \in [0, d] \;\Rightarrow\; k = p\frac{\pi}{d}, \quad p = 0, 1, 2, \cdots$

$$\Rightarrow \left.\begin{matrix} \mathbf{E}_t \\ \\ H_z \end{matrix}\right] = 0 \text{ at both } \begin{matrix} z = 0 \\ \text{and} \\ z = d \end{matrix} \;\Rightarrow\; \left[\begin{matrix} E_z = \psi(x, y) \cos \dfrac{p\pi z}{d} & \text{TM} \\ \\ H_z = \psi(x, y) \sin \dfrac{p\pi z}{d} & \text{TE} \end{matrix}\right. \qquad (9) \;\Leftarrow\; \left[\begin{matrix} \mathbf{E}_{\parallel} = 0 \\ \\ H_{\perp} = 0 \end{matrix}\right.$$

$$\Rightarrow \quad \mathbf{E}_t = -\sin\frac{p\pi z}{d}\left[\begin{matrix} \dfrac{p\pi}{d\gamma^2}\nabla_t \psi \\ \dfrac{i\mu\omega}{\gamma^2}\hat{\mathbf{z}} \times \nabla_t \psi \end{matrix}\right. , \quad \mathbf{H}_t = \cos\frac{p\pi z}{d}\left[\begin{matrix} \dfrac{i\epsilon\omega}{\gamma^2}\hat{\mathbf{z}} \times \nabla_t \psi & \text{TM} \\ \dfrac{p\pi}{d\gamma^2}\nabla_t \psi & \text{TE} \end{matrix}\right. \qquad (10)$$

$$\text{where } \gamma^2 = \mu\epsilon\omega^2 - \frac{p^2\pi^2}{d^2} \;\Rightarrow\; \omega_{\lambda p}^2 = \frac{\gamma_\lambda^2 + k_p^2}{\mu\epsilon} = \frac{1}{\mu\epsilon}\left(\gamma_\lambda^2 + \frac{p^2\pi^2}{d^2}\right)$$

For the TM modes $\left(H_z = 0\right)$: Consider $E_z = \psi\left(x\,,\,y\right)\left(A\sin k_z z + \cos k_z z\right)$

$$\left.\begin{aligned}\partial_z\,\mathbf{E}_t + i\,\mu\,\omega\,\hat{\mathbf{z}}\times\mathbf{H}_t &= \nabla_t\,E_z\\ \partial_z\,\mathbf{H}_t - i\,\epsilon\,\omega\,\hat{\mathbf{z}}\times\mathbf{E}_t &= \quad 0\end{aligned}\right. \quad\Rightarrow\quad \partial_z^2\,\mathbf{E}_t + \mu\,\epsilon\,\omega^2\,\mathbf{E}_t = \partial_z\,\nabla_t\,E_z$$

$$\Rightarrow\quad\left(\mu\,\epsilon\,\omega^2 - k_z^2\right)\mathbf{E}_t = \gamma^2\,\mathbf{E}_t = k_z\left(A\cos k_z z - \sin k_z z\right)\nabla_t\,\psi$$

$$\mathbf{E}_t\left(z=0\right) = \mathbf{E}_t\left(z=d\right) = 0 \quad\Rightarrow\quad A = 0\,,\quad k_z = \frac{p\,\pi}{d} \quad\Leftarrow\quad p = 1, 2, 3, \cdots$$

$$\Rightarrow\quad E_z = \psi\cos k_z z\,,\quad \mathbf{E}_t = -\frac{k_z}{\gamma^2}\sin k_z z\,\nabla_t\,\psi\,,\quad \mathbf{H}_t = i\,\frac{\epsilon\,\omega}{\gamma^2}\left(\hat{\mathbf{z}}\times\nabla_t\,\psi\right)\cos k_z z$$

For the TE modes $\left(E_z = 0\right)$: Consider $H_z = \psi\left(x\,,\,y\right)\left(\sin k_z z + A\cos k_z z\right)$

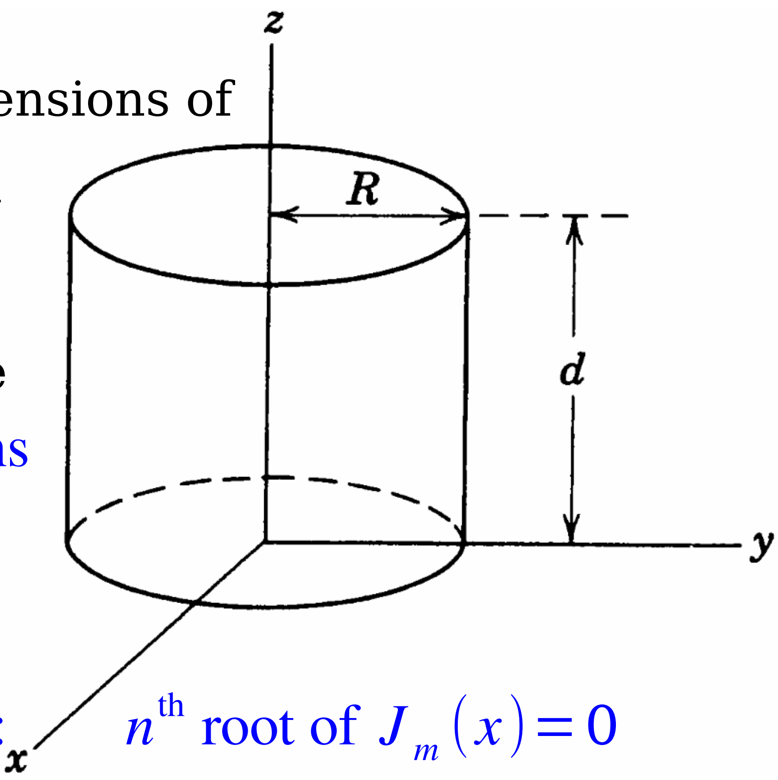
$$H_z\left(z=0\right) = H_z\left(z=d\right) = 0 \quad\Rightarrow\quad A = 0\,,\quad k_z = \frac{p\,\pi}{d} \quad\Leftarrow\quad p = 1, 2, 3, \cdots$$

$$\left.\begin{aligned}\partial_z\,\mathbf{E}_t + i\,\mu\,\omega\,\hat{\mathbf{z}}\times\mathbf{H}_t &= \quad 0\\ \partial_z\,\mathbf{H}_t - i\,\epsilon\,\omega\,\hat{\mathbf{z}}\times\mathbf{E}_t &= \nabla_t\,H_z\end{aligned}\right. \quad\Rightarrow\quad \partial_z^2\,\mathbf{H}_t + \mu\,\epsilon\,\omega^2\,\mathbf{H}_t = \partial_z\,\nabla_t\,H_z$$

$$\Rightarrow\quad\left(\mu\,\epsilon\,\omega^2 - k_z^2\right)\mathbf{H}_t = \gamma^2\,\mathbf{H}_t = k_z\cos k_z z\,\nabla_t\,\psi$$

$$\Rightarrow\quad H_z = \psi\sin k_z z\,,\quad \mathbf{H}_t = \frac{k_z}{\gamma^2}\left(\nabla_t\,\psi\right)\cos k_z z\,,\quad \mathbf{E}_t = -\,i\,\frac{\mu\,\omega}{\gamma^2}\left(\hat{\mathbf{z}}\times\nabla_t\,\psi\right)\sin k_z z$$

- It is usually expedient to choose the various dimensions of the cavity so that the resonant frequency lies well separated from other resonant freq. and the cavity will be stable and insensitive to perturbing effects.



- Consider a right circular cylinder, for a TM mode

$\psi = E_z\,, \quad E_z(\rho = R) = 0\,,$ modified Bessel functions

$$\Rightarrow \quad \psi(\rho\,,\phi) = E_0\, J_m(\gamma_{m\,n}\,\rho)\, e^{\pm i\,m\,\phi} \quad \Leftarrow \quad \gamma_{m\,n} = \frac{x_{m\,n}}{R}$$

$$\Rightarrow \quad \text{resonant frequency} \quad \omega_{m\,n\,p} = \frac{1}{\sqrt{\mu\,\epsilon}}\sqrt{\frac{x^2_{m\,n}}{R^2} + \frac{p^2\,\pi^2}{d^2}} \quad \Leftarrow \quad x_{m\,n} : \quad n^{\text{th}} \text{ root of } J_m(x) = 0$$

$$m : 0, 1, 2\,, \cdots, \quad n : 1, 2, 3\,, \cdots$$

$$\Rightarrow \quad \text{the lowest TM mode TM}_{010} \text{ has } m = 0,\, n = 1,\, p = 0 \quad \Rightarrow \quad \omega_{010} = \frac{2.405}{\sqrt{\mu\,\epsilon}\,R}$$

$$\Rightarrow \quad E_z = E_0\, J_0\left(\frac{2.405\,\rho}{R}\right) e^{-i\,\omega\,t}\,,\, E_\rho = 0\,, \quad H_\phi = -i\sqrt{\frac{\epsilon}{\mu}}\, E_0\, J_1\left(\frac{2.405\,\rho}{R}\right) e^{-i\,\omega\,t}$$

- The resonant frequency for this mode is independent of d. So simple tuning is impossible.

- For TE modes, $\psi = H_z$, $\frac{\partial\,\psi}{\partial\,\rho}\Big|_R = 0$ $\quad \Rightarrow \quad \psi(\rho\,,\phi) = E_0\, J_m(\gamma_{m\,n}\,\rho)\, e^{\pm i\,m\,\phi} \quad \Leftarrow \quad \gamma_{m\,n} = \frac{x'_{m\,n}}{R}$

$$x'_{m\,n} : n^{\text{th}} \text{ root of } J'_m(x) = 0$$

- $\omega_{m\,n\,p} = \frac{1}{\sqrt{\mu\,\epsilon}} \sqrt{\frac{x'^{\,2}_{m\,n}}{R^2} + \frac{p^2\,\pi^2}{d^2}} \Leftarrow \quad m:0,1,2\,,\cdots, \quad n\,,\,p:1,2,3\,,\cdots$

$$\Rightarrow \begin{array}{c}\text{the lowest TE}\\ \text{mode TE}_{111}\end{array} \text{ has } m=1,\, n=1,\, p=1 \quad \Rightarrow \quad \omega_{111} = \frac{1.841}{\sqrt{\mu\,\epsilon}\,d}\sqrt{\frac{d^2}{R^2}+2.912}$$

$$\Rightarrow \quad \psi = H_z = H_0\,J_1\left(\frac{1.841\,\rho}{R}\right)\cos\phi\,\sin\frac{\pi\,z}{d}\,e^{-i\,\omega\,t} \quad \Rightarrow \quad \mathbf{H}_t\,,\ \mathbf{E}_t$$

- For $d>2.03R$, the resonance frequency ω_{111} is smaller than that for the lowest TM mode. Then the TE_{111} mode is the fundamental oscillation of the cavity.

- Because the frequency depends on the ratio $\frac{d}{R}$ it is possible to provide easy tuning by making the separation of the end faces adjustable.

Power Losses in a Cavity; Q of a Cavity

- If one attempts to excite a particular mode in a cavity, no fields of the right sort could be built up unless the exciting frequency were exactly equal to the chosen resonance frequency.

- In fact there will not be a delta function singularity, but rather a narrow band of frequencies around the eigenfrequency over which excitation can occur.

- An important source of this smearing out of the sharp frequency is the energy dissipation in the cavity walls and/or in the dielectric filling the cavity.

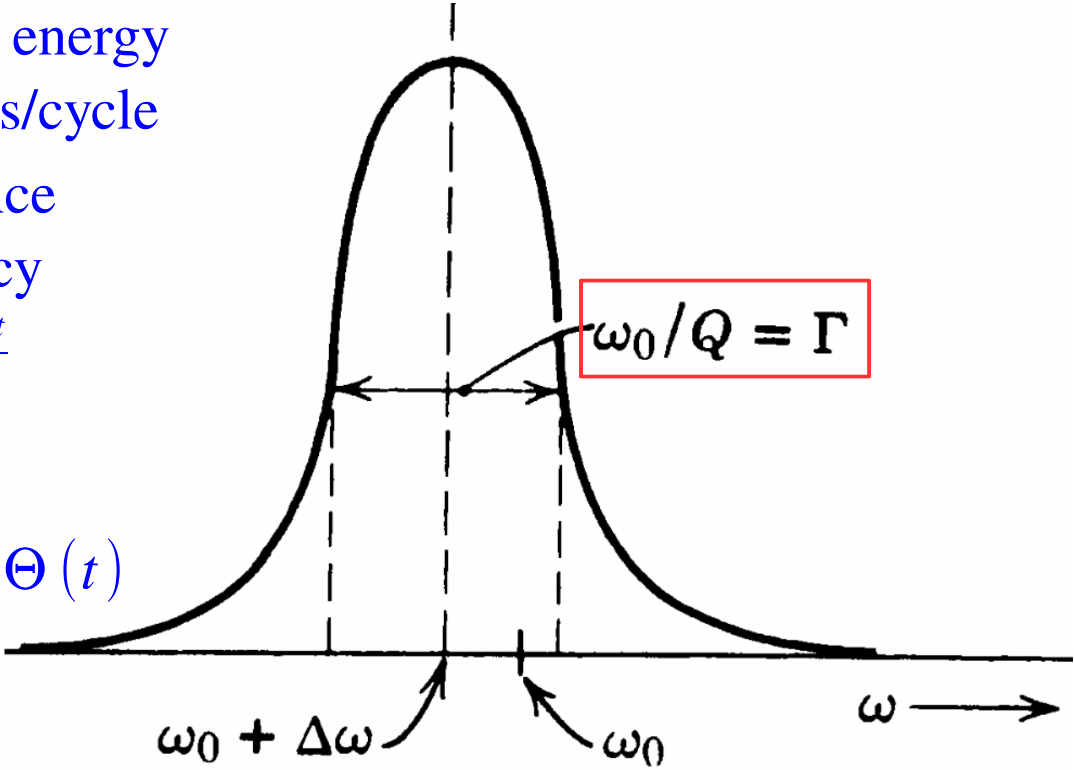
- A measure of the sharpness of response of the cavity to external excitation is

$Q =$ $2\pi \times$ the ratio of the time-averaged energy stored in the cavity to the energy loss/cycle

$$= \omega_0 \frac{\text{Stored energy}}{\text{Power loss}} \Leftarrow \quad \omega_0 : \text{resonance frequency}$$

$$\Rightarrow \quad \frac{dU}{dt} = -\frac{\omega_0}{Q} U \quad \Rightarrow \quad U(t) = U_0 e^{-\frac{\omega_0 t}{Q}}$$

$$\Rightarrow \quad \text{the field } E(t) = E_0 e^{-\frac{\omega_0 t}{2Q}} e^{-i(\omega_0 + \Delta\omega)t} \Theta(t) \qquad (12)$$



where $E(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} E(\omega)\, e^{-i\omega t}\, \mathrm{d}\omega$, $E(\omega)=\frac{E_0}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-\frac{\omega_0 t}{2Q}}\, e^{i(\omega-\omega_0-\Delta\omega)t}\, \mathrm{d}t$

$$\Rightarrow \quad |E(\omega)|^2 \propto \frac{1}{\frac{\omega_0^2}{4Q^2}+(\omega-\omega_0-\Delta\omega)^2} \qquad \Uparrow \frac{E_0}{\sqrt{2\pi}}\frac{1}{\frac{\omega_0}{2Q}-i(\omega-\omega_0-\Delta\omega)}$$

- The frequency separation between half-power points $Q=\frac{\omega_0}{\delta\omega}=\frac{\omega_0}{\Gamma} \Leftarrow \delta\omega=\frac{\omega_0}{Q}$
- To determine the Q of a cavity, consider only the cylindrical cavities

$$U=\frac{d}{4}\left[1+\left(\frac{p\pi}{\gamma_\lambda d}\right)^2\right]\begin{bmatrix}\epsilon\\ \mu\end{bmatrix}\int_{\mathcal{A}}|\psi|^2\,\mathrm{d}a \Leftarrow \begin{array}{l}\text{TM: } U=\text{the result}\times 2 \text{ for } p=0\\ \text{TE}\end{array} \Leftarrow \begin{array}{c}(9)\\(10)\end{array}$$

$$\Rightarrow \quad P_{\text{loss}}=\frac{1}{2\sigma\delta}\left(\oint_{\mathcal{C}}\int_0^d |\mathbf{n}\times\mathbf{H}|^2_{\text{sides}}\,\mathrm{d}z\,\mathrm{d}\ell+2\int_{\mathcal{A}}|\mathbf{n}\times\mathbf{H}|^2_{\text{ends}}\,\mathrm{d}a\right) \Leftarrow (11)$$

$$\Rightarrow \quad P_{\text{loss}}=\frac{\epsilon}{\sigma\delta\mu}\left[1+\left(\frac{p\pi}{\gamma_\lambda d}\right)^2\right]\left(1+\begin{bmatrix}1\\2\end{bmatrix}\xi_\lambda\frac{C\,d}{4A}\right)\int_{\mathcal{A}}|\psi|^2\,\mathrm{d}a \text{ for } \begin{bmatrix}p\neq 0\\ p=0\end{bmatrix} \text{ TM modes}$$

$$\Rightarrow \quad Q=2\frac{\mu}{\mu_c}\frac{d}{\delta}\left[\begin{array}{ll}\frac{A}{4A+\xi_\lambda C\,d} & \text{for } p\neq 0\\ \frac{A}{2A+\xi_\lambda C\,d} & \text{for } p=0\end{array}\right. \text{ of TM modes}$$

For the TM modes $\left(H_z = 0\right)$

$$\Rightarrow \quad E_z = \psi \cos k_z z \, , \quad \mathbf{E}_t = -\frac{k_z}{\gamma^2} \sin k_z z \, \nabla_t \psi \, , \quad \mathbf{H}_t = i \, \frac{\epsilon \, \omega}{\gamma^2} \left(\hat{\mathbf{z}} \times \nabla_t \psi \right) \cos k_z z$$

$$u = \frac{\epsilon \, |\mathbf{E}|^2 + \mu \, |\mathbf{H}|^2}{4} = \frac{\epsilon \left(|E_z|^2 + |\mathbf{E}_t|^2 \right) + \mu \, |\mathbf{H}_t|^2}{4}$$

$$= \frac{\epsilon}{4} \left(|\psi|^2 \cos^2 k_z z + \frac{k_z^2}{\gamma^4} |\nabla_t \psi|^2 \sin^2 k_z z \right) + \frac{\mu}{4} \frac{\epsilon^2 \omega^2}{\gamma^4} |\nabla_t \psi|^2 \cos^2 k_z z$$

$$U = \int u \, \mathrm{d}^3 x = \frac{\epsilon}{4} \int_0^d \cos^2 k_z z \, \mathrm{d} z \int_{\mathcal{A}} |\psi|^2 \, \mathrm{d} a$$

$$+ \frac{\epsilon}{4 \gamma^4} \int_0^d \left(k_z^2 \sin^2 k_z z + \mu \, \epsilon \, \omega^2 \cos^2 k_z z \right) \mathrm{d} z \int_{\mathcal{A}} |\nabla_t \psi|^2 \, \mathrm{d} a$$

$$= \frac{\epsilon \, d}{8} \left(1 + \frac{k_z^2 + \mu \, \epsilon \, \omega^2}{\gamma^2} \right) \int_{\mathcal{A}} |\psi|^2 \, \mathrm{d} a \quad \Leftarrow \quad \int_{\mathcal{A}} |\nabla_t \psi|^2 \, \mathrm{d} a = \gamma^2 \int_{\mathcal{A}} |\psi|^2 \, \mathrm{d} a$$

$$= \frac{\epsilon \, d}{4} \left(1 + \frac{k_z^2}{\gamma^2} \right) \int_{\mathcal{A}} |\psi|^2 \, \mathrm{d} a \quad \Leftarrow \quad \gamma^2 = \mu \, \epsilon \, \omega^2 - k_z^2 \, , \quad k_z = \frac{p \, \pi}{d} \, , \quad p \neq 0$$

If $p = 0 \quad \Rightarrow \quad U_{p=0} = \dfrac{\epsilon \, d}{2} \left(1 + \dfrac{k_z^2}{\gamma^2} \right) \displaystyle\int_{\mathcal{A}} |\psi|^2 \, \mathrm{d} a$

$$\Rightarrow \quad \begin{matrix}\text{physical}\\ \text{interpretation}\end{matrix} \quad Q = \frac{\mu}{\mu_c}\,\frac{V}{S\,\delta}\times(\text{Geometrical factor}) \quad \Leftarrow \quad \begin{matrix} V : \text{volume of the cavity} \\ S : \text{total surface area}\end{matrix}$$

- The Q of a cavity is the ratio of the volume occupied by the fields to the volume of the conductor in which the fields penetrate because of the finite conductivity.

- The expression for Q applies to cavities of arbitrary shape, with an appropriate geometrical factor of the order of unity.

- For the TE_{111} mode in the right circular cylinder cavity,

$$\begin{matrix}\text{Geometrical}\\ \text{factor}\end{matrix} = \frac{(R+d)(R^2+0.343\,d^2)}{R^3+0.209\,d\,R^2+0.244\,d^3} \quad \begin{matrix} \in [1, 2.13\,(\max), 1.42] \\ \text{for } \dfrac{d}{R} \in [0, 1.91, \infty] \end{matrix}$$

- Possible shifts in frequency cannot be calculated with the energy conservation, but the perturbation of boundary conditions again removes these deficiencies. With the similar procedures in Sec. 8.6.

$$\gamma_0^2-\gamma^2 \propto \omega_0^2-\omega^2 \simeq -2\,\omega_0\,(a_0+i\,b_0) \quad \Leftarrow \quad \omega=\omega_0+a_0+i\,b_0$$

$$\simeq -2\,\omega_0\,b_0\,(1+i)=(1+i)\,I \quad \Leftarrow \quad I : \text{ratio of appropriate integrals}$$

$$\Rightarrow \quad \Im[\omega]=b_0=-\frac{I}{2\,\omega_0} \quad \Leftarrow \quad I\rightarrow 0$$

$$= -\frac{\omega_0}{2\,Q} \quad \Leftarrow \quad E\propto e^{-i\,\omega\,t}=e^{-i\left(\omega_0+\Delta\,\omega-i\,\frac{\omega_0}{2\,Q}\right)t} \quad \Leftarrow \quad (12)$$

$$\Rightarrow \quad I = \frac{\omega_0^2}{Q} \quad \Rightarrow \quad \omega^2 \simeq \omega_0^2 \left(1 - \frac{1+i}{Q} \right) \quad \Leftarrow$$

Damping causes equal modifications
to the real and imaginary parts of ω^2

For large Q $\Rightarrow$ $\Delta \omega \simeq \Im[\omega] \simeq -\frac{\omega_0}{2Q}$ $\Leftarrow$

The resonant frequency is always lowered
by the presence of resistive losses

- The near equality of the real and imaginary parts of the change in ω^2 is a consequence of the boundary condition (13) appropriate for good conductors.

- It is possible for other system the relative magnitude of the real and imaginary parts of the change in ω^2 can be different.

Earth & Ionosphere as a Resonant Cavity: Schumann Resonances

- An example of a resonant cavity is the earth as one boundary surface and the ionosphere as the other.

- The lowest resonant modes are of very low frequency, λ~the earth's radius.

- Seawater $\sigma \sim \frac{1}{10\,\Omega\cdot\mathrm{m}}$ & ionosphere $\sigma < \frac{1}{10^4\,\Omega\cdot\mathrm{m}}$ are not perfect conducting.

- Idealize the physical reality and consider as a model two perfectly conducting, concentric spheres with radii a(=6400km, earth's radius) and $b=a+h$ (h=100km, the ionosphere's height.)

- Focus on the TM modes for the lowest frequency, with only tangential $\mathbf{B}$.

- The TM modes, with a radially directed $\mathbf{E}$, can satisfy the boundary condition of vanishing tangential $\mathbf{E}$ at boundaries without radial variation of the fields.

⇒ the frequency for the lowest TM modes $\omega_{\mathrm{TM}} = c\,k_{\mathrm{TM}} = \dfrac{2\pi c}{\lambda_{\mathrm{TM}}} \sim \dfrac{c}{a}$.

- The TE modes, with only tangential $\mathbf{E}$, have a radial variation about half a wavelength between the boundaries.

⇒ The lowest frequency for the TE modes $\omega_{\mathrm{TE}} = c\,k_{\mathrm{TE}} = \dfrac{2\pi c}{\lambda_{\mathrm{TE}}} \sim \dfrac{\pi c}{h}$

- Assume that $\mathbf{E}$ and $\mathbf{B}$ are independent of ϕ. For the TM mode: (spherical coor.)

$$B_r = 0 \quad \Rightarrow \quad B_\theta = 0 \;\; \text{if} \;\; \partial_\theta \left(B_\theta \sin\theta \right)_{\theta=0} = 0 \quad \Leftarrow \quad \nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad B_\phi \neq 0$$

$$\Rightarrow \quad E_\phi = 0 \;\; \text{if} \;\; \partial_\theta \left(E_\phi \sin\theta \right)_{\theta=0} = 0 \quad \Leftarrow \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\Rightarrow \quad B_\phi \neq 0 \,, \quad E_r \neq 0 \,, \quad E_\theta \neq 0 \quad \text{independent of } \phi$$

$$\text{Maxwell's equations} \quad \Rightarrow \quad \frac{\omega^2}{c^2} \mathbf{B} - \nabla \times \nabla \times \mathbf{B} = 0 \quad \Leftarrow \quad \mathbf{E} \;\; \& \;\; \mathbf{B} \;\; \propto \;\; e^{-i \omega t}$$

$$\Rightarrow \quad \frac{\omega^2}{c^2} r B_\phi + \frac{\partial^2}{\partial r^2} \left(r B_\phi \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(r B_\phi \sin\theta \right) \right) = 0 \quad \Leftarrow \quad \begin{array}{c} \phi \\ \text{component} \end{array}$$

$$\Rightarrow \quad \frac{\omega^2}{c^2} r B_\phi + \frac{\partial^2}{\partial r^2} \left(r B_\phi \right) + \frac{1}{r^2} \left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial r B_\phi}{\partial \theta} \right) - \frac{r B_\phi}{\sin^2\theta} \right] = 0$$

$$\Rightarrow \quad \text{the associated Legendre polynomials} \;\; P_\ell^m \left(\cos\theta \right) \;\; \text{with} \;\; m = \pm 1 \quad \Leftarrow \quad (3.6)$$

$$\Rightarrow \quad B_\phi \left(r \,, \theta \right) = \frac{u_\ell (r)}{r} \, p_\ell^1 \left(\cos\theta \right) \quad \Rightarrow \quad \frac{\mathrm{d}^2 u_\ell}{\mathrm{d} r^2} + \left(\frac{\omega^2}{c^2} - \frac{\ell \left(\ell + 1 \right)}{r^2} \right) u_\ell = 0$$

$$\Rightarrow \quad \begin{array}{l} E_r = \dfrac{i c^2}{\omega r \sin\theta} \dfrac{\partial}{\partial \theta} \left(B_\phi \sin\theta \right) = - \dfrac{i c^2}{\omega r} \ell \left(\ell + 1 \right) \dfrac{u_\ell}{r} \, p_\ell \left(\cos\theta \right) \\[2ex] E_\theta = - \dfrac{i c^2}{\omega r} \dfrac{\partial}{\partial r} \left(r B_\phi \right) = - \dfrac{i c^2}{\omega r} \dfrac{\mathrm{d} u_\ell}{\mathrm{d} r} \, p_\ell^1 \left(\cos\theta \right) \qquad \Nwarrow \qquad \nabla \times \mathbf{B} = \dfrac{1}{c^2} \dfrac{\partial \mathbf{E}}{\partial t} \end{array}$$

$$\mathbf{E}_{\parallel}=0 \quad \Leftarrow \quad E_{\theta}(a)=E_{\theta}(b)=0 \quad \Rightarrow \quad \boxed{\frac{\mathrm{d}\,u_{\ell}}{\mathrm{d}\,r}(a)=\frac{\mathrm{d}\,u_{\ell}}{\mathrm{d}\,r}(b)=0} \quad (14)$$

- $u_{\ell}=r\times$ (the spherical Bessel function) [see Section 9.6].

$$\frac{h}{a}\ll 1 \quad \Rightarrow \quad \frac{\ell(\ell+1)}{r^2}\simeq\frac{\ell(\ell+1)}{a^2} \quad \Rightarrow \quad u_{\ell}\simeq\left[\begin{array}{l}\sin(q\,r)\\ \cos(q\,r)\end{array}\right. \quad \Leftarrow \quad q^2=\frac{\omega^2}{c^2}-\frac{\ell(\ell+1)}{a^2}$$

$$\Rightarrow \quad u_{\ell}\simeq A\cos q(r-a) \quad \Leftarrow \quad q\,h=n\,\pi\,, \quad n=0,1,2,\cdots \quad \Leftarrow \quad (14)$$

- For $n>0$ the frequency of the modes > $\omega=\dfrac{n\,\pi\,c}{h}$ and are in the domain of frequency of the TE modes. Only for $n=0$ are there very-low-frequency modes.

$$q=0 \quad \Rightarrow \quad u_{\ell}=\text{const} \quad \Rightarrow \quad \omega_{\ell}\simeq\frac{c}{a}\sqrt{\ell(\ell+1)} \quad \text{Schumann resonances} \quad \Leftarrow \quad \frac{h}{a}\to 0$$

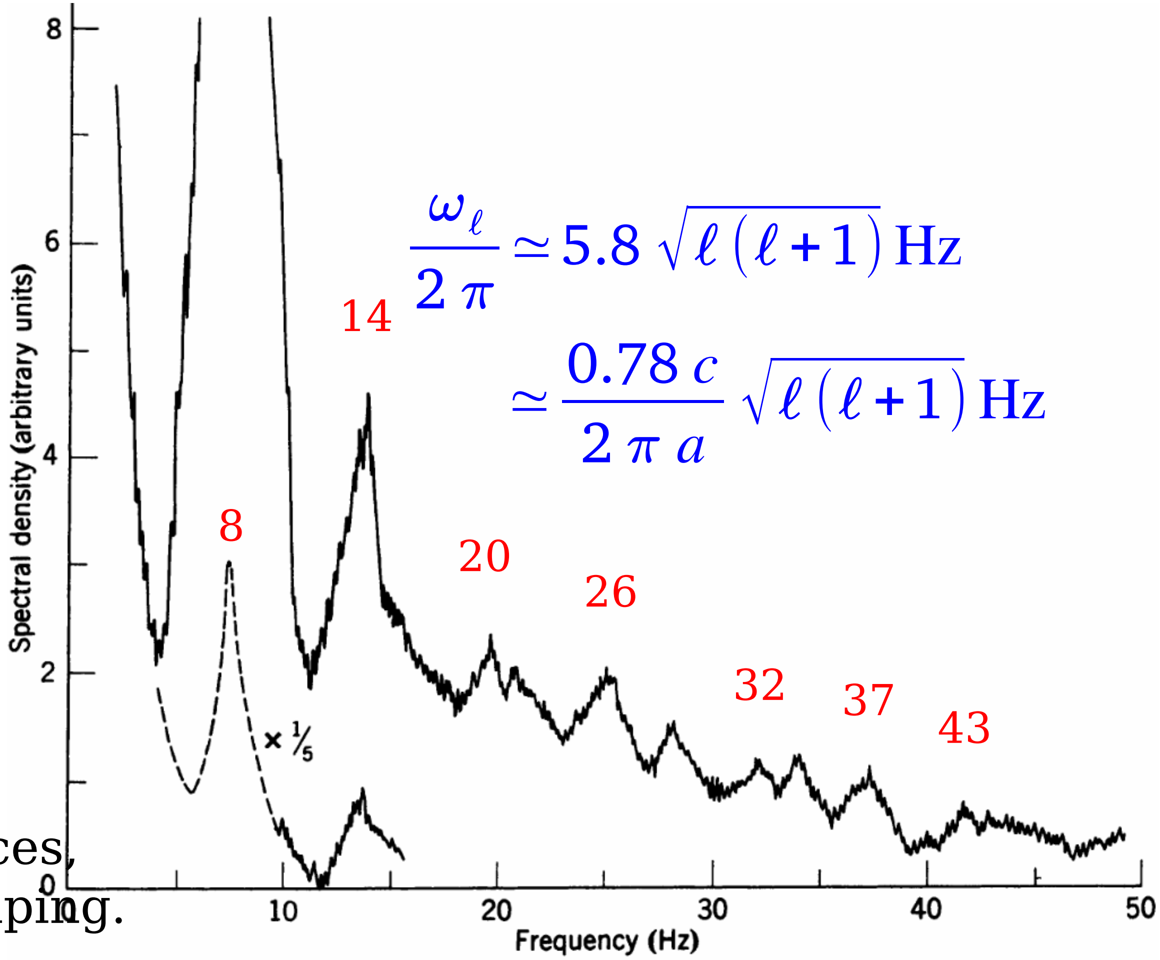
- To 1st-order in $\dfrac{h}{a}$ the correct result has a replaced by $a+\dfrac{h}{2}$.
- $E_{\theta}=0\,,\quad E_r\propto\dfrac{P_{\ell}(\cos\theta)}{r^2}\,,\ B_{\phi}\propto\dfrac{P_{\ell}^{1}(\cos\theta)}{r}$
- $\dfrac{\omega_{\ell}}{2\,\pi}=10.6,\ 18.3,\ 25.8,\ 33.4,\ 40.9\ \ \text{Hz} \quad \Leftarrow \quad$ extremely low frequencies (ELF)
- Schumann resonances manifest themselves as peaks in the noise power spectrum of ELFs propagating around the earth.

- Lightning bolts act as sources of radial electric fields E_r.

- The frequencies near the Schumann resonances are preferred because they are normal modes of the earth-ionosphere cavity.

- The lack of precise agreement is that the assumption of perfectly conducting walls is rather far from the truth.

- Q~4-10 for the first few resonances, corresponding to rather heavy damping.



$$\frac{\omega_\ell}{2\pi} \simeq 5.8\sqrt{\ell(\ell+1)}\,\mathrm{Hz}$$

$$\simeq \frac{0.78\,c}{2\pi a}\sqrt{\ell(\ell+1)}\,\mathrm{Hz}$$

- The effect of the damping looks alright, but the simple shift is only about half of what is observed.

- 1st curiosity: A nuclear explosion can decrease 3-5% in Schumann resonant frequencies by the alterations in the ionosphere.

- 2nd curiosity: Schumann resonances can serve as a global tropical thermometer, due to the Schumann resonant **B** depends on the frequency of lightning, and the frequency of lightning depends on temperature.

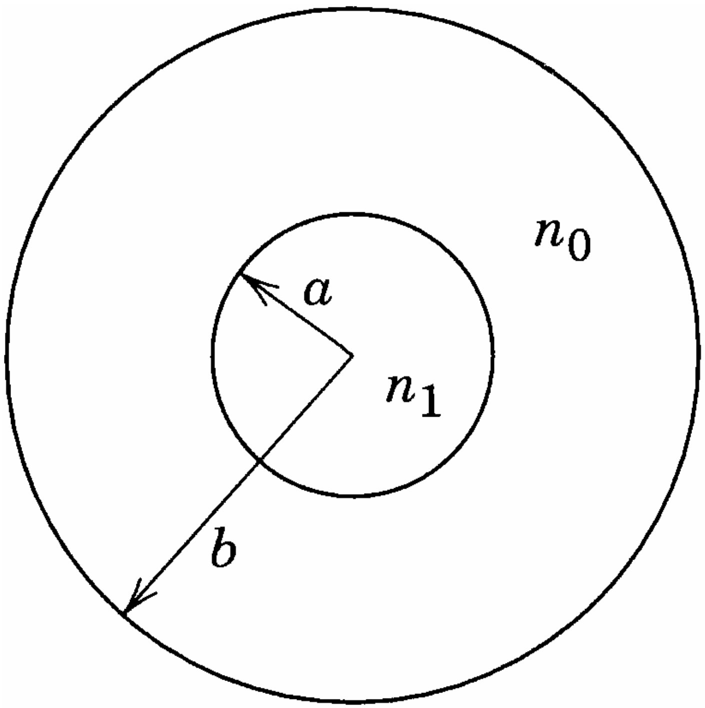
Multimode Propagation in Optical Fibers

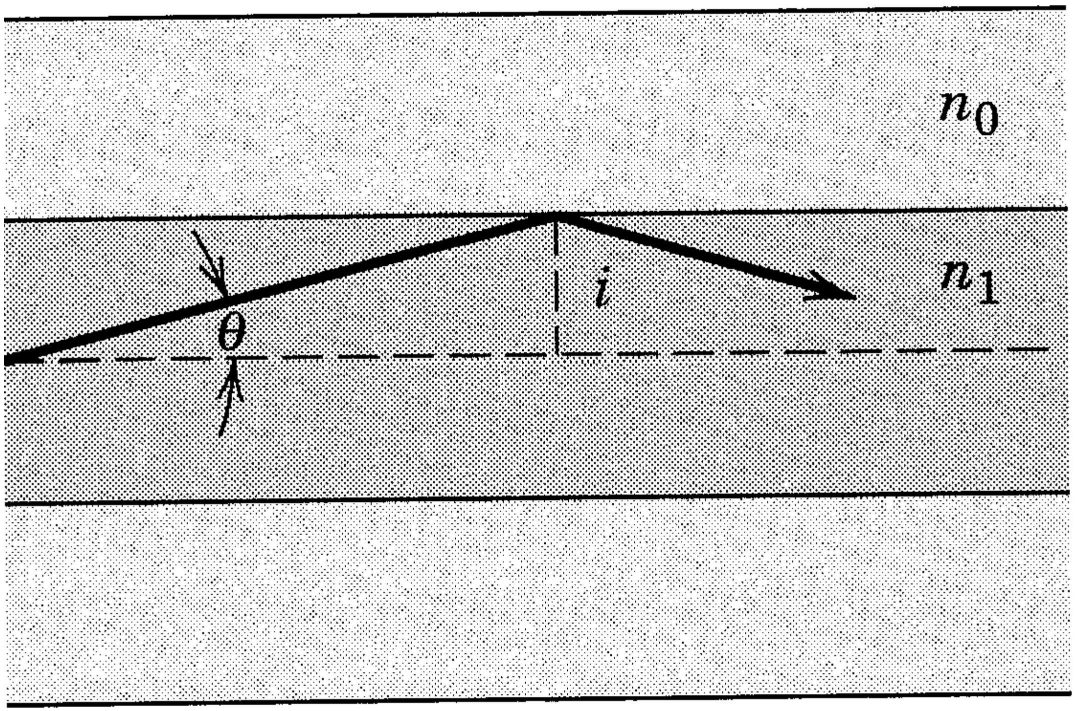
- Transmission via optical fibers falls approximately into 2 classes:
(1) multimode: cores are typically 50μm in diameter for a wavelength $\sim 1\mu$m;
(2) single-mode: cores are around 5μm diameters. Near InfraRed

- Consider multimode transmission with the semi-geometrical eikonal or WKB.

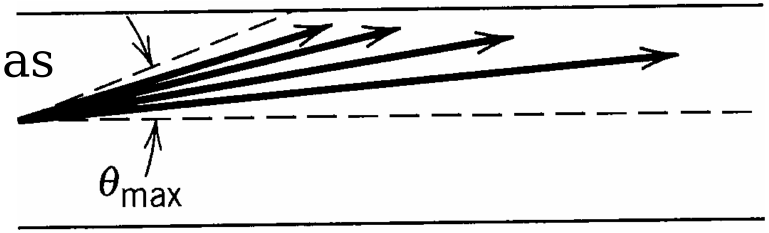
- Optical fiber cables, $\sim$2cm in diameter, are actually nests of smaller cables with 6 or 8 optical fibers protected by secondary coatings and buffer layers.

- The operative fiber consists of a cylindrical core of radius a [$2a=O(50\mu\text{m})$] and index of refraction n_1 surrounded by a cladding of outer radius b [$2b=O(150\mu\text{m})$] and index of refraction $n_0<n_1$.





- Since the wavelength of the light ~ 1μm, the ideas of geometrical optics apply; the interface between core and cladding can be treated as locally flat.



- If the angle of incidence i of a ray is greater than i_0 ($i_0 = \sin^{-1}\dfrac{n_0}{n_1}$, the critical angle for total internal reflection), the ray will be confined and thus propagate.

$$\text{Propagation occurs for rays with } \theta < \theta_{\max} = \cos^{-1}\frac{n_0}{n_1}$$

- Define $\Delta \equiv \dfrac{n_1^2 - n_0^2}{2\,n_1^2} \approx 1 - \dfrac{n_0}{n_1} \quad \Rightarrow \quad \Delta \leq 1\,\%\ \text{typically} \quad \Rightarrow \quad \theta_{\max} \approx \sqrt{2\,\Delta} \leq 0.14 = 8\,^\circ$

- $\begin{matrix}\text{2d phase-space}\\ \text{number density}\end{matrix}$ $\mathrm{d}\,N = \pi\,a^2\,\dfrac{\mathrm{d}^2\,k}{(2\,\pi)^2}\cdot 2 = \begin{matrix}\text{spatial}\\ \text{area}\end{matrix} \times \begin{matrix}\text{wave number}\\ \text{volume elememt}\end{matrix} \times 2\ \text{polarizations}$

$$k_\perp \approx k\,\theta < k\,\theta_{\max} \quad \Rightarrow \quad \mathrm{d}^2\,k = 2\,\pi\,k_\perp\,\mathrm{d}\,k_\perp = 2\,\pi\,k^2\,\theta\,\mathrm{d}\,\theta$$

$$\Rightarrow \quad N \approx a^2\,k^2 \int_0^{\theta_{\max}} \theta\,\mathrm{d}\,\theta \approx \frac{1}{2}\,V^2 \quad \Leftarrow \quad V \equiv k\,a\,\sqrt{2\,\Delta} \quad \text{fiber parameter}$$

Typically $\lambda = 0.85\,\mu\,\mathrm{m}\,, \quad a = 25\,\mu\,\mathrm{m}\,, \quad n_1 \approx 1.4$

$$\Rightarrow \quad k\,a \approx 260 \quad \Rightarrow \quad \Delta = 0.005 \quad \Rightarrow \quad N \approx 335$$

In contrast $a \sim 2.7\,\mu\,\mathrm{m}\,,\ \Delta \sim 0.0025\,,\ N \sim 2$ for each polarizatioin in single-mode

- If the indices of refraction decrease layer by layer out from the center, a ray at some angle is bent successively more toward the axis until it is totally reflected.

- For an arbitrary number of layers outside the core, the critical angle

$\theta_{\text{max}}=\cos^{-1}\dfrac{n_{\text{outer}}}{n_{\text{inner}}}$, just as for the simple 2-index fiber.

- The limit of many layers is a graded index fiber in which the index of refraction varies continuously with radius from the axis.

- With the eikonal approximation, assume the medium is linear, nonconducting, nonmagnetic with an index of refraction $n(\mathbf{r})=\sqrt{\dfrac{\epsilon(\mathbf{r})}{\epsilon_0}}$ varying in space slowly.

- With fields $\propto e^{-i\omega t}$, the Maxwell equations can give Helmholtz wave equations

$$[\nabla^2+\mu_0\omega^2\epsilon(\mathbf{r})]\begin{bmatrix}\mathbf{E}\\ \mathbf{H}\end{bmatrix}+\cancel{\begin{bmatrix}\nabla(\mathbf{E}\cdot\nabla\ln\epsilon)\\ i\omega\mathbf{E}\times\nabla\epsilon\end{bmatrix}}=0 \;\Leftarrow\; \begin{matrix}\epsilon\ \text{little}\\ \text{changes}\end{matrix} \;\Rightarrow\; \left(\nabla^2+\frac{\omega^2}{c^2}n^2(\mathbf{r})\right)\psi=0$$

$$\Rightarrow\ \text{plane wave}\ \Rightarrow\ |k(\mathbf{r})|=\frac{\omega}{c}n(\mathbf{r})\ \Rightarrow\ \psi=e^{i\frac{\omega}{c}S(\mathbf{r})}\ \Leftarrow\ S(\mathbf{r})\text{: eikonal}\ \propto\ \text{phase}$$

$$\Rightarrow\ \frac{\omega^2}{c^2}[n^2(\mathbf{r})-\nabla S\cdot\nabla S]+i\frac{\omega}{c}\cancel{\nabla^2}S=0\ \Leftarrow\ \begin{matrix}\text{slow variation of } n\\ \text{small change of } S\end{matrix}$$

$$\Rightarrow\ \nabla S\cdot\nabla S=n^2\ \Leftarrow\ \textit{eikonal approximation}\ \text{of quasi-geometrical optics}$$

The Maxwell equations without source: $\mathbf{D} = \epsilon(\mathbf{r})\,\mathbf{E}$, $\mathbf{B} = \mu\,\mathbf{H}$, all $\propto e^{-i\omega t}$

$$\left[\begin{array}{ll} \nabla\cdot\mathbf{D} = 0\,, & \nabla\times\mathbf{H} - \dfrac{\partial\,\mathbf{D}}{\partial\,t} = 0 \\ \nabla\cdot\mathbf{B} = 0\,, & \nabla\times\mathbf{E} + \dfrac{\partial\,\mathbf{B}}{\partial\,t} = 0 \end{array}\right. \Rightarrow \left[\begin{array}{ll} \nabla\cdot\mathbf{E} = -\mathbf{E}\cdot\nabla\ln\epsilon\,, & \nabla\times\mathbf{H} + i\,\epsilon\,\omega\,\mathbf{E} \quad = 0 \\ \nabla\cdot\mathbf{H} = \qquad 0\,, & \nabla\times\mathbf{E} - i\,\mu\,\omega\,\mathbf{H} = 0 \end{array}\right.$$

$$\Rightarrow \quad \nabla\times\begin{pmatrix} \nabla\times\mathbf{H} + i\,\epsilon\,\omega\,\mathbf{E} \quad = 0 \\ \nabla\times\mathbf{E} - i\,\mu\,\omega\,\mathbf{H} = 0 \end{pmatrix} \quad \Rightarrow \left[\begin{array}{l} \left(\nabla^2 + \mu\,\epsilon\,\omega^2\right)\mathbf{E} + \nabla\left(\mathbf{E}\cdot\nabla\ln\epsilon\right) = 0 \\ \left(\nabla^2 + \mu\,\epsilon\,\omega^2\right)\mathbf{H} + i\,\omega\,\mathbf{E}\times\nabla\,\epsilon = 0 \end{array}\right.$$

- $S(\mathbf{r}) = S(\mathbf{r}_0) + (\mathbf{r}-\mathbf{r}_0)\cdot\nabla S(\mathbf{r}_0) + \cdots \;\Rightarrow$ wave amplitude $\psi(\mathbf{r}) \approx e^{i\omega\frac{S(\mathbf{r}_0)}{c}}\, e^{i\omega\frac{\mathbf{r}-\mathbf{r}_0}{c}\cdot\nabla S(\mathbf{r}_0)}$
- The form of ψ is that of a plane wave with wave vector

$$\boldsymbol{k}(\mathbf{r}_0) = \frac{\omega}{c}\nabla S(\mathbf{r}_0) = \frac{\omega}{c} n(\mathbf{r}_0)\,\hat{\boldsymbol{k}}(\mathbf{r}_0) \;\Leftarrow\; \nabla S = n(\mathbf{r})\,\hat{\boldsymbol{k}}(\mathbf{r}) \;\Leftarrow\; \nabla S\cdot\nabla S = n^2$$

- ψ is a wave front being locally plane & propagating in the direction of $\boldsymbol{k}$.

- If we imagine advancing incrementally in the direction of $\boldsymbol{k}$, we trace out a path that is the geometrical ray associated with the wave.

- Let s be the distance of the path,

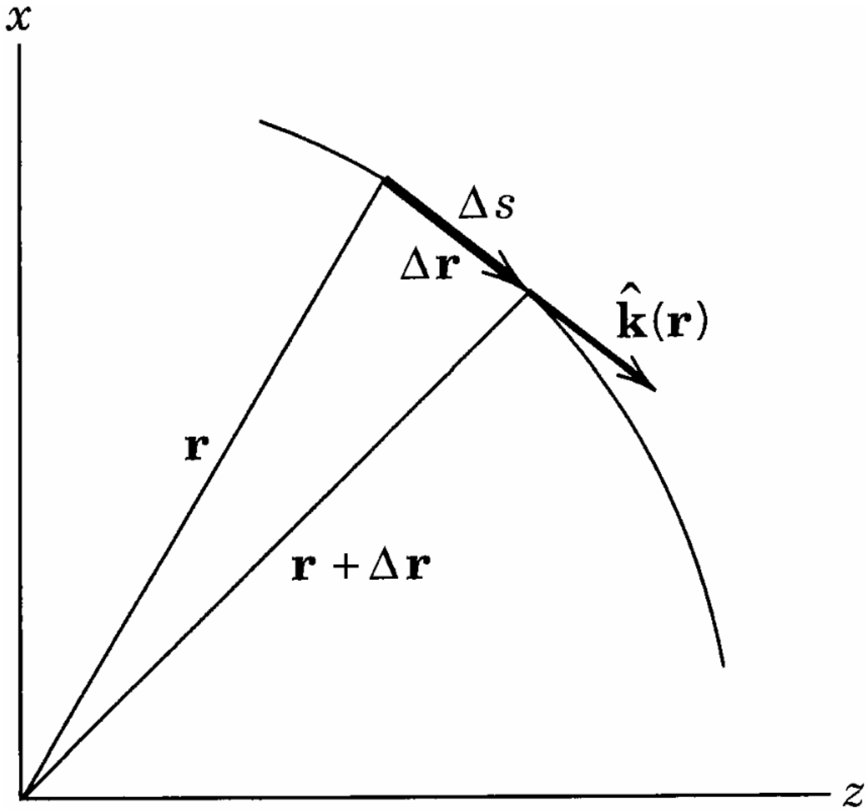
$$\hat{\boldsymbol{k}} \equiv \lim_{\delta s\to 0}\frac{\delta\mathbf{r}}{\delta s} = \frac{d\mathbf{r}}{ds} \;\Rightarrow\; n(\mathbf{r})\frac{d\mathbf{r}}{ds} = \nabla S$$

$$\Rightarrow \frac{d}{ds}\left(n(\mathbf{r})\frac{d\mathbf{r}}{ds}\right) = \frac{d}{ds}\nabla S = \nabla\frac{dS}{ds}$$

$$\frac{d}{ds} = \hat{\boldsymbol{k}}\cdot\nabla \;\Rightarrow\; \frac{dS}{ds} = \hat{\boldsymbol{k}}\cdot\nabla S = \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}\, n(\mathbf{r})$$

$$\Rightarrow \frac{d}{ds}\left(n(\mathbf{r})\frac{d\mathbf{r}}{ds}\right) = \nabla n(\mathbf{r}) \qquad (17)$$

generalization of Snell's law



- Rays in a circular fiber fall into 2 classes:

1. *Meridional rays*: rays that pass through the cylinder axis; they correspond to modes with vanishing m and nonvanishing intensity at $\rho = 0$.

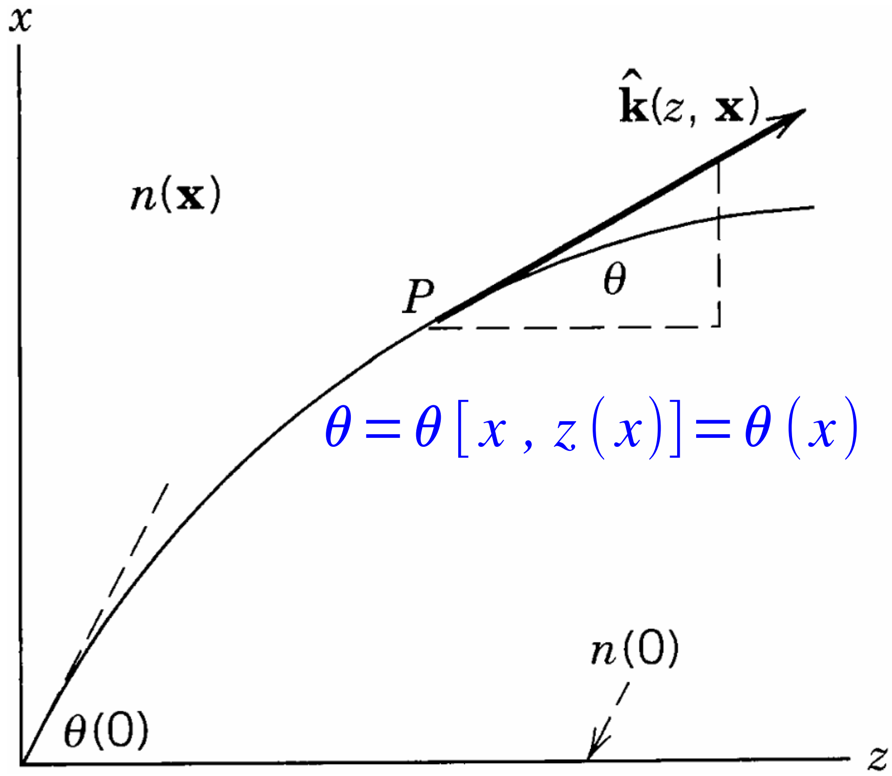
2. *Skew rays*: rays that originate off-axis and whose path is a spiral in space with inner and outer turning points in radius; they correspond to modes with nonvanishing m and vanishing intensity at $\rho = 0$.

- Apply (17) only to the transmission of meridional rays in an optical fiber, or to rays in a "slab" geometry.

- $$\frac{\mathrm{d}\,x}{\mathrm{d}\,s} = \sin\theta(x), \quad \frac{\mathrm{d}\,z}{\mathrm{d}\,s} = \cos\theta(x)$$

$$(17) \quad \Rightarrow \quad \begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\,s}[n(x)\sin\theta(x)] &= \frac{\mathrm{d}}{\mathrm{d}\,x} n(x) \\ \frac{\mathrm{d}}{\mathrm{d}\,s}[n(x)\cos\theta(x)] &= \quad 0 \end{aligned}$$

$$\Rightarrow \quad n(x)\cos\theta(x) = n(0)\cos\theta(0)$$



$$\theta = \theta[x, z(x)] = \theta(x)$$

If $n(x)$ is monotonically decreasing with respect to $|x|$, there is a max (min) value of x attained by the ray when $\cos\theta(x_{\max}) = 1 \;\Leftarrow\; \theta(x_{\max}) = 0$

$$|x| = x_{\max} \quad \Rightarrow \quad \bar{n} = n(x_{\max}) = n(0)\cos\theta(0)$$

- $\bar{n}$ is a characteristic of a given ray or trajectory. From $n(x)$ we can deduce $x_{\max}$ and so delimit the lateral extent of that trajectory.
- $\bar{n} = n\cos\theta = n\frac{\mathrm{d}z}{\mathrm{d}s} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}s} = \frac{\bar{n}}{n}\frac{\mathrm{d}}{\mathrm{d}z} \Rightarrow \frac{\bar{n}}{n}\frac{\mathrm{d}}{\mathrm{d}z}\left(\bar{n}\frac{\mathrm{d}x}{\mathrm{d}z}\right) = \frac{\mathrm{d}n}{\mathrm{d}x} \Leftarrow (17)$

$\Rightarrow \bar{n}^2\frac{\mathrm{d}^2x}{\mathrm{d}z^2} = \frac{1}{2}\frac{\mathrm{d}n^2}{\mathrm{d}x} \Rightarrow$ structure of Newton's equation of motion $\Rightarrow \begin{matrix} z \to t \\ \bar{n}^2 \to m \end{matrix}, \quad -\frac{n^2(x)}{2} \to V(x)$

$\Rightarrow \bar{n}^2\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 = n^2(x) - \bar{n}^2$ energy conservation $\Leftarrow \frac{\mathrm{d}x}{\mathrm{d}z} = 0$ for $n(x) = \bar{n}$

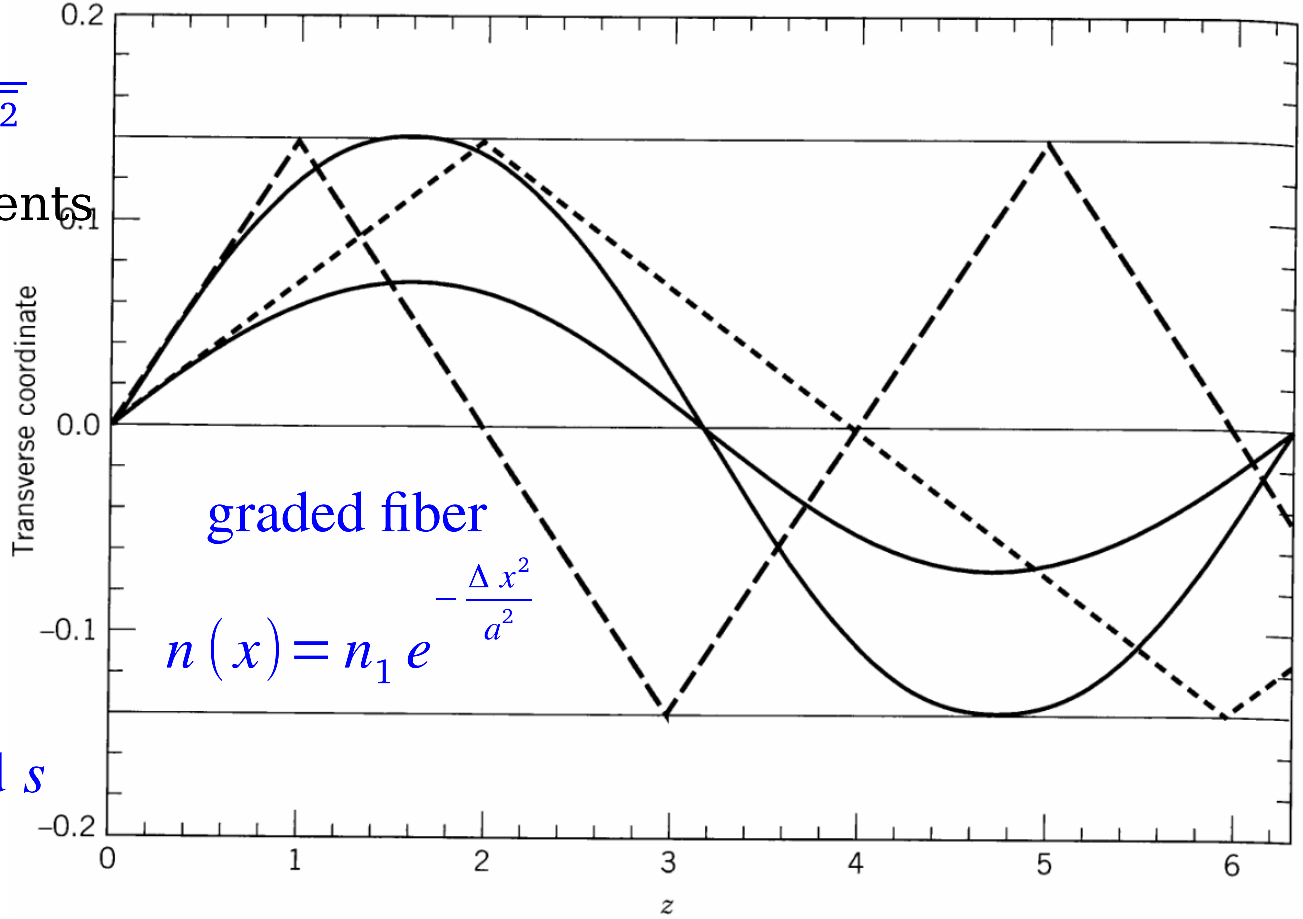
$$\Rightarrow z(x) = \bar{n}\int_0^x \frac{\mathrm{d}x}{\sqrt{n^2(x) - \bar{n}^2}}$$

- For $x \le x_{\max}$, the path represents 1/4 of a cycle of oscillation.
- $$Z = 2\bar{n}\int_0^{x_{\max}} \frac{\mathrm{d}x}{\sqrt{n^2(x) - \bar{n}^2}}$$

half-period

- The physical & optical path lengths from A to B are

$$L_{\mathrm{phy}} = \int_A^B \mathrm{d}s, \quad L_{\mathrm{opt}} = \int_A^B n\,\mathrm{d}s$$



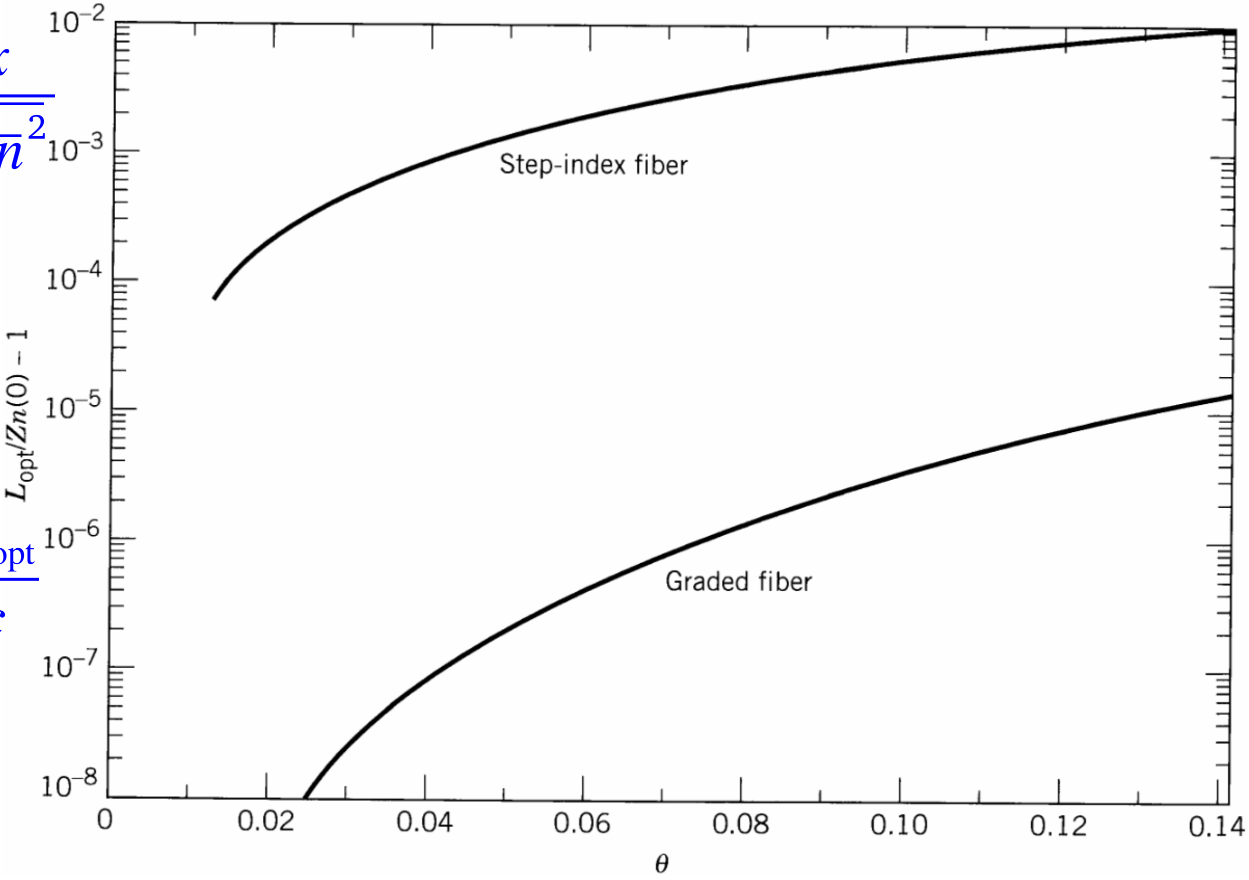
graded fiber

$n(x) = n_1 e^{-\frac{\Delta x^2}{a^2}}$

$$\mathrm{d}s = \frac{n}{\bar{n}}\,\mathrm{d}z = \frac{n}{\bar{n}}\frac{\mathrm{d}z}{\mathrm{d}x}\,\mathrm{d}x = \frac{n\,\mathrm{d}x}{\sqrt{n^2 - \bar{n}^2}}$$

$$\Rightarrow \quad L_{\mathrm{phy}} = 2\int_0^{x_{\max}} \frac{n(x)\,\mathrm{d}x}{\sqrt{n^2(x) - \bar{n}^2}}$$

$$L_{\mathrm{opt}} = 2\int_0^{x_{\max}} \frac{n^2(x)\,\mathrm{d}x}{\sqrt{n^2(x) - \bar{n}^2}}$$



- The transit time of a ray is $\frac{L_{\mathrm{opt}}}{c}$

$$\Rightarrow \quad T(z) = \frac{L_{\mathrm{opt}}}{Z}\frac{z}{c} \quad \text{for } z \gg Z$$

where $\frac{cZ}{L_{\mathrm{opt}}}$ group velocity

- Different rays, defined by different $\theta(0)$ or $\bar{n}$, have different transit times, a form of dispersion that is geometrical.

- A signal launched with a nonvanishing angular spread will be distorted unless $n(x)$ is chosen to make the transit time largely independent of $\bar{n}$.

- With a graded profile, rays with larger initial angles and so larger $x_{\max}$ will have longer physical path, but will have larger phase velocities $\frac{c}{n}$ in those longer arcs; an inherent tendency toward equalization of transit times. [problem 8.14]

Modes in Dielectric Waveguides

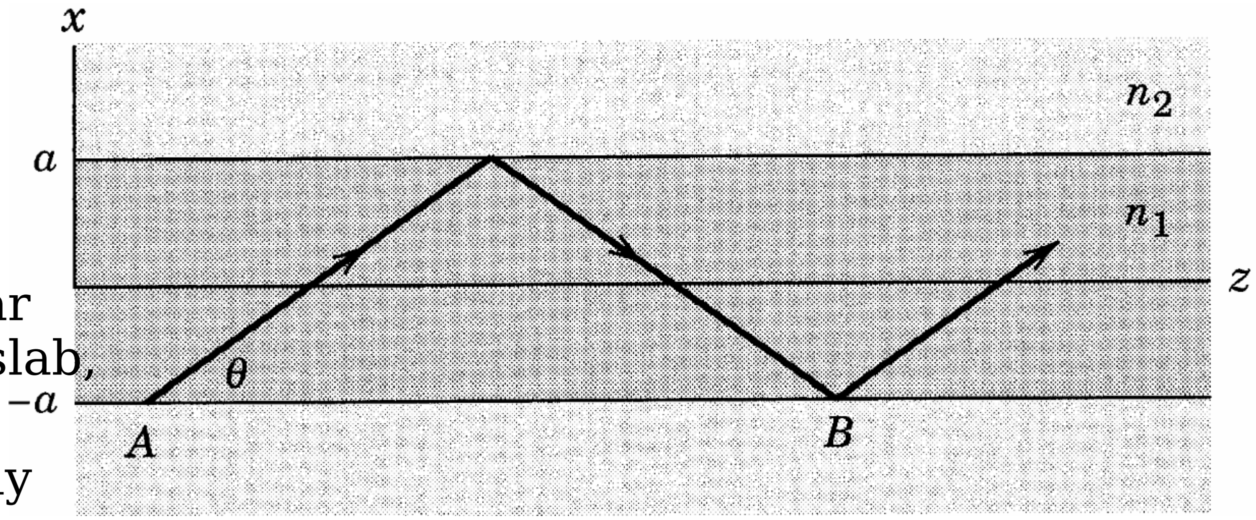
- The geometrical ray method of propagation is appropriate when the wavelength is short compared to the transverse dimensions of the waveguide, but the wave nature of the fields must be taken into account when the 2 scales are comparable.

- The bound rays ($\theta<\theta_{\text{max}}$) in the geometric description are in the bound modes, with fields outside the core that decrease exponentially in the radial direction.

- Unbound rays ($\theta>\theta_{\text{max}}$) correspond to the radiating modes, with oscillatory fields outside the core.

- Single-mode propagation is important in optical communication, just as it is in microwave transmission in metallic guides.

A. Modes in a Planar Slab Dielectric Waveguide



- Consider a "step-index" planar fiber consisting of a dielectric slab, any ray that makes an angle θ less than $\theta_{\max}$ is totally internally reflected; the light is confined and propagates in the z direction.

- The path can be thought of as the normal to the wave front of a plane wave, reflected back and forth or alternatively as 2 plane waves, with x component of wave number, $k_x = \pm k \sin\theta$.

- To have a stable transverse field configuration and coherent propagation, the *transverse* phase from A to B (with 2 internal reflections) must be $2p\pi$, $p \in \mathbb{N}^+$.

$$4\,k\,a\sin\theta + 2\,\phi = 2\,p\,\pi \qquad k = n_1\frac{\omega}{c}$$

$\Leftarrow$ ϕ : phase with total internal reflection $\Leftarrow$

$$\phi_{\mathrm{TE}} = -2\tan^{-1}\frac{\sqrt{2\,\Delta - \sin^2\theta}}{\sin\theta}$$

$$2\,\Delta = 1 - \frac{n_2^2}{n_1^2} \quad \Rightarrow \quad \phi_{\mathrm{TM}} = -2\tan^{-1}\frac{\sqrt{2\,\Delta - \sin^2\theta}}{(1 - 2\,\Delta)\sin\theta}$$

fiber parameter $V = k\,a\sqrt{2\,\Delta} = k\,a\sin\theta_{\max}$, transverse variable $\xi = \dfrac{\sin\theta}{\sqrt{2\,\Delta}} = \dfrac{\sin\theta}{\sin\theta_{\max}}$ $\Rightarrow$ $V\,\xi = k\,a\sin\theta$

symbol translation: $i = \frac{\pi}{2} - \theta$, $\begin{matrix} n = n_1 \\ n' = n_2 \end{matrix}$, $\mu' \simeq \mu$, $n' < n \sin i$ for total reflection

TE $(E_z = 0)$: $\mathbf{E} \perp$ incidence plane, so use Eq. (7.39) + $2\,\Delta = 1 - \frac{n_2^2}{n_1^2} = \sin^2\theta_{\max}$

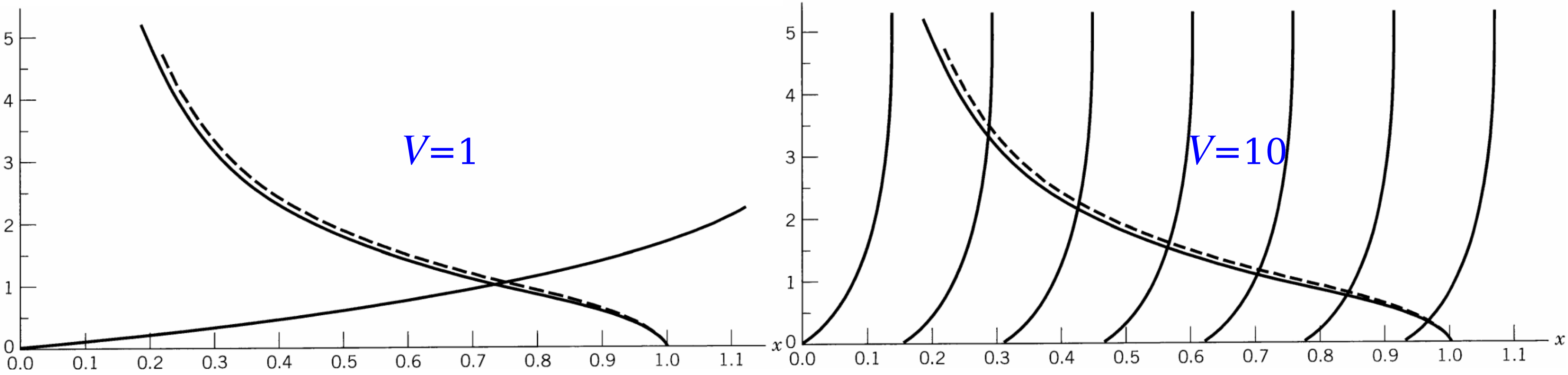
$$\frac{E_0''}{E_0} = \frac{\mu'\, n \cos i - \mu\,\sqrt{n'^{\,2} - n^2 \sin^2 i}}{\mu'\, n \cos i + \mu\,\sqrt{n'^{\,2} - n^2 \sin^2 i}} \simeq \frac{n_1 \sin\theta - i\,\sqrt{n_1^2 \cos^2\theta - n_2^2}}{n_1 \sin\theta + i\,\sqrt{n_1^2 \cos^2\theta - n_2^2}} = \frac{F\, e^{-i\varphi}}{F\, e^{i\varphi}} = e^{-2\,i\,\varphi}$$

$$= e^{i\,\phi_{\rm TE}} \quad \Rightarrow \quad \phi_{\rm TE} = -2\,\varphi = -2\,\tan^{-1}\frac{\sqrt{n_1^2\cos^2\theta - n_2^2}}{n_1 \sin\theta} = -2\,\tan^{-1}\frac{\sqrt{2\,\Delta - \sin^2\theta}}{\sin\theta}$$

TM $(B_z = 0)$: $\mathbf{B} \perp$ incidence plane, so use Eq. (7.41)

$$\frac{E_0''}{E_0} = \frac{\mu\, n'^{\,2} \cos i - \mu'\, n\,\sqrt{n'^{\,2} - n^2 \sin^2 i}}{\mu\, n'^{\,2} \cos i + \mu'\, n\,\sqrt{n'^{\,2} - n^2 \sin^2 i}} = \frac{n_2^2 \sin\theta - i\, n_1\,\sqrt{n_1^2 \cos^2\theta - n_2^2}}{n_2^2 \sin\theta + i\, n_1\,\sqrt{n_1^2 \cos^2\theta - n_2^2}} = e^{-2\,i\,\varphi} = e^{i\,\phi_{\rm TM}}$$

$$\Rightarrow \quad \phi_{\rm TM} = -2\,\varphi = -2\,\tan^{-1}\frac{n_1\,\sqrt{n_1^2\cos^2\theta - n_2^2}}{n_2^2 \sin\theta} = -2\,\tan^{-1}\frac{\sqrt{2\,\Delta - \sin^2\theta}}{(1 - 2\,\Delta)\sin\theta}$$



$$\Rightarrow \quad \tan\frac{2V\xi - p\pi}{2} = \frac{\sqrt{1-\xi^2}}{\xi} \times \left[\begin{array}{ll} 1 & \text{TE} \\ \dfrac{1}{1-2\Delta} & \text{TM} \end{array}\right. \qquad (14)$$

- For small Δ the TE & TM modes are almost degenerate.
- $(14) \Rightarrow$ No. of modes $N \approx \dfrac{4V}{\pi}$
- From 1d phase-space $N_{\text{TE}} \simeq N_{\text{TM}} \simeq 2a\displaystyle\int_{-k_{\max}}^{+k_{\max}} \frac{d\,k_x}{2\pi} = \frac{2ka}{\pi}\int_0^{\sqrt{2\Delta}} d\sin\theta = \frac{2V}{\pi}$
- The lowest approximation $\xi_p(\text{TE}) \approx \dfrac{\pi}{2}\dfrac{p+1}{V+1} \Leftarrow$ equal spacing in p $\Leftarrow$ valid for $V \gg 1$ small p

$$\text{Proof: } \tan\frac{2V\xi - p\pi}{2} = \frac{\sqrt{1-\xi^2}}{\xi} \approx \cot\xi = \tan\left(\frac{\pi}{2} - \xi\right) \quad \Rightarrow \quad V\xi - \frac{p\pi}{2} \approx \frac{\pi}{2} - \xi$$

- The fields outside the slab vary as (7.46) $\Rightarrow \ \psi_{\text{outside}} \propto e^{-\beta|x|} \ \Leftarrow \ \beta = k\sqrt{2\Delta - \sin^2\theta} = \dfrac{V}{a}\sqrt{1-\xi^2}$

V fixed $\Rightarrow$ β smaller as p increases $(\xi \to 1)$ $\Rightarrow$ the fields extend farther outside

$\theta > \theta_{\max} \ \Rightarrow \ \xi > 1 \ \Rightarrow \ \beta \in \Im \ \Rightarrow$ unconfined transverse fields
The slab radiates rather than confines the fields

$\Rightarrow$ part of the power propagates within the core and part outside [Problem 8.15]

$\Rightarrow \ P_{\text{inside}}(V=1) \simeq \dfrac{2}{3} P_{\text{TE}_0}, \ P_{\text{inside}}(V \gg 1) \approx P_{\text{total}}, \ P_{\text{outside}}$ appreciable for $p \approx p_{\max}$

- $\Delta \ll 1 \ \Rightarrow \ \theta_{\max} \approx \sqrt{2\Delta} \ll 1 \ \Rightarrow$ longitudinal propagation $k_z = k\cos\theta \approx k$

$$\Rightarrow \ \left|\frac{E_z}{E_x}\right| = \tan\theta \le \theta_{\max} = \sqrt{2\Delta} \ll 1 \text{ for the TM modes}$$

to 0th-order in Δ, the TM modes have transverse electric fields and are degenerate with the TE modes.

- The combination of 2 such degenerate modes can give a mode with arbitrary direction of polarization in the x-y plane, labeled LP (for linearly polarized).

- LP modes are approximate descriptions in a circular fiber, provided $\Delta \ll 1$.

B. Modes in Circular Fibers

- For a fiber of uniform cross section with unit relative magnetic permeability and an index of refraction that does not vary along the cylinder axis but may vary in the transverse directions.

$$\begin{array}{c}\text{Maxwell}\\ \text{equations}\end{array} + \begin{array}{c}\text{fields} \propto e^{i(k_z z - \omega t)}\\ \epsilon = n^2 \epsilon_0\end{array} \Rightarrow \left(\nabla^2 + \frac{n^2 \omega^2}{c^2} \right) \begin{bmatrix} \mathbf{H} \\ \mathbf{E} \end{bmatrix} = \left[\begin{array}{l} i\omega\epsilon_0 \nabla n^2 \times \mathbf{E} \\ -\nabla\left(\mathbf{E}\cdot\nabla \ln n^2\right) \end{array} \right.$$

$$(2) \Rightarrow \left. \begin{array}{c} \mathbf{E}_t \\ \mathbf{H}_t \end{array} \right] = \frac{i}{\gamma^2} \left(k_z \nabla_t \begin{bmatrix} E_z \\ H_z \end{bmatrix} + \begin{bmatrix} -\omega\mu_0 \\ n^2 \omega \epsilon_0 \end{bmatrix} \hat{\mathbf{z}} \times \nabla_t \begin{bmatrix} H_z \\ E_z \end{bmatrix} \right) \Leftarrow \gamma^2 = \frac{n^2 \omega^2}{c^2} - k_z^2$$

radial propagation const

$$\Rightarrow \begin{array}{l} \left(\nabla_t^2 + \gamma^2\right) H_z - \dfrac{\omega^2}{\gamma^2 c^2} \nabla_t H_z \cdot \nabla_t n^2 = -\dfrac{\omega k_z \epsilon_0}{\gamma^2} \hat{\mathbf{z}} \cdot \nabla_t n^2 \times \nabla_t E_z \\ \left(\nabla_t^2 + \gamma^2\right) E_z - \dfrac{k_z^2}{\gamma^2} \nabla_t E_z \cdot \nabla_t \ln n^2 = \dfrac{\omega k_z \mu_0}{\gamma^2} \hat{\mathbf{z}} \cdot \nabla_t \ln n^2 \times \nabla_t H_z \end{array} \Leftarrow \begin{array}{c} \text{assume} \\ \dfrac{\partial n^2}{\partial z} = 0 \end{array}$$

- In contrast to (4), the equations for E_z and H_z are coupled. In general there is no separation into purely TE or TM modes.

- Focus on a core of a circular cylinder of radius a with ϕ-symmetric index $n(\rho)$.

$$\Rightarrow \hat{\mathbf{z}} \cdot \nabla_t n^2 \times \nabla_t \begin{bmatrix} E_z \\ H_z \end{bmatrix} = \frac{\partial n^2}{\partial \rho} \frac{1}{\rho} \frac{\partial}{\partial \phi} \left[\begin{array}{c} E_z \\ H_z \end{array} \right.$$

- For a step-index fiber $\nabla_t n^2 = 0$, at least for $\rho < a$ and for $\rho > a$; but the change from $n = n_1$ to $n = n_2$ implies a transverse gradient, $\nabla_t n^2 = \left(n_2^2 - n_1^2 \right) \delta \left(\rho - a \right) \hat{\boldsymbol{\rho}}$

- Only if the fields have no azimuthal variation are these RHS=0; only in such circumstances are there separate TE & TM modes $\Leftarrow \quad \dfrac{\partial E_z}{\partial \phi} = \dfrac{\partial H_z}{\partial \phi} = 0$

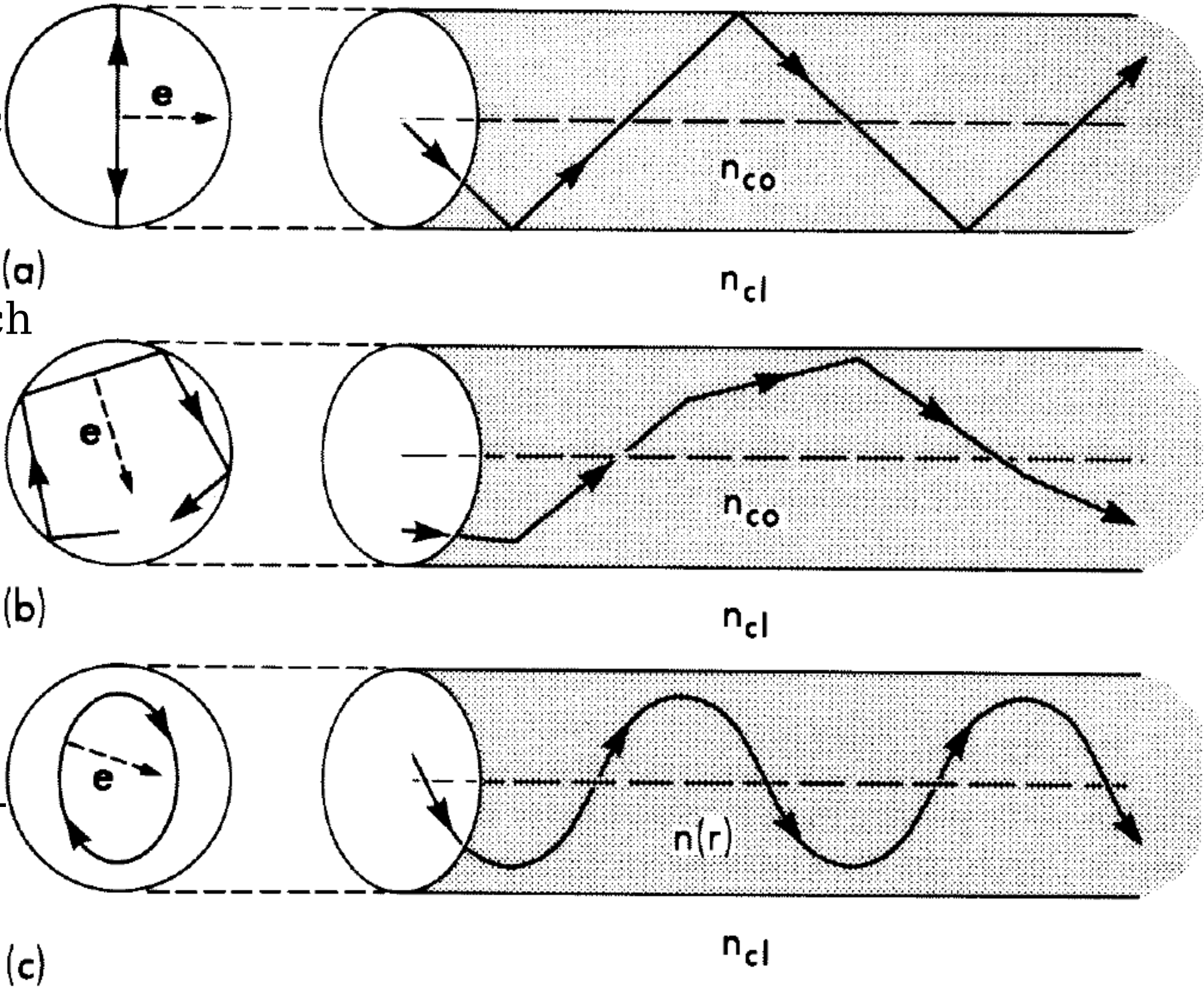
- The modes with both E_z and H_z nonzero are known as HE or EH hybrid modes.

continuity for $\begin{matrix} D_\perp \ \& \ B_\perp \\ \mathbf{E}_\parallel \ \& \ \mathbf{H}_\parallel \end{matrix}$ across $\rho = a$ + separation of variables

$$\Rightarrow \quad \begin{aligned} \begin{bmatrix} E_z \\ H_z \end{bmatrix} &= \begin{bmatrix} A_e \\ A_h \end{bmatrix} J_m(\gamma \rho)\, e^{i m \phi} \ \text{for} \ \rho < a \ \Leftarrow \ \gamma^2 = \frac{n_1^2 \omega^2}{c^2} - k_z^2 \\ \begin{bmatrix} E_z \\ H_z \end{bmatrix} &= \begin{bmatrix} B_e \\ B_h \end{bmatrix} K_m(\beta \rho)\, e^{i m \phi} \ \text{for} \ \rho > a \ \Leftarrow \ \beta^2 = k_z^2 - \frac{n_2^2 \omega^2}{c^2} \end{aligned}$$

- The TE & TM modes have nonvanishing cutoff frequencies, with the lowest for $V = \dfrac{\omega a}{c} \sqrt{n_1^2 - n_2^2} = 2.405$, the 1st root of $J_0(x)$. In contrast, the lowest HE mode (HE_{11}) has no cutoff frequency.

- For $0 < V < 2.405$, HE_{11} is the only mode that propagates in the fiber.



- The azimuthally symmetric TE or TM modes correspond to meridional rays;
- The HE or EH modes, which have azimuthal variation, correspond to skew rays.
- $\mathbf{E}$ after reflection will have a different projection on the z axis than before, as will $\mathbf{H}$.
- Successive reflections mix TE and TM waves; the eigenmodes have both E_z and H_z Nonvanishing.
- In fibers with small Δ, called weakly guiding waveguides, the fields have small longitudinal components and are closely transverse. The language of plane light waves can be employed.
- An HE$_{11}$ mode, with azimuthal dependence for E_z of $\cos\phi$, has fields that are approximately linearly polarized and vary as $J_0(\gamma\rho)$, labled as LP$_{01}$.

Expansion in Normal Modes; Fields Generated by a Localized Source in a Hollow Metallic Guide

- For any given finite frequency, only a finite number of the TE and TM modes can propagate; the rest are cutoff or evanescent modes.

- Far away from any source/obstacle/aperture in the guide, the fields are simple, with only the propagating modes (often just one) present.

- Near a source or obstacle, many modes, both propagating and evanescent, must be superposed in order to describe the fields correctly.

- The cutoff modes have sizable amplitudes only near the source or obstacle; their effects decay away over distances.

- Problems for a source/obstacle/aperture in a waveguide involves the expansion of the fields in terms of all normal modes of the guide in the vicinity, and a determination of the amplitudes for the propagating modes that will describe the fields far away.

Selected problems: 2, 5, 14

A. Orthonormal Modes

- Treat TE and TM modes on an equal footing.

- The fields for the λ mode propagating in the $\pm z$ direction:

$$\begin{aligned} \mathbf{E}_\lambda^{\pm}(x,y,z) &= [+\mathbf{E}_\lambda(x,y) \pm \mathbf{E}_{z\lambda}(x,y)]\, e^{\pm i k_\lambda z} \\ \mathbf{H}_\lambda^{\pm}(x,y,z) &= [\pm\mathbf{H}_\lambda(x,y) + \mathbf{H}_{z\lambda}(x,y)]\, e^{\pm i k_\lambda z} \end{aligned} \quad \Leftarrow \quad \begin{array}{ll} k_\lambda \in \mathbb{R}^+ & \text{propagating modes} \\ k_\lambda \in \Im & \text{cutoff modes} \end{array}$$

tranverse field + longitudinal field

- The sign in the equation is from the need to satisfy $\nabla\cdot\mathbf{E}=\nabla\cdot\mathbf{H}=0$ for each direction & the requirement of positive power flow in the propagation direction.

- Normalization condition by taking the transverse electric fields to be real and

$$\begin{aligned} \int_{\mathcal{A}} \mathbf{E}_\lambda\cdot\mathbf{E}_\mu \,\mathrm{d}\,a &= \delta_{\lambda\mu} \\ \int_{\mathcal{A}} \mathbf{H}_\lambda\cdot\mathbf{H}_\mu \,\mathrm{d}\,a &= \frac{\delta_{\lambda\mu}}{Z_\lambda^2} \\ \int_{\mathcal{A}} \frac{\mathbf{E}_\lambda\times\mathbf{H}_\mu}{2}\cdot\mathrm{d}\,\boldsymbol{a}_z &= \frac{\delta_{\mu\lambda}}{2\,Z_\lambda} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \int_{\mathcal{A}} E_{z\lambda}E_{z\mu}\,\mathrm{d}\,a &= -\frac{\gamma_\lambda^2}{k_\lambda^2}\,\delta_{\lambda\mu} \quad \text{TM Waves} \\ \int_{\mathcal{A}} H_{z\lambda}H_{z\mu}\,\mathrm{d}\,a &= -\frac{\gamma_\lambda^2}{k_\lambda^2}\,\frac{\delta_{\lambda\mu}}{Z_\lambda^2} \quad \text{TE Waves} \end{aligned}$$

time-averaged power flow

$$\mathbf{H}_t = \frac{\hat{\mathbf{z}} \times \mathbf{E}_t}{Z} \quad \Rightarrow \quad \mathbf{H}_\mu = \frac{\hat{\mathbf{z}} \times \mathbf{E}_\mu}{Z_\mu} \quad \Rightarrow \quad \begin{array}{l} \mathbf{H}_\lambda \cdot \mathbf{H}_\mu = \dfrac{\left(\hat{\mathbf{z}} \times \mathbf{E}_\lambda\right) \cdot \left(\hat{\mathbf{z}} \times \mathbf{E}_\mu\right)}{Z_\mu \, Z_\lambda} = \dfrac{\mathbf{E}_\lambda \cdot \mathbf{E}_\mu}{Z_\mu \, Z_\lambda} \\ \mathbf{E}_\lambda \times \mathbf{H}_\mu = \dfrac{\mathbf{E}_\lambda \times \left(\hat{\mathbf{z}} \times \mathbf{E}_\mu\right)}{Z_\mu} = \dfrac{\mathbf{E}_\lambda \cdot \mathbf{E}_\mu}{Z_\mu} \, \hat{\mathbf{z}} \end{array}$$

$$\delta_{\lambda\mu} \Leftarrow \int_{\mathcal{A}} \mathbf{E}_\lambda \cdot \mathbf{E}_\mu \, \mathrm{d}\, a = -\frac{k_\lambda \, k_\mu}{\gamma_\lambda^2 \, \gamma_\mu^2} \int_{\mathcal{A}} \nabla_t \, E_{z\lambda} \cdot \nabla_t \, E_{z\mu} \, \mathrm{d}\, a \quad \Leftarrow \quad \mathbf{E}_t = \frac{i\,k}{\gamma^2} \nabla_t \, E_z$$

$$= -\frac{k_\lambda \, k_\mu}{\gamma_\lambda^2 \, \gamma_\mu^2} \left(\oint_{c} \cancel{E_{z\lambda} \frac{\partial E_{z\mu}}{\partial n}} \, \mathrm{d}\,\ell - \int_{\mathcal{A}} E_{z\lambda} \, \nabla_t^2 \, E_{z\mu} \, \mathrm{d}\, a \right) \quad \Leftarrow \quad \begin{array}{r} E_{z|S} = 0 \\ \left(\nabla_t^2 + \gamma^2\right) E_z = 0 \end{array}$$

$$= -\frac{k_\lambda \, k_\mu}{\gamma_\lambda^2} \int_{\mathcal{A}} E_{z\lambda} \, E_{z\mu} \, \mathrm{d}\, a \quad \Rightarrow \quad \int_{\mathcal{A}} E_{z\lambda} \, E_{z\mu} \, \mathrm{d}\, a = -\frac{\gamma_\lambda^2}{k_\lambda^2} \, \delta_{\lambda\mu}$$

Similarly, $\displaystyle \int_{\mathcal{A}} H_{z\lambda} \, H_{z\mu} \, \mathrm{d}\, a = -\frac{\gamma_\lambda^2}{k_\lambda^2} \, \frac{\delta_{\lambda\mu}}{Z_\lambda^2} \qquad \Leftarrow \quad \frac{\partial H_z}{\partial n}_{|S} = 0$

- The normalized fields in a rectangular guide for $\gamma_{mn}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$

TM **TE**

$$E_{xmn} = \frac{2\pi m}{\gamma_{mn} a \sqrt{ab}} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad E_{xmn} = \frac{-2\pi n}{\gamma_{mn} b \sqrt{ab}} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$E_{ymn} = \frac{2\pi n}{\gamma_{mn} b \sqrt{ab}} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b}, \qquad E_{ymn} = \frac{2\pi m}{\gamma_{mn} a \sqrt{ab}} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b}$$

$$E_{zmn} = -i\frac{2\gamma_{mn}}{k_\lambda \sqrt{ab}} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad H_{zmn} = \frac{-2i\gamma_{mn}}{k_\lambda Z_\lambda \sqrt{ab}} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}$$

$$\Rightarrow \quad \mathbf{H}_t = \pm\frac{\hat{\mathbf{z}} \times \mathbf{E}_t}{Z}$$

- For TM modes, the lowest value is $m=n=1$. For TE modes, $m=0$ or $n=0$ is allowed. If $m=0$ or $n=0$, the normalization must be fixed by multiplying with $\frac{1}{\sqrt{2}}$.

B. Expansion of Arbitrary Fields

- An arbitrary EM field with time dependence $e^{-i\omega t}$ can be expanded in terms of the normal mode fields:

$$\left.\begin{matrix}\mathbf{E}\\ \mathbf{H}\end{matrix}\right] = \begin{bmatrix}\mathbf{E}^{+}\\ \mathbf{H}^{+}\end{bmatrix} + \begin{bmatrix}\mathbf{E}^{-}\\ \mathbf{H}^{-}\end{bmatrix} \quad \Leftarrow \quad \left.\begin{matrix}\mathbf{E}^{\pm}\\ \mathbf{H}^{\pm}\end{matrix}\right] = \sum A_{\lambda}^{\pm} \left[\begin{matrix}\mathbf{E}_{\lambda}^{\pm}\\ \mathbf{H}_{\lambda}^{\pm}\end{matrix}\right.$$

- **Theorem**: The fields everywhere in the guide are determined uniquely by specification of the transverse components of **E** and **H** in a plane, ie, z=constant.

Proof: Let $z = 0\ (= \text{const})$

$$\Rightarrow \quad \left.\begin{matrix}\mathbf{E}_t\\ \mathbf{H}_t\end{matrix}\right] = \sum_{\lambda'} \left(A_{\lambda'}^{+} \begin{bmatrix}+\\ -\end{bmatrix} A_{\lambda'}^{-} \right) \left[\begin{matrix}\mathbf{E}_{\lambda'}\\ \mathbf{H}_{\lambda'}\end{matrix}\right. \quad \Rightarrow \quad \begin{matrix} A_{\lambda}^{+} + A_{\lambda}^{-} = \displaystyle\int \mathbf{E}_{\lambda} \cdot \mathbf{E}_{t}\, \mathrm{d}\, a \\ A_{\lambda}^{+} - A_{\lambda}^{-} = Z_{\lambda}^{2} \displaystyle\int \mathbf{H}_{\lambda} \cdot \mathbf{H}_{t}\, \mathrm{d}\, a \end{matrix}$$

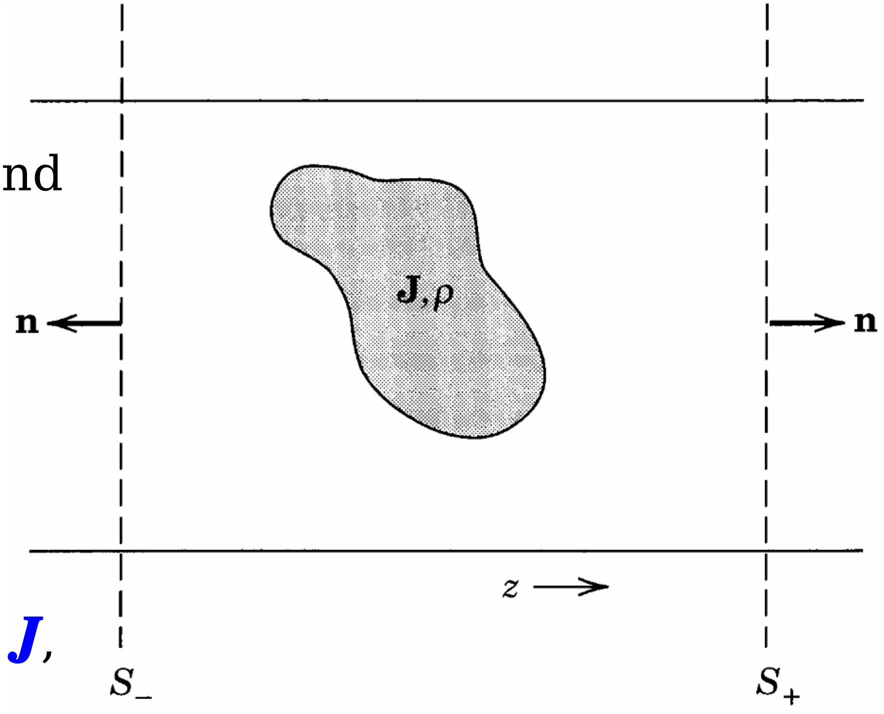
$$\Rightarrow \quad A_{\lambda}^{\pm} = \frac{1}{2} \int \left(\mathbf{E}_{\lambda} \cdot \mathbf{E}_{t} \pm Z_{\lambda}^{2} \mathbf{H}_{\lambda} \cdot \mathbf{H}_{t} \right) \mathrm{d}\, a$$

$\Rightarrow$ If $\mathbf{E}_t$ & $\mathbf{H}_t$ are given at $z = 0$, the coefficients in the expansion are determined.

- The completeness of the normal mode expansion assures the uniqueness of the representation for all z.

C. Fields Generated by a Localized Source

- Let the current density varies in time $e^{-i\omega t}$ and fields propagate to the left and to the right.



- At and to the right/left of the surfaces $S_\pm$,

varying as $e^{\pm i k_\lambda z}$ $\Rightarrow$ $\left.\begin{matrix}\mathbf{E}^\pm \\ \mathbf{H}^\pm\end{matrix}\right] = \sum A_{\lambda'}^\pm \left[\begin{matrix}\mathbf{E}_{\lambda'}^\pm \\ \mathbf{H}_{\lambda'}^\pm\end{matrix}\right.$

- To determine the coefficients $A^\pm_\lambda$ in terms of $\boldsymbol{J}$,

source-free Maxwell equations for $\mathbf{E}_\lambda^\pm, \mathbf{H}_\lambda^\pm$ $\quad + \quad \nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$

Maxwell equations with source for $\mathbf{E}, \mathbf{H}$

$$\Rightarrow \quad \nabla\cdot(\mathbf{E}\times\mathbf{H}_\lambda^\pm - \mathbf{E}_\lambda^\pm\times\mathbf{H}) = \mathbf{H}_\lambda^\pm\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{H}_\lambda^\pm) - \mathbf{H}\cdot(\nabla\times\mathbf{E}_\lambda^\pm) + \mathbf{E}_\lambda^\pm\cdot(\nabla\times\mathbf{H}) \quad \Leftarrow \quad \begin{matrix}\mathbf{E}_\lambda^\pm, \mathbf{H}_\lambda^\pm \\ \mathbf{E}, \mathbf{H}\end{matrix} \propto e^{-i\omega t}$$

$$= -\mu\,\cancel{\mathbf{H}_\lambda^\pm\cdot\frac{\partial\mathbf{H}}{\partial t}} - \epsilon\,\cancel{\mathbf{E}\cdot\frac{\partial\mathbf{E}_\lambda^\pm}{\partial t}} + \mu\,\cancel{\mathbf{H}\cdot\frac{\partial\mathbf{H}_\lambda^\pm}{\partial t}} + \epsilon\,\cancel{\mathbf{E}_\lambda^\pm\cdot\frac{\partial\mathbf{E}}{\partial t}} + \boldsymbol{J}\cdot\mathbf{E}_\lambda^\pm = \boldsymbol{J}\cdot\mathbf{E}_\lambda^\pm$$

$$\Rightarrow \boxed{\nabla\cdot(\mathbf{E}\times\mathbf{H}_\lambda^\pm - \mathbf{E}_\lambda^\pm\times\mathbf{H}) = \boldsymbol{J}\cdot\mathbf{E}_\lambda^\pm}$$

$$\Rightarrow \int_S (\mathbf{E}\times\mathbf{H}_\lambda^\pm - \mathbf{E}_\lambda^\pm\times\mathbf{H})\cdot d\boldsymbol{a} = \int_{S_+}(\cdots)\cdot d\boldsymbol{a}_z - \int_{S_-}(\cdots)\cdot d\boldsymbol{a}_z + \int_{\text{other}}\cancel{(\cdots)\cdot d\boldsymbol{a}}$$

Assume perfectly conducting walls containing no sources or apertures

$$= \hat{\mathbf{z}}\cdot\sum_{\lambda'}\left(A^{+}_{\lambda'}\int_{\mathcal{S}_+}\left(\mathbf{E}^{+}_{\lambda'}\times\mathbf{H}^{\pm}_{\lambda}-\mathbf{E}^{\pm}_{\lambda}\times\mathbf{H}^{+}_{\lambda'}\right)\mathrm{d}\,a - A^{-}_{\lambda'}\int_{\mathcal{S}_-}\left(\mathbf{E}^{-}_{\lambda'}\times\mathbf{H}^{\pm}_{\lambda}-\mathbf{E}^{\pm}_{\lambda}\times\mathbf{H}^{-}_{\lambda'}\right)\mathrm{d}\,a\right)$$

$$= -\frac{2}{Z_\lambda}A^{\mp}_{\lambda} \quad\Rightarrow\quad A^{\pm}_{\lambda} = -\frac{Z_\lambda}{2}\int_{\mathcal{V}}\boldsymbol{J}\cdot\mathbf{E}^{\mp}_{\lambda}\,\mathrm{d}^3 x$$

- The amplitude for propagation in the $+z$ direction comes from integration of the scalar product of the current with the mode propagating in the $-z$ direction, and vice versa.

- For the presence of apertures in the walls between the 2 planes S_+ and S_-

$$\mathbf{E}^{\pm}_{\lambda|S} = 0 \quad\Rightarrow\quad \mathbf{E}^{\pm}_{\lambda|S}\times\mathbf{H} = 0$$

$$\Rightarrow\quad A^{\pm}_{\lambda} = \frac{Z_\lambda}{2}\int_{\text{apertures}}\mathbf{E}\times\mathbf{H}^{\mp}_{\lambda}\cdot\mathrm{d}\,\boldsymbol{a} - \frac{Z_\lambda}{2}\int_{\mathcal{V}}\boldsymbol{J}\cdot\mathbf{E}^{\mp}_{\lambda}\,\mathrm{d}^3 x \quad\Rightarrow\quad \text{Sec. } 9.5$$