Chapter 7 Plane EM Waves & Wave Propagation

Plane Waves in a Nonconducting Medium

- A feature of the Maxwell equations is the existence of traveling wave solutions.
- The simplest and most fundamental EM waves are transverse, plane waves.
- If no sources, $\nabla \cdot \mathbf{B} = 0\,, \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0\,, \quad \nabla \cdot \mathbf{D} = 0\,, \quad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 0$
- Assume solutions with harmonic time dependence $e^{-i\omega t}$

$$\Rightarrow \begin{array}{l} \nabla \cdot \mathbf{B} = 0\,, \quad \nabla \times \mathbf{E} - i\,\omega\,\mathbf{B} = 0 \\ \nabla \cdot \mathbf{D} = 0\,, \quad \nabla \times \mathbf{H} + i\,\omega\,\mathbf{D} = 0 \end{array} + \begin{array}{l} \mathbf{D} = \epsilon\,\mathbf{E} \\ \mathbf{B} = \mu\,\mathbf{H} \end{array} \Leftarrow \epsilon(\omega)\,,\ \mu(\omega) \in \mathbb{R}^+ \Leftarrow \text{no losses of energy}$$

$$\Rightarrow \begin{array}{l} \nabla \times \mathbf{E} - i\,\omega\,\mathbf{B} = 0 \\ \nabla \times \mathbf{B} + i\,\omega\,\mu\,\epsilon\,\mathbf{E} = 0 \end{array} \Rightarrow \left(\nabla^2 + \mu\,\epsilon\,\omega^2\right)\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix} = 0 \quad \text{Helmholtz wave equation}$$

- If the plane wave travels in the x-direction, $e^{i\,k\,x - i\,\omega\,t} \Rightarrow k = \sqrt{\mu\,\epsilon}\,\omega$

$$\Rightarrow v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\,\epsilon}} = \frac{c}{n} \quad \text{phase velocity} \quad \Leftarrow \quad n(\omega) = \sqrt{\frac{\mu}{\mu_0}\,\frac{\epsilon}{\epsilon_0}} \quad \text{index of refraction}$$

- The primordial solution in 1d

$$u(x\,,t) = a\,e^{i(k\,x - \omega\,t)} + b\,e^{i(-k\,x - \omega\,t)} \Rightarrow u_k(x\,,t) = a\,e^{i\,k(x - v\,t)} + b\,e^{-i\,k(x + v\,t)} \Leftarrow v = \frac{\omega}{k}$$

- If the medium is nondispersive ($\mu\,\epsilon$ independent of frequency)

$$\Rightarrow \quad u(x,t) = f(x - v\,t) + g(x + v\,t) \quad \Leftarrow \quad f\,,\, g \text{ are arbitrary}$$

waves traveling in the positive and negative x directions with phase velocity.

- If the medium is dispersive, the wave changes shape as it propagates.

- EM plane waves should satisfy both the Helmholtz wave & Maxwell equations

$$\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix}(\mathbf{r}, t) = \begin{bmatrix}\mathcal{E}\\ \mathcal{B}\end{bmatrix} e^{i\,\boldsymbol{k}\cdot\mathbf{r} - i\,\omega\,t} \quad \Rightarrow \quad |\boldsymbol{k}|^2 = k^2 = \mu\,\epsilon\,\omega^2 \quad \Leftarrow \quad \text{Helmholtz wave equation}$$

$$\nabla\cdot\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix} = \mathbf{0} \quad \Rightarrow \quad \hat{\boldsymbol{k}}\cdot\begin{bmatrix}\mathcal{E}\\ \mathcal{B}\end{bmatrix} = 0 \quad \Rightarrow \quad \mathbf{E}\ \&\ \mathbf{B} \text{ are both } \perp \text{ to the direction of propagation } \hat{\boldsymbol{k}} \quad \Rightarrow \quad \text{transverse wave}$$

$$\Rightarrow \quad \mathcal{B} = \sqrt{\mu\,\epsilon}\;\hat{\boldsymbol{k}}\times\mathcal{E} \quad \Leftarrow \quad \nabla\times\mathbf{E} = i\,\omega\,\mathbf{B} \quad \Rightarrow \quad \mathcal{E}\perp\mathcal{B}$$

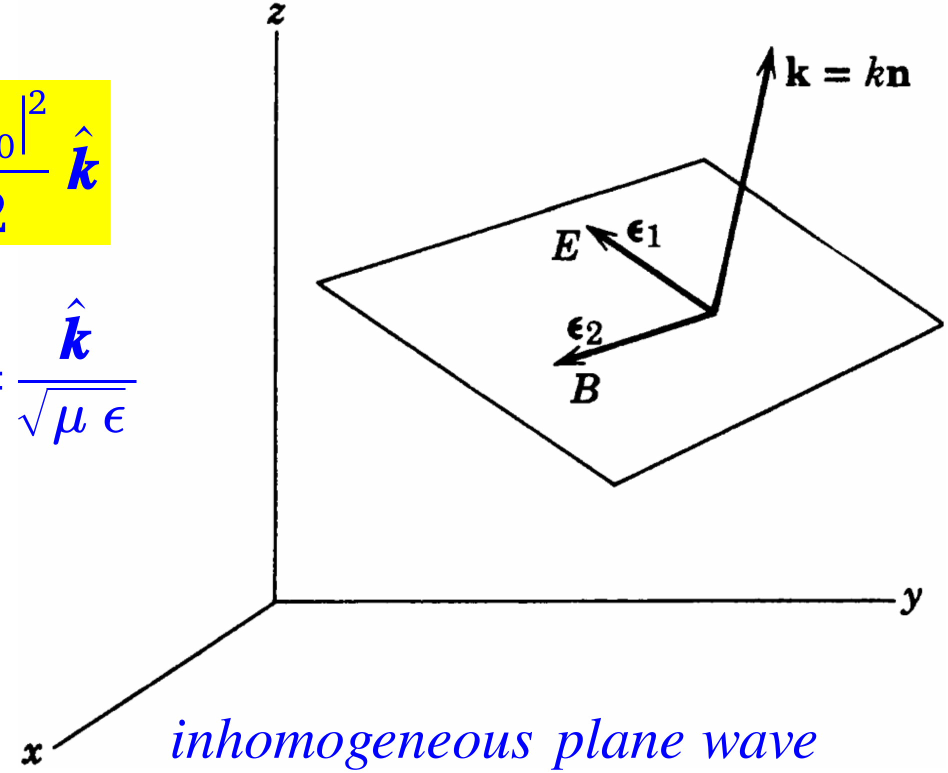
$$\Rightarrow \quad \mathcal{H} = \frac{\hat{\boldsymbol{k}}\times\mathcal{E}}{Z} \quad \Leftarrow \quad Z = \sqrt{\frac{\mu}{\epsilon}} \text{ impedance} \quad \Rightarrow \quad Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 376.7 \text{ ohms}$$

- $c\,\mathbf{B}$ and $\mathbf{E}$ have the *same dimensions*, ie, the *same magnitude* for plane EM waves in free space and differ by the index of refraction in ponderable media.

- If $\hat{\boldsymbol{k}} \in \mathbb{R} \quad \Rightarrow \quad \mathcal{E}\ \&\ \mathcal{B}$ have the same phase $+$ real orthonormal vectors $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\boldsymbol{k}})$

$$\Rightarrow \quad \mathcal{E} = E_0\,\hat{\mathbf{e}}_1,\ \mathcal{B} = \sqrt{\mu\,\epsilon}\,E_0\,\hat{\mathbf{e}}_2, \text{ or } \quad \mathcal{E} = E_0'\,\hat{\mathbf{e}}_2,\ \mathcal{B} = -\sqrt{\mu\,\epsilon}\,E_0'\,\hat{\mathbf{e}}_1$$

$E_0,\ E_0' \in \mathbb{C}$ are constant



- The Poynting vector $\mathbf{S} = \dfrac{\mathbf{E}\times\mathbf{H}^*}{2} = \sqrt{\dfrac{\epsilon}{\mu}}\,\dfrac{|E_0|^2}{2}\,\hat{\boldsymbol{k}}$
- The time-averaged energy density

$$u = \frac{\epsilon\,\mathbf{E}\cdot\mathbf{E}^*}{4} + \frac{\mathbf{B}\cdot\mathbf{B}^*}{4\,\mu} = \frac{\epsilon}{2}|E_0|^2 \quad\Rightarrow\quad \mathbf{v} = \frac{\mathbf{S}}{u} = \frac{\hat{\boldsymbol{k}}}{\sqrt{\mu\,\epsilon}}$$

- If $\hat{\boldsymbol{k}} \in \mathbb{C} \quad\Rightarrow\quad \hat{\boldsymbol{k}} = \hat{\boldsymbol{k}}_{\mathrm{R}} + i\,\hat{\boldsymbol{k}}_{\mathrm{I}}$

$$\Rightarrow\quad e^{i(\hat{\boldsymbol{k}}\cdot\mathbf{r}-\omega t)} = e^{-k\,\hat{\boldsymbol{k}}_{\mathrm{I}}\cdot\mathbf{r}}\; e^{i(k\,\hat{\boldsymbol{k}}_{\mathrm{R}}\cdot\mathbf{r}-\omega t)}$$

$\Rightarrow$ The wave exponentially grows or decays in some directions $\Leftarrow$ *inhomogeneous plane wave*

- The surfaces of constant amplitude and constant phase are still planes, but they are no longer parallel.

- $$\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}} = 1 \;\left(\text{not } |\hat{\boldsymbol{k}}|^2 = 1\right) \quad\Rightarrow\quad \begin{matrix} \hat{k}_{\mathrm{R}}^2 - \hat{k}_{\mathrm{I}}^2 = 1 \\ \hat{\boldsymbol{k}}_{\mathrm{R}}\cdot\hat{\boldsymbol{k}}_{\mathrm{I}} = 0 \end{matrix} \quad\Rightarrow\quad \hat{\boldsymbol{k}}_{\mathrm{R}} \perp \hat{\boldsymbol{k}}_{\mathrm{I}} \quad\Rightarrow\quad \hat{\boldsymbol{k}} = \cosh\theta\,\hat{\mathbf{e}}_1 + i\sinh\theta\,\hat{\mathbf{e}}_2$$

$$\Rightarrow\quad \mathcal{E} = A\left(i\sinh\theta\,\mathbf{e}_1 - \cosh\theta\,\mathbf{e}_2\right) + A'\,\mathbf{e}_3 \quad\Leftarrow\quad \hat{\boldsymbol{k}}\cdot\mathcal{E} = 0 \quad\Leftarrow\quad A,\; A' \in \mathbb{C}$$

- For $\theta \neq 0$, $\boldsymbol{\mathcal{E}}$ in general has components in the direction(s) of $\hat{\boldsymbol{k}}$.

- Inhomogeneous plane waves form a general basis for treatments of boundary-value problems for waves and are useful in the solution of diffraction in 2d.

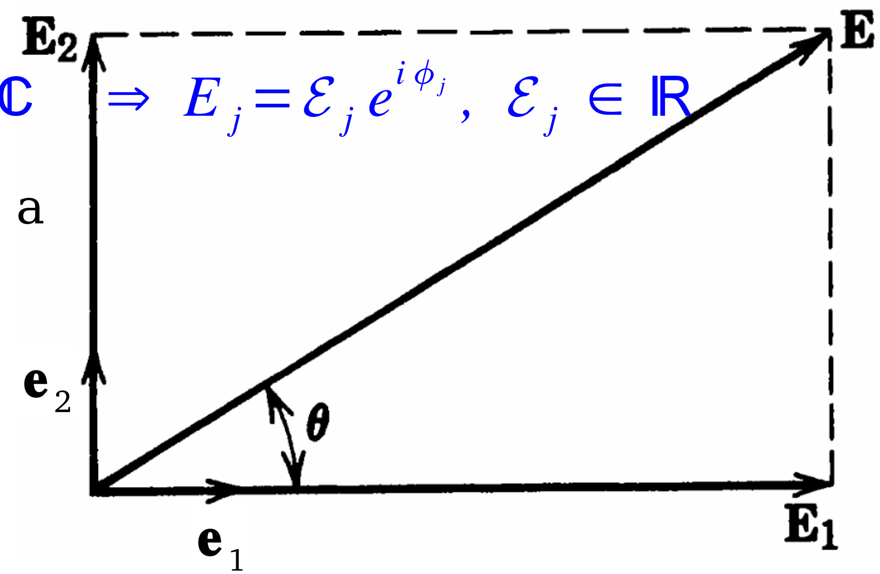
Linear and Circular Polarization; Stokes Parameters

- If a plane wave with its electric field vector is always in one direction, it is linearly polarized.

- 2 waves can be combined to give the most general homogeneous plane wave propagating in one direction,

$$\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\end{bmatrix}=\begin{bmatrix}E_1\,\hat{\mathbf{e}}_1\\ E_2\,\hat{\mathbf{e}}_2\end{bmatrix}e^{i(\boldsymbol{k}\cdot\mathbf{r}-\omega t)}\quad \text{with}\ \mathbf{B}_j=\sqrt{\mu\epsilon}\,\hat{\boldsymbol{k}}\times\mathbf{E}_j\,,\quad j=1,2\ \Leftarrow\ \boldsymbol{k}=k\,\hat{\boldsymbol{k}}$$

$$\Rightarrow\ \mathbf{E}(\mathbf{r},t)=\left(E_1\,\hat{\mathbf{e}}_1+E_2\,\hat{\mathbf{e}}_2\right)e^{i(\boldsymbol{k}\cdot\mathbf{r}-\omega t)}\ \Leftarrow\ E_i\in\mathbb{C}\ \Rightarrow\ E_j=\mathcal{E}_j\,e^{i\phi_j},\ \mathcal{E}_j\in\mathbb{R}$$



- If E_1 and E_2 have the *same phase*, the wave is a *linearly polarized* wave, with its polarization vector making an angle $\theta=\tan^{-1}\dfrac{E_2}{E_1}$ with $\hat{\mathbf{e}}_1$ and magnitude $E=\sqrt{E_1^2+E_2^2}$

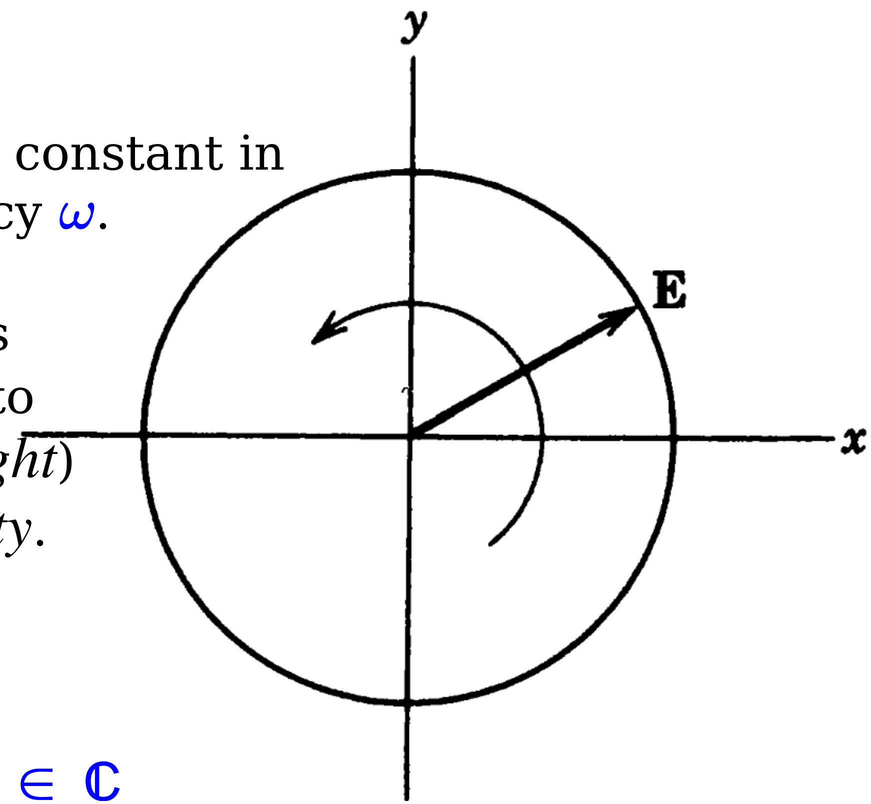
- If E_1 and E_2 have *different phases*, the wave is *elliptically polarized*.

- If $|E_1|=|E_2|=E_0$, but the phase difference is $90°$, this is *circular polarization*.

$$\mathbf{E}(\mathbf{r},t)=E_0\left(\hat{\mathbf{e}}_1\pm i\,\hat{\mathbf{e}}_2\right)e^{i(\boldsymbol{k}\cdot\mathbf{r}-\omega t)}\ \Rightarrow\ \begin{aligned}E_x(\mathbf{r},t)&=\ E_0\cos(kz-\omega t)\\ E_y(\mathbf{r},t)&=\mp E_0\sin(kz-\omega t)\end{aligned}$$

- At a *fixed point in space*, the electric vector is constant in magnitude, but sweeps in a circle at a frequency ω.

- For $(\hat{\mathbf{e}}_1 + i\,\hat{\mathbf{e}}_2)\,[(\hat{\mathbf{e}}_1 - i\,\hat{\mathbf{e}}_2)]$, the rotation of $\mathbf{E}$ is counter-clockwise (clockwise) when looking into the oncoming wave. This wave is called *left* (*right*) *circularly polarized*, or *positive* (*negative*) *helicity*.



$$\mathbf{E}(\mathbf{r},t) = E_0(\hat{\mathbf{e}}_1 + i\,\hat{\mathbf{e}}_2)\,e^{i(\boldsymbol{k}\cdot\mathbf{r} - \omega t)}$$

- $\hat{\mathbf{e}}_\pm \equiv \dfrac{\hat{\mathbf{e}}_1 \pm i\,\hat{\mathbf{e}}_2}{\sqrt{2}} \quad \Rightarrow \quad \hat{\mathbf{e}}_\pm^* \cdot \hat{\mathbf{e}}_\mp = 0\,, \quad \hat{\mathbf{e}}_\pm^* \cdot \hat{\mathbf{e}}_3 = 0$

  $\hat{\mathbf{e}}_\pm^* \cdot \hat{\mathbf{e}}_\pm = 1$

$$\Rightarrow \quad \mathbf{E}(\mathbf{r},t) = (E_+\,\hat{\mathbf{e}}_+ + E_-\,\hat{\mathbf{e}}_-)\,e^{i(\boldsymbol{k}\cdot\mathbf{r} - \omega t)} \quad \Leftarrow \quad E_\pm \in \mathbb{C}$$

- If $E_+ \neq E_-$, but the same phase, the wave represents an elliptically polarized wave with principal axes of the ellipse in the directions of $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$

$$\frac{\text{semimajor axis}}{\text{semiminor axis}} = \left|\frac{1+r}{1-r}\right| \quad \Leftarrow \quad r \equiv \frac{E_-}{E_+}$$

- If $\dfrac{E_-}{E_+} = r\,e^{i\alpha}$, the ellipse traced out by the $\mathbf{E}$ vector has its axes rotated by an angle $\dfrac{\alpha}{2}$.

- For $r = \pm 1$ we get back a linearly polarized wave.

$$\mathbf{E} = \left(E_+ \hat{\mathbf{e}}_+ + E_- \hat{\mathbf{e}}_-\right) e^{i(\boldsymbol{k}\cdot\mathbf{r} - \omega t)} = \frac{\left(E_+ + E_-\right)\hat{\mathbf{e}}_1 + i\left(E_+ - E_-\right)\hat{\mathbf{e}}_2}{\sqrt{2}} e^{i(\boldsymbol{k}\cdot\mathbf{r} - \omega t)}$$

$$= \frac{E_+}{\sqrt{2}}\left[\left(1+r\right)\hat{\mathbf{e}}_1 + i\left(1-r\right)\hat{\mathbf{e}}_2\right] e^{i(\boldsymbol{k}\cdot\mathbf{r} - \omega t)} \quad \Leftarrow \quad \frac{E_-}{E_+} = r > 0 \in \mathbb{R}$$

$$= \mathcal{E}' \left(\left(1+r\right)\hat{\mathbf{e}}_1 + \left(1-r\right) e^{i\frac{\pi}{2}} \hat{\mathbf{e}}_2 \right) e^{-i(\omega t - \phi)} \quad \Leftarrow \quad \frac{E_+}{\sqrt{2}} e^{i\boldsymbol{k}\cdot\mathbf{r}} = \mathcal{E}' e^{i\phi}, \quad \mathcal{E}' \in \mathbb{R}$$

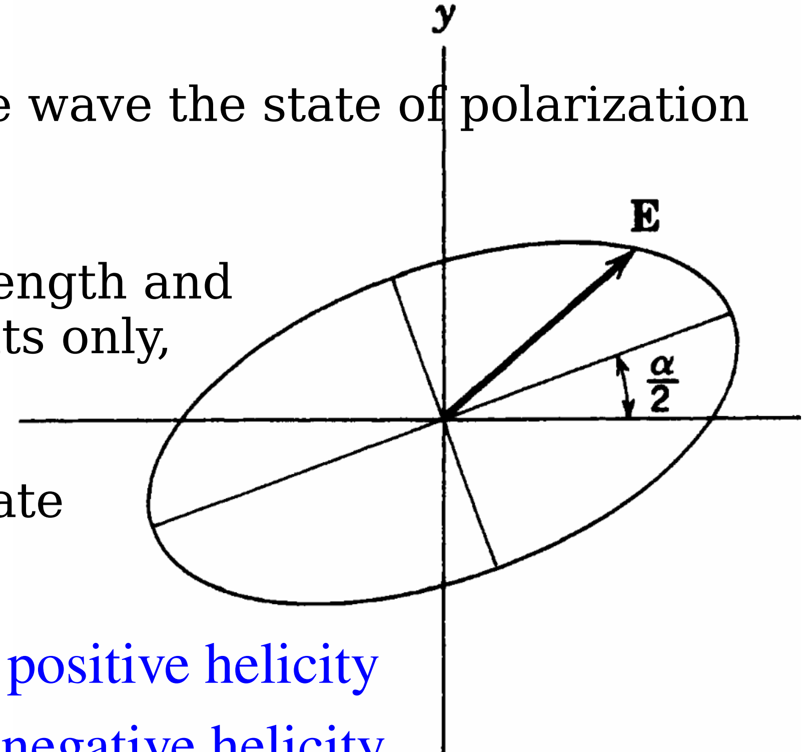
$$\Rightarrow \quad \begin{matrix} \mathcal{E}'_x = \mathcal{E}'\left(1+r\right)\cos\left(\phi - \omega t\right) \\ \mathcal{E}'_y = \mathcal{E}'\left(1-r\right)\sin\left(\phi - \omega t\right) \end{matrix} \quad \Rightarrow \quad \frac{\mathcal{E}'^2_x}{\left(1+r\right)^2} + \frac{\mathcal{E}'^2_y}{\left(1-r\right)^2} = \mathcal{E}'^2 \quad \Rightarrow \quad \frac{a}{b} = \left|\frac{1+r}{1-r}\right|$$

$$\frac{E_-}{E_+} = r e^{i\alpha} \quad \Rightarrow \quad \mathscr{E} \equiv E_+ e^{i\frac{\alpha}{2}}$$

$$\Rightarrow \quad \mathbf{E} = \frac{\mathscr{E}}{\sqrt{2}}\left[\left(e^{-i\frac{\alpha}{2}} + r e^{i\frac{\alpha}{2}}\right)\hat{\mathbf{e}}_1 + i\left(e^{-i\frac{\alpha}{2}} - r e^{i\frac{\alpha}{2}}\right)\hat{\mathbf{e}}_2\right] e^{i(\boldsymbol{k}\cdot\mathbf{r} - \omega t)}$$

$$= \frac{\mathscr{E}}{\sqrt{2}}\left[\left(1+r\right)\hat{\mathbf{e}}'_1 + i\left(1-r\right)\hat{\mathbf{e}}'_2\right] e^{i(\boldsymbol{k}\cdot\mathbf{r} - \omega t)} \quad \Leftarrow \quad \begin{bmatrix} \hat{\mathbf{e}}'_1 \\ \hat{\mathbf{e}}'_2 \end{bmatrix} = \begin{bmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \end{bmatrix}$$

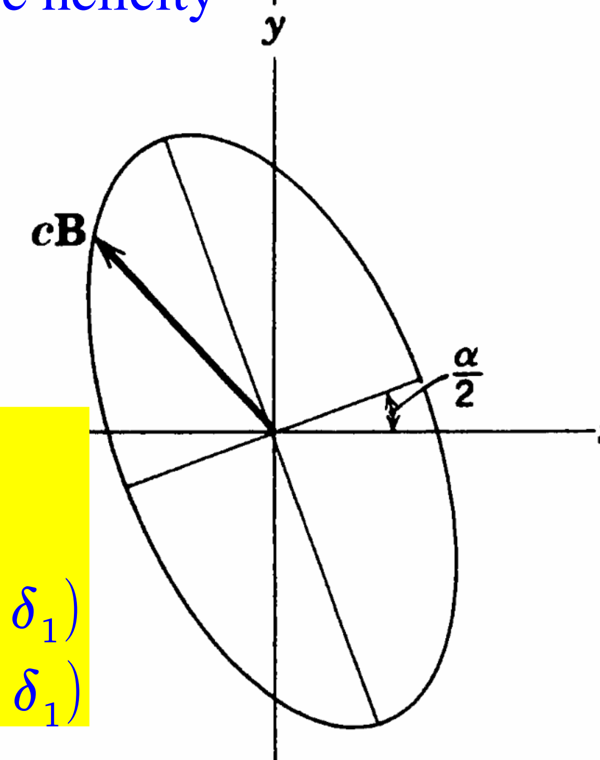
- How can we determine from observations on the wave the state of polarization in all its particulars? ⇒ *Stokes parameters*



- Stokes parameters are quadratic in the field strength and can be determined through intensity measurements only, with a linear polarizer and a quarter-wave plate.

- The measurement determines completely the state of polarization of the wave.

- $\hat{\mathbf{e}}_1 \cdot \mathbf{E}$: linear polarization in the x-axis , $\hat{\mathbf{e}}_+^* \cdot \mathbf{E}$: positive helicity
  $\hat{\mathbf{e}}_2 \cdot \mathbf{E}$: linear polarization in the y-axis , $\hat{\mathbf{e}}_-^* \cdot \mathbf{E}$: negative helicity



- The squares of these amplitudes give a measure of the intensity of each type of polarization. Phase information can be obtained from cross products.

- $E_1 \equiv a_1 e^{i\delta_1}, \; E_2 \equiv a_2 e^{i\delta_2}, \; E_+ \equiv a_+ e^{i\delta_+}, \; E_- \equiv a_- e^{i\delta_-}$

⇒ Stokes parameters in $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$

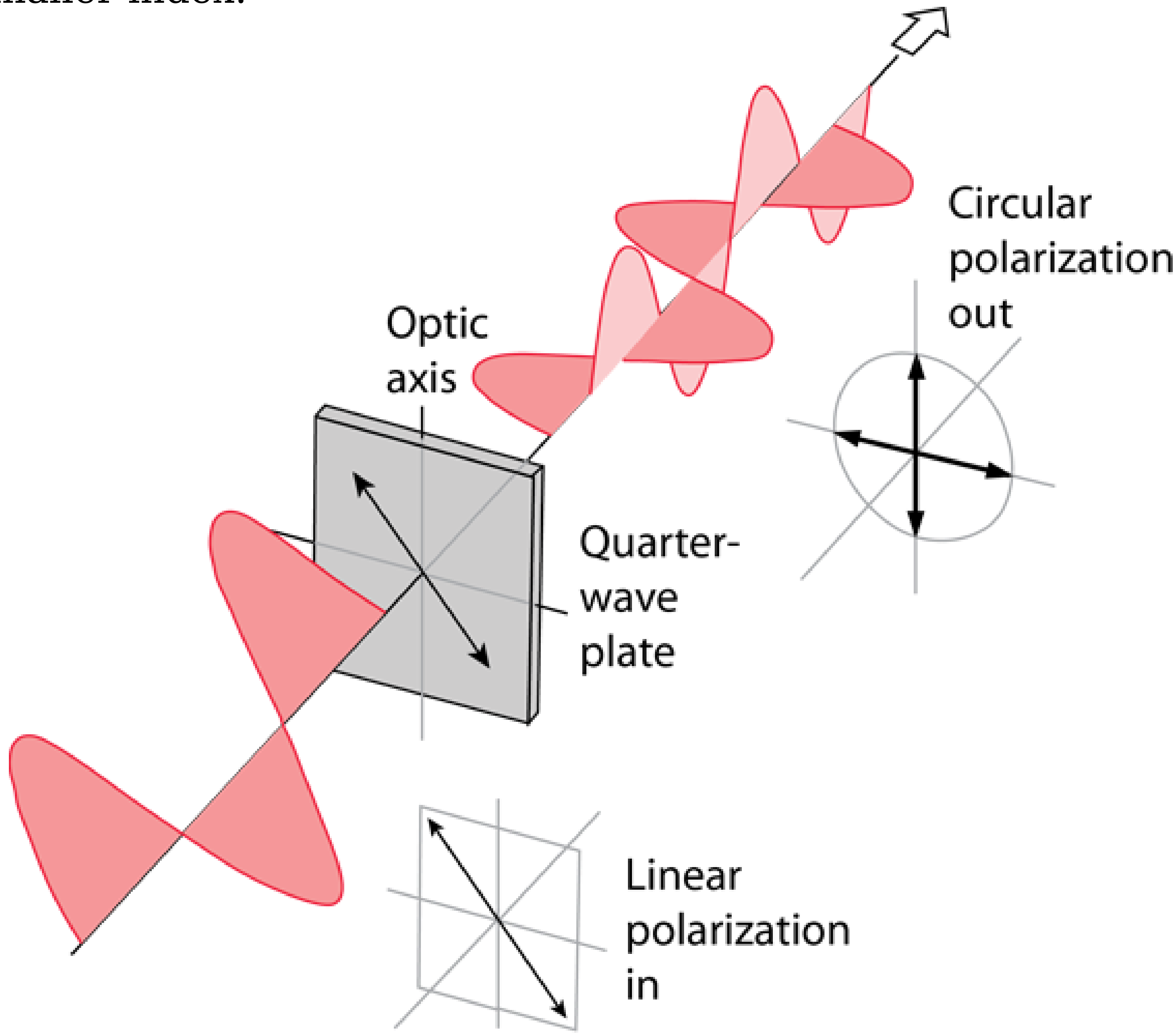
$$s_0 = |\hat{\mathbf{e}}_1 \cdot \mathbf{E}|^2 + |\hat{\mathbf{e}}_2 \cdot \mathbf{E}|^2 = a_1^2 + a_2^2$$

$$s_1 = |\hat{\mathbf{e}}_1 \cdot \mathbf{E}|^2 - |\hat{\mathbf{e}}_2 \cdot \mathbf{E}|^2 = a_1^2 - a_2^2$$

$$s_2 = 2\,\Re\left[(\hat{\mathbf{e}}_1 \cdot \mathbf{E})^* (\hat{\mathbf{e}}_2 \cdot \mathbf{E})\right] = 2\,a_1\,a_2 \cos(\delta_2 - \delta_1)$$

$$s_3 = 2\,\Im\left[(\hat{\mathbf{e}}_1 \cdot \mathbf{E})^* (\hat{\mathbf{e}}_2 \cdot \mathbf{E})\right] = 2\,a_1\,a_2 \sin(\delta_2 - \delta_1)$$

A **quarter-wave plate** consists of a carefully adjusted thickness of a birefringent material such that the light associated with the larger index of refraction is retarded by 90° in phase (a quarter wavelength) with respect to that associated with the smaller index.



$\Rightarrow$ Stokes parameters in $(\hat{\mathbf{e}}_+, \hat{\mathbf{e}}_-)$

$$s_0 = |\hat{\mathbf{e}}_+^* \cdot \mathbf{E}|^2 + |\hat{\mathbf{e}}_-^* \cdot \mathbf{E}|^2 = a_+^2 + a_-^2$$
$$s_1 = 2\,\Re\left[(\hat{\mathbf{e}}_+^* \cdot \mathbf{E})^* (\hat{\mathbf{e}}_-^* \cdot \mathbf{E})\right] = 2\, a_+ a_- \cos(\delta_- - \delta_+)$$
$$s_2 = 2\,\Im\left[(\hat{\mathbf{e}}_+^* \cdot \mathbf{E})^* (\hat{\mathbf{e}}_-^* \cdot \mathbf{E})\right] = 2\, a_+ a_- \sin(\delta_- - \delta_+)$$
$$s_3 = |\hat{\mathbf{e}}_+^* \cdot \mathbf{E}|^2 - |\hat{\mathbf{e}}_-^* \cdot \mathbf{E}|^2 = a_+^2 - a_-^2$$

- s_0 measures the relative intensity of the wave. In the linear basis, s_1 gives the preponderance of x-linear polarization over y-linear polarization, s_2 & s_3 give phase information. In the circular basis, s_3 tells the difference in relative intensity of positive and negative helicity, s_1 & s_2 concern the phases.

- 4 Stokes parameters are not independent, they depend on only 3 quantities,

$$a_1,\; a_2,\; \delta_2 - \delta_1 \quad \Rightarrow \quad s_0^2 = s_1^2 + s_2^2 + s_3^2$$

- No beams of radiation are completely monochromatic
  + the magnitudes and phases vary slowly in time.

$\Rightarrow$ the observable Stokes parameters become averages over a time interval,

$$\text{eg} \quad s_2 = 2\,\langle a_1 a_2 \cos(\delta_2 - \delta_1)\rangle$$

$$\Rightarrow \quad s_0^2 \geq s_1^2 + s_2^2 + s_3^2 \quad \Rightarrow \quad \text{it could be } \; s_1 = s_2 = s_3 = 0 \; \text{ in 'Natural light'}$$

- An astrophysical example of the use of Stokes parameters is to study of optical and radio frequency radiation from the Crab nebula. The optical light shows small linear polarization, the radio emission shows a high degree of linear polarization. In both frequency there is no evidence for circular polarization. The information helps to decide the mechanism of radiation.

Reflection and Refraction of Electromagnetic Waves at a Plane Interface Between Dielectrics

- The phenomena of the reflection and refraction of light at a plane surface between 2 media can be divided themselves into 2 classes:

1. Kinematic properties: $\boldsymbol{k}\cdot\mathbf{r}=\boldsymbol{k}'\cdot\mathbf{r}=\boldsymbol{k}''\cdot\mathbf{r}=\omega\, t=\text{phase}$

(a) Angle of reflection = angle of incidence; (b) Snell's law: $\dfrac{\sin i}{\sin r}=\dfrac{n'}{n}$

2. Dynamic properties:

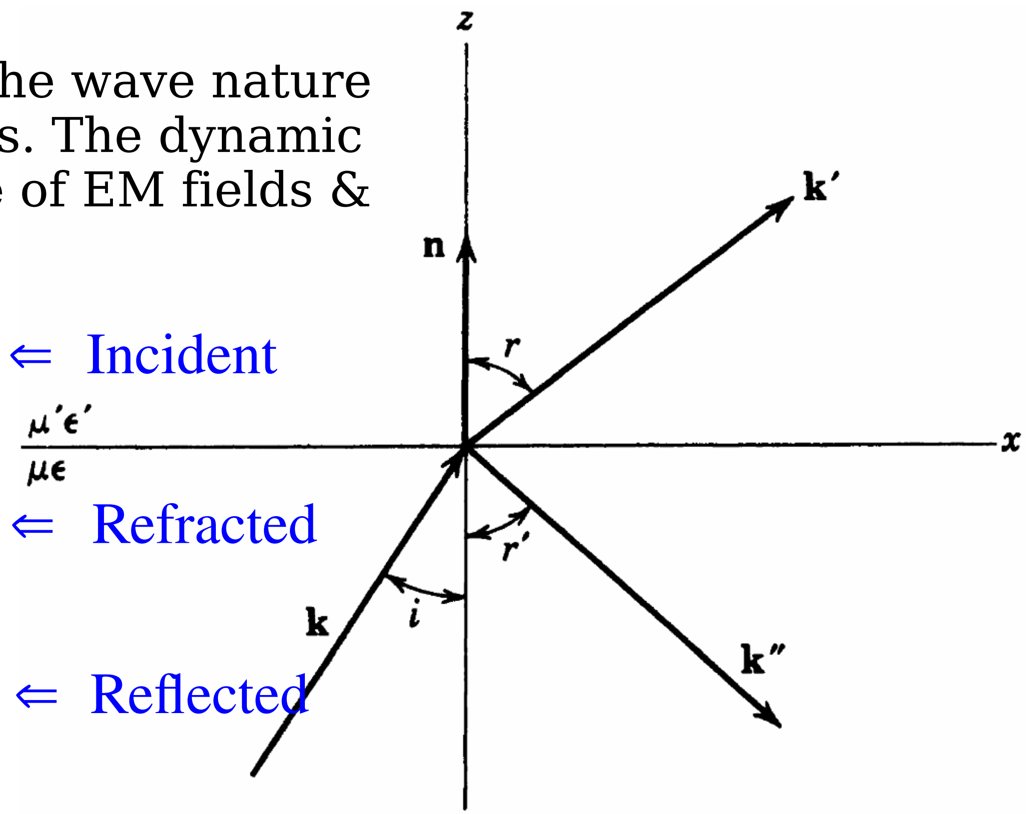
(a) Intensities of reflected & refracted radiation. (b) Phase change & polarization.

- The kinematic properties follow from the wave nature of the phenomena & boundary conditions. The dynamic properties depend on the specific nature of EM fields & their boundary conditions.

- $\mathbf{E}=\mathbf{E}_0\, e^{i\boldsymbol{k}\cdot\mathbf{r}-i\omega t},\quad \mathbf{B} = \sqrt{\mu\,\epsilon}\,\hat{\boldsymbol{k}}\times\mathbf{E} \;\Leftarrow\;$ Incident

$\mathbf{E}'=\mathbf{E}_0'\, e^{i\boldsymbol{k}'\cdot\mathbf{r}-i\omega t},\quad \mathbf{B}' = \sqrt{\mu'\,\epsilon'}\,\hat{\boldsymbol{k}}'\times\mathbf{E}' \;\Leftarrow\;$ Refracted

$\mathbf{E}''=\mathbf{E}_0''\, e^{i\boldsymbol{k}''\cdot\mathbf{r}-i\omega t},\quad \mathbf{B}'' = \sqrt{\mu\,\epsilon}\,\hat{\boldsymbol{k}}''\times\mathbf{E}'' \;\Leftarrow\;$ Reflected



$$\Rightarrow \quad |\boldsymbol{k}| = |\boldsymbol{k}''| = k = \omega\sqrt{\mu\,\epsilon}\,, \quad |\boldsymbol{k}'| = k' = \omega\sqrt{\mu'\,\epsilon'}$$

$$\Rightarrow \quad (\boldsymbol{k}\cdot\mathbf{r})_{z=0} = (\boldsymbol{k}'\cdot\mathbf{r})_{z=0} = (\boldsymbol{k}''\cdot\mathbf{r})_{z=0} \quad \Leftarrow \quad \text{the phase factors all equal at } z=0$$

$$\Rightarrow \quad k\sin i = k'\sin r = k''\sin r' \quad \Rightarrow \quad i = r' \quad \Leftarrow \quad k'' = k$$

$$\Rightarrow \quad \frac{\sin i}{\sin r} = \frac{k'}{k} = \sqrt{\frac{\mu'\,\epsilon'}{\mu\,\epsilon}} = \frac{n'}{n} \quad \text{Snell's law} \quad \Leftarrow \quad n = \sqrt{\frac{\mu\,\epsilon}{\mu_0\,\epsilon_0}}\,, \quad n' = \sqrt{\frac{\mu'\,\epsilon'}{\mu_0\,\epsilon_0}}$$

- The dynamic properties are contained in the boundary conditions

(1) normal components of **D** and **B**, $D_\perp$ & $B_\perp$, are continuous;

(2) tangential components of **E** and **H** , $\mathbf{E}_\parallel$ & $\mathbf{H}_\parallel$,are continuous.

$$\Rightarrow \quad \begin{aligned} \left[\epsilon\,(\mathbf{E}_0 + \mathbf{E}_0'') - \epsilon'\,\mathbf{E}_0'\right]\cdot\mathbf{n} = 0 \qquad (1) \\ (\boldsymbol{k}\times\mathbf{E}_0 + \boldsymbol{k}''\times\mathbf{E}_0'' - \boldsymbol{k}'\times\mathbf{E}_0')\cdot\mathbf{n} = 0 \qquad (2) \\ (\mathbf{E}_0 + \mathbf{E}_0'' - \mathbf{E}_0')\times\mathbf{n} = 0 \qquad (3) \\ \left(\frac{\boldsymbol{k}\times\mathbf{E}_0 + \boldsymbol{k}''\times\mathbf{E}_0''}{\mu} - \frac{\boldsymbol{k}'\times\mathbf{E}_0'}{\mu'}\right)\times\mathbf{n} = 0 \qquad (4) \end{aligned}$$

- Consider 2 separate situations:

(1) the incident plane wave is linearly polarized with its polarization vector ⊥ to the plane of incidence (defined by $\boldsymbol{k}$ & $\mathbf{n}$), ie, $\mathbf{E} \perp$ plane of incidence.

(2) the polarization vector is ∥ to the plane of incidence, ie, $\mathbf{B} \perp$ plane of incidence.

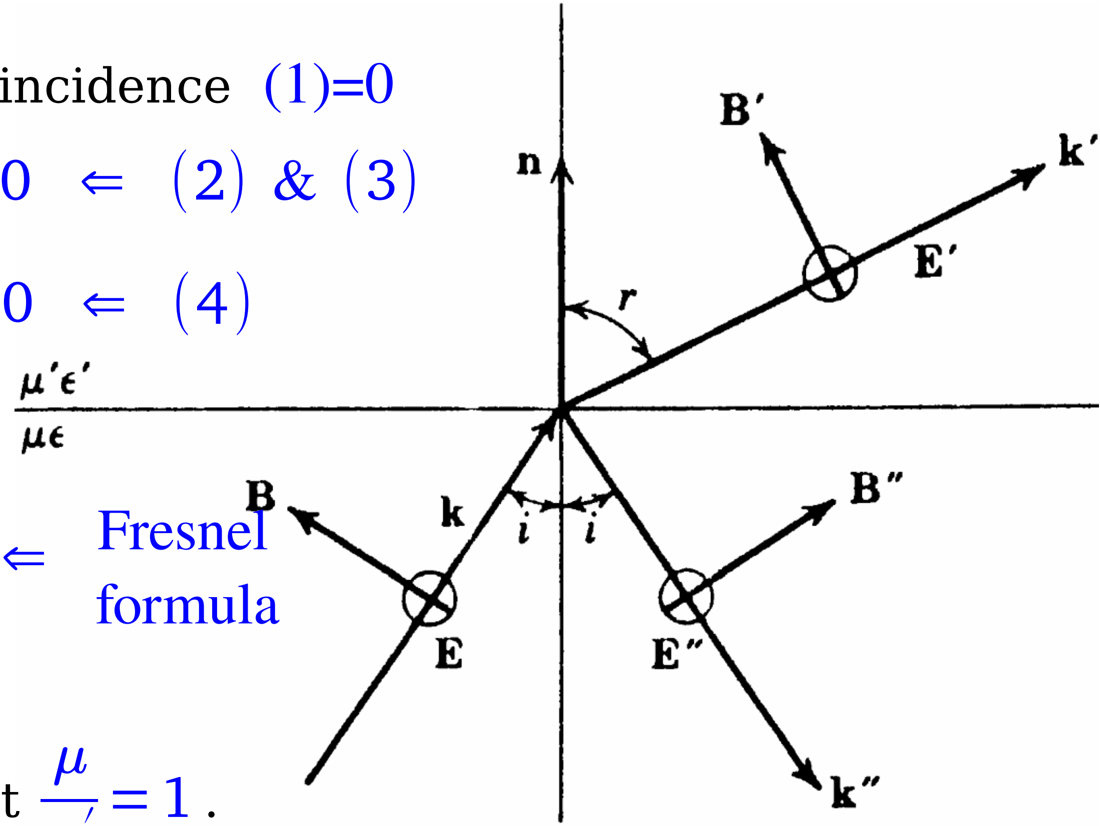
- The general case of arbitrary elliptic polarization can be obtained by linear combinations of the 2 results.

- For the electric field ⊥ the plane of incidence (1)=0

$$E_0 + E_0'' - E_0' = 0 \quad \Leftarrow \quad (2)\ \&\ (3)$$

$$\Rightarrow \quad \sqrt{\frac{\epsilon}{\mu}}\,(E_0 - E_0'')\cos i - \sqrt{\frac{\epsilon'}{\mu'}}\,E_0'\cos r = 0 \quad \Leftarrow \quad (4)$$

$$\Rightarrow \quad \frac{E_0'}{E_0} = \frac{2\,\mu'\,n\cos i}{\mu'\,n\cos i + \mu\sqrt{n'^2 - n^2\sin^2 i}} \qquad \frac{E_0''}{E_0} = \frac{\mu'\,n\cos i - \mu\sqrt{n'^2 - n^2\sin^2 i}}{\mu'\,n\cos i + \mu\sqrt{n'^2 - n^2\sin^2 i}} \quad \Leftarrow \quad \text{Fresnel formula}$$



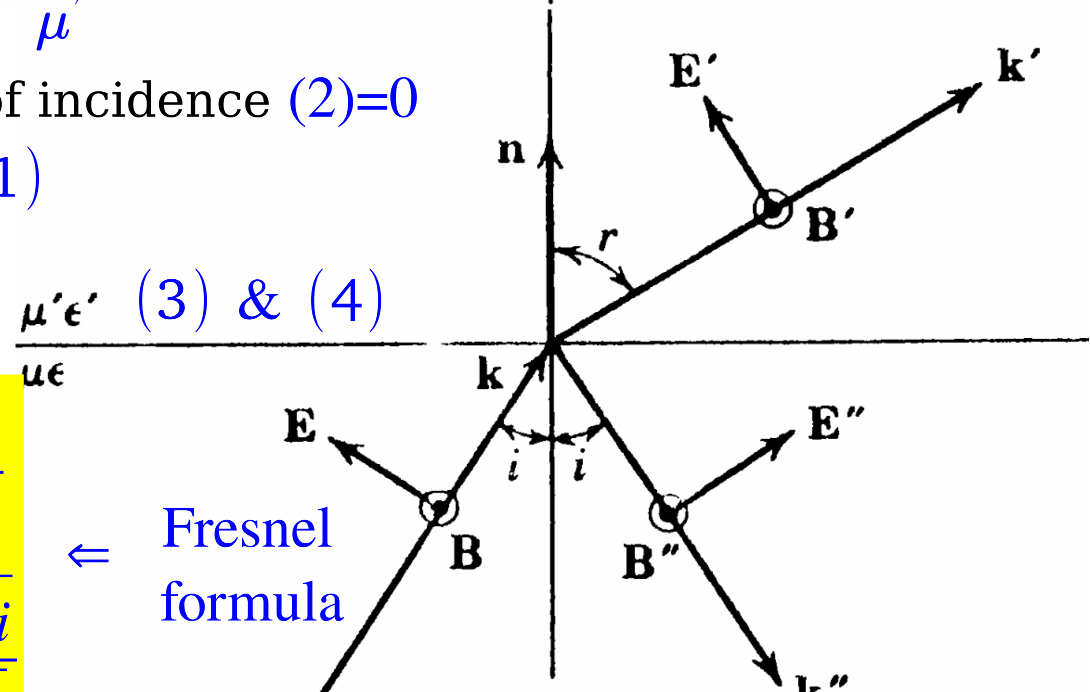
- For optical frequencies it usually put $\frac{\mu}{\mu'} = 1$.

- For the electric field ∥ to the plane of incidence (2)=0

$$\cos i\,(E_0 - E_0'') - \cos r\,E_0' = 0 \quad \Leftarrow \quad (1)$$

$$\Rightarrow \quad \sqrt{\frac{\epsilon}{\mu}}\,(E_0 + E_0'') - \sqrt{\frac{\epsilon'}{\mu'}}\,E_0' = 0 \quad \Leftarrow \quad (3)\ \&\ (4)$$

$$\Rightarrow \quad \frac{E_0'}{E_0} = \frac{2\,\mu'\,n\,n'\cos i}{\mu\,n'^2\cos i + \mu'\,n\sqrt{n'^2 - n^2\sin^2 i}} \qquad \frac{E_0''}{E_0} = \frac{\mu\,n'^2\cos i - \mu'\,n\sqrt{n'^2 - n^2\sin^2 i}}{\mu\,n'^2\cos i + \mu'\,n\sqrt{n'^2 - n^2\sin^2 i}} \quad \Leftarrow \quad \text{Fresnel formula}$$



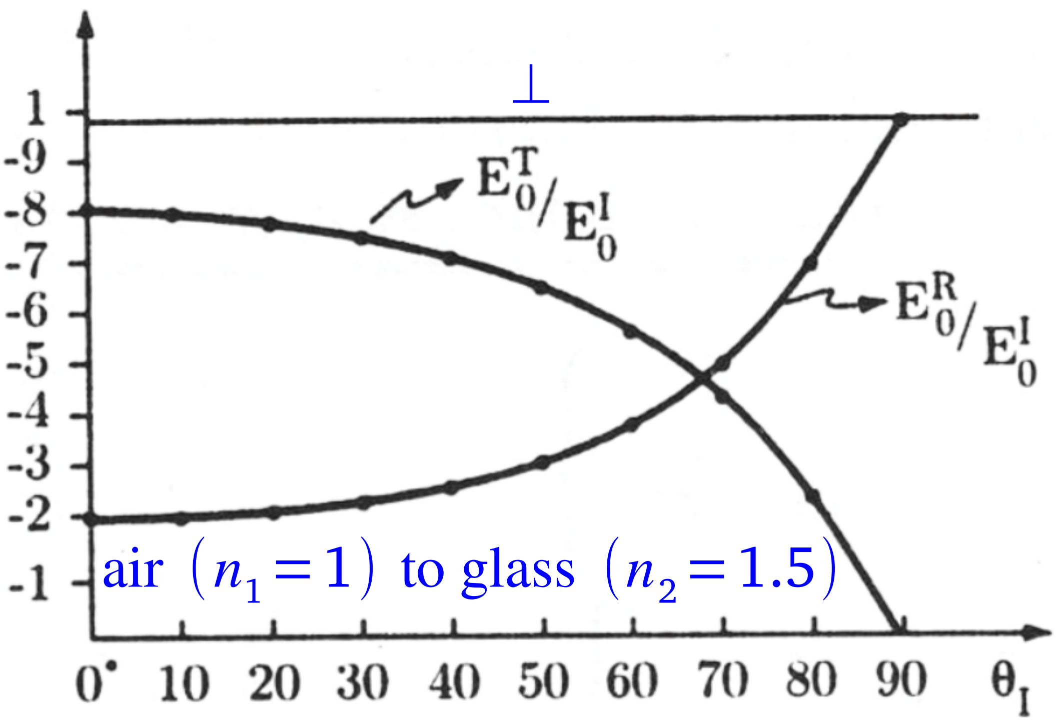
- For normal incidence ($i=0$), both sets of equations reduce to

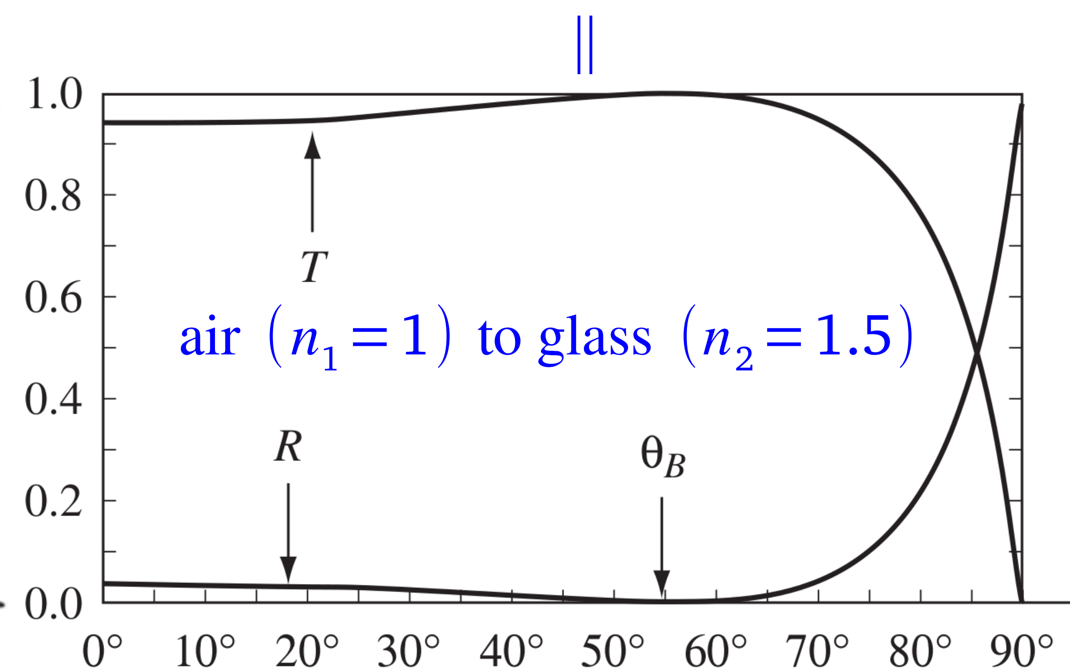
$$\frac{E_0'}{E_0} = \frac{2\sqrt{\mu'\epsilon}}{\sqrt{\epsilon'\mu}+\sqrt{\mu'\epsilon}} \rightarrow \frac{2n}{n'+n}$$

$$\frac{E_0''}{E_0} = \pm\frac{\sqrt{\epsilon'\mu}-\sqrt{\mu'\epsilon}}{\sqrt{\epsilon'\mu}+\sqrt{\mu'\epsilon}} \rightarrow \pm\frac{n'-n}{n'+n} \quad \Leftarrow \quad \begin{array}{l} \mu' \approx \mu \\ +:\ \parallel \\ -:\ \perp \end{array}$$

- For $n'>n$ there is a phase reversal for the reflected wave at normal incidence.

- $I_I \equiv u\,v = \frac{v\,\epsilon}{2}E_0^2,\quad I_R = \frac{v\,\epsilon}{2}E_0''^2,\quad I_T = \frac{v'\,\epsilon'}{2}E_0'^2,\quad R = \frac{I_R}{I_I},\quad T = \frac{I_T}{I_I}$





Polarization by Reflection and Total Internal Reflection; Goos-Hänchen Effect

- For polarization ∥ the plane of incidence there is an angle of incidence, called *Brewster's angle*, for which there is no reflected wave.

$$E''=0\,,\quad \mu'=\mu \;\Rightarrow\; n'^2\cos i_B = n\sqrt{n'^2-n^2\sin^2 i_B} \;\Rightarrow\; i_B=\tan^{-1}\frac{n'}{n}$$

- If a plane wave of mixed polarization is incident on a plane interface at the Brewster angle, the reflected radiation is *completely* plane-*polarized* with polarization vector ⊥ the plane of incidence.

- Even if the angles is other than the Brewster angle, there is a tendency for the reflected wave to be predominantly polarized ⊥ the plane of incidence.

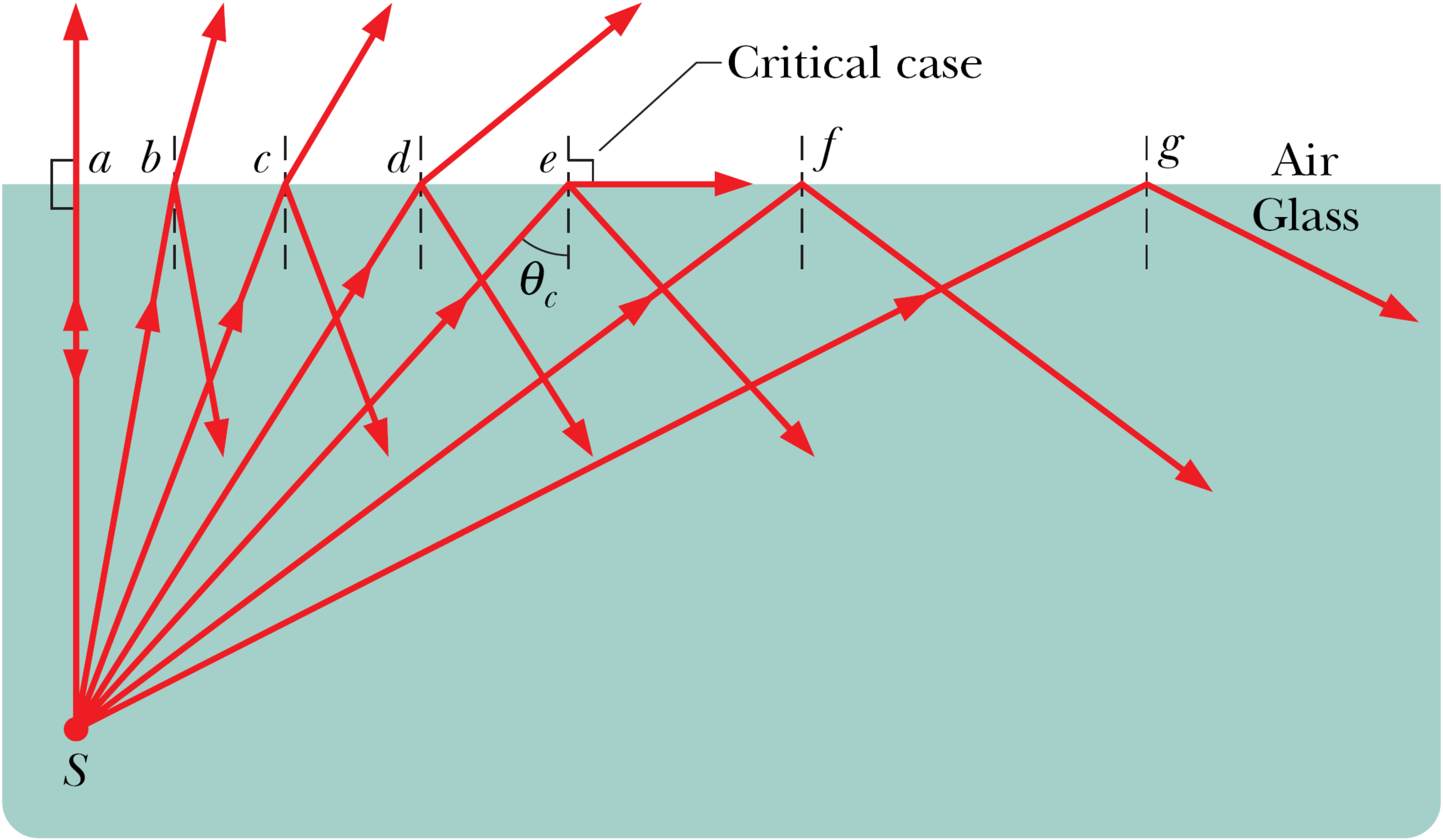
- This behavior can be used to produce beams of plane-polarized light.

- Receiving antennas can be oriented to discriminate against noises in favor of the directly transmitted wave.

- **Total internal reflection**: the refracted wave is propagated ∥ to the surface.

- “internal”: the incident wave is in a medium of larger index of refraction, $n>n'$.

$$\text{Snell's law}\quad \frac{\sin i}{\sin r}=\frac{n'}{n}<1 \;\Rightarrow\; i_0=\sin^{-1}\frac{n'}{n}\quad \text{for}\;\; r=\frac{\pi}{2}$$



Critical case
a
b
c
d
e
f
g
Air
Glass
θ_c
S

- At $i=i_0$, it is a total internal reflection. Thus no energy flows across the surface.

$$\text{At } i>i_0 \quad \Rightarrow \quad \sin r>1 \quad \Rightarrow \quad r\in\mathbb{C} \quad \Rightarrow \quad \cos r=i\sqrt{\frac{\sin^2 i}{\sin^2 i_0}-1} \quad \Leftarrow \quad \cos^2 r+\sin^2 r=1$$

$$\Rightarrow \quad e^{i\,\boldsymbol{k}'\cdot\mathbf{r}}=e^{i\,k'(x\sin r+z\cos r)}=e^{-k'z\sqrt{\frac{\sin^2 i}{\sin^2 i_0}-1}}\,e^{i\,k'x\frac{\sin i}{\sin i_0}}$$

- for $i>i_0$, the refracted wave is propagated only parallel to the surface and is attenuated exponentially beyond the interface.

- The attenuation occurs in a few wavelengths of the boundary, except for $i\approx i_0$.

- Even though fields exist on the other side of the surface there is no energy flow through the surface. Hence total internal reflection occurs for $i\geq i_0$.

$$\mathbf{S}\cdot\mathbf{n}=\frac{\Re\left[\mathbf{n}\cdot\mathbf{E}'\times\mathbf{H}'^*\right]}{2}=\frac{\Re\left[\mathbf{n}\cdot\boldsymbol{k}'|\mathbf{E}_0'|^2\right]}{2\,\omega\,\mu'} \quad \Leftarrow \quad \mathbf{H}'=\frac{\boldsymbol{k}'\times\mathbf{E}'}{\omega\,\mu'}$$

$$=\frac{\Re\left[k'\cos r\,|\mathbf{E}_0'|^2\right]}{2\,\omega\,\mu'}=0 \quad \Leftarrow \quad \cos r\in\mathbf{I}$$

- $\left|\frac{E_0''}{E_0}\right|=1$ for total internal reflection, but the reflected wave suffers a phase change that is different for the 2 kinds of incidence and depends on the angle of incidence and on $\frac{n}{n'}$ $\quad \Rightarrow \quad \frac{E_0''}{E_0}=e^{i\,\phi(i,i_0)}$.

- The phase changes can be used to convert one kind of polarization into another.

- Fresnel's rhombus is to convert linearly polarized light with equal amplitudes in the plane of incidence and ⊥ to it into circularly polarized light by 2 successive internal reflections, each involving a relative phase change of 45°.

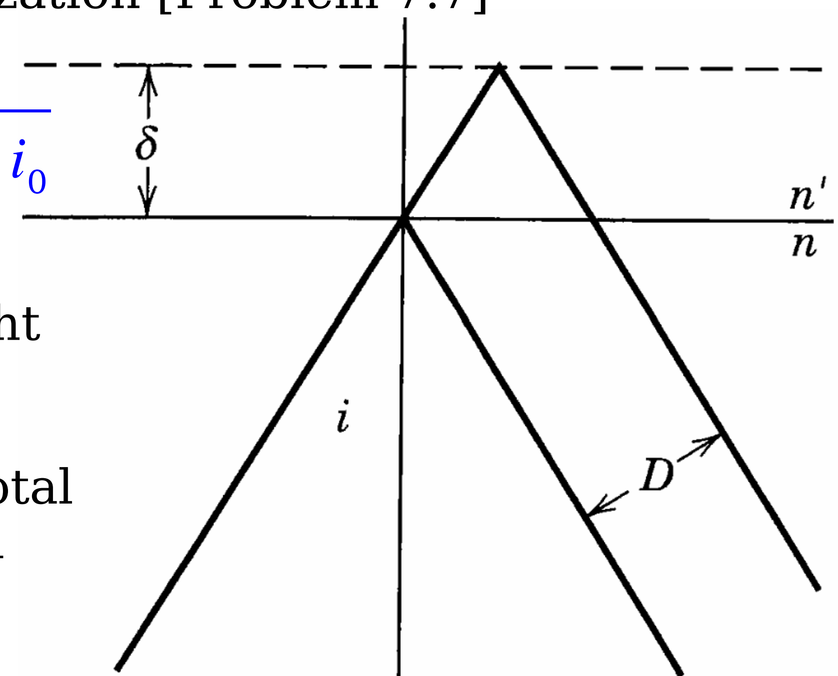
- The wave penetrating into the region $z>0$ has an exponential decay in the perpendicular direction, $e^{-\frac{z}{\delta}} \Leftarrow \frac{1}{\delta} = k\sqrt{\sin^2 i - \sin^2 i_0} \Leftarrow k' = k\frac{\sin i}{\sin r} = k \sin i_0$

- **Goos-Hänchen effect**: If a beam of radiation having a finite transverse extent undergoes total internal reflection, the reflected beam emerges displaced laterally with respect to the prediction of a geometrical ray at the boundary.

- The 1st-order D for the 2 states of linear polarization [Problem 7.7]

$$D_\perp = \frac{\lambda}{\pi}\frac{\sin i}{\sqrt{\sin^2 i - \sin^2 i_0}}, \quad D_\parallel = \frac{D_\perp \sin^2 i_0}{\sin^2 i - \cos^2 i \sin^2 i_0}$$

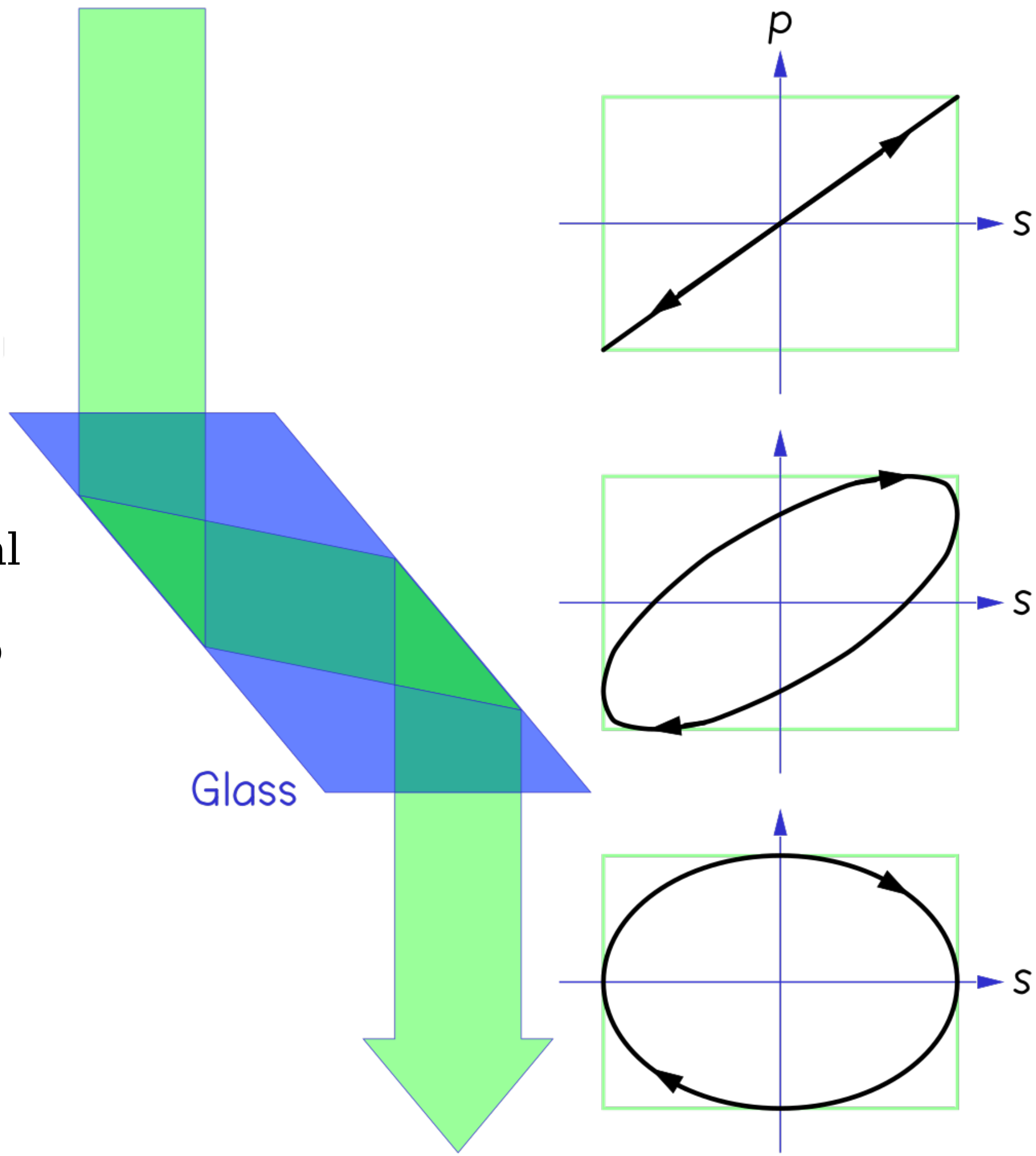
$$\Rightarrow \quad D \approx 2\,\delta \sin i$$



- The internal reflection is useful to transmit light without loss in intensity.

- In telecommunications, optical fibers exploit total internal reflection for transmission of modulated light signals over long distances.

A **Fresnel rhomb** is an optical prism that introduces a 90° phase difference between two perpendicular components of polarization, by means of two total internal reflections.



$$e^{i\phi_\perp(\theta,\theta_0)} = \frac{\mu' n\cos\theta - \mu\sqrt{n'^2 - n^2\sin^2\theta}}{\mu' n\cos\theta + \mu\sqrt{n'^2 - n^2\sin^2\theta}} \approx \frac{\cos\theta - i\sqrt{\sin^2\theta - \sin^2\theta_0}}{\cos\theta + i\sqrt{\sin^2\theta - \sin^2\theta_0}} = \frac{e^{-i\varphi}}{e^{i\varphi}},$$

where $\mu' \approx \mu$, $\sin\theta_0 \equiv \dfrac{n'}{n}$. Therefore $\phi_\perp = -2\varphi = -2\tan^{-1}\dfrac{\sqrt{\sin^2\theta - \sin^2\theta_0}}{\cos\theta}$. So $\dfrac{\mathrm{d}\phi_\perp}{\mathrm{d}\theta} = \dfrac{-2\sin\theta}{\sqrt{\sin^2\theta - \sin^2\theta_0}}$, therefore, for $k = \dfrac{\lambda}{2\pi}$,

$$D_\perp = \delta x_\perp = -\frac{1}{k}\frac{\mathrm{d}\phi_\perp}{\mathrm{d}\theta} = \frac{\lambda}{\pi}\frac{\sin\theta}{\sqrt{\sin^2\theta - \sin^2\theta_0}}.$$

Frequency Dispersion Characteristics of Dielectrics, Conductors, and Plasmas

- All the results of the preceding sections involving a single frequency component are valid in the presence of dispersion.

- Where a superposition of a range of frequencies occurs, new effects arise as a result of the frequency dependence of ϵ and μ.

A. Simple Model for $\epsilon(\omega)$

- Almost all of the physics of dispersion is illustrated by an extension to time-varying fields of the classical model for the molecular polarizability.

- Without differing between the applied electric field and the local field, the model is appropriate only for substances of low density.

- The equation of motion for an electron $m\left(\ddot{\mathbf{r}}+\gamma\,\dot{\mathbf{r}}+\omega_0^2\,\mathbf{r}\right)=-e\,\mathbf{E}\left(\mathbf{r},t\right)$

- Magnetic forces are neglected. The amplitude of oscillation is small enough to permit evaluation of the electric field at the average position of the electron.

- The dipole moment of the electron $\mathbf{p}=-e\,\mathbf{r}=\dfrac{e^2}{m}\dfrac{\mathbf{E}}{\omega_0^2-\omega^2-i\,\omega\,\gamma} \quad \Leftarrow \quad \mathbf{E}\propto e^{-i\,\omega\,t}$

$$\mathbf{P}=\sum N_i\langle\mathbf{p}_i\rangle=\epsilon_0\,\chi_e\,\mathbf{E}$$
$$\mathbf{D}=\epsilon\,\mathbf{E}=\epsilon_0\,\mathbf{E}+\mathbf{P}$$
$$\Rightarrow \quad \frac{\epsilon(\omega)}{\epsilon_0}=1+\chi_e=1+\frac{N\,e^2}{\epsilon_0\,m}\sum_j\frac{f_j}{\omega_j^2-\omega^2-i\,\omega\,\gamma_j} \quad (\star)$$

where $\sum f_j=Z \quad \Leftarrow \quad f_j$: oscillator strengths

B. Anomolous Dispersion and Resonant Absorption

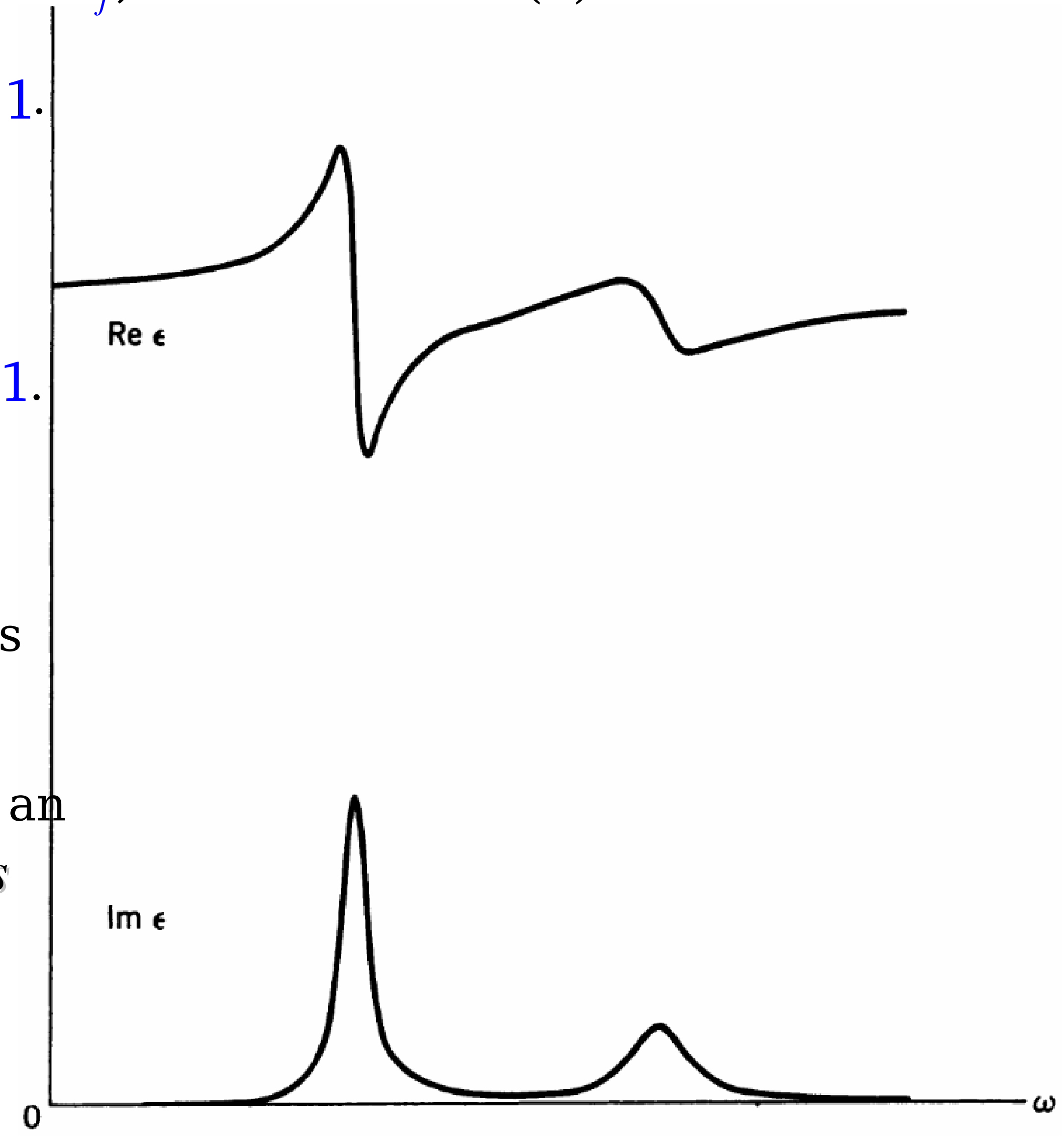
- Generally $\gamma_j \ll \omega_j \quad \Rightarrow \quad \epsilon(\omega) \in \mathbb{R}$ for most frequencies $\Rightarrow$ $\begin{matrix} \omega_j^2 - \omega^2 > 0 \text{ for } \omega < \omega_j \\ \omega_j^2 - \omega^2 < 0 \text{ for } \omega > \omega_j \end{matrix}$

- At low frequencies, below the smallest ω_j, all the terms in (✶) contribute with the same positive sign and $\frac{\Re[\epsilon(\omega)]}{\epsilon_0} > 1$.

- As successive ω_j values are passed, more negative terms occur, until the whole sum is negative and $\frac{\Re[\epsilon(\omega)]}{\epsilon_0} < 1$.

- Near any ω_j, the real part of the denominator in (✶) vanishes for $\omega = \omega_j$ and the term is large & imaginary, thus there is rather violent behavior.

- *Normal dispersion* is associated with an increase in Re[$\epsilon(\omega)$] with ω, *anomalous dispersion* with the reverse.



- Normal dispersion occurs everywhere except near a resonant frequency. And only where there is anomalous dispersion is the imaginary part of ϵ appreciable.

- $\mathrm{Im}[\epsilon]>0$ represents dissipation of energy from the EM wave into the medium, the regions where $\mathrm{Im}[\epsilon]$ is large are called regions of *resonant absorption*.

- If $\mathrm{Im}[\epsilon]<0$, the medium gives energy to the wave; amplification occurs, as in a maser or laser.

- $k=\beta+i\frac{\alpha}{2} \quad \Leftarrow \quad \alpha$: attenuation constant or absorption coefficient

$$\Rightarrow \quad \text{The } \textit{intensity} \text{ of the wave } I \propto |\mathbf{E}|^2 \text{ falls off as } e^{-\alpha z}$$

$$\frac{\omega}{k}=\frac{c}{n} \quad \Leftarrow \quad n=\sqrt{\frac{\mu\,\epsilon}{\mu_0\,\epsilon_0}}, \quad \mu\approx\mu_0 \quad \Rightarrow \quad \frac{k^2}{\omega^2}=\frac{n^2}{c^2}\approx\frac{1}{c^2}\frac{\epsilon}{\epsilon_0}=\frac{\Re[\epsilon]+i\,\Im[\epsilon]}{c^2\,\epsilon_0}$$

$$\Rightarrow \quad \beta^2-\frac{\alpha^2}{4}=\frac{\omega^2}{c^2}\frac{\Re[\epsilon]}{\epsilon_0}, \quad \beta\,\alpha=\frac{\omega^2}{c^2}\frac{\Im[\epsilon]}{\epsilon_0} \quad \Leftarrow \quad k^2=\beta^2-\frac{\alpha^2}{4}+i\,\alpha\,\beta$$

$$\Rightarrow \quad \alpha\simeq\frac{\Im[\epsilon(\omega)]}{\Re[\epsilon(\omega)]}\beta \quad \text{for} \quad \alpha\ll\beta=\frac{\omega}{c}\sqrt{\frac{\Re[\epsilon]}{\epsilon_0}}$$

$$\Rightarrow \quad -\ln\frac{I(z+\lambda)}{I(z)}=\alpha\,\lambda=2\,\pi\frac{\Im[\epsilon]}{\Re[\epsilon]} \quad \Leftarrow \quad I\propto e^{-\alpha z}, \quad \beta\simeq k=\frac{2\,\pi}{\lambda}$$

- Use the decrease of intensity to measure the ratio.

C. Low-Frequency Behavior, Electric Conductivity

- For insulators the lowest resonant frequency is different from 0.

$$\omega = 0 \quad \Rightarrow \quad \chi_e = \frac{N e^2}{\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2} \ \text{ from } (\ast) \ \text{ vs } \ \gamma_{\text{mol}} = \frac{1}{\epsilon_0} \sum \frac{e_j^2}{m_i \omega_j^2} \ \text{ in Sec. 4.6}$$

- If some fraction f_0 of the electrons per molecule are "free", ie, $\omega_0 = 0$, the dielectric constant is singular at $\omega = 0$,

$$\epsilon(\omega) = \epsilon_b(\omega) + i \frac{N e^2 f_0}{m \omega (\gamma_0 - i \omega)} \quad \Leftarrow \quad \epsilon_b : \text{ the contribution of the other dipoles}$$

- Check the singular behavior in the Maxwell-Ampere equation

$$\nabla \times \mathbf{H} = \boldsymbol{J} + \frac{\partial \mathbf{D}}{\partial t} = -i\omega \left(\epsilon_b + i \frac{\sigma}{\omega} \right) \mathbf{E} = -i \omega \epsilon \mathbf{E} \quad \Leftarrow \quad \boldsymbol{J} = \sigma \mathbf{E}, \ \ \mathbf{D} = \epsilon_b \mathbf{E} \propto e^{-i\omega t}$$

$$\Rightarrow \quad \sigma = \frac{f_0 N e^2}{m(\gamma_0 - i\omega)} \to \frac{f_0 N e^2}{m \gamma_0} \ \text{ for } \ \omega = 0$$

$f_0 N$: free electron number density $\quad \Rightarrow \quad$ Drude's model

- For copper: $N \simeq 8 \times 10^{28} \text{ atom/m}^3, \quad \sigma \simeq 5.9 \times 10^7 / (\Omega \cdot \text{m})$

$$\Rightarrow \quad \text{damping coefficient } \frac{\gamma_0}{f_0} \simeq 4 \times 10^{13}/\text{s} \quad \Rightarrow \quad \text{assuming } f_0 \sim 1$$

$$\Rightarrow \quad \sigma \in \mathbb{R} \ \ \& \ \text{ independent of frequency for } \ \omega \leq 10^{11}/\text{s (microwave) for metals}$$

- At higher frequencies the conductivity is complex and like the above equation for σ.

- The problem of electrical conductivity is really a quantum-mechanical one.

- The free electrons are actually valence electrons of the isolated atoms that become quasi-free and move relatively unimpeded through the lattice.

- The damping effects come from collisions involving appreciable momentum transfer between the electrons and lattice vibrations, etc.

- If the medium possesses free electrons it is a conductor at low frequencies; otherwise, an insulator.

- At nonzero frequencies the conductivity only appears as a resonant amplitude.

- The dispersive properties of the medium can be attributed as a complex dielectric constant or a frequency-dependent conductivity & a dielectric constant.

D. High-Frequency Limit, Plasma Frequency

- At frequencies far above the highest resonant frequency $\omega \gg \omega_p$

$$\frac{\epsilon(\omega)}{\epsilon_0} \simeq 1 - \frac{\omega_p^2}{\omega^2} \quad (\#) \quad \Leftarrow \quad \omega_p^2 \equiv \frac{N Z e^2}{\epsilon_0 m} \quad \Rightarrow \quad \begin{array}{c} c\,k = \sqrt{\omega^2 - \omega_p^2} \\ \omega^2 = \omega_p^2 + c^2 k^2 \end{array} \quad \Leftarrow \quad \text{dispersion relation}$$

ω_p, depending only on the total number NZ of electrons/volume, is called the *plasma frequency* of the medium.

- In the case the dielectric constant is close to 1, and increases with frequency. The wave number is real and varies with frequency.

- In the ionosphere or in a tenuous electronic plasma, electrons are free ($\omega_0 = 0$) and the damping is negligible, (#) holds even for $\omega < \omega_p$. k is purely imaginary.

- Such waves incident on a plasma are reflected and the fields inside fall off exponentially with distance from the surface. $\alpha_{\text{plasma}} \simeq \frac{2\,\omega_p}{c}$ for $\omega \to 0$ $\Rightarrow \quad \omega_p^2 + c^2 k^2 \simeq 0$

- If $N Z = 10^{18} - 10^{22}\ \text{electrons}/\text{m}^3$

$$\Rightarrow \quad \omega_p = 6 \times 10^{10} - 6 \times 10^{12}/\text{s} \quad \Rightarrow \quad \text{attenuation lengths } \frac{1}{\alpha} \sim 0.002 - 0.2\ \text{cm}$$

- The expulsion of fields from within a plasma is a well-known effect in controlled thermonuclear processes and in attempts at confinement of hot plasma.

- For the reflectivity of metals, at high frequencies $\omega \gg \gamma_0$

$$\Rightarrow \quad \epsilon(\omega) \simeq \epsilon_b(\omega) - \frac{\omega_p^2}{\omega^2}\epsilon_0 \quad \Leftarrow \quad \omega_p^2 = \frac{n\, e^2}{m^\star \epsilon_0} \quad \Leftarrow \quad m^\star :$$ the effecitve mass including the binding effect

$$\Rightarrow \quad \text{similar to } (\#) \text{ for } \omega \ll \omega_p$$

- The light penetrates a short distance in the metal & is almost entirely reflected.

- When the frequency is increased into the domain where $\epsilon(\omega)>0$, the metal suddenly can transmit light and its reflectivity changes drastically.

- Occurs typically in the ultraviolet leading to "ultraviolet transparency of metals."

- Determination of the critical frequency gives information on the density or the effective mass of the conduction electrons.

E. Index of Refraction and Absorption Coefficient of Liquid Water as a Function of Frequency

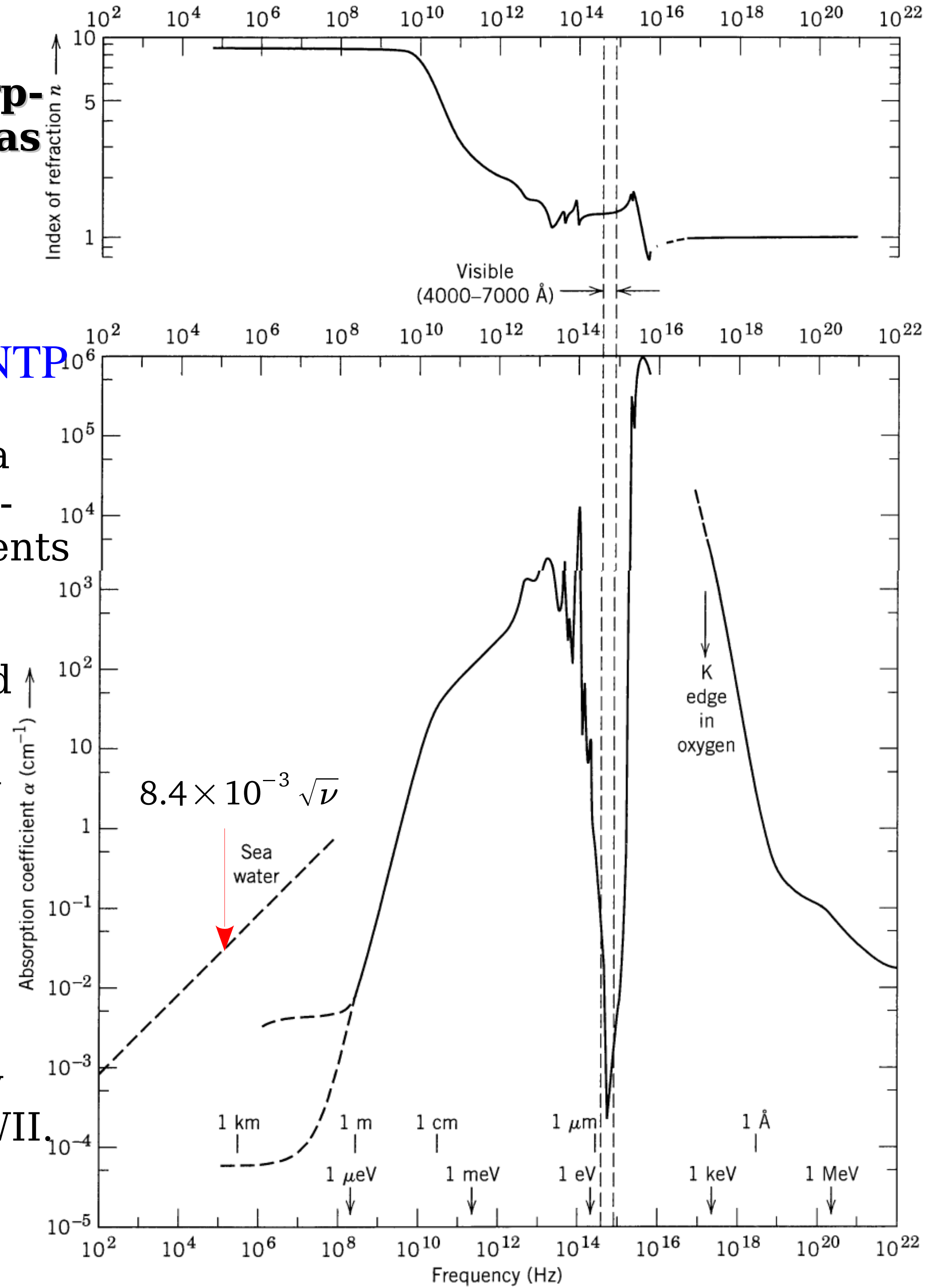
- $n(\omega) = \Re\left[\sqrt{\dfrac{\mu\,\epsilon}{\mu_0\,\epsilon_0}}\right]$ $\qquad \dfrac{k}{\omega} = \dfrac{c}{n}$

  $\alpha(\omega) = 2\,\Im\left[\sqrt{\mu\,\epsilon}\right]\omega$ for water at NTP

- At very low frequencies, $n(\omega) \simeq 9$, a value arising from the partial orientation of the permanent dipole moments of the water molecules.

- As the frequency increases toward 10^{11} Hz, the absorption coefficient increases to $\alpha \simeq 10^4$/m, leading to an attenuation length of $100\ \mu$m in liquid water.

- This is the microwave absorption by water. It is the phenomenon (in moist air) that terminated the trend toward better resolution in radar by going to shorter wavelengths in WWII.



- In the infrared region absorption bands associated with vibrational modes and oscillations of the molecule cause the absorption to reach peak values~ $\alpha \simeq 10^6/\mathrm{m}$.

- $4\times10^{14}\mathrm{Hz}-8\times10^{14}\ \mathrm{Hz}$ is a dramatic absorption *window* called the *visible region*.

- In the far ultraviolet the absorption has a peak value of $\alpha\simeq1.1\times10^{8}/\mathrm{m}$ at $\nu\simeq5\times10^{15}\mathrm{Hz}$ (21eV).

- This is exactly at the plasmon energy $\hbar\omega_p$, corresponding to a collective excitation of all the electrons in the molecule.

- At higher frequencies the photoelectric effect will shows, and then Compton scattering and other high-energy processes take over.

- At low frequencies, seawater has an electrical conductivity $\sigma\simeq4.4/\Omega/\mathrm{m}$,

$$\begin{matrix} \epsilon=\epsilon_b+i\dfrac{\sigma}{\omega} \\ k=\omega\sqrt{\mu\,\epsilon}=\beta+i\dfrac{\alpha}{2} \end{matrix} \Rightarrow \begin{matrix} \beta^2-\dfrac{\alpha^2}{4}=\omega^2\mu\,\epsilon_b\approx0 \\ \alpha\,\beta=\omega\,\mu\,\sigma=\omega\,\mu_0\,\sigma \end{matrix} \Leftarrow \begin{matrix} \omega\approx0 \\ \mu\simeq\mu_0 \end{matrix} \Rightarrow \alpha\simeq\sqrt{2\,\mu_0\,\omega\,\sigma}$$

- At $10^2\mathrm{Hz}$, the attenuation length in seawater is $\dfrac{1}{\alpha}\simeq 10$ meters. It means that

1% of the intensity at the surface will survive at 50 meters below the surface.

- One can consider extremely low-frequency (ELF) communications to send message to submarines.

- The resonances of the earth-ionosphere cavity from 8Hz to a few hundred hertz reduce the attenuation & make the region of the frequency spectrum attractive. With wavelengths of the order of $5 \times 10^{3}\,\text{km}$, very large antennas are needed.

Simplified Model of Propagation in Ionosphere & Magnetosphere

- The **ionosphere** is the uppermost part of the atmosphere, distinguished because it is ionized by solar radiation.

- The propagation of EM waves in the ionosphere is described approximately by $\frac{\epsilon}{\epsilon_b} \simeq 1 - \frac{\omega_p^2}{\omega^2}$, but Earth's magnetic field modifies the behavior significantly.
- Consider a tenuous electronic plasma of uniform density with a static, uniform, $\mathbf{B}_0$ and transverse waves propagating parallel to the direction of $\mathbf{B}_0$.

$$\Rightarrow \quad m\,\ddot{\mathbf{r}} - e\,\mathbf{B}_0 \times \dot{\mathbf{r}} = -e\,\mathbf{E} = -e\,\mathcal{E}\,e^{-i\omega t} \quad \Leftarrow \quad \mathbf{r} \text{ is small and neglect collisions}$$

$$\mathcal{E} = E\left(\hat{\mathbf{e}}_1 \pm i\,\hat{\mathbf{e}}_2\right)$$

$$\Rightarrow \quad \mathbf{r} = r\left(\hat{\mathbf{e}}_1 \pm i\,\hat{\mathbf{e}}_2\right) e^{-i\omega t} \quad \Rightarrow \quad \mathbf{r} = \frac{e\,\mathbf{E}}{m\,\omega\left(\omega \mp \omega_B\right)} \quad \Leftarrow \quad \omega_B \equiv \frac{e\,B_0}{m} \quad \text{precession frequency}$$

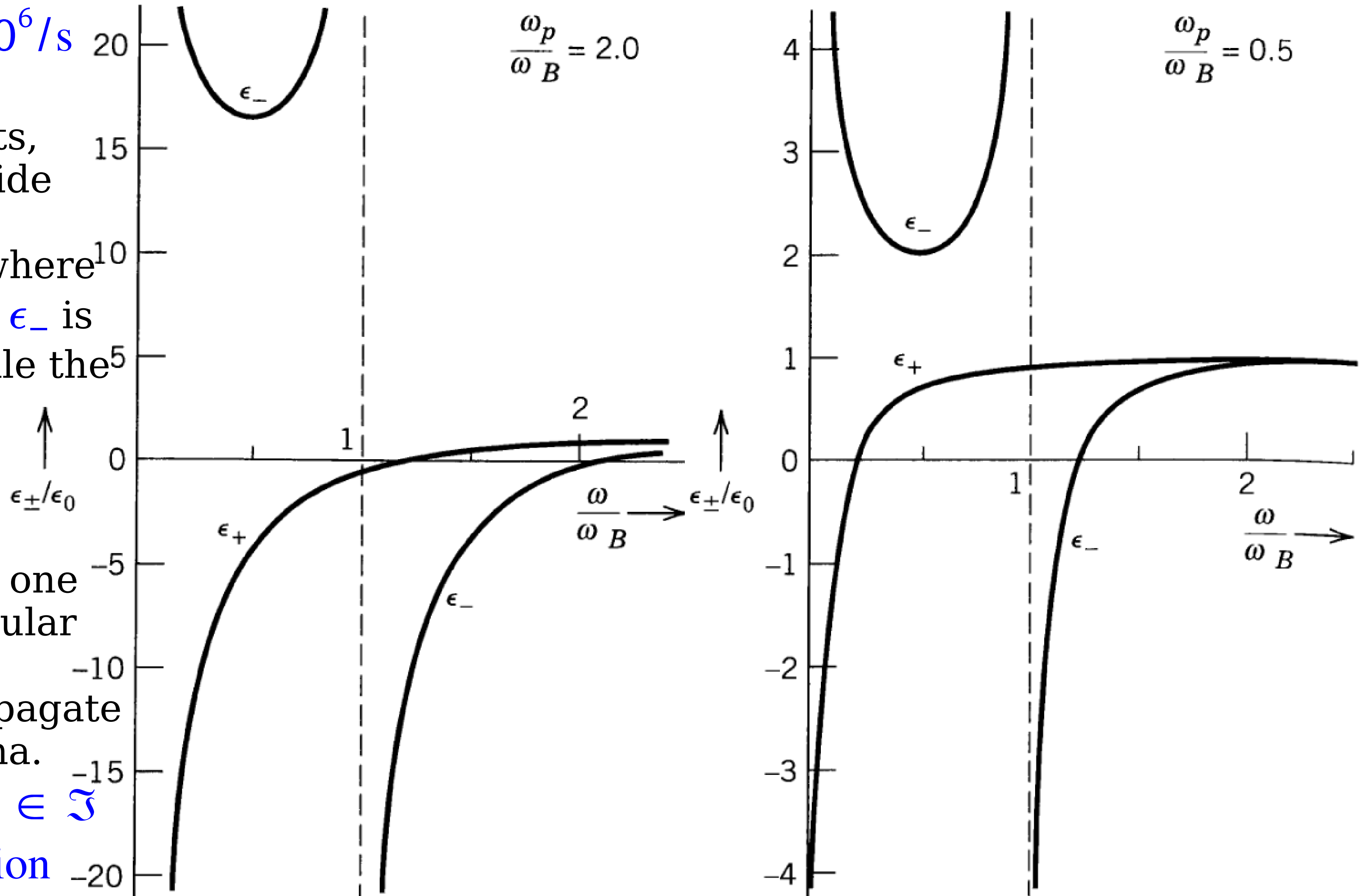
- The frequency dependence of $\mathbf{r}$ can be understood by transforming the EOM to a coordinate system precessing with frequency ω_B about the direction of $\mathbf{B}_0$.
- $\frac{\epsilon_\mp}{\epsilon_0} \simeq 1 - \frac{\omega_p^2}{\omega_{\text{eff}}^2} = 1 - \frac{\omega_p^2}{\omega\left(\omega \mp \omega_B\right)} \quad \Leftarrow \quad$ $-$: positive helicity; $+$: negative helicity
- For propagation antiparallel to the magnetic field $\mathbf{B}_0$, the signs are reversed.

- Waves of right-handed and left-handed circular polarizations propagate differently with magnetic field. The ionosphere is birefringent.

- For propagation other than ∥ to $\mathbf{B}_0$ the precession frequency is interpreted as that due to only the component of $\mathbf{B}_0$ ∥ the direction of propagation. This means $\omega_B=\omega_B(\text{angle})$—the medium is not only birefringent, but also anisotropic.

- For $n \simeq 10^{10} - 10^{12}\ \text{electrons}/\text{m}^3 \Rightarrow \omega_p \simeq 6\times10^6 - 6\times10^7/\text{s}$ + $B_{\text{earth}} = 30\,\mu\,\text{T}$

$\Rightarrow \omega_B \simeq 6\times10^6/\text{s}$



- In the plots, there are wide intervals of frequency where one of ϵ_+ or ϵ_- is positive while the other is negative.

- At such frequencies one state of circular polarization can not propagate in the plasma.

$\epsilon < 0 \Rightarrow \nu \in \Im$

$\Rightarrow$ dissipation

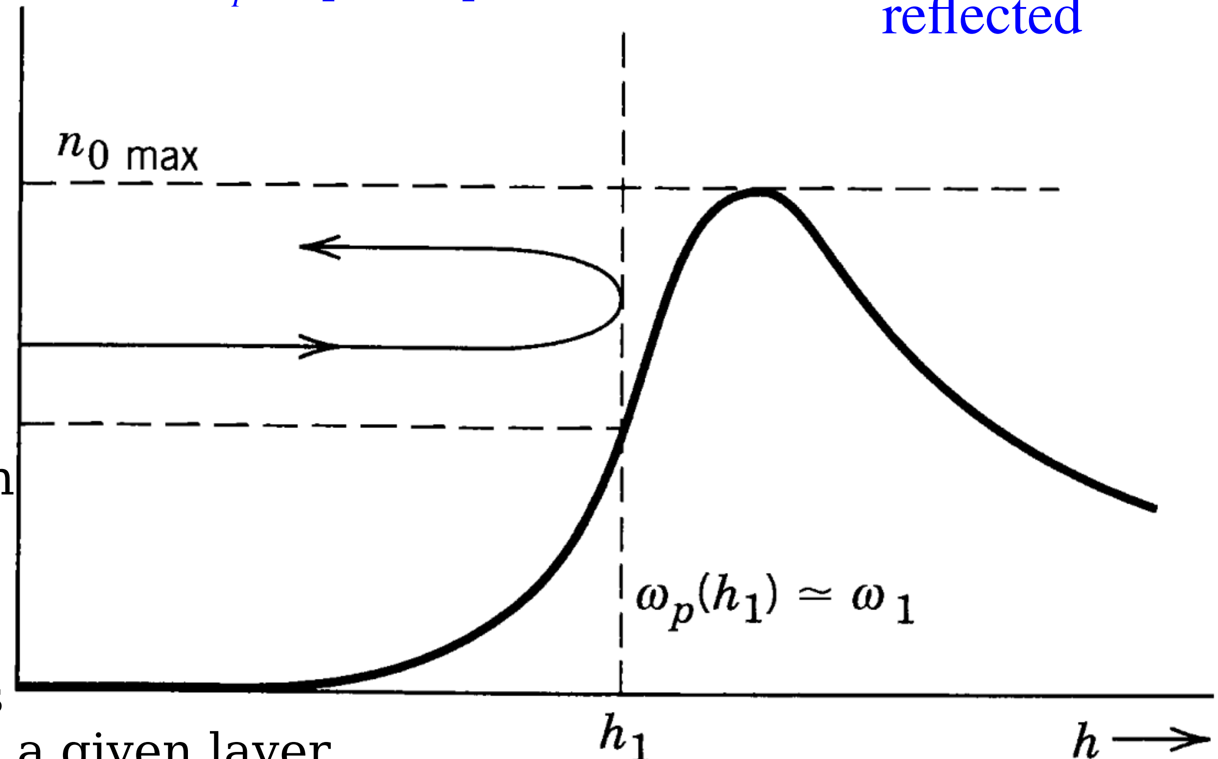
- Consequently the wave of one polarization incident on the plasma will be totally reflected. The other state of polarization will be partially transmitted.

- When a linearly polarized wave is incident on a plasma, the reflected wave will be elliptically polarized.

- The ionosphere can reflect radio waves.

- The electron densities at various heights can be inferred by studying the reflection of pulses of radiation transmitted vertically upwards.

- When the density is large enough $\Rightarrow \quad \omega_p(h_1) \simeq \omega_1 \quad \Rightarrow \quad \epsilon = 0 \quad \Rightarrow$ the pulse reflected $\Rightarrow \quad n_0 \quad \Leftarrow \quad \omega_p^2 = \omega_1(\omega_1 \pm \omega_B)$



- h_1: the time interval between the initial transmission & reception of the reflected signal.

- Varying the frequency and studying the time intervals the electron density as a function of height can be determined.

- The frequency above which reflections disappear determines the maximum electron density in a given layer.

- The behavior of $\epsilon_-(\omega)$ at low frequencies is responsible for a magnetospheric propagation phenomenon called "whistlers."

- $\omega \to 0 \quad \Rightarrow \quad \frac{\epsilon_-}{\epsilon_0} \simeq \frac{\omega_p^2}{\omega\,\omega_B} \to \infty \quad \Rightarrow \quad k_- \simeq \frac{\omega_p}{c}\sqrt{\frac{\omega}{\omega_B}} \quad \Rightarrow$ highly dispersive medium

  $\Rightarrow$ group velocity $v_g(\omega) \simeq 2\, v_p(\omega) \simeq 2\, c\, \frac{\sqrt{\omega\,\omega_B}}{\omega_p} \quad \Leftarrow \quad v_g \equiv \frac{d\,\omega}{d\,k}$ mentioned later

- Pulses of radiation at different frequencies travel at different speeds: the lower the frequency, the slower the speed.

- The radiation from a thunderstorm gives rise at 10^5 Hz and below to whistlers, a whistlelike sound beginning at high audio frequencies and falling rapidly through the audible range.

Magnetohydrodynamic Waves

- In conducting fluids or dense ionized gases, collisions are rapid that Ohm's law holds for a wide range of frequencies.

- Under the applied fields the electrons and ions move in such a way that there is no separation of charge, although there can be current flow.

- The nonrelativistic mechanical motion of charge is described as a conducting fluid with the hydrodynamic variables of density, velocity, and pressure, with EM and gravitational forces. It is called as *magnetohydrodynamics* (*MHD*).

- Ohm's law: $\boldsymbol{J} = \sigma\,\mathbf{E}$ is generalized to $\boldsymbol{J} = \sigma\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)$

- Diffusion equation $\nabla^2\mathbf{B} = \mu\,\sigma\,\dfrac{\partial\,\mathbf{B}}{\partial\,t}$ is generalized to

$$\nabla\times\mathbf{H} = \boldsymbol{J} \quad\Rightarrow\quad \nabla\times\mathbf{B} = \mu\,\boldsymbol{J} = \mu\,\sigma\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)$$

$$\Rightarrow\quad \nabla\times\left(\nabla\times\mathbf{B}\right) = \mu\,\sigma\left[\nabla\times\mathbf{E} + \nabla\times\left(\mathbf{v}\times\mathbf{B}\right)\right] = \mu\,\sigma\left(-\frac{\partial\,\mathbf{B}}{\partial\,t} + \nabla\times\left(\mathbf{v}\times\mathbf{B}\right)\right)$$

$$\Rightarrow\quad \boxed{\frac{\partial\,\mathbf{B}}{\partial\,t} = \nabla\times\left(\mathbf{v}\times\mathbf{B}\right) + \frac{1}{\mu\,\sigma}\nabla^2\mathbf{B}} \quad\Leftarrow\quad \nabla\times\left(\nabla\times\mathbf{B}\right) = \nabla\left(\nabla\cdot\mathbf{B}\right) - \nabla^2\mathbf{B}$$

assumed that the conductivity and permeability are independent of position.

- Consider a compressible, nonviscous, "perfectly conducting"(ie, $\sigma\to\infty$) fluid without gravity, but in an external magnetic field—the diffusion time is long.

- The hydrodynamic equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \text{conservation of matter} \qquad ⇙ \qquad \boldsymbol{J} \times \mathbf{B} = (\nabla \times \mathbf{H}) \times \mathbf{B}$$

$$\rho \frac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}\,t} = \rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p - \frac{1}{\mu} \mathbf{B} \times (\nabla \times \mathbf{B}) \quad \text{Newtonian equation of motion}$$

- The magnetic force can be written as

$$-\frac{1}{\mu} \mathbf{B} \times (\nabla \times \mathbf{B}) = -\nabla \frac{\mathbf{B}^2}{2\mu} + \frac{1}{\mu} (\mathbf{B} \cdot \nabla) \mathbf{B} \quad \Leftarrow \quad \nabla (\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$

$$= \text{the gradient of magnetic pressure} + \text{additional tension}$$

- The hydrodynamic equations must be supplemented by an equation of state.

- Without magnetic field, the mechanical equations can describe small-amplitude, longitudinal, compressional (sound) waves with a sound speed,

$$s^2 = \frac{\mathrm{d}\,p}{\mathrm{d}\,\rho}\Big|_{\text{entropy} = \text{const}} \quad \Rightarrow \quad s^2 = \gamma \frac{p_0}{\rho_0} \quad \Leftarrow \quad p = K \rho^{\gamma} \quad \Leftarrow \quad \text{adiabatic gas law}$$

- By analogy, longitudinal MHD waves in a conducting fluid in an external field, with a speed $\mathrm{v}_{\text{MHD}} = O\left(\frac{B_0}{\sqrt{2 \mu \rho_0}} \right)$

- For small-amplitude departures from equilibrium,

$$\begin{aligned}
\mathbf{B} &= \mathbf{B}_0 + \mathbf{B}_1(\mathbf{r},t) \\
\rho &= \rho_0 + \rho_1(\mathbf{r},t) \\
\mathbf{v} &= \mathbf{v}_1(\mathbf{r},t)
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
&\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0 \\
&\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + s^2 \nabla \rho_1 + \frac{\mathbf{B}_0}{\mu} \times (\nabla \times \mathbf{B}_1) = 0 \\
&\frac{\partial \mathbf{B}_1}{\partial t} - \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0) - \frac{1}{\mu \sigma} \nabla^2 \mathbf{B} = 0 \quad \Leftarrow \quad \sigma \to \infty
\end{aligned}$$

$$\Rightarrow \quad \frac{\partial^2 \mathbf{v}_1}{\partial t^2} - s^2 \nabla (\nabla \cdot \mathbf{v}_1) + \mathbf{v}_A \times \nabla \times [\nabla \times (\mathbf{v}_1 \times \mathbf{v}_A)] = 0 \quad \Leftarrow \quad \mathbf{v}_A = \frac{\mathbf{B}_0}{\sqrt{\mu \rho_0}} \text{ Alfven velocity}$$

$$\mathbf{v}_1(\mathbf{r},t) = \mathbb{v}_1 e^{i(\boldsymbol{k}\cdot\mathbf{r} - \omega t)}$$

$$\Rightarrow \quad -\omega^2 \mathbb{v}_1 + (s^2 + v_A^2)(\boldsymbol{k}\cdot\mathbb{v}_1)\boldsymbol{k} + (\mathbf{v}_A \cdot \boldsymbol{k})[(\mathbf{v}_A \cdot \boldsymbol{k})\mathbb{v}_1 - (\mathbf{v}_A \cdot \mathbb{v}_1)\boldsymbol{k} - (\boldsymbol{k}\cdot\mathbb{v}_1)\mathbf{v}_A] = 0$$

- If $\boldsymbol{k} \perp \mathbf{v}_A \quad \Rightarrow \quad -\omega^2 \mathbb{v}_1 + (s^2 + v_A^2)(\boldsymbol{k}\cdot\mathbb{v}_1)\boldsymbol{k} = 0$

  $\Rightarrow \quad \mathbf{v}_1$: *longitudinal* magnetosonic wave with phase velocity:

  $u_{\text{long}} = \sqrt{s^2 + v_A^2}$ depends on the sum of hydrostatic and magnetic pressures

- If $\boldsymbol{k} \parallel \mathbf{v}_A \quad \Rightarrow \quad (k^2 v_A^2 - \omega^2)\mathbb{v}_1 + k^2 \left(\frac{s^2}{v_A^2} - 1 \right) (\mathbf{v}_A \cdot \mathbb{v}_1)\mathbf{v}_A = 0$

- 2 types of wave motion possible in this case:
  (1) an ordinary longitudinal wave ($\mathbf{v}_1 \| \boldsymbol{k}$ & $\mathbf{v}_A$) with phase velocity $u_{\text{long}} = s$.
  (2) a transverse wave ($\mathbf{v}_1 \cdot \mathbf{v}_A = 0$) with a phase velocity $u_{\text{trans}} = v_A$.

- This Alfvén wave is a purely MHD phenomenon, which depends only on the magnetic field (tension) and the density (inertia).

- In usual laboratories the Alfvén velocity is much less than the speed of sound.

- In astrophysical problems, the Alfvén velocity can become very large because of the much smaller densities.

- In the sun's photosphere,

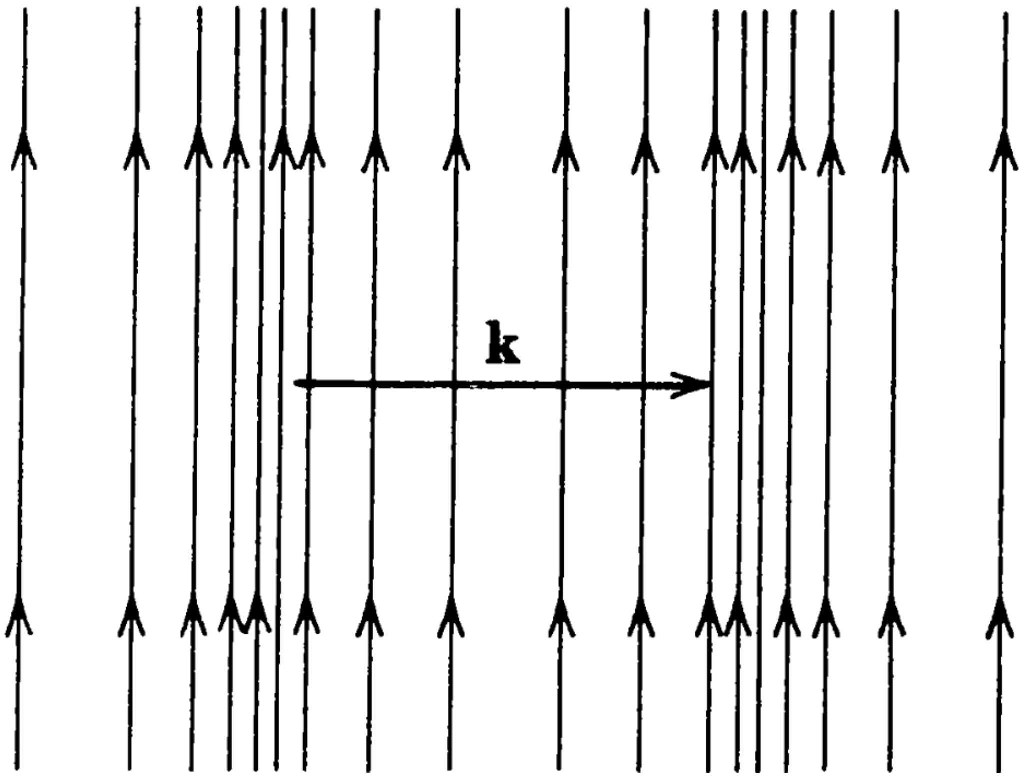
$$\rho \simeq 10^{-4}\ \text{kg}/\text{m}^3 \quad \Rightarrow \quad v_A \simeq 10^5\, B\ \text{m}/\text{s} \quad \Leftarrow \quad B \simeq 10^{-4}\, T$$

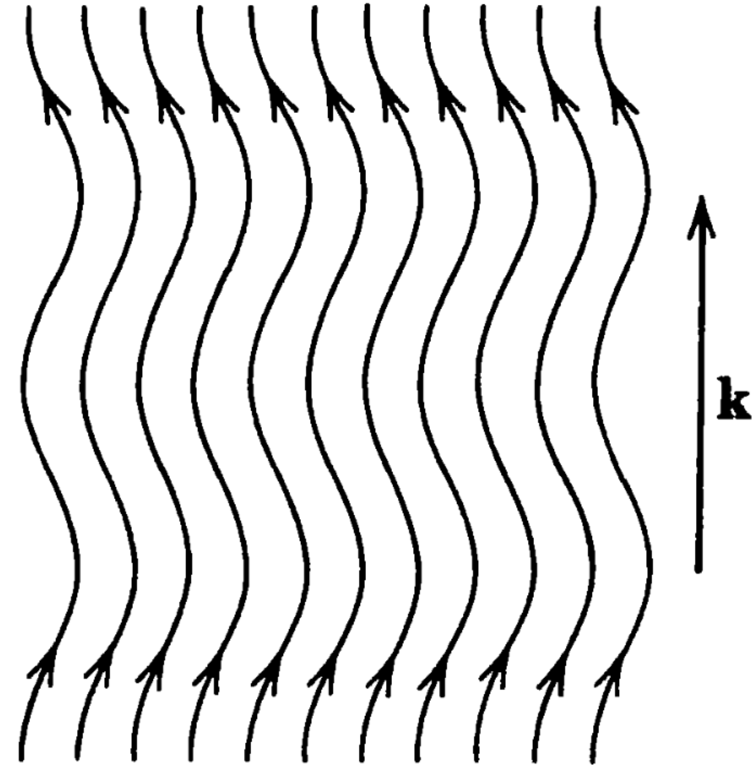
$$\text{vs} \quad s \simeq 10^4\ \text{m}/\text{s}$$

- The magnetic fields of different waves

$$\mathbf{B}_1 = \begin{cases} \dfrac{k}{\omega} v_1 \mathbf{B}_0 & \text{for } \mathbf{k} \perp \mathbf{B}_0 \\ 0 & \text{for the longitudinal } \mathbf{k} \| \mathbf{B}_0 \\ -\dfrac{k}{\omega} B_0 \mathbf{v}_1 & \text{for the transverse } \mathbf{k} \| \mathbf{B}_0 \end{cases}$$

- The magnetosonic wave moving $\perp \mathbf{B}_0$ causes compressions and rarefactions in the lines of force without changing their direction.





- The Alfvén wave $\parallel \mathbf{B}_0$ causes the lines of force to oscillate back and forth laterally.

- In either case the lines of force are "frozen in" and move with the fluid.

Superposition of Waves in 1D; Group Velocity

- Even in the most monochromatic light source or the most sharply tuned radio transmitter/receiver, there is still a finite spread of frequencies or wavelengths.

- Since the basic equations are linear, the linear superposition of solutions with different frequencies works.

- New features:

1. If the medium is dispersive, ie, $\epsilon=\epsilon(\omega)$, the phase velocity is not the same for each frequency component of the wave. Thus different components of the wave travel with different speeds to change phase with respect to one another.

2. In a dispersive medium the velocity of energy flow may differ from the phase velocity, or may even lack precise meaning.

3. In a dissipative medium, a pulse of radiation will be attenuated as it travels with or without distortion.

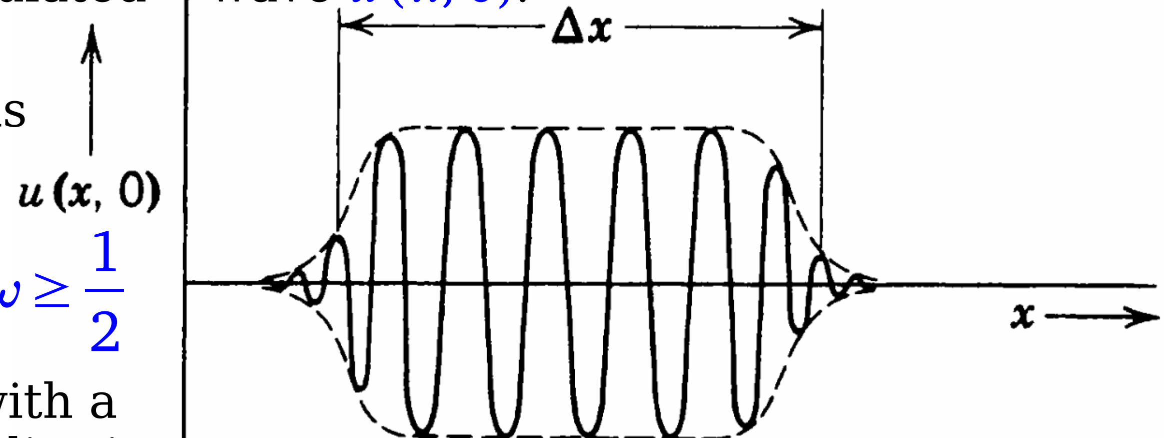
- Consider $\omega=\omega(k) \quad \Rightarrow \quad \omega(-k)=\omega(k) \quad \Leftarrow$ independent of the wave traveling to the left/right

$k, \omega \in \mathbb{R} \quad \Rightarrow$ general solution for wave $\quad u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} A(k)\, e^{i[kx-\omega(k)t]}\,\mathrm{d}k$

$$\Rightarrow \quad A(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} u(x,0)\, e^{-ikx}\,\mathrm{d}x$$

- If $u(x,0) \propto e^{i k_0 x} \Rightarrow A(k) = \sqrt{2\pi}\,\delta(k-k_0) \Rightarrow u(x,t) = e^{i[k_0 x - \omega(k_0) t]}$
  monochromatic

- If $u(x,0)$ is a finite wave train with a length of Δx, the amplitude $A(k)$ is a peaked function with a breadth of Δk, centered around a wave number k_0, dominant wave number in the modulated wave $u(x,0)$.

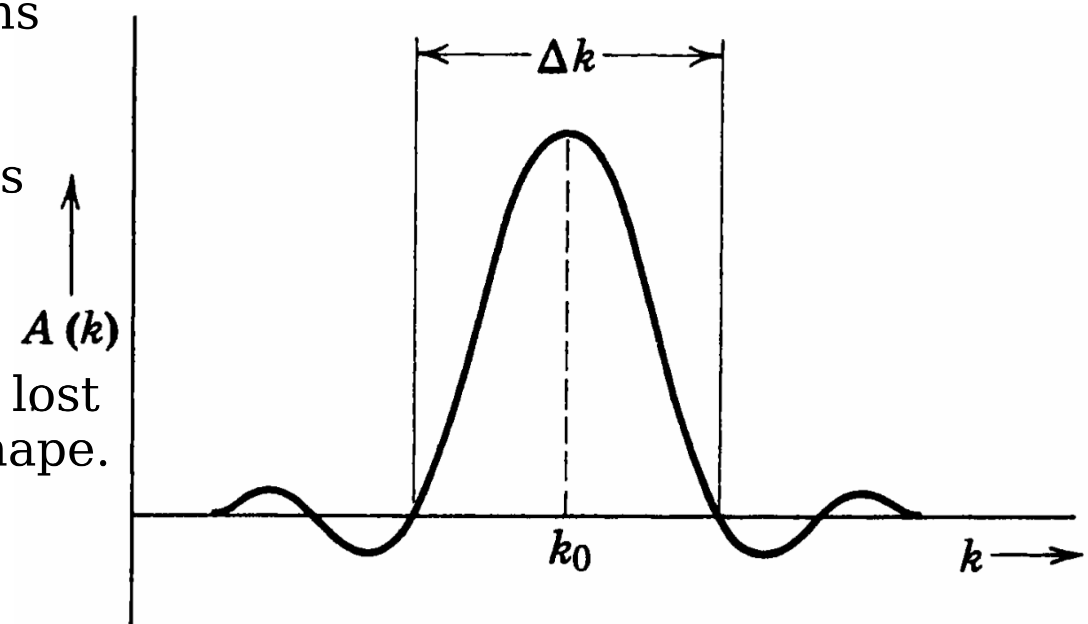


- If Δx & Δk are defined as the rms deviations from the average values of x & k $\Rightarrow \quad \delta x \, \delta k \geq \frac{1}{2} \quad$ vs $\quad \delta t \, \delta \omega \geq \frac{1}{2}$

- It means that short wave trains with a few wavelengths have a very wide distribution of wave numbers of monochromatic waves, and long sinusoidal wave trains are almost monochromatic.

- The different frequency components in the wave move at different phase velocities.

- Thus the original coherence will be lost & the pulse to become distorted in shape.



- $\omega(k) = \omega_0 + \frac{d\omega}{dk}_{|0}(k - k_0) + \cdots \quad (1) \quad \Leftarrow \quad \frac{d\omega}{dk}_{|0} = \frac{d\omega}{dk}_{|k=k_0}$

$$\Rightarrow \quad u(x,t) \approx \frac{e^{i\left(k_0 \frac{d\omega}{dk}_{|0} - \omega_0\right)t}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A(k)\, e^{ik\left(x - \frac{d\omega}{dk}_{|0} t\right)} dk$$

$$= e^{i\left(k_0 \frac{d\omega}{dk}_{|0} - \omega_0\right)t} u\left(x - \frac{d\omega}{dk}_{|0} t, 0\right)$$

the pulse travels along undistorted in shape with the *group velocity*: $v_g = \frac{d\omega}{dk}_{|0}$

- In this approximation the transport of energy occurs with the group velocity, since that is the rate at which the pulse travels along.

- For light waves $\omega(k) = \frac{ck}{n(k)} \quad \Rightarrow \quad v_g = \frac{c}{n + \omega \frac{dn}{d\omega}}$ vs $v_p = \frac{\omega}{k} = \frac{c}{n}$

- For normal dispersion $\frac{dn}{d\omega} > 0$, $n > 1$, then the velocity of energy flow is less than the phase velocity and also less than c.

- In regions of anomalous dispersion, $\frac{dn}{d\omega}$ can become large & negative, then v_g differs greatly from v_p, often becoming larger than c or even negative.

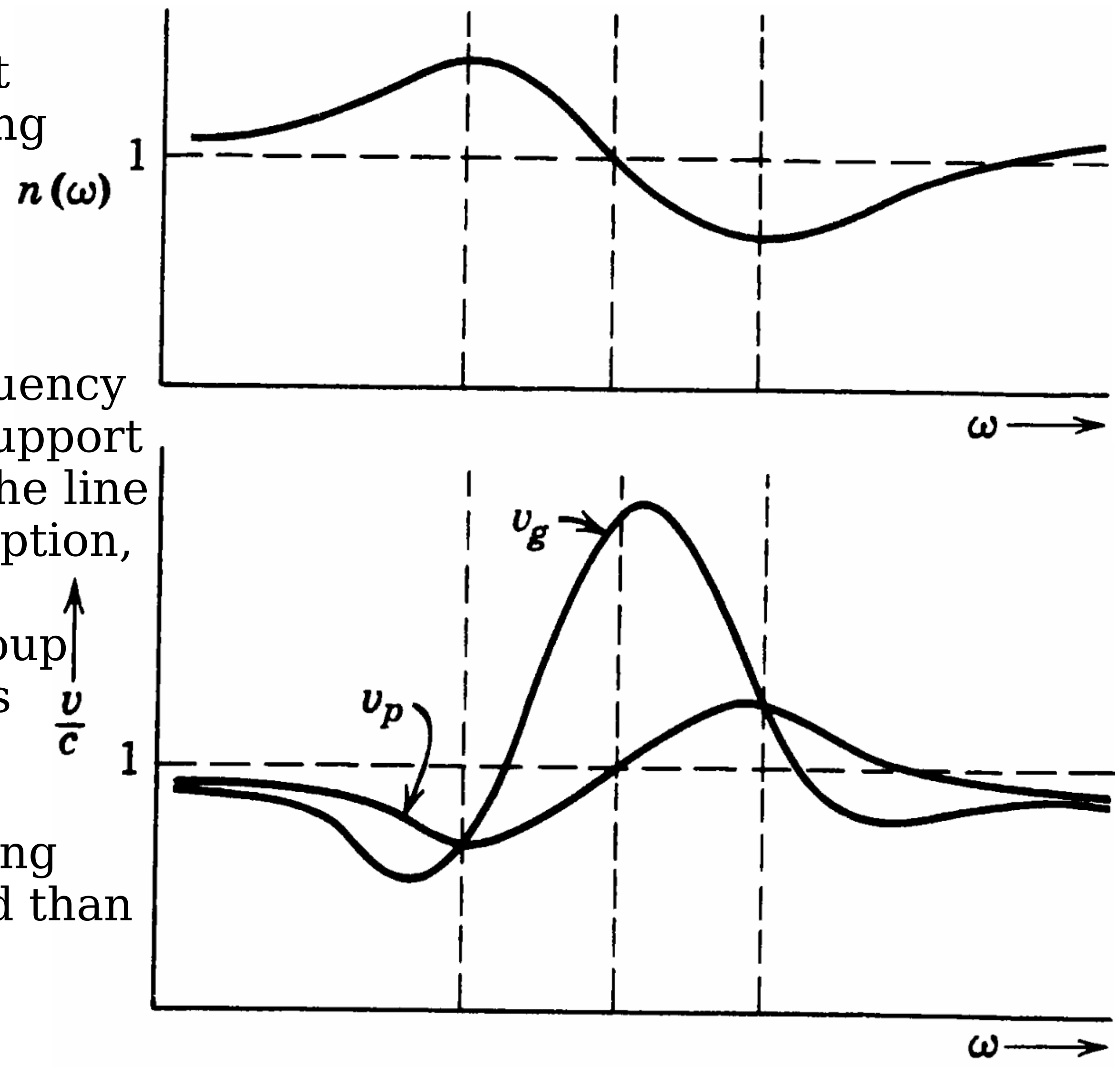
- Group velocity is *generally* not a useful concept in regions of anomalous dispersion; no idea of special relativity will be violated.

- A large $\frac{\mathrm{d}\,n}{\mathrm{d}\,\omega}$ means a rapid variation of ω with n. Equation (1) & the following equations in this section then are no longer valid.

- Usually a pulse with its dominant frequency components near a strong absorption line is absorbed and distorted as it travels.

- If absorbers are not too thick, a Gaussian pulse with a central frequency near an absorption line and with support narrow compared to the width of the line propagates with appreciable absorption, but more or less retains its shape, the peak of which moves at the group velocity, even when that quantity is Negative.

- It is a pulse reshaping—the leading edge of the pulse is less attenuated than the trailing edge.



- Conditions can be that the peak of the attenuated pulse emerges from the absorber before the peak of the incident pulse has entered it! (That is the meaning of negative group velocity.)

- Since a Gaussian pulse does not have a sharply defined front edge, no question of causality violation.

- Some experiments show that photons travel faster than the speed of light through optical devices. While it is true that the centroid of the very small transmitted Gaussian pulse appears slightly in advance of the vacuum transit time, no signal or information travels faster than the speed of light.

Illustration of the Spreading of a Pulse as It Propagates in a Dispersive Medium

- The proper specification of an initial-value problem for the wave equation demands the initial values of both the function & its time derivative

$$u(x,t)=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(A(k)\,e^{i[kx-\omega(k)t]}+A^*(k)\,e^{-i[kx-\omega(k)t]}\right)\mathrm{d}k$$

$$\Rightarrow\quad A(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(u(x,0)+\frac{i}{\omega(k)}\frac{\partial u}{\partial t}(x,0)\right)e^{-ikx}\,\mathrm{d}x\quad\Leftarrow\quad A(k)\in\mathbb{C}$$

Let $u(x,0)=e^{-\frac{x^2}{2L^2}}\cos k_0 x$ Gaussian modulated oscillation , $\frac{\partial u}{\partial t}(x,0)=0$

$$\Rightarrow\quad A(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-ikx}\,e^{-\frac{x^2}{2L^2}}\cos k_0 x\,\mathrm{d}x=\frac{L}{2}\left(e^{-\frac{L^2}{2}(k-k_0)^2}+e^{-\frac{L^2}{2}(k+k_0)^2}\right)$$

$$\Rightarrow\quad A(-k)=A(k)\quad\Leftarrow\quad \text{reflection of 2 pulses traveling away from the origin}$$

- Assume $\omega(k)=\left(1+\frac{a^2k^2}{2}\right)\nu \quad\Leftarrow$ ν : constant frequency; a : typical wavelength where dispersive effects become important

approximation to the dispersion equation of the tenuous plasma: $\omega^2=\omega_p{}^2+c^2k^2$

$$\Rightarrow\quad v_g=\frac{\mathrm{d}\omega}{\mathrm{d}k}(k_0)=\nu\,a^2k_0\quad\Rightarrow$$ the pulse will be unaltered in shape if the pulse is not too narrow in space

$$\Rightarrow \quad u(x,t) = \frac{L}{2\sqrt{2\pi}} \Re\left[\int_{-\infty}^{\infty} \left(e^{-\frac{L^2}{2}(k-k_0)^2} + e^{-\frac{L^2}{2}(k+k_0)^2} \right) e^{i\left(k\,x - \nu \frac{2+a^2 k^2}{2} t\right)} \mathrm{d}k \right]$$

$$= \frac{L}{2} \Re\left[\frac{e^{-\frac{(x-\nu a^2 k_0 t)^2}{2(L^2+i a^2 \nu t)}}}{\sqrt{L^2+i a^2 \nu t}} e^{i\left(k_0 x - \nu \frac{2+a^2 k_0^2}{2} t\right)} + \frac{e^{-\frac{(x+\nu a^2 k_0 t)^2}{2(L^2+i a^2 \nu t)}}}{\sqrt{L^2+i a^2 \nu t}} e^{-i\left(k_0 x + \nu \frac{2+a^2 k_0^2}{2} t\right)} \right]$$

The equation represents 2 pulses traveling in opposite directions.

- The peak amplitude of each pulse travels with the group velocity, while the modulation envelop remains Gaussian in shape.

- The width of the Gaussian is not constant, but increases with time

width of the envolope $\quad L(t) = \sqrt{L^2 + \left(\frac{a^2 \nu t}{L}\right)^2} \quad (**)$ $\quad \Leftarrow \quad L^2(t) = (L_R + i L_I)^2 = L^2 + i a^2 \nu t$

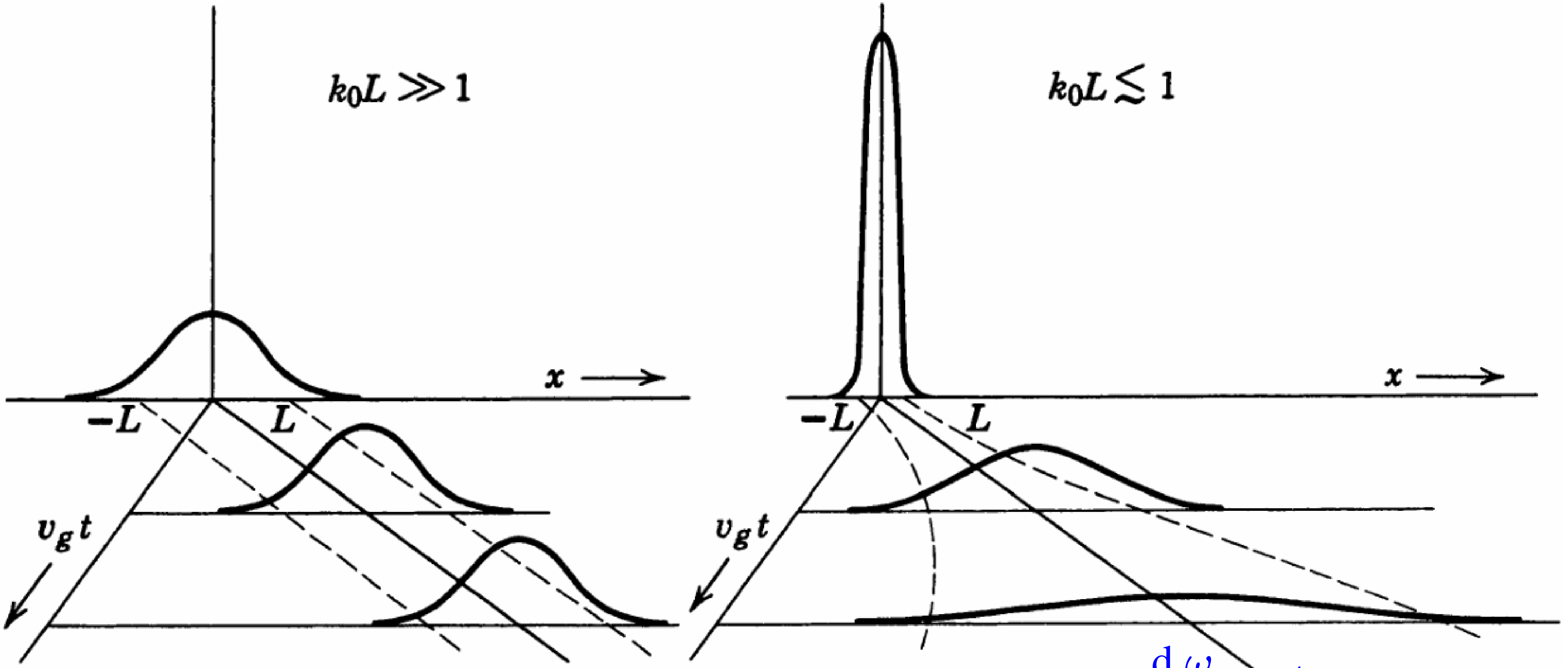
$\Rightarrow \quad L(t) \to L$ for $L \gg a$

the dispersive effects on the pulse are greater, the sharper the envelope.

- At long time the width of the Gaussian increases linearly with time $L(t) \to \frac{a^2 \nu t}{L}$

but the time of attainment of this asymptotic form depends on the ratio $\frac{L}{a}$.

- Although the results above have been derived for a special choice, their implications are of a more general nature.



- The average velocity of a pulse is the group velocity. $v_g = \frac{\mathrm{d}\,\omega}{\mathrm{d}\,k} = \omega'$
- The spreading of the pulse must have a spread of wave numbers $\Delta k \sim \frac{1}{\Delta x_0}$

$$\Rightarrow \quad \Delta v_g \sim \frac{\mathrm{d}^2\omega}{\mathrm{d}\,k^2}\Delta k \sim \frac{\omega''}{\Delta x_0} \;\Rightarrow\; \Delta x(t) \simeq \sqrt{(\Delta x_0)^2 + \left(\frac{\omega'' t}{\Delta x_0}\right)^2} \;\Leftarrow$$ RMS for position uncertainty

$$\Rightarrow \quad \text{set } \Delta x_0 = L \quad \Rightarrow \quad (\ast\ast)$$

a narrow pulse spreads rapidly because of its broad spectrum of wave numbers, and vice versa.

- All these ideas form the basis of the Heisenberg uncertainty principle.

Causality in the Connection Between D and E; Kramers-Kronig Relations

A. Nonlocality in Time

$$\mathbf{D}(\mathbf{r},\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\mathbf{D}(\mathbf{r},t')\,e^{i\omega t'}\,\mathrm{d}t',\quad \mathbf{E}(\mathbf{r},\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\mathbf{E}(\mathbf{r},t')\,e^{i\omega t'}\,\mathrm{d}t'$$

$$\Rightarrow\quad \mathbf{D}(\mathbf{r},t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\mathbf{D}(\mathbf{r},\omega)\,e^{-i\omega t}\,\mathrm{d}\omega\quad\Leftarrow\quad \mathbf{D}(\mathbf{r},\omega)=\epsilon(\omega)\,\mathbf{E}(\mathbf{r},\omega)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\epsilon(\omega)\,\mathbf{E}(\mathbf{r},\omega)\,e^{-i\omega t}\,\mathrm{d}\omega\quad\Leftarrow\quad \epsilon(\omega)=\epsilon_0[1+\chi_e(\omega)]$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\epsilon(\omega)\,e^{-i\omega t}\left(\int_{-\infty}^{+\infty}\mathbf{E}(\mathbf{r},t')\,e^{i\omega t'}\,\mathrm{d}t'\right)\mathrm{d}\omega$$

$$=\epsilon_0\left(\mathbf{E}(\mathbf{r},t)+\int_{-\infty}^{+\infty}G(\tau)\,\mathbf{E}(\mathbf{r},t-\tau)\,\mathrm{d}\tau\right)\quad\Leftarrow\quad \tau=t-t'$$

$$\text{where}\quad G(\tau)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\chi_e\,e^{-i\omega\tau}\,\mathrm{d}\omega$$

- The equations give a nonlocal connection between **D** and **E**, ie, **D** at time t depends on the electric field at times other than t.

- If $\epsilon(\omega)$ is independent of ω, then $G(\tau)\propto\delta(\tau)$ and the instantaneous connection is obtained, but if $\epsilon(\omega)$ varies with ω, $G(\tau)$ is nonvanishing for some $\tau\neq 0$.

B. Simple Model for $G(t)$, Limitations

↓ susceptibility kernel

- Consider $\frac{\epsilon(\omega)}{\epsilon_0} - 1 = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega} \Rightarrow G(\tau) = \frac{\omega_p^2}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{-i\omega\tau}}{\omega_0^2 - \omega^2 - i\gamma\omega}\, d\omega$

$$\Rightarrow G(\tau) = \frac{\omega_p^2}{2\pi}\int_{-\infty}^{+\infty} \frac{-e^{-i\omega\tau}\, d\omega}{\left(\omega + \nu_0 + i\frac{\gamma}{2}\right)\left(\omega - \nu_0 + i\frac{\gamma}{2}\right)} \Leftarrow \nu_0^2 = \omega_0^2 - \frac{\gamma^2}{4}$$

- For $\tau < 0$ the contour can be closed in the upper half-plane. Since the integrand is regular inside the closed contour, the integral vanishes.

- For $\tau > 0$, the contour is closed in the lower half-plane and the integral is $-2\pi i$ times the residues at the 2 poles $\Rightarrow G(\tau) = \omega_p^2 e^{-\frac{\gamma\tau}{2}} \frac{\sin \nu_0 \tau}{\nu_0} \Theta(\tau) \Leftarrow \Theta(\tau)$: step function

- The kernel $G(\tau)$ is oscillatory with the characteristic frequency of the medium and damped in time with the damping constant of the electronic oscillators.

- The nonlocality in time of the connection between $\mathbf{D}$ & $\mathbf{E}$ is of the order of $\frac{1}{\gamma}$.

- γ is the width in frequency of spectral lines and typically $10^7 - 10^9$/s, thus the departure from simultaneity is of the order of $10^{-7} - 10^{-9}$s.

- For frequencies > the microwave's many cycles of the electric field oscillations contribute an average weighed by $G(\tau)$ to $\mathbf{D}$ at a given instant of time.

Cauchy's Integral Formula: Let $f(z)$ is analytic on/inside the contour $\mathcal{C}$, then

$$\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{f(z)}{z-z_0} = \begin{cases} f(z_0), & z_0 \text{ within the contour } \mathcal{C} \\ 0, & z_0 \text{ outside the contour } \mathcal{C} \end{cases} \Leftarrow \text{ counterclockwise}$$

$$\int \frac{e^{-i\omega\tau}}{\omega_0^2-\omega^2-i\gamma\omega}\,\mathrm{d}\omega = \int \frac{-e^{-i\omega\tau}\,\mathrm{d}\omega}{\left(\omega+\nu_0+i\frac{\gamma}{2}\right)\left(\omega-\nu_0+i\frac{\gamma}{2}\right)} \quad \Leftarrow \quad \nu_0^2=\omega_0^2-\frac{\gamma^2}{4}$$

$$=\frac{1}{2\nu_0}\int\left(\frac{e^{-i\omega\tau}}{\omega+\nu_0+i\frac{\gamma}{2}}-\frac{e^{-i\omega\tau}}{\omega-\nu_0+i\frac{\gamma}{2}}\right)\mathrm{d}\omega$$

$$=-\frac{i\pi}{\nu_0}\left(e^{-i\left(-\nu_0-i\frac{\gamma}{2}\right)\tau}-e^{-i\left(\nu_0-i\frac{\gamma}{2}\right)\tau}\right) \quad \text{for the lower half-plane}$$

$$=\frac{2\pi}{\nu_0}e^{-\frac{\gamma\tau}{2}}\sin\nu_0\tau$$

- The equation is nonlocal in time, but not in space. This approximation is valid if the spatial variation of the applied fields has a scale that is large compared with the scale of the atomic/molecular polarization.

- For bound charges the scale of polarization is of the order of atomic dimension, so the concept of a dielectric constant that is a function only of ω holds can be expected to hold for frequencies well beyond the visible range..

- For conductors, the free charges with macroscopic mean free paths makes the assumption of a simple $\epsilon(\omega)$ break down at much lower frequencies.

- For copper $\gamma_0 \sim 3\times10^{13}$/s at 300K; $\gamma_0 \sim 3\times10^{10}$/s at 4K.

- The electron's velocity in metals is $\dfrac{c}{137}$ (Bohr velocity in hydrogen), so the mean free path $L \sim \dfrac{c}{137\,\gamma_0} \sim 10^{-4}\ \mathrm{m}$, the conventional skin depth is 10^{-8}–10^{-7}m.

- In the circumstances, Ohm's law must be replaced by a nonlocal expression. The conductivity becomes a tensorial quantity depending on $\boldsymbol{k}$ and frequency ω.

- The departures from the standard behavior are the *anomalous skin effect*. They can be used to map out the Fermi surfaces in metals.

- Similar nonlocal effects occur in superconductors where the EM properties involve a coherence length of the order of 10^{-6} m.

C. Causality and Analyticity Domain $\epsilon(\omega)$

- The most obvious and fundamental feature of the kernel is $G(\tau < 0) = 0$.

- Only values of the electric field prior to the time t determine the displacement in accord with causality $\Rightarrow$ $\mathbf{D}(\mathbf{r}, t) = \epsilon_0 \left(\mathbf{E}(\mathbf{r}, t) + \int_0^\infty G(\tau)\, \mathbf{E}(\mathbf{r}, t - \tau)\, \mathrm{d}\tau \right)$

The most general spatially local, linear, and causal relation that can be written between $\mathbf{D}$ and $\mathbf{E}$ in a uniform isotropic medium.

$$\Rightarrow \frac{\epsilon(\omega)}{\epsilon_0} = 1 + \int_0^\infty G(\tau)\, e^{i\omega\tau}\, \mathrm{d}\tau \Rightarrow \epsilon(-\omega) = \epsilon^*(\omega^*) \Leftarrow G \in \mathbb{R} \Leftarrow \mathbf{D},\ \mathbf{E} \in \mathbb{R}$$

- $\epsilon(\omega)$ is an analytic function in the upper half-plane of the complex ω plane.

- For the physical requirement, $G(\tau \to \infty) \to 0$ to assure $\epsilon(\omega)$ is also analytic.

- It is true for dielectrics, but not for conductors, where $G(\tau \to \infty) \to \dfrac{\sigma}{\epsilon_0}$ and $\epsilon(\omega)$ has a simple pole at $\omega = 0$, ie, $\epsilon(\omega \to 0) \to i\dfrac{\sigma}{\omega}$. Apart from a possible pole at $\omega = 0$, the dielectric constant $\epsilon(\omega)$ is analytic for $\mathrm{Im}[\omega] > 0$.

- The behavior of $\epsilon(\omega \gg 0)$ can be related to the behavior of $G(\tau \to 0)$

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \int_0^\infty G(\tau)\, e^{i\omega\tau}\, \mathrm{d}\tau = 1 + \frac{1}{i\omega}\left(G\, e^{i\omega\tau}\Big|_0^\infty - \int_0^\infty G'(\tau)\, e^{i\omega\tau}\, \mathrm{d}\tau \right)$$

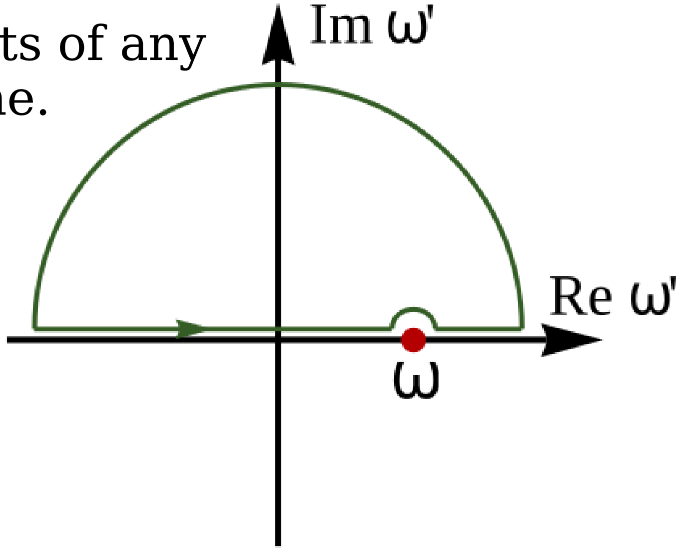
$$= 1 + \frac{i\,G(0^+)}{\omega} - \frac{G'(0^+)}{\omega^2} + \cdots = 1 - \frac{G'(0^+)}{\omega^2} + \cdots \Rightarrow (\#) \quad \Leftarrow \quad G(0^+) = G(0^-) = 0$$

$$\Rightarrow \quad \frac{\Re[\epsilon(\omega)]}{\epsilon_0} = 1 + O\left(\frac{1}{\omega^2}\right), \quad \frac{\Im[\epsilon(\omega)]}{\epsilon_0} = O\left(\frac{1}{\omega^3}\right)$$

- These asymptotic forms depend only upon the existence of $G'(\tau \to 0)$.

D. Kramers-Kronig (KK) Relations

- The KK relations connect the real and imaginary parts of any complex function that is analytic in the upper half-plane.



- $f(z) = f_R(z) + i\, f_I(z), \quad z \in \mathbb{C}, \quad f_R, \; f_I \in \mathbb{R}$

- Suppose f is analytic in the closed upper half-plane of z and tends to 0 as $|z| \to \infty$.

- The Kramers–Kronig relations are given by

$$f_R(\omega) = \frac{1}{\pi} \wp \left[\int_{-\infty}^{+\infty} \frac{f_I(\omega')}{\omega' - \omega} \, d\omega' \right], \quad f_I(\omega) = -\frac{1}{\pi} \wp \left[\int_{-\infty}^{+\infty} \frac{f_R(\omega')}{\omega' - \omega} \, d\omega' \right]$$

where $\omega \in \mathbb{R}$ on the real axis, $\wp$: Cauchy principal value

f_R and f_I are dependent, allowing f to be reconstructed given only f_R or f_I.

Proof: According to Cauchy's residue theorem for complex integration

$$f(\omega) = \frac{1}{i\pi} \wp \left[\int_{-\infty}^{+\infty} \frac{f(\omega')}{\omega' - \omega} \, d\omega' \right] \Rightarrow f_R + i\, f_I = \frac{1}{i\pi} \wp \left[\int_{-\infty}^{+\infty} \frac{f_R + i\, f_I}{\omega' - \omega} \, d\omega' \right]$$

$$\Rightarrow \quad f_R(\omega) = \frac{1}{\pi} \wp \left[\int_{-\infty}^{+\infty} \frac{f_I(\omega')}{\omega' - \omega} \, d\omega' \right], \quad f_I(\omega) = -\frac{1}{\pi} \wp \left[\int_{-\infty}^{+\infty} \frac{f_R(\omega')}{\omega' - \omega} \, d\omega' \right]$$

as an integral analog of the Cauchy-Riemann differential equations for a complex function

- For any point z inside a closed contour C in the upper half-ω-plane, Cauchy's theorem gives

$$\frac{\epsilon(z)}{\epsilon_0}-1=\frac{1}{2\pi i}\oint_c \frac{\dfrac{\epsilon(\omega')}{\epsilon_0}-1}{\omega'-z}\,d\omega'=\frac{1}{2\pi i\,\epsilon_0}\int_{-\infty}^{+\infty}\frac{\epsilon(\omega')-\epsilon_0}{\omega'-z}\,d\omega' \quad \Leftarrow \quad \epsilon(\infty)\to\epsilon_0$$

- For $z=\omega$ being real, the Cauchy principal value should be considered

$$\epsilon(\omega)-\epsilon_0=\frac{1}{\pi i}\wp\left[\int_{-\infty}^{+\infty}\frac{\epsilon(\omega')-\epsilon_0}{\omega'-\omega}\,d\omega'\right] \quad \Leftarrow \quad \wp : \text{principal part}$$

$$\Rightarrow \begin{aligned}\Re[\epsilon(\omega)]&=\epsilon_0+\frac{1}{\pi}\wp\left[\int_{-\infty}^{+\infty}\frac{\Im[\epsilon(\omega')]}{\omega'-\omega}\,d\omega'\right]\\ \Im[\epsilon(\omega)]&=-\frac{1}{\pi}\wp\left[\int_{-\infty}^{+\infty}\frac{\Re[\epsilon(\omega')]-\epsilon_0}{\omega'-\omega}\,d\omega'\right]\end{aligned} \quad (a) \quad \Leftarrow \quad \begin{array}{c}\text{KK relations}\\ \text{dispersion relations}\end{array}$$

$$\epsilon(-\omega)=\epsilon^*(\omega^*) \quad \Rightarrow \quad \begin{array}{ll}\Re[\epsilon(\omega)] & \text{even}\\ \Im[\epsilon(\omega)] & \text{odd}\end{array}$$

$$\Rightarrow \begin{aligned}\Re[\epsilon(\omega)]&=\epsilon_0+\frac{2}{\pi}\wp\left[\int_0^{\infty}\frac{\omega'\,\Im[\epsilon(\omega')]}{\omega'^2-\omega^2}\,d\omega'\right]\\ \Im[\epsilon(\omega)]&=-\frac{2\omega}{\pi}\wp\left[\int_0^{\infty}\frac{\Re[\epsilon(\omega')]-\epsilon_0}{\omega'^2-\omega^2}\,d\omega'\right]\end{aligned} \quad (b) \quad \Leftarrow \quad \begin{array}{l}\text{here assume } \epsilon(\omega) \text{ is}\\ \text{regular at } \omega=0\end{array}$$

$\Re\left[\epsilon\left(\omega\right)\right]$ even , $\quad \Im\left[\epsilon\left(\omega\right)\right]$ odd

$$\int_{-\infty}^{+\infty} \frac{\Im\left[\epsilon\left(\omega'\right)\right]}{\omega' - \omega} \, d\omega' = \int_{-\infty}^{0} \frac{\Im\left[\epsilon\left(\omega'\right)\right]}{\omega' - \omega} \, d\omega' + \int_{0}^{\infty} \frac{\Im\left[\epsilon\left(\omega'\right)\right]}{\omega' - \omega} \, d\omega'$$
$$= \int_{0}^{\infty} \frac{\Im\left[\epsilon\left(\omega'\right)\right]}{\omega' - \omega} \, d\omega' + \int_{0}^{\infty} \frac{\Im\left[\epsilon\left(\omega'\right)\right]}{\omega' + \omega} \, d\omega'$$
$$= 2 \int_{0}^{\infty} \frac{\omega' \, \Im\left[\epsilon\left(\omega'\right)\right]}{\omega'^{2} - \omega^{2}} \, d\omega'$$

Similarly,

$$\int_{-\infty}^{+\infty} \frac{\Re\left[\epsilon\left(\omega'\right)\right] - \epsilon_0}{\omega' - \omega} \, d\omega' = \int_{-\infty}^{0} \frac{\Re\left[\epsilon\left(\omega'\right)\right] - \epsilon_0}{\omega' - \omega} \, d\omega' + \int_{0}^{\infty} \frac{\Re\left[\epsilon\left(\omega'\right)\right] - \epsilon_0}{\omega' - \omega} \, d\omega'$$
$$= \int_{0}^{\infty} \frac{\Re\left[\epsilon\left(\omega'\right)\right] - \epsilon_0}{\omega' - \omega} \, d\omega' - \int_{0}^{\infty} \frac{\Re\left[\epsilon\left(\omega'\right)\right] - \epsilon_0}{\omega' + \omega} \, d\omega'$$
$$= 2\,\omega \int_{0}^{\infty} \frac{\Re\left[\epsilon\left(\omega'\right)\right] - \epsilon_0}{\omega'^{2} - \omega^{2}} \, d\omega'$$

$$\Im\left[\epsilon\left(\omega'\right)\right]\simeq\frac{\pi K}{2\omega_0}\delta\left(\omega'-\omega_0\right)+\cdots \quad \Leftarrow \quad \begin{array}{c}\text{from experiments}\\ \text{of absorption}\end{array} \quad \Rightarrow \quad \Re\left[\epsilon\left(\omega\right)\right]\simeq\bar{\epsilon}+\frac{K}{\omega_0^2-\omega^2}$$

- The approximation exhibits the rapid variation of $\Re[\epsilon(\omega)]$ in the neighborhood of an absorption line.

- Relations (a) & (b) connecting the dispersive & absorptive aspects of a process are useful in all areas of physics since very small number of physical assumptions are necessary for their derivation.

- The plasma frequency can be defined as

$$\omega_p^2=\lim_{\omega\to\infty}\omega^2\frac{\epsilon_0-\epsilon\left(\omega\right)}{\epsilon_0} \quad \Leftarrow \quad \frac{\epsilon\left(\omega\right)}{\epsilon_0}\simeq 1-\frac{\omega_p^2}{\omega^2} \quad + \quad \frac{\Im\left[\epsilon\left(\omega\right)\right]}{\epsilon_0}=O\left(\frac{1}{\omega^3}\right)$$

$$=-\lim_{\omega\to\infty}\frac{2\omega^2}{\pi\epsilon_0}\wp\left[\int_0^\infty\frac{\omega'\,\Im\left[\epsilon\left(\omega'\right)\right]}{\omega'^2-\omega^2}\,\mathrm{d}\omega'\right]=\frac{2}{\pi\epsilon_0}\int_0^\infty\omega'\,\Im\left[\epsilon\left(\omega'\right)\right]\mathrm{d}\omega'$$

$$\Rightarrow \quad \omega_p^2=\frac{2}{\pi\epsilon_0}\int_0^\infty\omega\,\Im\left[\epsilon\left(\omega\right)\right]\mathrm{d}\omega$$ 1st sum rule or the 1st Kramers-Kronig relation

- The relation is the sum rule for oscillator strengths f_i in Sec 7.5 but more general.

- With the assumption $\dfrac{\Re\left[\epsilon\left(\omega'\right)\right]}{\epsilon_0}=1-\dfrac{\omega_p^2}{\omega'^2}+O\left(\dfrac{1}{\omega'^4}\right)$ for $\omega'>N \Leftarrow \Re[\epsilon]$ even

$$\frac{\Im[\epsilon(\omega)]}{\epsilon_0} = -\frac{2\,\omega}{\pi\,\epsilon_0}\,\wp\left[\int_0^\infty \frac{\Re[\epsilon(\omega')]-\epsilon_0}{\omega'^2-\omega^2}\,d\,\omega'\right] \qquad \Leftarrow \quad \omega \gg N$$

$$\approx \frac{2}{\pi\,\epsilon_0\,\omega}\left(\int_0^N \Re[\epsilon(\omega')-\epsilon_0]\,d\,\omega' + \int_N^\infty \Re[\epsilon(\omega')-\epsilon_0]\,d\,\omega'\right)$$

$$\approx \frac{2}{\pi\,\omega}\left(\int_0^N \frac{\Re[\epsilon(\omega')]-\epsilon_0}{\epsilon_0}\,d\,\omega' + \left.\frac{\omega_p^2}{\omega'}\right|_N^\infty\right) = O\left(\frac{1}{\omega}\right)$$

$$\Rightarrow \quad \frac{1}{\epsilon_0}\int_0^N \Re[\epsilon(\omega')]\,d\,\omega' - N - \frac{\omega_p^2}{N} = 0 \quad \Leftarrow \quad \frac{\Im[\epsilon(\omega)]}{\epsilon_0} = O\left(\frac{1}{\omega^3}\right)$$

$$\Rightarrow \quad \boxed{\frac{1}{N\,\epsilon_0}\int_0^N \Re[\epsilon(\omega')]\,d\,\omega' = 1 + \frac{\omega_p^2}{N^2}} \quad \text{2nd sum rule or superconvergence relation}$$

$$\Rightarrow \quad \lim_{N\to\infty}\frac{1}{N\,\epsilon_0}\int_0^N \Re[\epsilon(\omega')]\,d\,\omega' = 1 \quad \Leftarrow$$

for $N \to \infty$, the average $\Re[\epsilon(\omega)]$ over all frequencies is equal to ϵ_0

- For conductors, the 1st sum rule still holds, but the 2nd sum rule has an added term $-\dfrac{\pi\,\sigma}{2\,\epsilon_0\,N}$ on the right hand side [Problem 7.23].

- Selected problems: 5, 18, 23

Airy's integearl $\quad \mathrm{Ai}(x) \equiv \dfrac{1}{2\,\pi}\displaystyle\int_{-\infty}^{+\infty} e^{i\left(p\,x+\frac{p^3}{3}\right)}\,d\,p$

Arrival of a Signal After Propagation Through a Dispersive Medium

- Consider a plane wave train normally incident from vacuum on a semi-infinite uniform medium of index of refraction $n(\omega)$

$$u(x,t)=\int_{-\infty}^{+\infty}\frac{2A(\omega)}{1+n(\omega)}e^{i[k(\omega)x-\omega t]}\,\mathrm{d}\omega \quad \Leftarrow \quad A(\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}u_i(0,t)\,e^{i\omega t}\,\mathrm{d}t$$

$$\Rightarrow \quad k(\omega)=\frac{\omega}{c}n(\omega)\in\mathbb{C} \quad \Rightarrow \quad \Im[k]>0 \text{ for energy absorption}$$

- $u(0,t<0)=0$ assures that $A(\omega)$ is analytic in the upper half-ω-plane. Generally, $A(\omega)$ have singularities in the lower half-ω-plane. Assume $A(|\omega|\to\infty)$ is bounded.

$$t=-|t|<0$$

$$\Rightarrow \quad e^{-i\omega t}=e^{i|t|(\omega_{\mathrm{R}}+i\omega_{\mathrm{I}})}$$

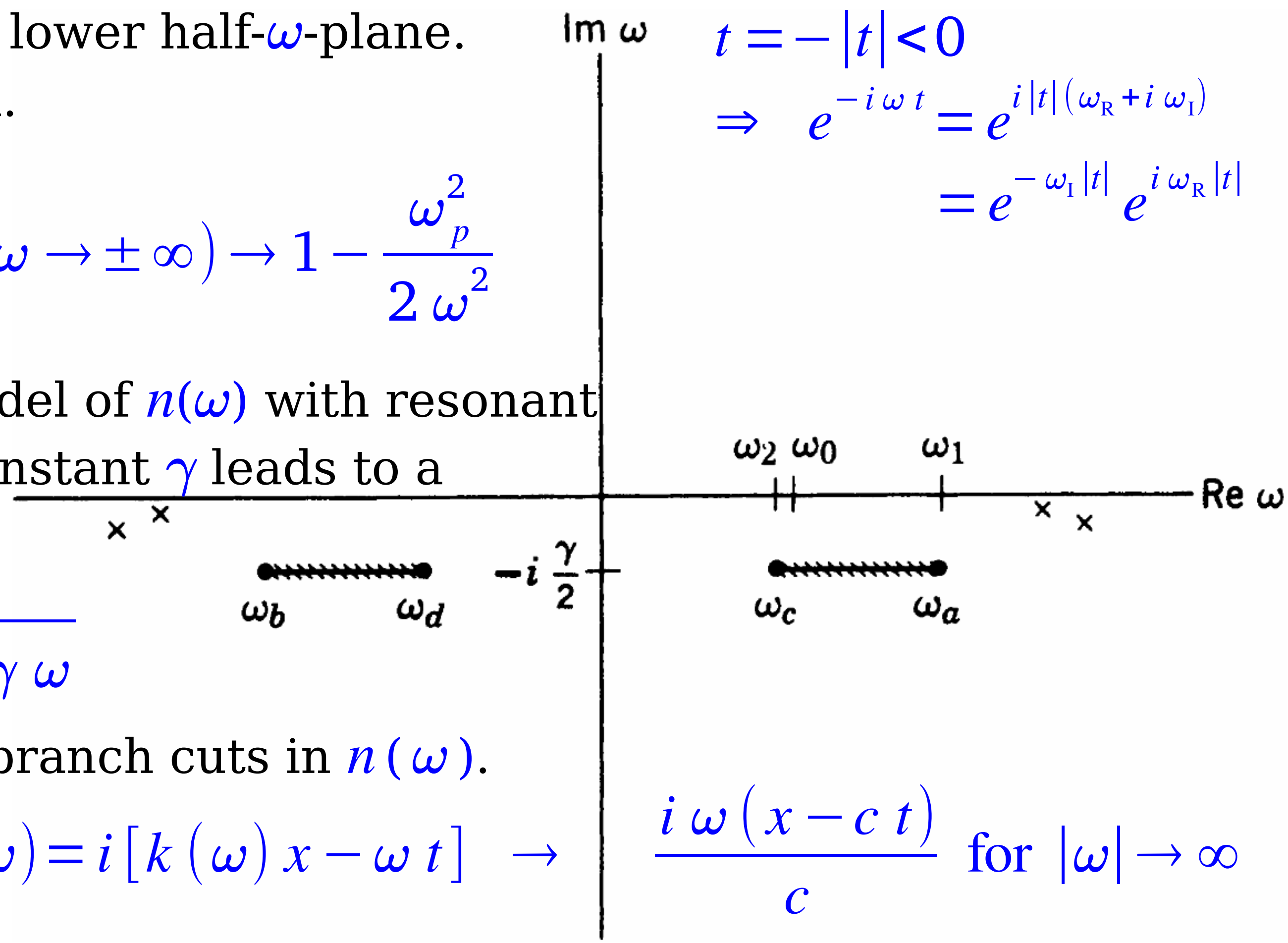
$$=e^{-\omega_{\mathrm{I}}|t|}\,e^{i\omega_{\mathrm{R}}|t|}$$

- $n(\omega)$ is analytic in the upper half-ω-plane, as ϵ

$$n(\omega\to\pm\infty)\to 1-\frac{\omega_p^2}{2\omega^2}$$

- A simple one-resonance model of $n(\omega)$ with resonant frequency ω_0 and damping constant γ leads to a singularity structure.

$$\epsilon(\omega)\sim\bar{\epsilon}+\frac{1}{\omega_0^2-\omega^2-i\gamma\omega}$$



- The poles of $\epsilon(\omega)$ become branch cuts in $n(\omega)$.

- $$n(\omega\to\pm\infty)\to 1 \quad \Rightarrow \quad i\phi(\omega)=i[k(\omega)x-\omega t] \quad \to \quad \frac{i\omega(x-ct)}{c} \text{ for } |\omega|\to\infty$$

- $u(x,t) = \oint_c \frac{2}{1+n(\omega)} A(\omega) e^{ik(\omega)x - i\omega t} \, d\omega$

obtain a zero contribution to the integral by closing the contour with a semicircle at infinity in the upper (lower) half-plane for $x > ct$ ($x < ct$)

$\Rightarrow \quad u(x,t) = 0$ for $x - ct > 0$ + $n(\omega)$ & $A(\omega)$ are analytic in the upper half-plane

- Since the specific form of $n(\omega)$ does not enter, we have a general proof that no signal propagates with a velocity greater than c, whatever the medium.

- For $x < ct$, the contour is closed in the lower half-plane, having the singularities.

- The method of stationary phase is based on the idea that the phase $\phi(\omega)$ in an integral is usually large and rapidly varying. So the integrand averages almost to 0. Exceptions occur when $\phi(\omega)$ is "stationary," ie, when $\phi(\omega)$ has an extremum. The integral can therefore be estimated by approximating the integral at each of the points of stationary phase and summing these contributions.

- $\phi(\omega) = k(\omega)x - \omega t \quad \Rightarrow \quad \frac{\partial \phi}{\partial \omega} = 0 \quad \Rightarrow \quad c\frac{dk}{d\omega} = n(\omega) + \omega\frac{dn}{d\omega} = \frac{t}{t_0}$ for $t > t_0 = \frac{x}{c}$

- The earliest part of the wave occurs when $t \to t_0$, the point of stationary phase is

$$n(|\omega| \to \infty) \to 1 \quad \Rightarrow \quad c\frac{dk}{d\omega} \simeq 1 + \frac{\omega_p^2}{2\omega^2} = \frac{t}{t_0} \quad \text{for } t \geq t_0$$

$$\Rightarrow \quad \omega_s \approx \frac{\omega_p}{\sqrt{2}} \sqrt{\frac{t_0}{t - t_0}} \quad \text{the frequency of stationary phase}$$

- The incident wave's $A(\omega_s)$ should be very small. So the earliest part of the signal is small and of high frequency, bearing no resemblance to incident wave. This part of the signal is called the *1st or Sommerfeld precursor*.

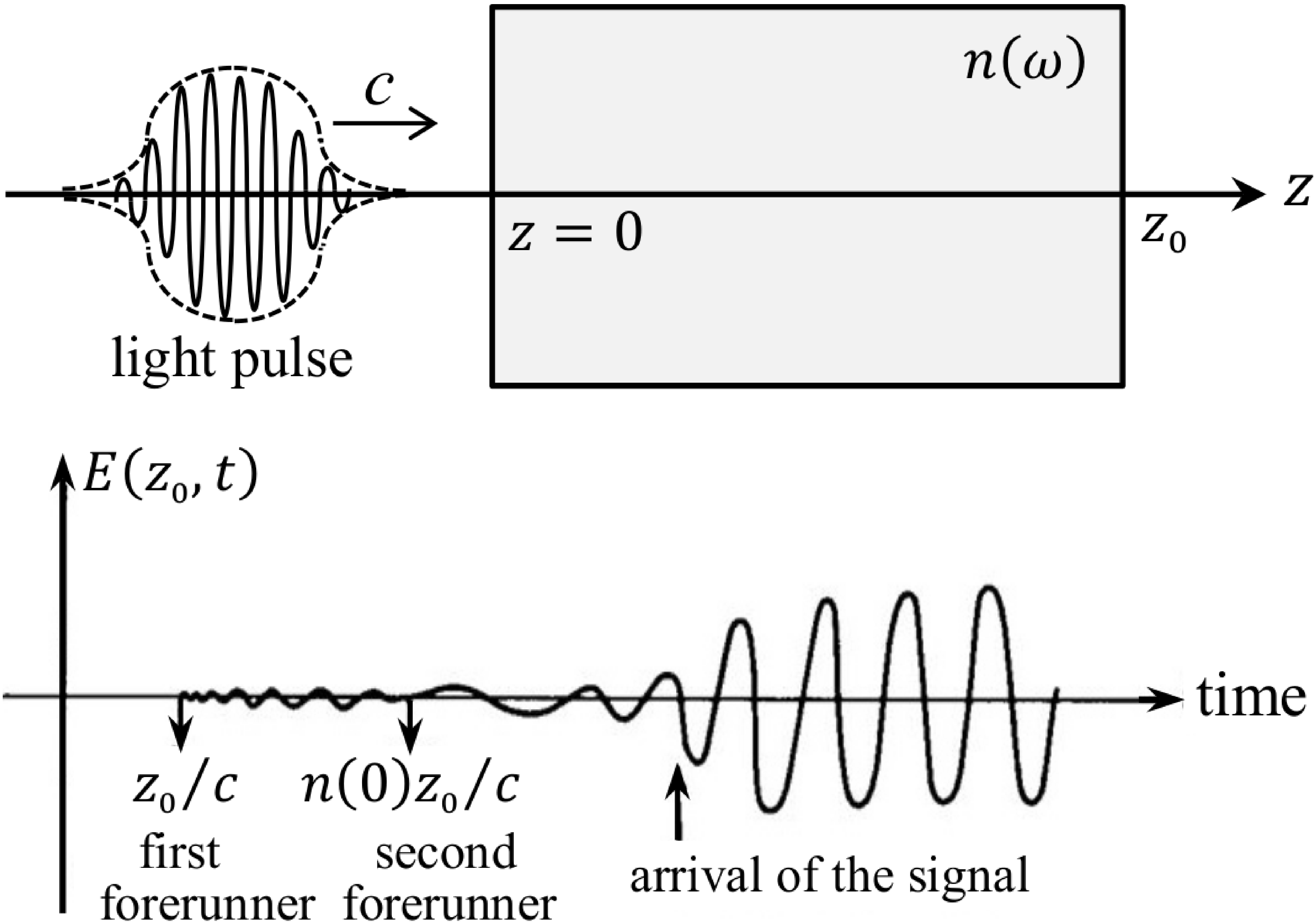
- Only when $\frac{t}{t_0}$ reaches $n(0)$ is a qualitative change in the amplitude.

- $\omega = 0$ is now a point of stationary phase, the high frequency of oscillation is replaced by lower frequencies. More important is $\frac{d^2 k}{d\omega^2}(0) = 0$ ($\Leftarrow \frac{dn}{d\omega}(0) = 0$ for symmetry.) In such situations, $\phi'' = 0$, thus the stationary phase approximation fails, giving an infinite result.

- Improve the approximation to include cubic terms in $\phi(\omega \approx \omega_s)$. The amplitude is expressible in terms of Airy integrals. The wave becomes large in amplitude and of long period for $t \geq n(0)\, t_0$. This phase is called the *2nd or Brillouin precursor*.

- Later, the behavior of $A(\omega)$ dominate the integral. Thus the main part of the wave has arrived. The amplitude behaves as if it were the initial wave with the appropriate phase velocity and attenuation.

- The sequence of arrival of Sommerfeld precursor, Brillouin precursor, and the main signal depends upon the $n(\omega)$, $A(\omega)$, and the position of observation.



$n(\omega)$
c
z
$z = 0$
z_0
light pulse
$E(z_0, t)$
time
z_0/c
first
forerunner
$n(0)z_0/c$
second
forerunner
arrival of the signal