Chapter 12 Electrodynamics and Relativity

The Special Theory of Relativity

Einstein's Postulates

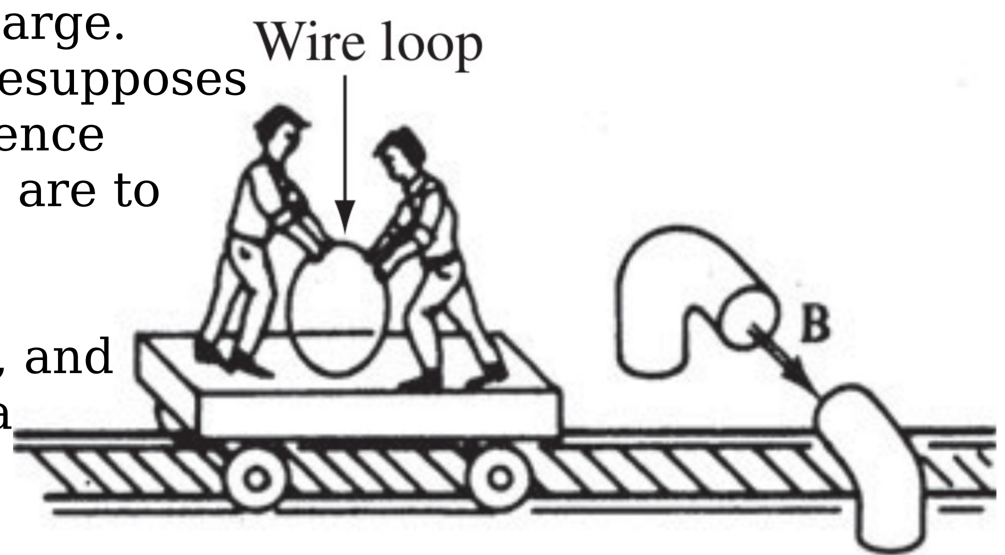
- Classical mechanics obeys the **principle of relativity**: the same laws apply in any **inertial reference frame**. By "inertial" it means that the system is at rest or moving with constant velocity.

- By contrast you know *immediately* if the train accelerates. The laws of mechanics are certainly *not* the same in *accelerating* reference frames.

- Does the principle of relativity also apply to the laws of electrodynamics? At first glance, the answer would seem to be *no*. After all, a charge in motion produces a magnetic field, whereas a charge at rest does not.

- Many of the equations of electrodynamics with the Lorentz force law make explicit reference to the velocity of the charge.
It appears that electromagnetic theory presupposes the existence of a unique stationary reference frame, with respect to which all velocities are to be measured.

- If we mount a wire loop on a freight car, and have the train pass between the poles of a giant magnet.



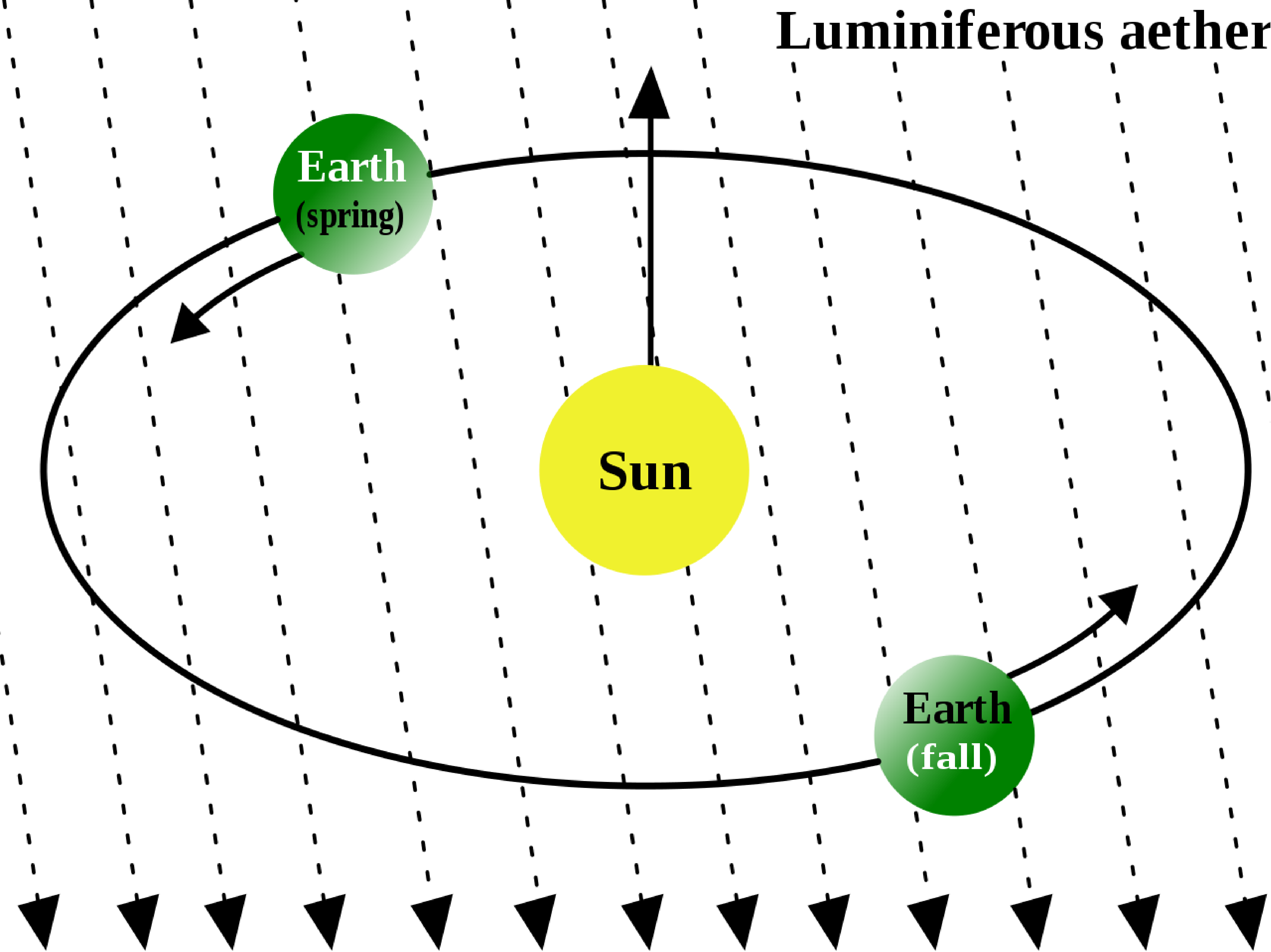
- As the loop rides through the magnetic field, a motional emf is $\mathcal{E} = -\frac{d\Phi}{dt}$
- This emf is due to the magnetic force on charges in the wire loop, which are moving along with the train.

- On the other hand, someone on the car will find no *magnetic* force because the loop is at rest. But as the magnet flies by, the magnetic field in the car changes, and a changing magnetic field induces an electric field, by Faraday's law.
- The resulting *electric* force would generate an emf: $\mathcal{E} = \oint \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi}{dt}$
- Because Faraday's law and the flux rule predict exactly the same emf, people on the train will get the right answer, *even though their physical interpretation of the process is completely wrong*!

- Before Einstein's special relativity, people thought of electric and magnetic fields as strains in an invisible jellylike medium called **ether**, which permeated all of space. The speed of the charge was to be measured *with respect to the ether*—only then would the laws of electrodynamics be valid.

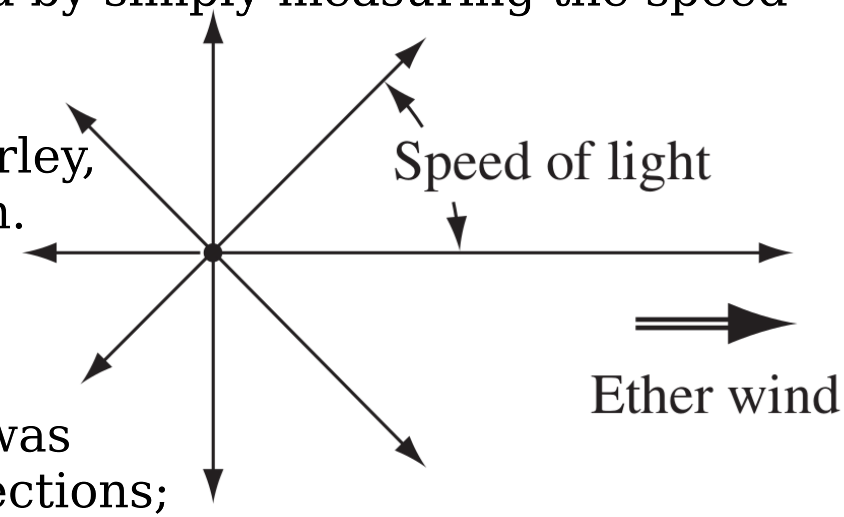
- With this assumption, it becomes a matter of crucial importance to *find* the ether frame, experimentally, or else *all* our calculations will be invalid.

- The problem becomes to determine our motion through the ether—to measure the speed and direction of the "ether wind."


Luminiferous aether
Earth
(spring)
Sun
Earth
(fall)

- Among the results of classical electrodynamics is the prediction that EM waves travel through the vacuum at a speed $\frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3 \times 10^8 \text{ m/s}$ *relative to the ether*.

- So one should be able to detect the ether wind by simply measuring the speed of light in various directions.



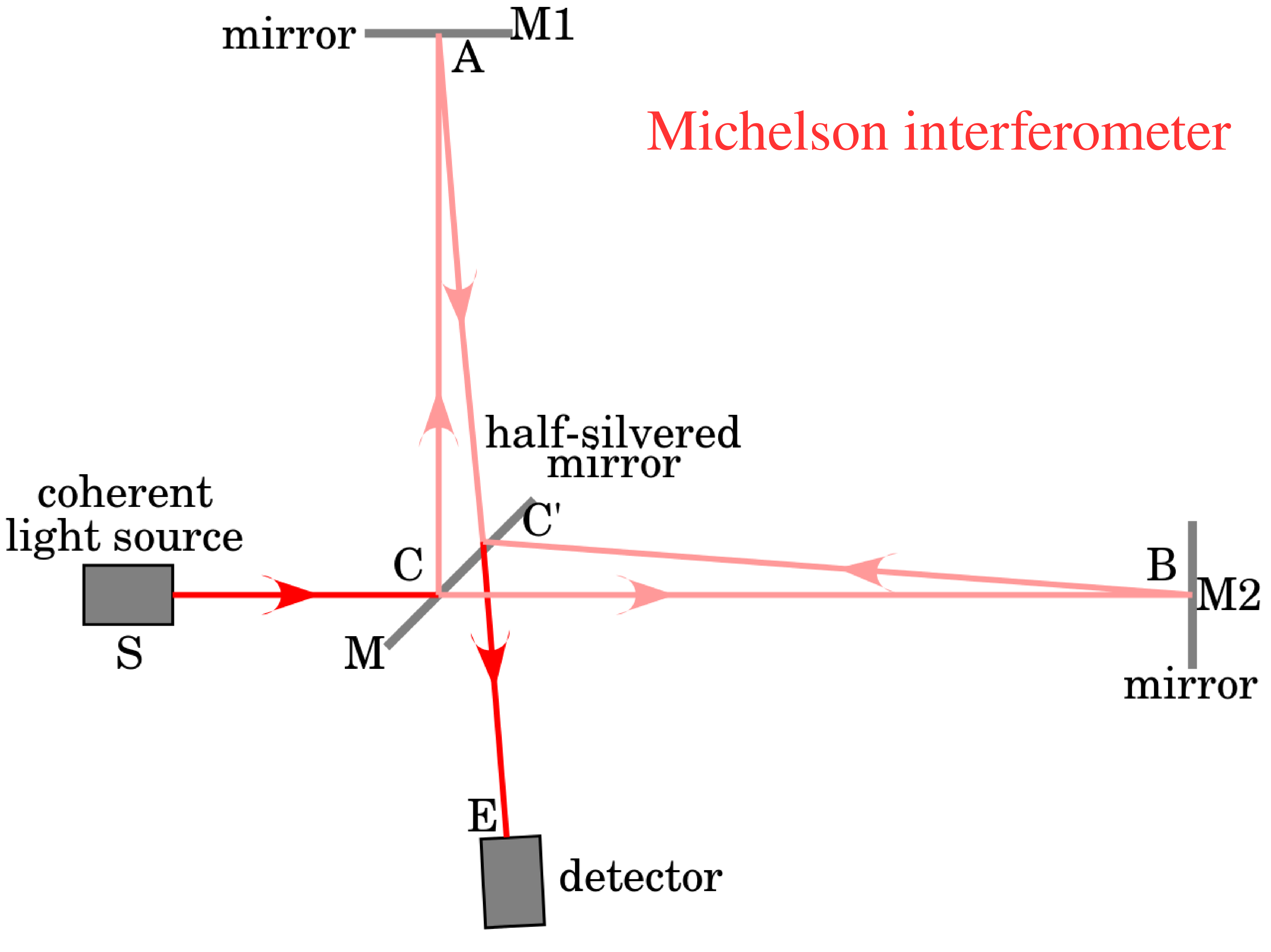
- The experiment was done by Michelson & Morley, using an optical interferometer of nice precision.

- 2 essential points:

  (1) all Michelson & Morley were trying to do was compare the speed of light in different directions;
  (2) what they *discovered* was that this speed is *exactly the same in all directions*.

- At that time all other waves travel at a prescribed speed *relative to the propagating medium*, and if this medium is in motion with respect to the observer, the net speed is always greater "downstream" than "upstream." This made light's propagation difficult to understand.

- Michelson & Morley interpreted their experiment as confirmation of the "ether drag" hypothesis, ie, the earth pulls the ether along with it. But this was found to be inconsistent with other observations, eg, the aberration of starlight.


Michelson interferometer
mirror
M1
A
half-silvered
mirror
coherent
light source
S
C
C'
M
B
M2
mirror
E
detector

- Einstein took the Michelson-Morley result at face value, and suggested that the speed of light is a universal constant, the same in all directions, regardless of the motion of the observer or the source. There *is* no ether wind because there is no ether.

- Also *any* inertial system is a suitable reference frame for the application of Maxwell's eqns, and the velocity of a charge is to be measured *not* with respect to an absolute rest frame, nor with respect to an ether, but simply with respect to the particular inertial system you happen to have chosen.

- Inspired both by internal theoretical hints and by external empirical evidence, Einstein proposed his 2 famous postulates:

  1. **The principle of relativity**: The laws of physics apply in all inertial reference systems.
  2. **The universal speed of light**: The speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source.

- The 1st elevates Galileo's observation about classical mechanics to the status of a general law, applying to *all* of physics. Also there is no absolute rest system.

- The 2nd is the response to the result of the Michelson-Morley experiment. It means that there is no ether.

- The universal speed of light was radically new and preposterous:

$$v_{AC} = v_{AB} + v_{BC}$$ Galileo's velocity addition rule

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + v_{AB}\, v_{BC} / c^2}$$ Einstein's velocity addition rule $\Leftarrow$ $v_{AC} = v_{AB} = c$ for light

- For "ordinary" speeds ($v_{AB} \ll c$, $v_{BC} \ll c$), the denominator is so close to 1 that the discrepancy between Galileo's formula and Einstein's formula is negligible.

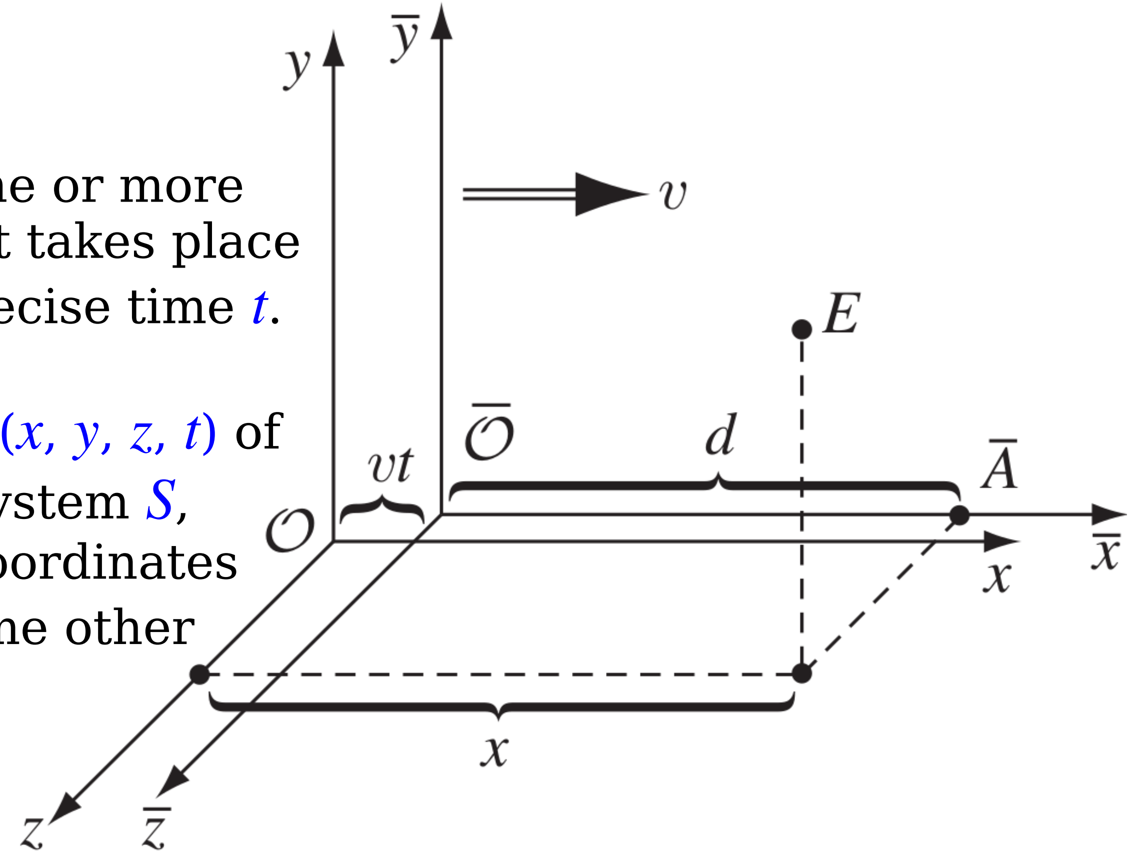
- Einstein's formula has the desired property that if $v_{AB}=c$, then automatically

$$v_{AC} = \frac{c + v_{BC}}{1 + c\, v_{BC} / c^2} = c$$

- Special relativity compels us to alter our notions of space and time themselves, and therefore also of such derived quantities as velocity, momentum, and energy.

- Special theory is not limited to any particular class, it is a description of the space-time "arena" in which all physical phenomena take place. Thus relativity defines the structure of space and time.

The Lorentz Transformations



- Any physical process consists of one or more **events**. An "event" is something that takes place at a specific location (x, y, z), at a precise time t.

- Suppose we know the coordinates (x, y, z, t) of a particular event E in *one* inertial system S, and we would like to calculate the coordinates $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ of that *same event* in some other inertial system $\bar{S}$.

- If we set $t=0$ at the moment the origins coincide, then at time t, $\bar{O}$ will be a distance vt from O: $x = d + v\,t$

- Before Einstein, $d = \bar{x}$ $\Rightarrow$

$$\bar{x} = x - v\,t \qquad \bar{y} = y \qquad \bar{z} = z \qquad \bar{t} = t$$

Galilean transformations

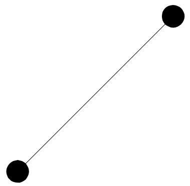
- In relativity,

$$\bar{x} = \gamma\,(x - v\,t) \qquad \bar{y} = y \qquad \bar{z} = z \qquad \bar{t} = \gamma\left(t - \frac{v}{c^2}\,x\right)$$

Lorentz transformations $\Leftarrow$ $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$

$$c=\bar{c} \quad \Rightarrow \quad 0=c^2\left(t_2-t_1\right)^2-\left(x_2-x_1\right)^2-\left(y_2-y_1\right)^2-\left(z_2-z_1\right)^2$$

$$=c^2\left(\bar{t}_2-\bar{t}_1\right)^2-\left(\bar{x}_2-\bar{x}_1\right)^2-\left(\bar{y}_2-\bar{y}_1\right)^2-\left(\bar{z}_2-\bar{z}_1\right)^2 \quad \text{for light}$$



$\Rightarrow$ define an (infinitesimal) interval $\mathrm{d}s^2=-c^2\,\mathrm{d}t^2+\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2$ for 2 events

$$\Rightarrow \quad \boxed{\mathrm{d}s^2=\mathrm{d}\bar{s}^2} \quad \Leftarrow \quad \mathrm{d}\bar{s}^2=a(v)\,\mathrm{d}s^2, \quad \mathrm{d}s^2=a(v)\,\mathrm{d}\bar{s}^2 \quad \Rightarrow \quad a=\pm 1 \quad \Rightarrow \quad a=1$$

$$\Rightarrow \quad -c^2\,\mathrm{d}t^2+\mathrm{d}x^2=-c^2\,\mathrm{d}\bar{t}^2+\mathrm{d}\bar{x}^2 \quad \Leftarrow \quad \mathrm{d}y=\mathrm{d}\bar{y}, \quad \mathrm{d}z=\mathrm{d}\bar{z}$$

$$\left[\begin{array}{l} \mathrm{d}\bar{x}=A\left(\mathrm{d}x-v\,\mathrm{d}t\right) \\ \mathrm{d}\bar{t}=\quad B\,\mathrm{d}t+D\,\mathrm{d}x \end{array}\right. \quad \Leftarrow \quad \text{generalized Galilean transformation}$$

$$\Rightarrow \quad -c^2\,\mathrm{d}t^2+\mathrm{d}x^2=-c^2\left(B\,\mathrm{d}t+D\,\mathrm{d}x\right)^2+A^2\left(\mathrm{d}x-v\,\mathrm{d}t\right)^2$$

$$=-\left(\underline{c^2 B^2-A^2 v^2}\right)\mathrm{d}t^2-2\left(\underline{c^2 B\,D+A^2 v}\right)\mathrm{d}t\,\mathrm{d}x+\left(\underline{A^2-c^2 D^2}\right)\mathrm{d}x^2$$

$$\Rightarrow \quad \begin{array}{c} B^2-\beta^2 A^2=1 \\ c\,B\,D+A^2\beta=0 \\ A^2-c^2 D^2=1 \end{array} \quad \Rightarrow \quad A^2=-\frac{c}{\beta}B\,D \quad \Rightarrow \quad \begin{array}{c} B\left(B+c\,\beta\,D\right)=1 \\ -\dfrac{c}{\beta}D\left(B+c\,\beta\,D\right)=1 \end{array} \quad \Leftarrow \quad \beta\equiv\frac{v}{c}$$

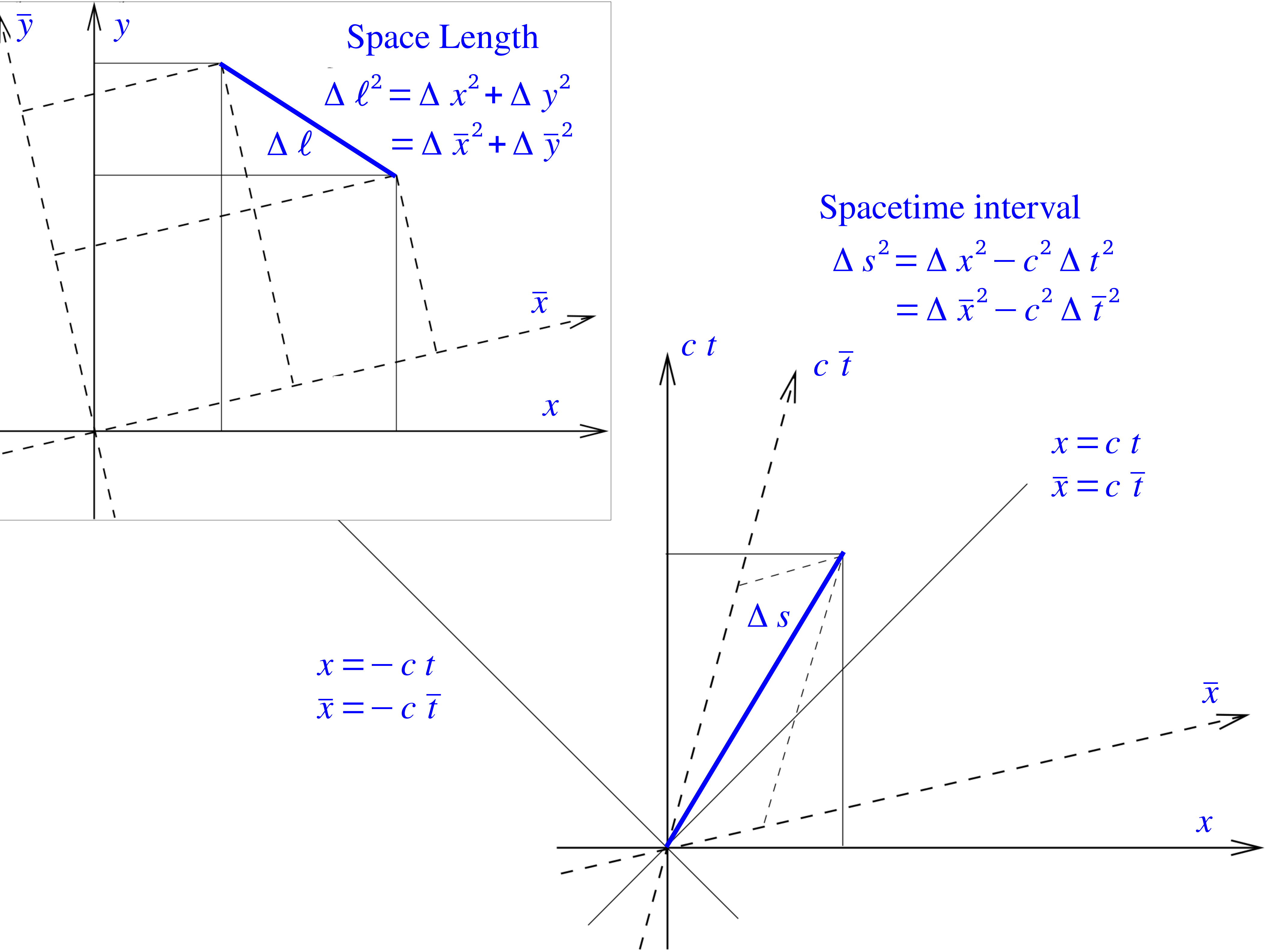
$$\Rightarrow \quad D = -\frac{\beta}{c} B \quad \Rightarrow \quad B^2 \left(1-\beta^2\right) = 1 \quad \Rightarrow \quad B = \pm\gamma \quad \Rightarrow \quad A^2 = B^2 \quad \Rightarrow \quad A = \pm\gamma$$

Choose $+$ for v approaching 0 continuously. $\Uparrow$ $\gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}} = \dfrac{1}{\sqrt{1-v^2/c^2}}$

$$\Rightarrow \quad \begin{aligned} c\,\mathrm{d}\,\bar{t} &= \gamma\left(c\,\mathrm{d}\,t - \beta\,\mathrm{d}\,x\right) \\ \mathrm{d}\,\bar{x} &= \gamma\left(\mathrm{d}\,x - \beta\,c\,\mathrm{d}\,t\right) \\ \mathrm{d}\,\bar{y} &= \mathrm{d}\,y \\ \mathrm{d}\,\bar{z} &= \mathrm{d}\,z \end{aligned}$$

- The reverse transformation from $\bar{S}$ to S is obtained algebraically by solving for x & t, or by switching v's sign:

$$\begin{aligned} x &= \gamma\left(\bar{x} + v\,\bar{t}\right) \\ y &= \bar{y} \\ z &= \bar{z} \\ t &= \gamma\left(\bar{t} + \frac{v\,\bar{x}}{c^2}\right) \end{aligned} \quad \Leftarrow \quad \begin{aligned} \mathrm{d}\,x &= \gamma\left(\mathrm{d}\,\bar{x} + \beta\,c\,\mathrm{d}\,\bar{t}\right) \\ \mathrm{d}\,y &= \mathrm{d}\,\bar{y} \\ \mathrm{d}\,z &= \mathrm{d}\,\bar{z} \\ c\,\mathrm{d}\,t &= \gamma\left(c\,\mathrm{d}\,\bar{t} + \beta\,\mathrm{d}\,\bar{x}\right) \end{aligned}$$



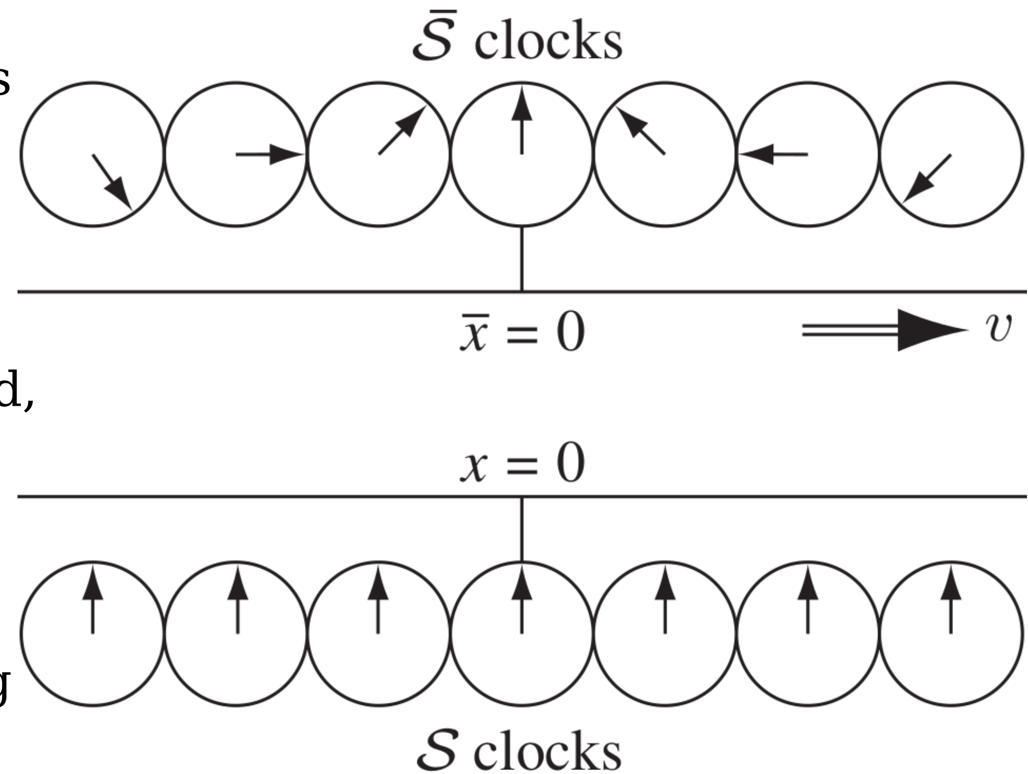
$\bar{y}$
y
Space Length
$\Delta \ell^2 = \Delta x^2 + \Delta y^2$
$= \Delta \bar{x}^2 + \Delta \bar{y}^2$
$\Delta \ell$
$\bar{x}$
x
Spacetime interval
$\Delta s^2 = \Delta x^2 - c^2 \Delta t^2$
$= \Delta \bar{x}^2 - c^2 \Delta \bar{t}^2$
$c\,t$
$c\,\bar{t}$
$x = c\,t$
$\bar{x} = c\,\bar{t}$
Δs
$x = -c\,t$
$\bar{x} = -c\,\bar{t}$
$\bar{x}$
x

Simultaneity, synchronization, and time dilation

- Suppose event A occurs at $x_A=0$, $t_A=0$, and event B occurs at $x_B=b$, $t_B=0$. The 2 events are simultaneous in S. But they are *not* simultaneous in $\bar{S}$ for the Lorentz transformations give $\bar{x}_A=0$, $\bar{t}_A=0$ and $\bar{x}_B=\gamma b$, $c\bar{t}_B=-\gamma\beta b$

- According to the $\bar{S}$ clocks, B occurred *before* A—the relativity of simultaneity.

- Suppose that at $t=0$ observer S decides to examine *all* the clocks in $\bar{S}$. He finds that they read *different* times, depending on their location $c\bar{t}=-\gamma\beta x$.



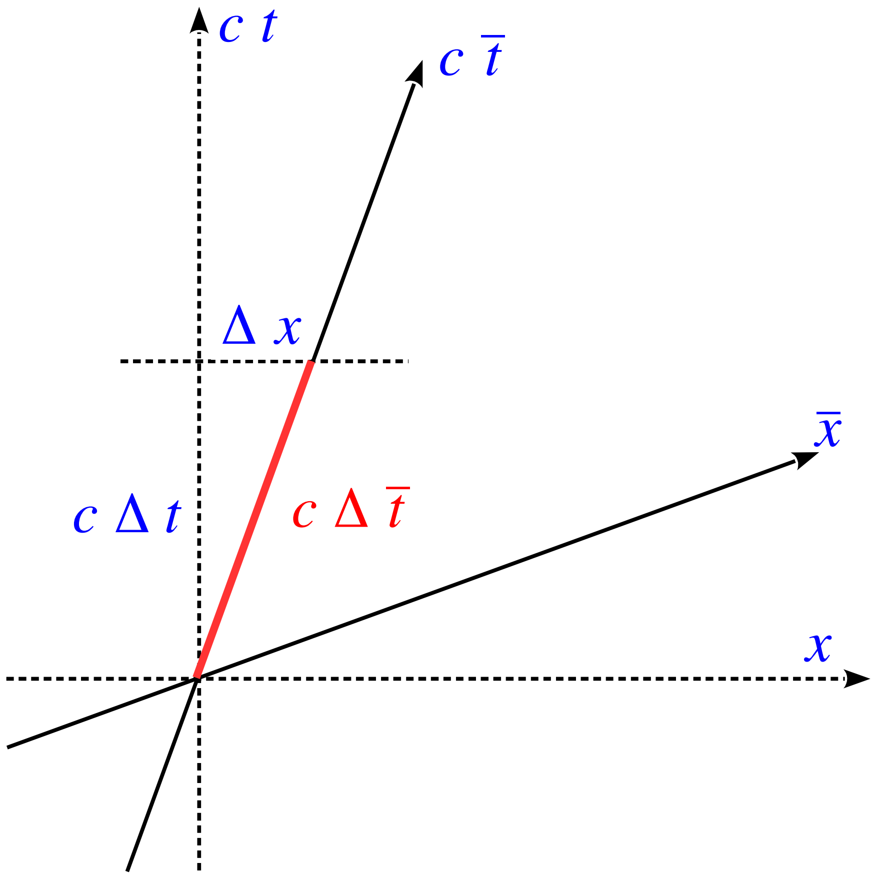
- Those to the left of the origin are ahead, and those to the right are behind, by an amount that increases $\propto$ distance. Only the master clock at the origin $\bar{t}=0$.

- Thus the nonsynchronization of moving clocks follows directly from the Lorentz transformations. And from $\bar{S}$ it is the S clocks that are out of synchronization.

- Let S focus his attention on a single clock at rest in the $\bar{S}$ frame (eg, $\bar{x}=a$), and watches it over some interval Δt. Then

$$\Delta\bar{x}=0 \quad\Rightarrow\quad \Delta t=\gamma\Delta\bar{t} \;\text{ time dilation } \Leftarrow \quad \gamma\geq 1 \quad\Rightarrow\quad \boxed{\Delta\bar{t}=\frac{\Delta t}{\gamma}}$$

Time Dilation



$$\Delta\,\bar{x} = 0 \quad \Rightarrow \quad \Delta\,x = v\,\Delta\,t$$

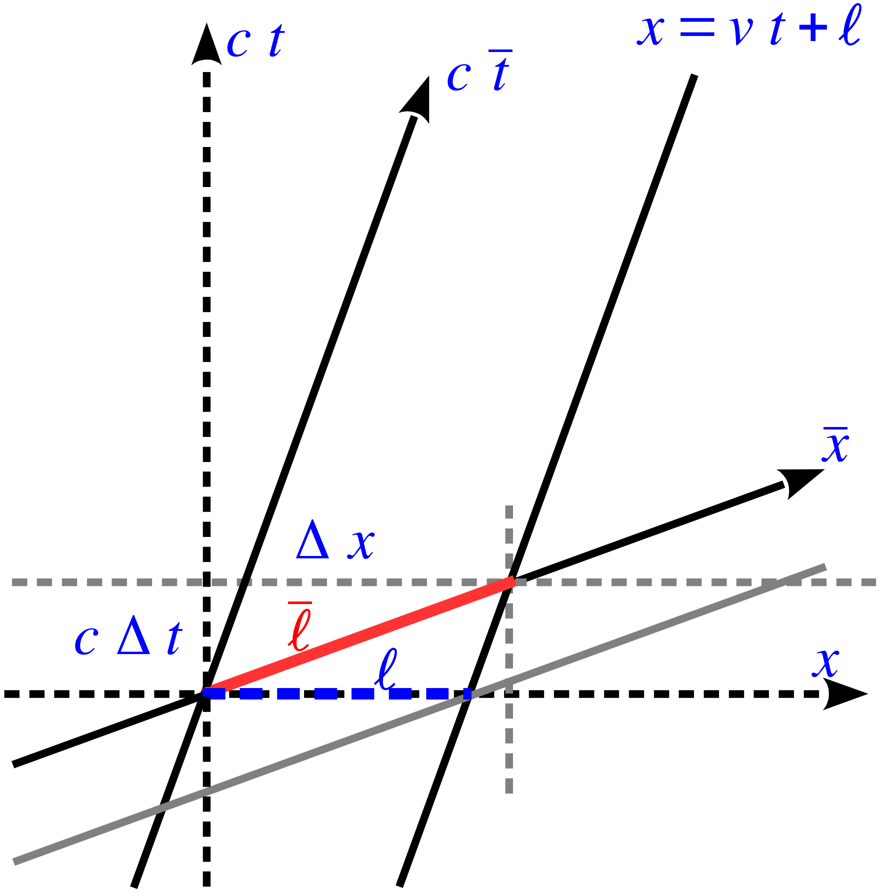
$$\Rightarrow \quad \Delta\,s^2 = -c^2\,\Delta\,\bar{t}^2 = -c^2\,\Delta\,t^2 + \Delta\,x^2$$

$$\Rightarrow \quad \Delta\,\bar{t}^2 = \Delta\,t^2\left[1 - \frac{1}{c^2}\left(\frac{\Delta\,x}{\Delta\,t}\right)^2\right]$$

$$= \Delta\,t^2\left(1 - \frac{v^2}{c^2}\right) = \frac{\Delta\,t^2}{\gamma^2}$$

$$\Rightarrow \quad \Delta\,t = \gamma\,\Delta\,\bar{t} \quad \Rightarrow \quad \Delta\,t \geq \Delta\,\bar{t}$$

Length Contraction



$$\Delta\,\bar{t} = 0 \quad \Rightarrow \quad c\,\Delta\,t = \beta\,\Delta\,x$$

$$\Rightarrow \quad \Delta\,s^2 = \Delta\,\bar{x}^2 = -\,c^2\,\Delta\,t^2 + \Delta\,x^2$$

$$\left[\begin{array}{r} c\,\Delta\,t = \beta\,\Delta\,x \\ \Delta\,x = \ell + v\,\Delta\,t \end{array}\right. \quad \Rightarrow \quad \left[\begin{array}{r} \Delta\,x = \gamma^2\,\ell \\ c\,\Delta\,t = \gamma^2\,\beta\,\ell \end{array}\right.$$

$$\Rightarrow \quad \bar{\ell}^2 = -\,\gamma^4\,\beta^2\,\ell^2 + \gamma^4\,\ell^2$$
$$= \ell^2\,\gamma^4\left(1 - \beta^2\right) = \gamma^2\,\ell^2$$

$$\Rightarrow \quad \bar{\ell} = \gamma\,\ell \quad \Rightarrow \quad \bar{\ell} \geq \ell$$

Lorentz contraction

- Let a stick at rest in $\bar{S}$ (moving to the right at speed v in S). Its rest length (as measured in $\bar{S}$) is $\Delta \bar{x} = \bar{x}_R - \bar{x}_L$.

- If an observer in S were to measure the stick $\Delta x = x_R - x_L$ at *his* time t,

$$\boxed{\Delta x = \frac{\Delta \bar{x}}{\gamma}} \quad \text{Lorentz contraction} \quad \Leftarrow \quad \Delta \bar{x} = \gamma \left(\Delta x - v \Delta t \right) \; + \; \Delta t = 0$$

- Note that S's t hold fixed here, because we're talking about a measurement made by S, and he marks off the 2 ends at the same instant of his time.

Einstein's velocity addition rule

- Let the particle move with constant velocity parallel to the x & $\bar{x}$ axes, send out 2 signals as it moves. Each observer measures the space interval and the time interval between these 2 events.

$$\begin{aligned} \Delta x &= \gamma \left(\Delta \bar{x} + v \Delta \bar{t} \right) \\ \Delta t &= \gamma \left(\Delta \bar{t} + \frac{v}{c^2} \Delta \bar{x} \right) \end{aligned} \quad \Rightarrow \quad \frac{\Delta x}{\Delta t} = \frac{\Delta \bar{x} + v \Delta \bar{t}}{\Delta \bar{t} + \frac{v}{c^2} \Delta \bar{x}} = \frac{\frac{\Delta \bar{x}}{\Delta \bar{t}} + v}{1 + \frac{v}{c^2} \frac{\Delta \bar{x}}{\Delta \bar{t}}}$$

To the limit $u \equiv \dfrac{dx}{dt}$ the velocity of the particle in S , $\bar{u} \equiv \dfrac{d\bar{x}}{d\bar{t}}$ the velocity of the particle in $\bar{S}$

$$\Rightarrow \quad \boxed{u = \frac{\bar{u} + v}{1 + \bar{u} v / c^2}} \quad \Rightarrow \quad \beta_u = \frac{\beta_{\bar{u}} + \beta}{1 + \beta \beta_{\bar{u}}} \quad \Leftarrow \quad \beta_u \equiv \frac{u}{c}, \quad \beta_{\bar{u}} \equiv \frac{\bar{u}}{c}, \quad \beta = \beta_v \equiv \frac{v}{c}$$

The Geometry of Relativity

- Present thought experiments to introduce the 3 most striking geometrical consequences of Einstein's postulates: time dilation, Lorentz contraction, and the relativity of simultaneity.

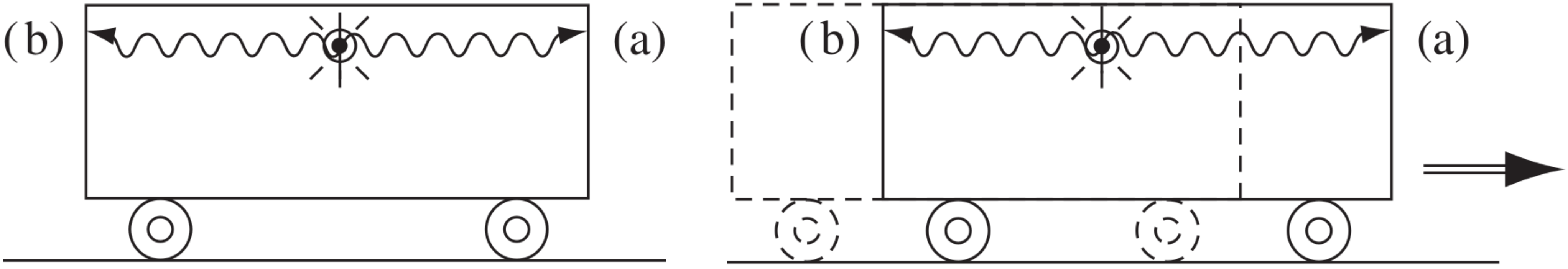
(i) The relativity of simultaneity

- The 2 events observed from the car—(a) light reaches the front end and (b) light reaches the back end—occur *simultaneously*.

- To an observer on the *ground* these same 2 events are *not* simultaneous. For as the light travels out from the bulb, the train moves forward, so the beam going to the back end has a shorter distance to travel than the one going forward. Thus, event (b) happens *before* event (a).

- An observer passing by on an express train would report that (a) preceded (b).

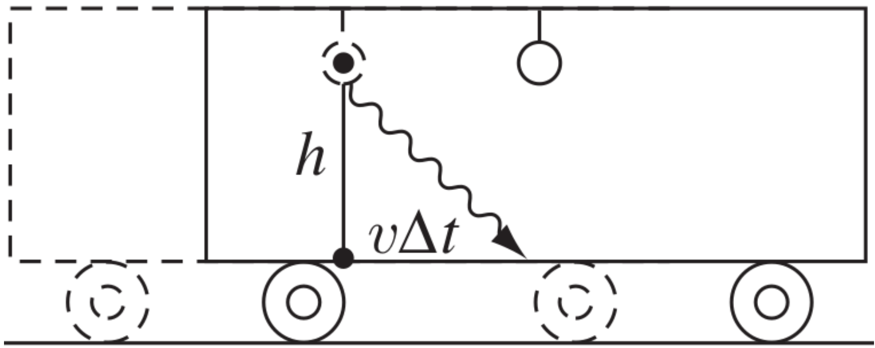
- *Conclusion*: **2 events that are simultaneous in one inertial system are not, in general, simultaneous in another.**



- An **observation** means that the correction of the time for the signal travel to the observer is made, and then records after doing so. So all the data shouldn't depend on where the observer is located.

(ii) Time dilation

- Consider a light ray that leaves the bulb and strikes the floor of the car directly below.



- From an observer on the train, the time is $\Delta \bar{t} = \dfrac{h}{c}$

- As observed from the ground, this same ray must travel farther, because the train itself is moving:

$$\Delta t = \frac{\sqrt{h^2 + (c\,\Delta t)^2}}{c} \;\Rightarrow\; \Delta t = \frac{h}{c}\frac{1}{\sqrt{1 - v^2/c^2}} = \gamma\,\Delta \bar{t} \;\Rightarrow\; \Delta \bar{t} = \frac{\Delta t}{\gamma}$$

- The time elapsed between the *same 2 events* is different for the 2 observers. In fact, the interval recorded on the train clock $\bar{t}$, is shorter by the factor γ.

- *Conclusion*: **Moving clocks run slow—time dilation.**

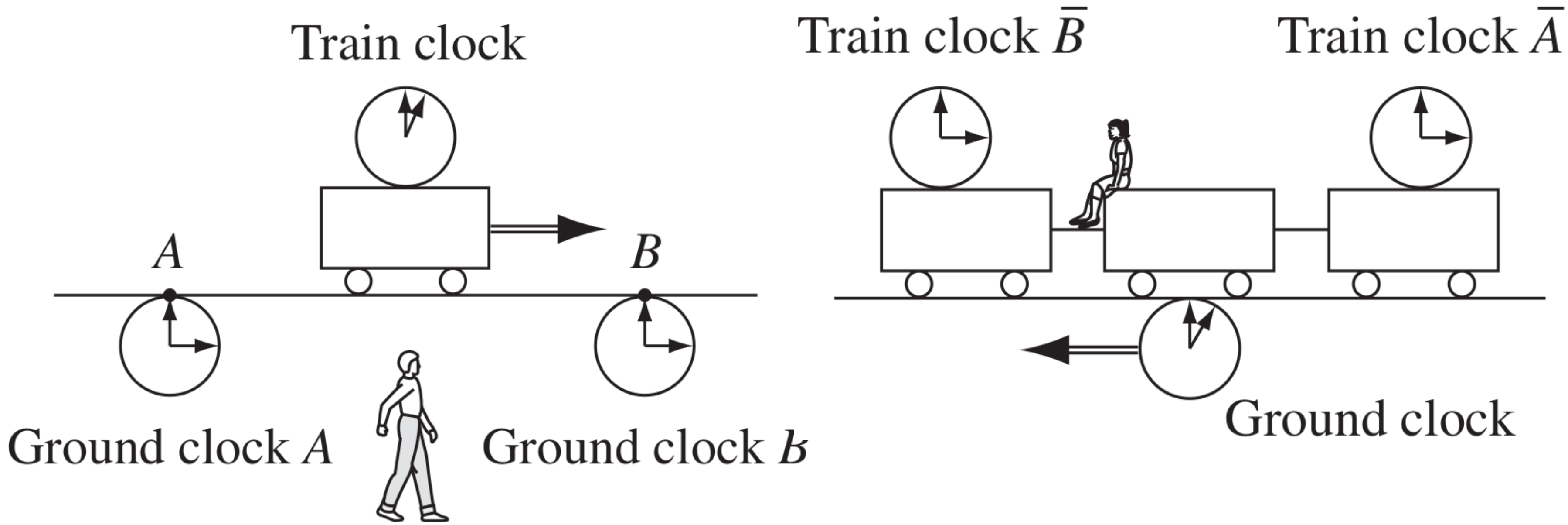
- It doesn't have anything to do with the mechanics of clocks; it's a statement about the nature of time, and works for any clock.

- When particles are moving at speeds close to c they last much longer, for their internal clocks are running slow, in accordance with Einstein's time dilation.

Example 12.1: A muon is traveling through the laboratory at 3/5 of the speed of light. How long does it last?

- The lifetime of a muon $\bar{\tau} = 2\times 10^{-6}\,\text{s} \;\Rightarrow\; \tau = \gamma\,\bar{\tau} = 2.5\times 10^{-6}\,\text{s} > \bar{\tau}$

- Time dilation seems inconsistent with the principle of relativity. But it doesn't. The observations from different frames are *all* correct.



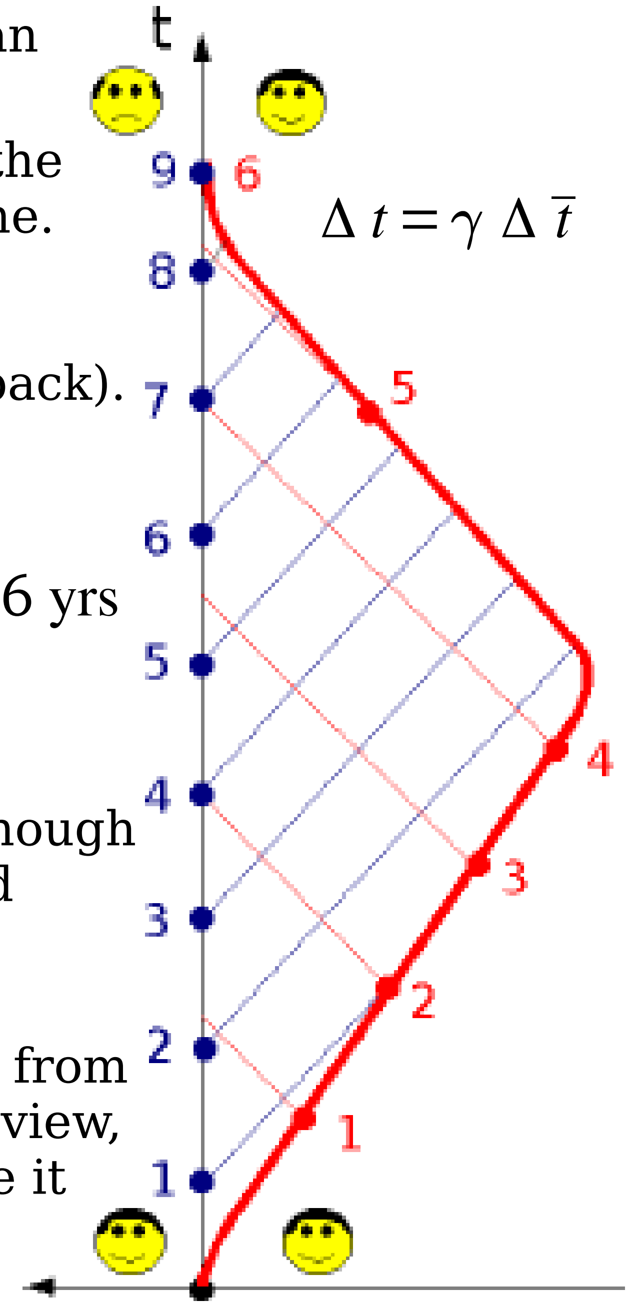
- There is *no* contradiction since the 2 observers have measured *different things*.

- *Clocks that are properly synchronized in one system will not be synchronized when observed from another system.*

- Whereas each observer conducted a perfect sound measurement from his own point of view, the other observer considers that he used 2 unsynchronized clocks.

- Because moving clocks are not synchronized, you can use as many stationary clocks (to you) as you please, for they are properly synchronized.

<u>Example 12.2</u>: **The twin paradox**. On her 21st birthday, an astronaut takes off in a rocket ship at a speed of $12c/13$. After 5 years on her watch, she turns and heads back at the same speed to rejoin her twin brother, who stayed at home. How old is each twin at their reunion?



- The traveling twin has aged 10 years (5 yrs out+5 yrs back).

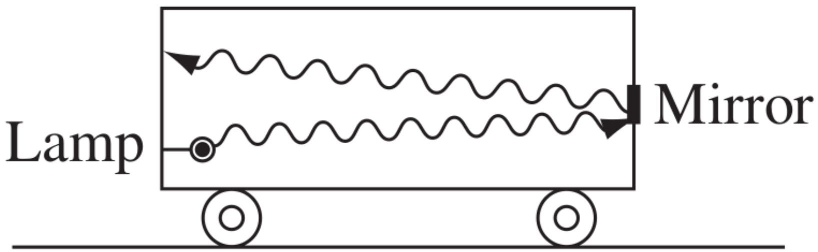
- As viewed from earth, the moving clock runs slow by

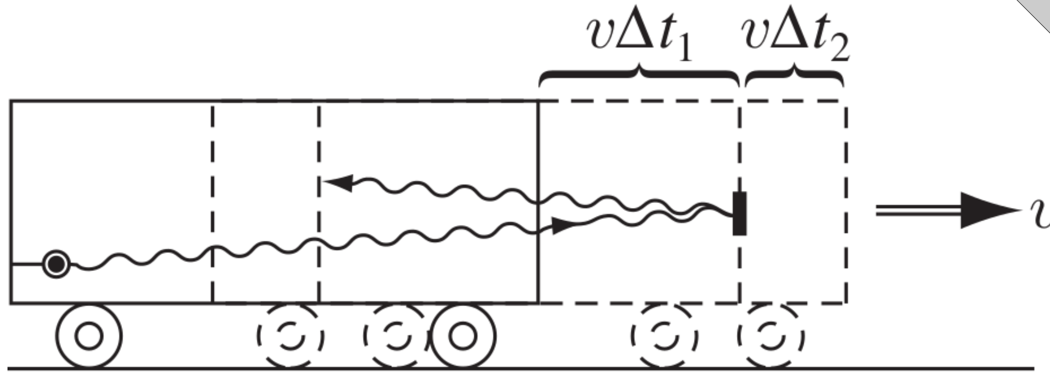
$$\gamma = \frac{1}{\sqrt{1-(12/13)^2}} = \frac{13}{5} \;\Rightarrow\; \begin{matrix}\text{the elapsed}\\ \text{time on earth}\end{matrix} = \gamma \times 10 \text{ yrs} = 26 \text{ yrs}$$

Her brother is now 16 years older than her!

- This is no fountain of youth for the traveling twin, for though she may die later than her brother, she will not have lived any *more*—she's just done it *slower*.

- The **twin paradox** arises when you try to tell this story from the point of view of the traveling twin. From her point of view, she's at rest, whereas her brother is in motion, and hence it is he who should be younger at the reunion. ✕





- The two twins are not equivalent. The traveling twin experiences *acceleration* when she turns around to head home, but her brother does not.

- The *traveling twin cannot claim to be a stationary observer* because you can't undergo acceleration and remain stationary.

(iii) Lorentz contraction

- To an observer on the train, the time for the signal to complete the round trip is $\Delta \bar{t} = 2\dfrac{\Delta \bar{x}}{c}$

- To a ground observer, the process is complicated because of the moving of train

$$\Delta t_1 = \frac{\Delta x + v\Delta t_1}{c}, \quad \Delta t_2 = \frac{\Delta x - v\Delta t_2}{c} \quad \Rightarrow \quad \Delta t_1 = \frac{\Delta x}{c-v}, \quad \Delta t_2 = \frac{\Delta x}{c+v}$$

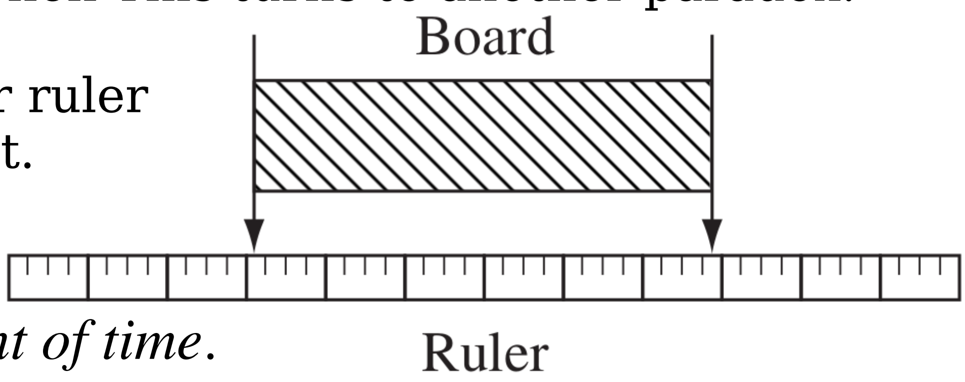
$$\Rightarrow \quad \Delta t = \Delta t_1 + \Delta t_2 = 2\frac{\Delta x}{c}\frac{1}{1-v^2/c^2} \quad \Rightarrow \quad \Delta x = \frac{\Delta \bar{x}}{\gamma} \quad \Leftarrow \quad \Delta t = \gamma \Delta \bar{t}$$

- The length of the boxcar is not the same when measured by an observer on the ground, as it is when measured by an observer on the train—from the ground point of view, it is somewhat *shorter*.

- *Conclusion*: **Moving objects are shortened—Lorentz contraction.**

- Moving clocks run slow, moving sticks are shortened, and the factor is γ.

- The observer on the train doesn't think her car is shortened—her meter sticks are contracted by that same factor, so all her measurements come out the same as when the train was standing in the station. In fact, from her point of view it is objects on the ground that are shortened. Then This turns to another paradox.



- To find the length of a board, you lay your ruler next to the board and measure, if it's at rest.

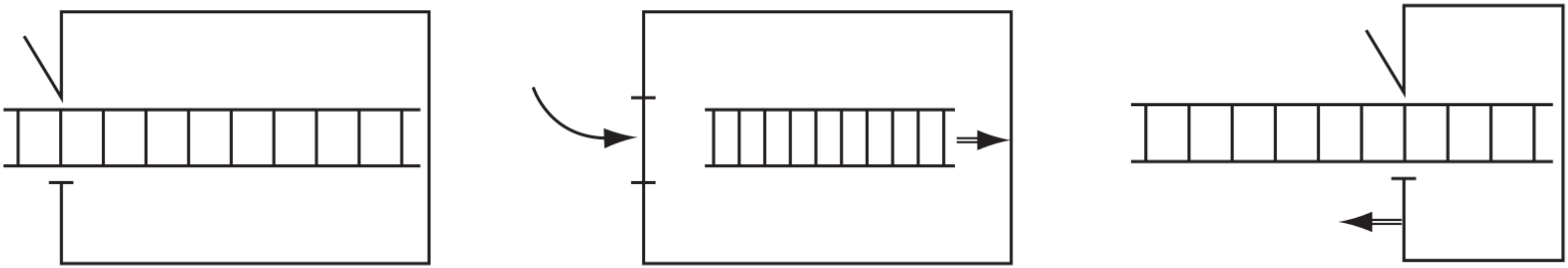
- If the board is *moving*, you do the same thing and read the 2 ends *at the same instant of time*.

- Here is the problem: Because of the relativity of simultaneity the 2 observers disagree on what constitutes "the same instant of time."

- When the person on the ground measures the length of the boxcar, he reads the position of the 2 ends at the same instant *in his system*. But the person on the train, watching him do it, complains that he read the front end first, then waited a moment before reading the back end. *Naturally*, he came out short.

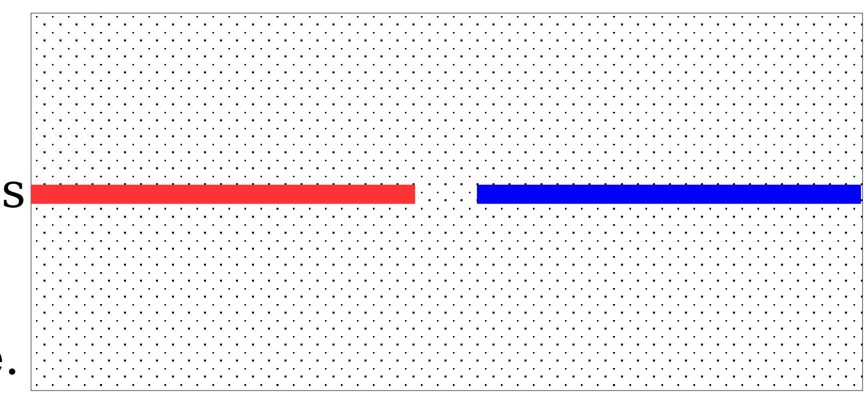
- Yet there is no inconsistency, for they are measuring different things, and each considers the other's method improper.

The barn and ladder paradox

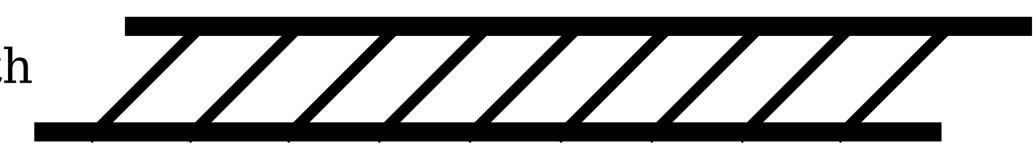


- A moving object is shortened *only along the direction of its motion*:
  Dimensions perpendicular to the velocity are not contracted.

- Taylor & Wheeler's thought experiment



- If the rule were that perpendicular directions contract, the person on the ground would predict that the red line is lower, while the person on the train would say it's the blue one.



- The principle of relativity says that both observers are equally justified, but they cannot both be right.

- No simultaneity or synchronization can rationalize this contradiction; either the blue line is higher or the red one is—*unless they exactly coincide*.

The Structure of Spacetime

(i) Four-vectors

- Change the unit of time from the *second* to the *meter* to make the Lorentz transformation simple: $x^0 \equiv c\,t\,,\quad x^1 = x\,,\quad x^2 = y\,,\quad x^3 = z$

$$\Rightarrow \begin{aligned} \bar{x}^0 &= \gamma\left(x^0 - \beta\,x^1\right) \\ \bar{x}^1 &= \gamma\left(x^1 - \beta\,x^0\right) \\ \bar{x}^2 &= x^2 \\ \bar{x}^3 &= x^3 \end{aligned} \quad \Rightarrow \begin{bmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\,\beta & 0 & 0 \\ -\gamma\,\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \quad \Rightarrow \bar{x}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu\, x^\nu$$

$$\Rightarrow \vec{\bar{x}} = \mathbf{\Lambda}\,\vec{x}$$

$\mathbf{\Lambda}$: Lorentz transformation matrix

- In this abstract manner we can handle in the same format a more general transformation, in which the relative motion is not along a common $x\ \bar{x}$ axis.

- Define a **4-vector** as any set of 4 components that transform in the same manner as (x^0, x^1, x^2, x^3) under Lorentz transformations:

$$\bar{a}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu\, a^\nu \quad \Rightarrow \begin{aligned} \bar{a}^0 &= \gamma\left(a^0 - \beta\,a^1\right) \\ \bar{a}^1 &= \gamma\left(a^1 - \beta\,a^0\right) \\ \bar{a}^2 &= a^2 \\ \bar{a}^3 &= a^3 \end{aligned}$$

as the case of a transformation along the x axis

- **4d scalar product**: $\vec{a}\cdot\vec{b} \equiv -a^0\,b^0 + a^1\,b^1 + a^2\,b^2 + a^3\,b^3$

- The scalar product has the same value in all inertial systems:

$$\vec{\bar{a}} \cdot \vec{\bar{b}} = \vec{a} \cdot \vec{b} \quad \Rightarrow \quad -\bar{a}^0 \bar{b}^0 + \bar{a}^1 \bar{b}^1 + \bar{a}^2 \bar{b}^2 + \bar{a}^3 \bar{b}^3 = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

- As the 3d dot product is **invariant** (unchanged) under rotations, this combination is invariant under Lorentz transformations.

- To keep track of the "−" sign, it is convenient to introduce the **covariant** vector a_μ, different from the **contravariant** a^μ only in the sign of the 0th component:

$$a_\mu = (a_0, a_1, a_2, a_3) \equiv (-a^0, a^1, a^2, a^3)$$

- *Upper/Lower* indices designate *contravariant/covariant* vectors.

- Formally, $a_\mu = \sum_{\nu=0}^{3} \eta_{\mu\nu} a^\nu \quad \Leftarrow \quad \eta_{\mu\nu} \equiv \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Minkowski metric

$$\Rightarrow \quad \vec{a} \cdot \vec{b} = \sum_{\mu=0}^{3} a_\mu b^\mu \quad \Rightarrow \quad a_\mu b^\mu \quad \text{in a compact way}$$

- Summation is *implied* whenever an index is repeated in a product—once as a covariant index and once as contravariant—**Einstein summation convention**.

- $a_\mu b^\mu = a^\mu b_\mu = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$

(ii) The invariant interval

- The scalar product of a 4-vector with *itself*, $a_\mu a^\mu = -(a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2$

$$\text{If } a_\mu a^\mu > 0, \quad a^\mu \text{ is called } \textit{spacelike},$$
$$\text{If } a_\mu a^\mu < 0, \quad a^\mu \text{ is called } \textit{timelike},$$
$$\text{If } a_\mu a^\mu = 0, \quad a^\mu \text{ is called } \textit{lightlike}.$$

- Let event A occurs at $(x_A^0, x_A^1, x_A^2, x_A^3)$, and event B at $(x_B^0, x_B^1, x_B^2, x_B^3)$. The difference, $\Delta x^\mu = x_A^\mu - x_B^\mu$ is the **displacement 4-vector**.

- The scalar product of Δx^μ with itself isthe **invariant interval** between 2 events:

$$I \equiv \Delta x_\mu \Delta x^\mu = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = -c^2 t^2 + d^2$$

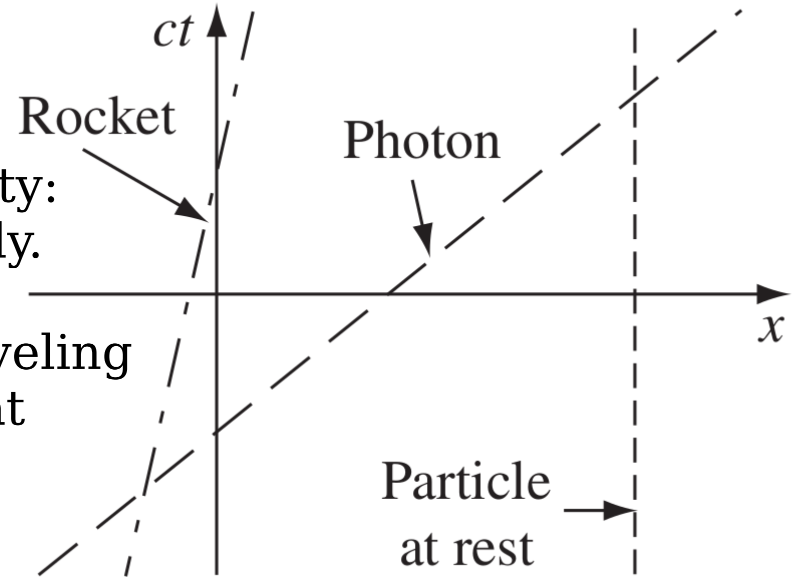
- When you transform to a moving system, the *time* between A & B is altered $\bar{t} \neq t$ and so is the *spatial separation* $\bar{d} \neq d$, but the interval I remains the same.

- If the displacement between 2 events is timelike ($I<0$), there exists an inertial system (using Lorentz transformation) in which they occur at the same point. You cannot do this for a *spacelike* interval because v would have to be greater than c, and no observer can exceed the speed of light.

- If I is spacelike ($I>0$), there exists a system in which the 2 events occur at the same time.

- If I is lightlike ($I=0$), the 2 events could be connected by a light signal.

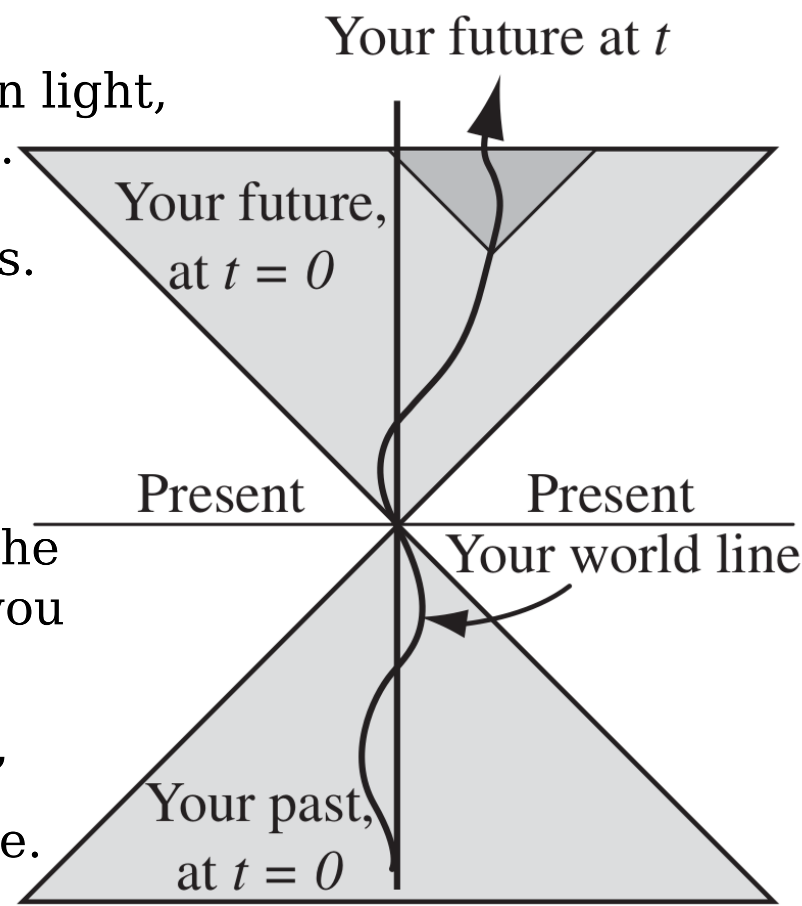
(iii) Space-time diagrams



- The convention in a graph is reversed in relativity: people plot position horizontally and time vertically.

- A particle at rest is a vertical line; a photon, traveling at the speed of light, is a 45° line; a rocket going at some intermediate speed follows a line of slope $c/v=1/\beta$ —**Minkowski diagrams**.

- The trajectory of a particle on a Minkowski diagram is called a **world line**.



- Because no material object can travel faster than light, your world line can never have a slope less than 1. Accordingly, your motion is restricted to the wedge-shaped region bounded by the two 45° lines.

- We call this your "future," in the sense that it is the locus of all points accessible to you.

- the *backward* wedge represents your "past," in the sense that it is the locus of all points from which you might have come.

- As for the rest, this is the generalized "present." You can't *get* there, and you didn't *come* from there.

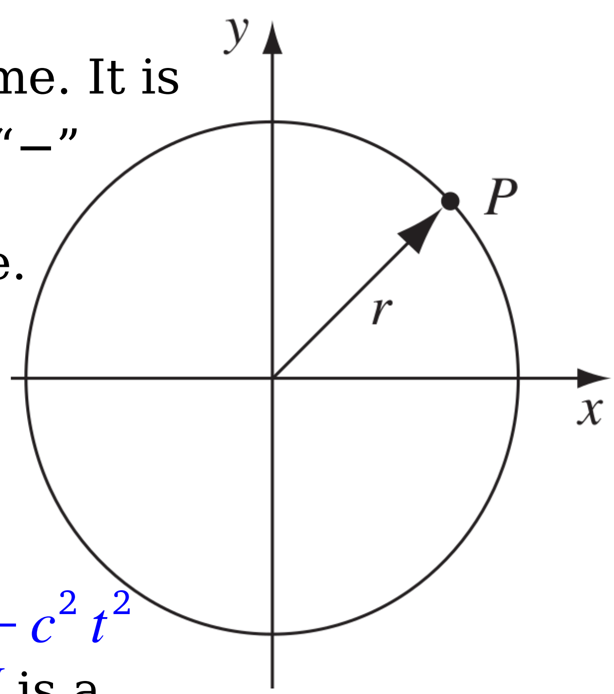
- There's no way you can influence any event in the present; it's a vast expanse of spacetime that is absolutely inaccessible to you.

- Because the boundaries are the trajectories of light rays, we call them the **forward light cone** and the **backward light cone**. Your future lies within your forward light cone, your past within your backward light cone.

- The slope of the line connecting 2 events on a space-time diagram tells you whether the displacement between them is timelike, spacelike, or lightlike.

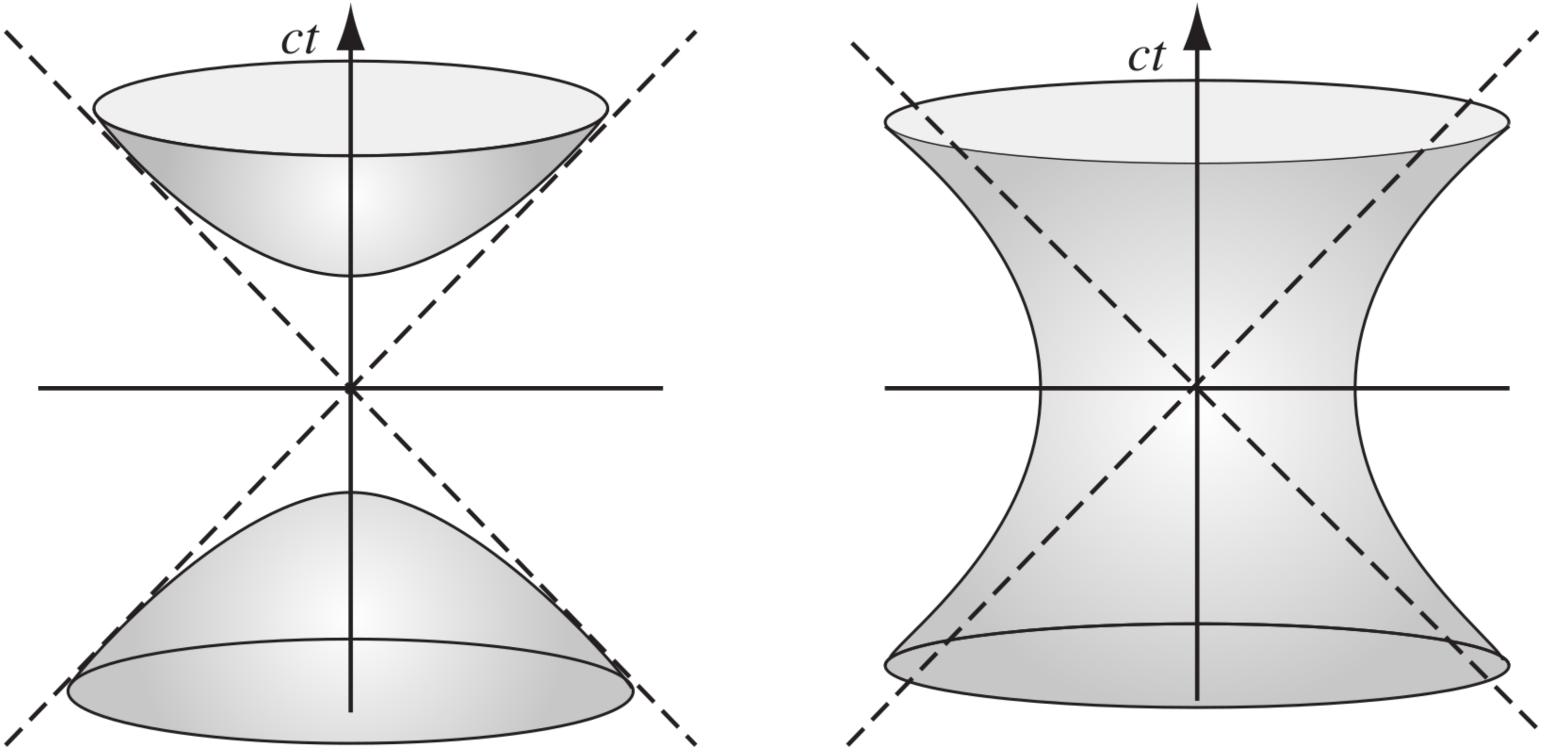
- All points in the past & future are timelike to your present location, whereas points in the present are spacelike, and points on the light cone are lightlike.

- Time is not just another coordinate, as x, y, z in spacetime. It is *utterly different* from the others with its distinction of the "−" sign in the invariant interval. The minus sign imparts to spacetime a hyperbolic geometry instead of a circular one.



- Under rotations a point P in the xy plane describes a *circle*: the locus of all points a fixed distance $r = \sqrt{x^2 + y^2}$ from the origin.

- Under Lorentz transformations, it is the interval $I = x^2 - c^2 t^2$ that is preserved, and the locus of all points with a fixed I is a *hyperbola*—or, a *hyperboloid of revolution*.



- When the displacement is *timelike*, it's a "hyperboloid of two sheets"; when the displacement is *spacelike*, it's a "hyperboloid of one sheet."

- When you perform a Lorentz transformation, the coordinates (x,t) of a given event will change to $\left(\bar{x}\ ,\ \bar{t}\ \right)$, but these new coordinates will lie on the same hyperbola as (x,t).

- It is impossible to make a transformation from the upper sheet of the timelike hyperboloid to the lower sheet, or to a spacelike hyperboloid.

- *If the displacement 4-vector between 2 events is timelike, their ordering is absolute; if the interval is spacelike, their ordering depends on the inertial system from which they are observed.*

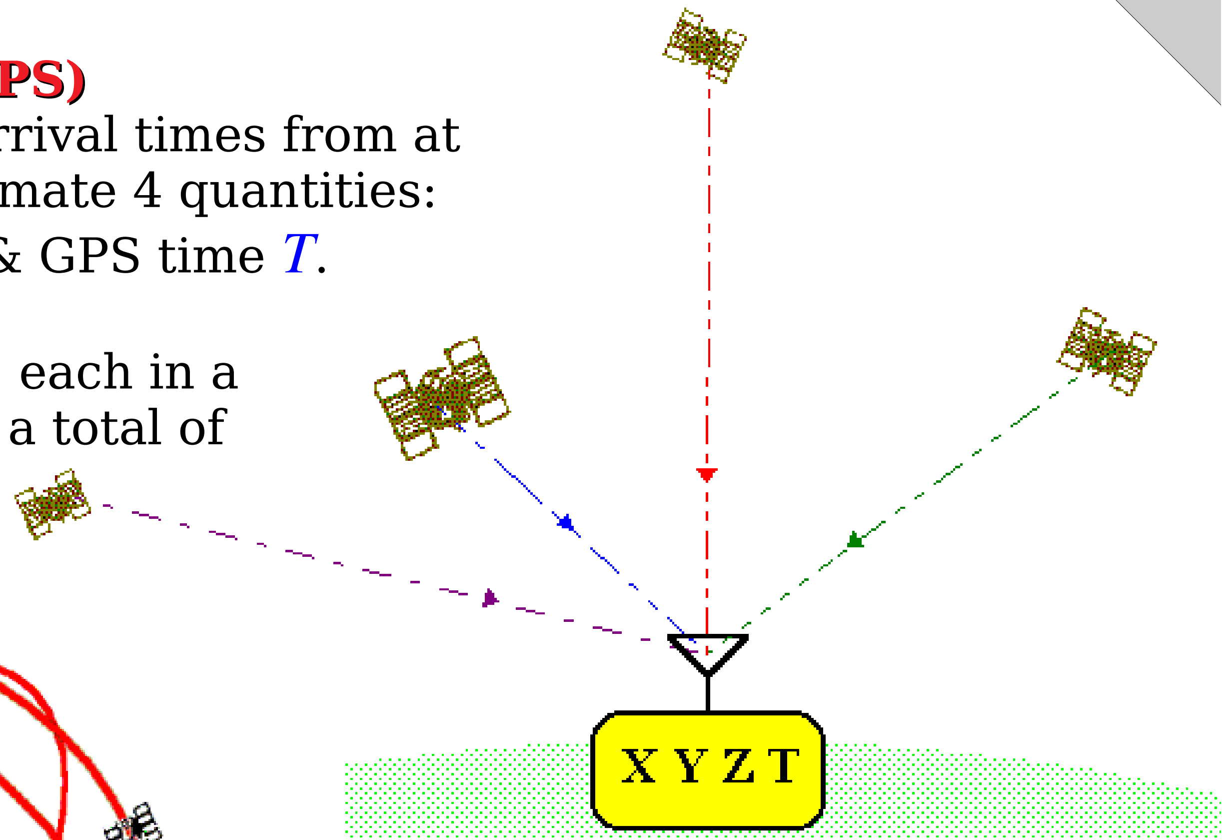
- An event on the upper sheet of a timelike hyperboloid definitely occurred after (0,0), and one on the lower sheet certainly occurred before; it is the notion of **causality**, on which all physics is based.

- Causality is preserved if the 2 events are timelike or lightlike seperated.

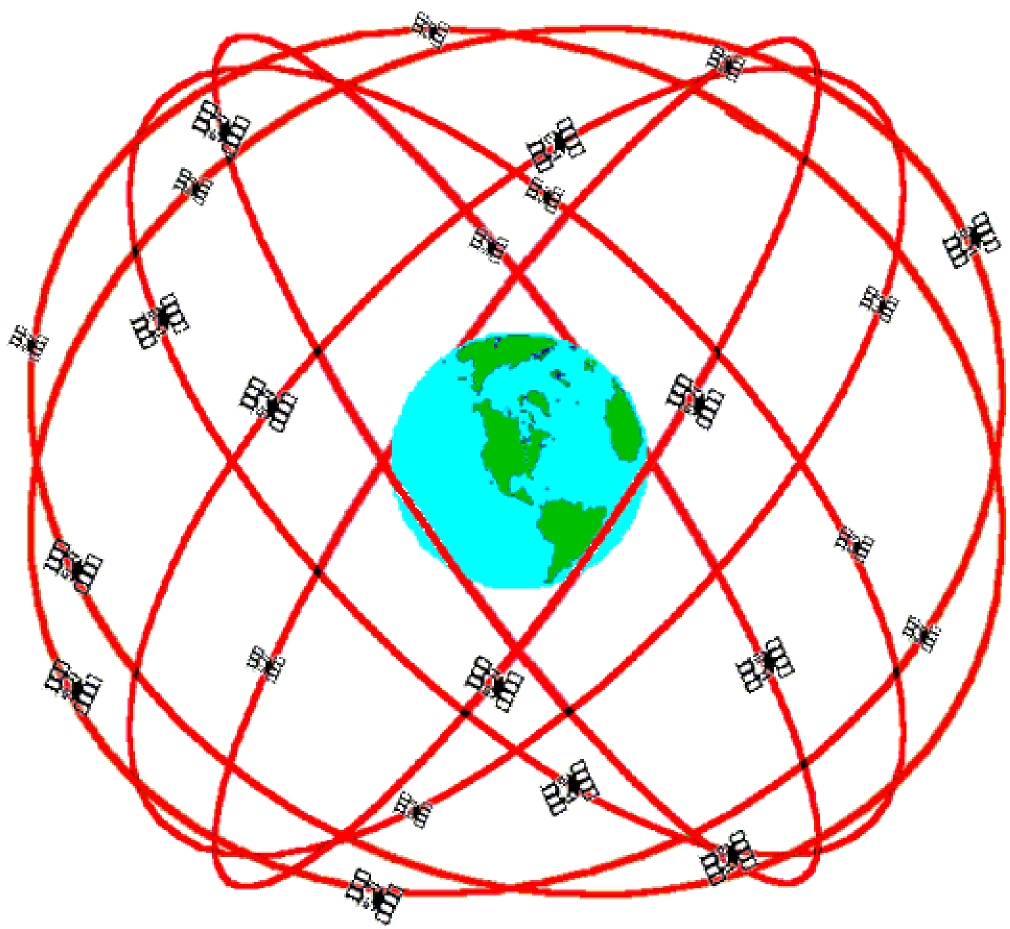
- *Conclusion*: The displacement between causally related events is always timelike, and their temporal ordering is the same for all inertial observers.

Global Positioning System (GPS)

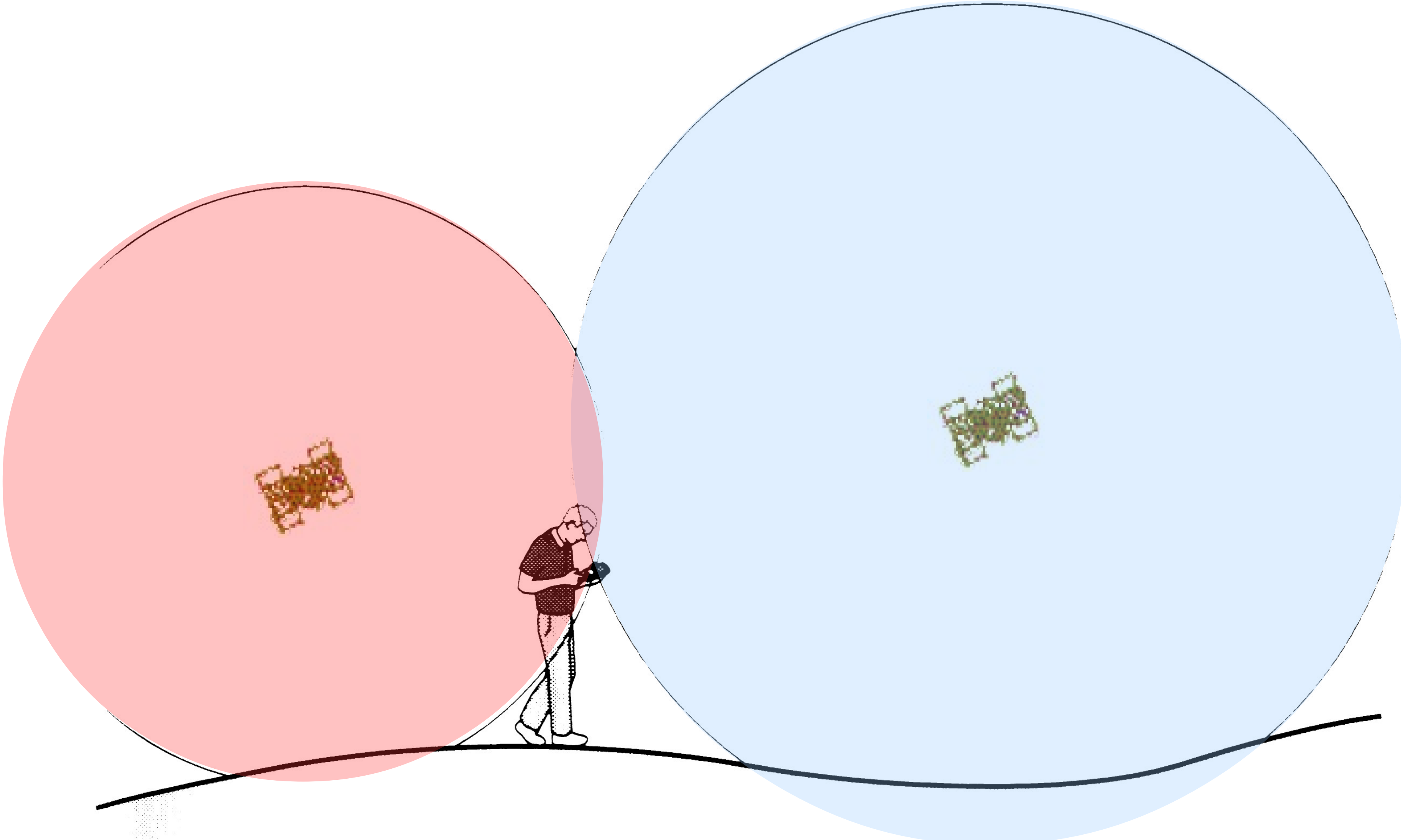
- Measurement of code-phase arrival times from at least 4 satellites are used to estimate 4 quantities: position in 3 dimension (x, y, z) & GPS time T.



- A constellation of 24 satellites, each in a 12-hour orbit about the Earth in a total of 6 orbital planes.



- Each satellite carries accurate atomic clocks that keep proper time on a satellite to accuracy up to 10^{-13} over a few weeks.



- A GPS satellite emits a signal encoded with its time of emission, t_e, and the location of the satellite. An observer who receives the signal at a time t_r that is an interval $\Delta t = t_r - t_e$ later knows that he or she is located somewhere on a sphere of radius $c\Delta t$ centered on the satellite. Signals from 2 satellites narrow the location down to the intersection of 2 spheres.

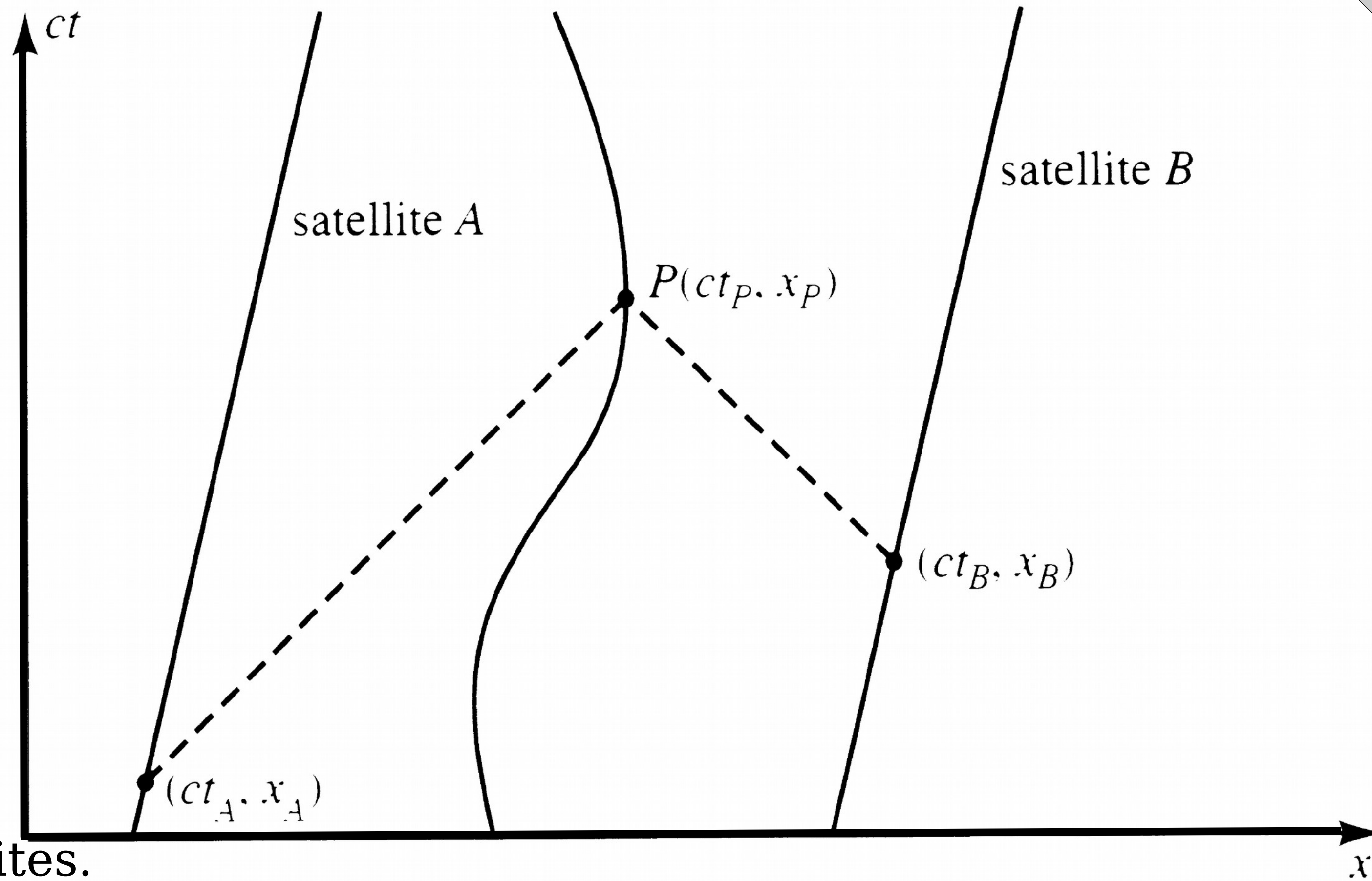
- With 4 satellites, the observer's position in spacetime can be fixed.

- In 1+1 spacetime, the signals from 2 satellites are sufficient to locate a point P in spacetime where they are received simultaneously:

$$ct_P = \frac{c\,t_B + x_B + c\,t_A - x_A}{2}$$

$$x_P = \frac{c\,t_B + x_B - c\,t_A + x_A}{2}$$

- In a 4d spacetime, a spacetime point can be similarly located with the signals from 4 satellites.



Corrections due to the relativistic effects

- Proper time on the satellite clocks has to be corrected to give the time of the inertial frame on the Earth for 2 reasons:

(1) time dilation of special relativity; (SR)

(2) the effects of the Earth's gravitational field. (GR)

- Suppose the radius of the orbit is R_s, then the satellite's speed V_s is

$$\frac{V_s^2}{R_s} = \frac{G\,M_E}{R_s^2}$$

where $R_s \approx 2.7 \times 10^4 \text{ km} \approx 4.2\, R_E = 4.2 \times 6.4 \times 10^3 \text{ km}$

$$\Rightarrow \quad V_s \approx 3.9 \text{ km/s} \quad \Rightarrow \quad \frac{V_s}{c} \approx 1.3 \times 10^{-5}$$

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{V_s^2}{c^2}} \quad \Rightarrow \quad \left(\begin{array}{c} \text{fractional correction in} \\ \text{rate for time dilation} \end{array} \right) \approx \frac{1}{2} \frac{V_s^2}{c^2} \approx 0.84 \times 10^{-10}$$

$$\frac{d\tau_s}{d\tau_E} \approx 1 + \frac{\dfrac{G\,M_E}{R_E} - \dfrac{G\,M_E}{R_s}}{c^2} \quad \Rightarrow \quad \left(\begin{array}{c} \text{fractional correction} \\ \text{in rate for the} \\ \text{gravitational potential} \end{array} \right) \approx \frac{G\,M_E}{R_s\, c^2} \approx 1.6 \times 10^{-10}$$

- The gravitational correction is bigger than the correction for time dilation.

- The GPS is a practical application of both SR & GR.

- Further corrections: rotating frame of the Earth, the relativistic Doppler effect, the relativity simultaneity, the Earth's rotation, the asphericity of the Earth's gravitational potential, the time delays from the index of refraction of the Earth's ionosphere, satellite clock errors, etc.

Relativistic Mechanics

Proper Time and Proper Velocity

- As you progress along your world line, your watch runs slow; while the clock on the wall ticks off an interval $\mathrm{d}\,t$, your watch only advances $\mathrm{d}\,\tau$: $\mathrm{d}\,\tau = \sqrt{1-u^2/c^2}\,\mathrm{d}\,t$ where u is your velocity. The time τ your watch registers is called **proper time**.

- τ is a more useful quantity than t, since proper time is invariant, whereas "ordinary" time depends on the particular reference frame you have in mind.

- The **ordinary velocity**: $\mathbf{u} = \frac{\mathrm{d}\,\boldsymbol{\ell}}{\mathrm{d}\,t}$, $\mathrm{d}\,t$ is measured by an arbitrary observer.

- The **proper velocity**: $\mathbf{U} \equiv \frac{\mathrm{d}\,\boldsymbol{\ell}}{\mathrm{d}\,\tau} \quad \Rightarrow \quad \mathbf{U} = \frac{\mathrm{d}\,t}{\mathrm{d}\,\tau}\frac{\mathrm{d}\,\boldsymbol{\ell}}{\mathrm{d}\,t} = \gamma_u\,\mathbf{u} \quad \Leftarrow \quad \gamma_u = \frac{1}{\sqrt{1-u^2/c^2}}$

- From a theoretical standpoint, proper velocity has an advantage over ordinary velocity: it transforms simply, when you go from one inertial system to another:

$$U^\mu \equiv \frac{\mathrm{d}\,x^\mu}{\mathrm{d}\,\tau} \quad \Leftarrow \quad U^0 = \frac{\mathrm{d}\,x^0}{\mathrm{d}\,\tau} = c\,\frac{\mathrm{d}\,t}{\mathrm{d}\,\tau} = \gamma_u\,c \quad \Leftarrow \quad \mathrm{d}\,\tau \text{ is invariant}$$

- When you go from system S to system $\bar{S}$, moving at speed v along the common $x\ \bar{x}$ axis: $\bar{U}^\mu = \Lambda^\mu{}_\nu\,U^\nu \quad \Rightarrow$

$$\bar{U}^0 = \gamma_v\left(U^0 - \beta\,U^1\right)$$
$$\bar{U}^1 = \gamma_v\left(U^1 - \beta\,U^0\right)$$
$$\bar{U}^2 = U^2$$
$$\bar{U}^3 = U^3$$

- U^μ is called the **proper velocity 4-vector**,

or the **4-velocity**; $\vec{U}^2 \equiv U_\mu\,U^\mu = -\,c^2 \quad \Leftarrow \quad \vec{U}^2 = -\,\gamma_u^2\,c^2 + \gamma_u^2\,u^2$

$$\vec{U}=\left(U^0,U^1,U^2,U^3\right)=\left(\gamma_u c,\gamma_u u^x,\gamma_u u^y,\gamma_u u^z\right)=\left(\gamma_u c,\gamma_u \mathbf{u}\right)$$

$$\Lambda^\mu{}_\nu(v)=\begin{bmatrix}\gamma & -\gamma\beta & 0 & 0\\ -\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \Leftarrow \begin{array}{l}\mathbf{v}=(v,\ 0,\ 0)\\ \boldsymbol{\beta}=\dfrac{\mathbf{v}}{c},\quad \beta_v=\beta=|\boldsymbol{\beta}|,\quad \gamma_v=\gamma=\dfrac{1}{\sqrt{1-\beta^2}}\end{array}$$

$$\vec{\bar{U}}=\left(\bar{U}^0,\bar{U}^1,\bar{U}^2,\bar{U}^3\right)=\left(\gamma_{\bar{u}} c,\gamma_{\bar{u}} \bar{u}^x,\gamma_{\bar{u}} \bar{u}^y,\gamma_{\bar{u}} \bar{u}^z\right)=\left(\gamma_{\bar{u}} c,\gamma_{\bar{u}} \bar{\mathbf{u}}\right)$$

$$\bar{U}^\mu=\Lambda^\mu{}_\nu U^\nu \Rightarrow \begin{array}{lll}\bar{U}^0=\gamma\left(U^0-\beta U^1\right)=\gamma\gamma_u\left(c-\beta u^x\right)=\gamma\gamma_u\left(1-\beta\beta_u^x\right)c & \Rightarrow \gamma_{\bar{u}} c\\ \bar{U}^1=\gamma\left(U^1-\beta U^0\right)=\gamma\gamma_u\left(u^x-\beta c\right)=\gamma\gamma_u\left(u^x-v\right) & \Rightarrow \gamma_{\bar{u}}\bar{u}^x\\ \bar{U}^2=U^2 \qquad =\gamma_u u^y & \Rightarrow \gamma_{\bar{u}}\bar{u}^y\\ \bar{U}^3=U^3 \qquad =\gamma_u u^z & \Rightarrow \gamma_{\bar{u}}\bar{u}^z\end{array}$$

where $\boldsymbol{\beta}_u=\dfrac{\mathbf{u}}{c},\ \beta_u=|\boldsymbol{\beta}_u|,\ \gamma_u=\dfrac{1}{\sqrt{1-\beta_u^2}},\quad \boldsymbol{\beta}_{\bar{u}}=\dfrac{\bar{\mathbf{u}}}{c},\ \beta_{\bar{u}}=|\boldsymbol{\beta}_{\bar{u}}|,\ \gamma_{\bar{u}}=\dfrac{1}{\sqrt{1-\beta_{\bar{u}}^2}}$

$$\Rightarrow\quad \gamma_{\bar{u}}=\gamma\gamma_u\left(1-\beta\beta_u^x\right)\quad\Rightarrow\quad \bar{u}^x=\frac{\gamma\gamma_u\left(u^x-v\right)}{\gamma_{\bar{u}}}=\frac{u^x-v}{1-\beta\beta_u^x}$$

$$\Rightarrow\quad \bar{u}^y=\frac{\gamma_u u^y}{\gamma_{\bar{u}}}=\frac{u^y}{\gamma\left(1-\beta\beta_u^x\right)},\quad \bar{u}^z=\frac{\gamma_u u^z}{\gamma_{\bar{u}}}=\frac{u^z}{\gamma\left(1-\beta\beta_u^x\right)}$$

- By contrast, the transformation rule for *ordinary* velocities is cumbersome,

$$\bar{u}^x = \frac{\mathrm{d}\,\bar{x}}{\mathrm{d}\,\bar{t}} = \frac{u^x - v}{1 - \beta\,\beta_u^x}$$

$$\bar{u}^y = \frac{\mathrm{d}\,\bar{y}}{\mathrm{d}\,\bar{t}} = \frac{u^y}{\gamma\left(1 - \beta\,\beta_u^x\right)} \quad \Leftarrow \quad \beta_v \equiv \beta = \frac{v}{c}\,,\ \gamma_v = \gamma = \frac{1}{\sqrt{1-\beta^2}}\,, \quad \boldsymbol{\beta}_u \equiv \frac{\mathbf{u}}{c}$$

$$\bar{u}^z = \frac{\mathrm{d}\,\bar{z}}{\mathrm{d}\,\bar{t}} = \frac{u^z}{\gamma\left(1 - \beta\,\beta_u^x\right)}$$

- The reason for the added complexity is that it needs to transform both $\mathrm{d}\,\boldsymbol{\ell}$ & $\mathrm{d}\,t$, whereas for *proper* velocity, $\mathrm{d}\,\tau$ is invariant, the ratio inherits the transformation rule of the numerator alone.

Relativistic Energy and Momentum

- Momentum is mass times velocity. In relativity, we should use *proper* velocity instead of ordinary velocity, for the law of conservation of momentum would be inconsistent with the principle of relativity if we define momentum as $m\,\mathbf{u}$,

$\mathbf{p} \equiv m\,\mathbf{U} = \gamma_u\, m\,\mathbf{u}$ relativistic momentum $\Rightarrow$ $p^\mu \equiv m\,U^\mu$ 4-vector $\Rightarrow$ $\vec{p} = m\,\vec{U}$

$\Rightarrow$ $p^0 = m\,U^0 = \gamma_u\, m\,c$ $\Rightarrow$ $E \equiv p^0 c = \gamma_u\, m\,c^2$ relativistic energy $\Rightarrow$ $p^0 = \dfrac{E}{c}$

- p^μ is called the **energy-momentum 4-vector** or the **momentum 4-vector**.

- $$E = \gamma_u\, m\,c^2 = \frac{m\,c^2}{\sqrt{1-\beta_u^2}} = m\,c^2 + \frac{1}{2} m\,u^2 + \frac{3}{8}\frac{m\,u^4}{c^2} + \cdots \quad \Rightarrow \quad K_E = (\gamma_u - 1)\, m\,c^2$$

rest energy kinetic energy K_E

- The relativistic energy is nonzero *even when the object is stationary*—**rest energy**. The remainder, which is attributable to the *motion*, is **kinetic energy**. And the 2nd term reproduces the classical formula.

- The experimental fact: **In every closed system, the total relativistic energy and momentum are conserved**.

- Relativistic mass $m_\gamma \equiv \gamma\, m$

- Note the distinction between an **invariant** quantity (same value in all inertial systems) and a **conserved** quantity (same value before and after some process).

- Mass is invariant but not conserved; energy is conserved but not invariant; electric charge is both conserved *and* invariant; velocity is neither conserved *nor* invariant.

$$\Downarrow \quad m^2 \vec{U}^2 = -m^2 c^2$$

- $\vec{p}^{\,2} = p_\mu p^\mu = -(p^0)^2 + \mathbf{p}\cdot\mathbf{p} = -m^2 c^2 \quad \Rightarrow \quad E^2 - p^2 c^2 = m^2 c^4 \quad (@) \quad \Leftarrow \quad p \equiv |\mathbf{p}|$

- This result is extremely useful, for it enables you to calculate E (knowing p), or p (knowing E), without ever having to determine the velocity.

- Simple derivation for $E = m_\gamma c^2 = \gamma m c^2$:

$$\mathrm{d}E \Leftarrow \mathrm{d}W = \mathbf{F}\cdot\mathrm{d}\mathbf{x} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\cdot\mathbf{u}\,\mathrm{d}t = \mathbf{u}\cdot\mathrm{d}\mathbf{p} = \mathbf{u}\cdot\mathrm{d}(\gamma_u m\,\mathbf{u}) = \gamma_u m\,\mathbf{u}\cdot\mathrm{d}\mathbf{u} + m u^2\,\mathrm{d}\gamma_u$$

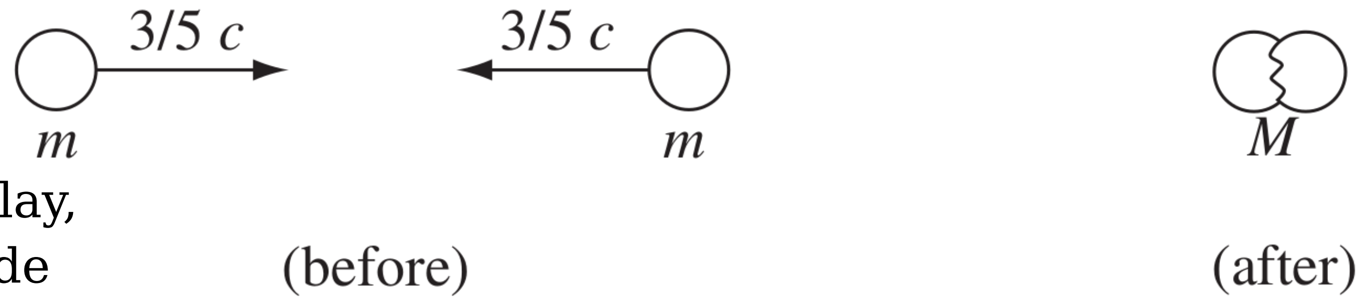
$$= m\left(\gamma_u \mathbf{u}\cdot\mathrm{d}\mathbf{u} + \gamma_u^3 \frac{u^2}{c^2}\mathbf{u}\cdot\mathrm{d}\mathbf{u}\right) \quad \Leftarrow \quad \mathrm{d}\gamma_u = \mathrm{d}\frac{1}{\sqrt{1-u^2/c^2}} = \frac{\gamma_u^3}{c^2}\mathbf{u}\cdot\mathrm{d}\mathbf{u}$$

$$= \gamma_u m\left(1+\gamma_u^2\beta_u^2\right)\mathbf{u}\cdot\mathrm{d}\mathbf{u} = m\,\gamma_u^3\,\mathbf{u}\cdot\mathrm{d}\mathbf{u} = m c^2\,\mathrm{d}\gamma_u = \mathrm{d}\left(\gamma_u m c^2\right)$$

$$\Rightarrow \quad E = \gamma_u m c^2 + \text{constant} \quad \Rightarrow \quad \Delta E = \Delta m\, c^2 \quad \Leftarrow \quad \text{nuclear experiment}$$

Selected problems: 6, 16, 25, 35, 45, 51, 54, 59, 64, 68

Relativistic Kinematics



Example 12.7: 2 lumps of clay, each of (rest) mass m, collide headon at $3c/5$. They stick together. What is the mass (M) of the composite lump?

- Conservation of momentum is trivial: 0=0.
- The energy of each lump prior to the collision is $\dfrac{mc^2}{\sqrt{1-(3/5)^2}} = \dfrac{5}{4}mc^2$. The energy of the composite lump after the collision is Mc^2. So conservation of energy

$$\frac{5}{4}mc^2 + \frac{5}{4}mc^2 = Mc^2 \quad \Rightarrow \quad M = \frac{5}{2}m > 2m$$

- Mass was not conserved in this collision; kinetic energy was converted into rest energy, so the mass increased. In fact, kinetic energy is converted into *thermal* (internal) energy.

- These internal energies are represented in the *mass* of the composite object: a hot potato is *heavier* than a cold potato, and a compressed spring is *heavier* than a relaxed spring (due to the potential energy).

- Internal energy (U) contributes an amount U/c^2 to the mass. The effect can be very striking in the realm of elementary particles.

- The neutral π meson (2.4×10^{-28}kg) decays into an electron and a positron (each 9.11×10^{-31}kg), the rest energy is converted almost entirely into kinetic energy—less than 1% of the original mass remains.

- In classical mechanics, there's no such thing as a massless ($m=0$) particle—its kinetic energy and its momentum would be 0, you couldn't apply a force to it, and hence (by Newton's 3rd law) it couldn't exert a force on anything else.

- In relativity, a massless particle could carry energy and momentum, *provided it always travels at the speed of light*, as photon: $(@) \;\Rightarrow\; E = p\,c$

- Relativity cannot tell what E & $\mathbf{p}$ of a photon are, but quantum mechanics can: According to the Planck formula, $E=h\nu$, where h is **Planck's constant** and ν is the frequency. So a blue photon is more energetic than a red one!

<u>Example 12.8</u>: A pion at rest decays into a muon and a neutrino. Find the energy of the outgoing muon, in terms of the 2 masses, m_π & m_μ (assume m_ν=0).

$$E_{\text{before}} = m_\pi c^2 \;,\quad \mathbf{p}_{\text{before}} = 0$$

$$E_{\text{after}} = E_\mu + E_\nu \;,\quad \mathbf{p}_{\text{after}} = \mathbf{p}_\mu + \mathbf{p}_\nu$$

$$\Rightarrow \quad \mathbf{p}_\nu = -\mathbf{p}_\mu \;,\quad E_\mu + E_\nu = m_\pi c^2$$

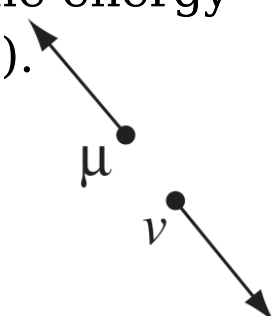
$$p_\nu c = E_\nu$$

$$p_\mu c = \sqrt{E_\mu^2 - m_\mu^2 c^4}$$

$$\Rightarrow \quad E_\mu + \sqrt{E_\mu^2 - m_\mu^2 c^4} = m_\pi c^2$$

$$\Rightarrow \quad E_\mu = \frac{m_\pi^2 + m_\mu^2}{2\,m_\pi} c^2$$

(before)



(after)

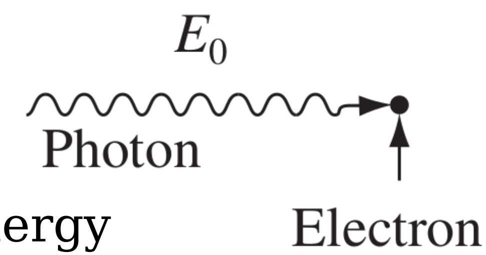
- In a classical collision, momentum and mass are always conserved, whereas kinetic energy, in general, is not. In the relativistic case, momentum and total energy are always conserved, but mass and kinetic energy, in general, are not.

- We call the process **elastic** if kinetic energy is conserved. In such a case the rest energy (the total − the kinetic) is also conserved, and so is the mass.

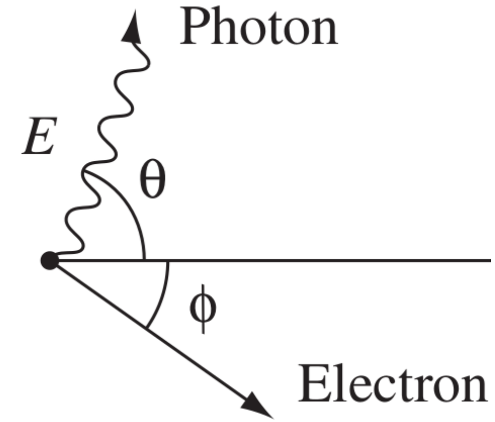
- In fact, it means that the *same particles* come out as went in.

Compton scattering

- A photon of energy E_0 bounces off an electron, initially at rest. Find the energy E of the outgoing photon, as a function of the **scattering angle** θ.



(before)



(after)

- vertical: $$p_e \sin\phi = p_p \sin\theta \;\Rightarrow\; \sin\phi = \frac{E}{p_e c}\sin\theta \;\Leftarrow\; p_p = \frac{E}{c}$$

horizontal: $$p_{p0} = \frac{E_0}{c} = p_p\cos\theta + p_e\cos\phi = \frac{E}{c}\cos\theta + p_e\sqrt{1-\left(\frac{E}{p_e c}\sin\theta\right)^2}$$

$$\Rightarrow\; p_e^2 c^2 = (E_0 - E\cos\theta)^2 + E^2\sin^2\theta = E_0^2 - 2E_0 E\cos\theta + E^2$$

energy: $E_0 + m\,c^2 = E + E_e = E + \sqrt{m^2\,c^4 + p_e^2\,c^2} = E + \sqrt{m^2\,c^4 + E_0^2 - 2\,E_0\,E\cos\theta + E^2}$

$$\Rightarrow \quad \left(E_0 + m\,c^2 - E\right)^2 = m^2\,c^4 + E_0^2 - 2\,E_0\,E\cos\theta + E^2$$

$$\Rightarrow \quad E_0^2 + m^2 c^4 + E^2 + 2\,m\,c^2\,E_0 - 2\,E\left(m\,c^2 + E_0\right) = m^2 c^4 + E_0^2 - 2\,E_0\,E\cos\theta + E^2$$

$$\Rightarrow \quad 2\,E\left(E_0 - E_0\cos\theta + m\,c^2\right) = 2\,m\,c^2\,E_0 \qquad \Downarrow \quad E = h\,\nu = \frac{h\,c}{\lambda}$$

$$\Rightarrow \quad E = \frac{m\,c^2\,E_0}{E_0\left(1-\cos\theta\right) + m\,c^2} = \frac{1}{\dfrac{1-\cos\theta}{m\,c^2} + \dfrac{1}{E_0}} \quad \Rightarrow \quad \frac{h\,c}{\lambda} = \frac{1}{\dfrac{1-\cos\theta}{m\,c^2} + \dfrac{\lambda_0}{h\,c}}$$

$$\Rightarrow \quad \boxed{\lambda = \lambda_0 + \frac{h}{m\,c}\left(1-\cos\theta\right)}$$

- The quantity $\dfrac{h}{m\,c}$ is called the **Compton wavelength** of the electron.

$$\Rightarrow \quad \Delta\lambda = \lambda_C\left(1-\cos\theta\right)$$

Relativistic Dynamics

- Newton's 1st law is built into the principle of relativity. His 2nd law, $\mathbf{F}=\dfrac{\mathrm{d}\,\mathbf{p}}{\mathrm{d}\,t}$

retains its validity in relativistic mechanics, *provided we use the relativistic momentum*.

<u>Example 12.10</u>: **Motion under a constant force**. A particle of mass *m* is subject to a const force *F*. If it starts from rest at the origin at t=0, find its position $x(t)$.

- $$\frac{\mathrm{d}\,p}{\mathrm{d}\,t}=F \quad \Rightarrow \quad p=F\,t+\text{constant}=F\,t \quad \Leftarrow \quad p\left(t=0\right)=0$$

$$\Rightarrow \quad p=\frac{m\,u}{\sqrt{1-u^2/c^2}}=F\,t \quad \Rightarrow \quad u=\frac{F\,t/m}{\sqrt{1+\left(F\,t/m\,c\right)^2}}$$

- The relativistic denominator ensures that u never exceeds c; as $t\to\infty$, $u\to c$.

- $$x\left(t\right)=\frac{F}{m}\int_0^t\frac{t'\,\mathrm{d}\,t'}{\sqrt{1+\left(F\,t'/m\,c\right)^2}}=\frac{m\,c^2}{F}\sqrt{1+\left(\frac{F\,t'}{m\,c}\right)^2}\Bigg|_0^t$$

$$=\frac{m\,c^2}{F}\left[\sqrt{1+\left(\frac{F\,t}{m\,c}\right)^2}-1\right] \quad \Rightarrow \quad \left(\frac{F\,x}{m\,c^2}+1\right)^2-\left(\frac{F\,t}{m\,c}\right)^2=1$$

- Instead of the classical parabola, $x\left(t\right)=\dfrac{F}{2\,m}\,t^2$, the graph is a hyperbola—

hyperbolic motion, eg, a charged particle placed in a uniform electric field.

- Work is the line integral of the force:

$$W \equiv \int \mathbf{F} \cdot \mathrm{d}\,\boldsymbol{\ell}$$

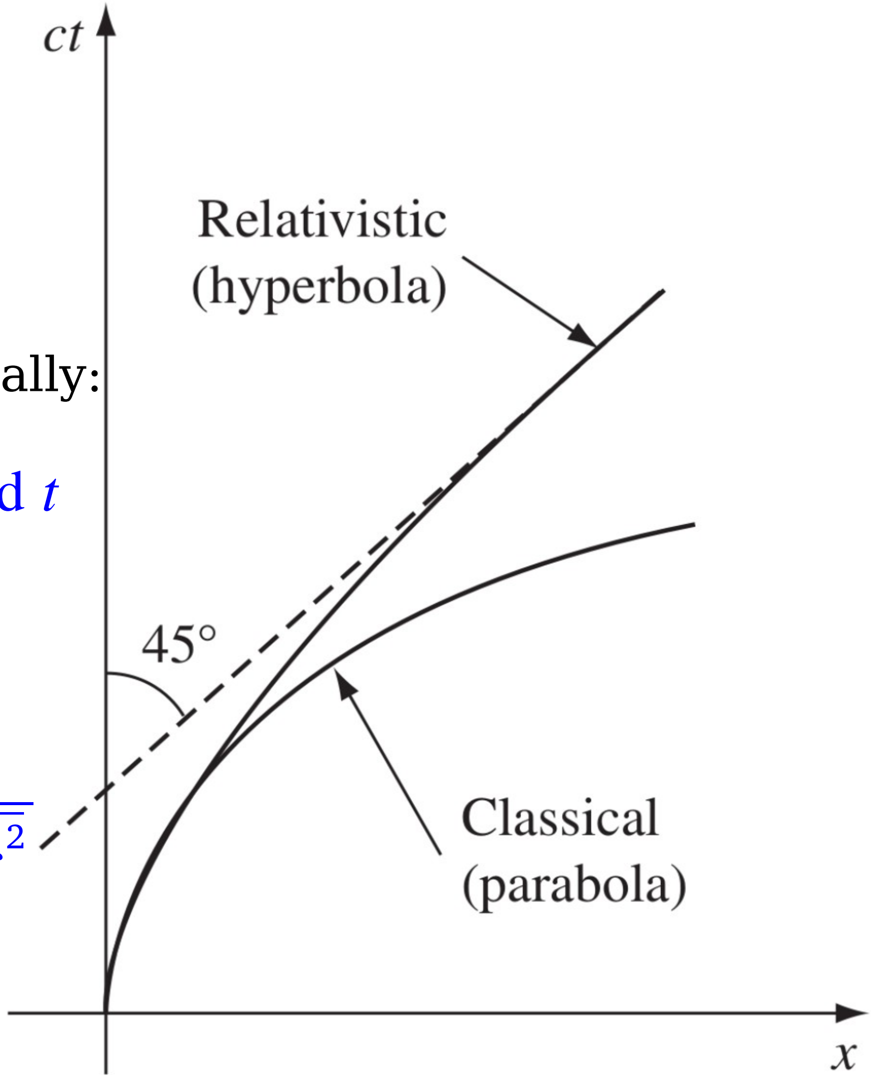
- The **work-energy theorem** holds relativistically:

$$W \equiv \int \frac{\mathrm{d}\,\mathbf{p}}{\mathrm{d}\,t} \cdot \mathrm{d}\,\boldsymbol{\ell} = \int \frac{\mathrm{d}\,\mathbf{p}}{\mathrm{d}\,t} \cdot \frac{\mathrm{d}\,\boldsymbol{\ell}}{\mathrm{d}\,t}\,\mathrm{d}\,t = \int \frac{\mathrm{d}\,\mathbf{p}}{\mathrm{d}\,t} \cdot \mathbf{u}\,\mathrm{d}\,t$$

$$\frac{\mathrm{d}\,\mathbf{p}}{\mathrm{d}\,t} \cdot \mathbf{u} = \mathbf{u} \cdot \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\gamma_u m\,\mathbf{u}\right) = \gamma_u^3 m\,\mathbf{u} \cdot \frac{\mathrm{d}\,\mathbf{u}}{\mathrm{d}\,t}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\gamma_u m c^2\right) = \frac{\mathrm{d}\,E}{\mathrm{d}\,t} \quad \Leftarrow \quad \gamma_u = \frac{1}{\sqrt{1 - u^2/c^2}}$$

$$\Rightarrow \quad W = \int \frac{\mathrm{d}\,E}{\mathrm{d}\,t}\,\mathrm{d}\,t = E_{\text{final}} - E_{\text{initial}}$$



- Newton's 3^{rd} law does not, in general, extend to the relativistic domain.

- If 2 objects are separated in space, the 3^{rd} law is incompatible with the relativity of simultaneity.

- If the force of A on B at some instant t is $\mathbf{F}(t)$, and the force of B on A at the same instant is $-\mathbf{F}(t)$; then the 3^{rd} law applies *in this reference frame*.

- But a moving observer will report that these equal and opposite forces occurred at different times; in his system, therefore, the 3rd law is violated.

- Only in the case of contact interactions, where the 2 forces are applied at the same physical point can the 3rd law be retained.

- Because $\mathbf{F}$ is the derivative of momentum with respect to *ordinary* time, it shares the ugly behavior of (ordinary) velocity, when you go from one inertial system to another:

$$\bar{F}^x = \frac{d\,\bar{p}^x}{d\,\bar{t}} = \frac{\gamma\left(d\,p^x - \beta\,d\,p^0\right)}{\gamma\left(d\,t - \frac{\beta}{c}\,d\,x\right)} = \frac{\frac{d\,p^x}{d\,t} - \beta\frac{d\,p^0}{d\,t}}{1 - \frac{\beta}{c}\frac{d\,x}{d\,t}} = \frac{F^x - \frac{\beta}{c}\frac{d\,E}{d\,t}}{1 - \beta\frac{u^x}{c}} = \frac{F^x - \beta\,\boldsymbol{\beta}_u \cdot \mathbf{F}}{1 - \beta\,\beta_u^x}$$

$$\bar{F}^{y,z} = \frac{d\,\bar{p}^{y,z}}{d\,\bar{t}} = \frac{d\,p^{y,z}}{\gamma\left(d\,t - \frac{\beta}{c}\,d\,x\right)} = \frac{\frac{d\,p^{y,z}}{d\,t}}{\gamma\left(1 - \frac{\beta}{c}\frac{d\,x}{d\,t}\right)} = \frac{F^{y,z}}{\gamma\left(1 - \beta\,\beta_u^x\right)} \quad \Leftarrow \quad \boldsymbol{\beta}_u \equiv \frac{\mathbf{u}}{c}, \quad \beta_u^x = \frac{u^x}{c}$$

- *If the particle is* (instantaneously) *at rest in* S, $\mathbf{u}=0$, then $\bar{\mathbf{F}}_\perp = \frac{\mathbf{F}_\perp}{\gamma}, \quad \bar{\mathbf{F}}_\parallel = \mathbf{F}_\parallel$.
The component of $\mathbf{F}$ ∥ the motion of $\bar{S}$ is unchanged, whereas ⊥ components are divided by γ.

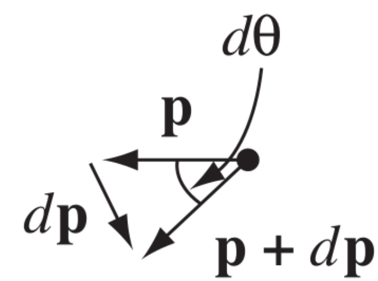
- Avoid the bad transformation behavior of $\mathbf{F}$ by using a “proper” force, similar to proper velocity, the derivative of momentum with respect to *proper* time:

$$K^{\mu} \equiv \frac{d\,p^{\mu}}{d\,\tau} \quad \text{Minkowski force} \quad \Leftarrow \quad \text{4-vector} \quad \Rightarrow \quad \vec{K} = \frac{d\,\vec{p}}{d\,\tau}$$

$$\Rightarrow \quad \mathbf{K} = \frac{d\,t}{d\,\tau}\frac{d\,\mathbf{p}}{d\,t} = \frac{\mathbf{F}}{\sqrt{1-\beta_u^2}} = \gamma_u \mathbf{F}, \quad K^0 = \frac{d\,p^0}{d\,\tau} = \frac{1}{c}\frac{d\,E}{d\,\tau} \propto \text{the proper } \textit{power} \text{ to the particle}$$

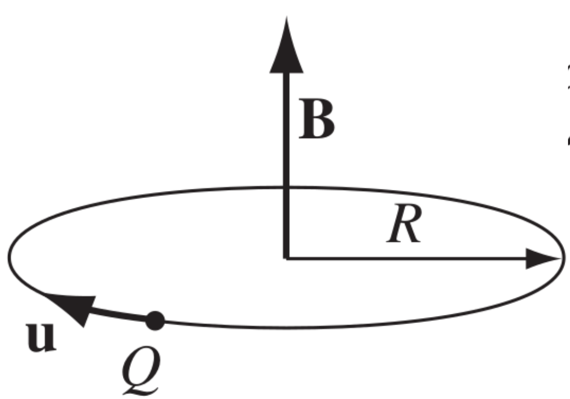
- Relativistic dynamics can be formulated in terms of the ordinary force or in terms of the Minkowski force. The latter is neater, but since we are interested in the particle’s trajectory as a function of *ordinary* time, the former is more useful.

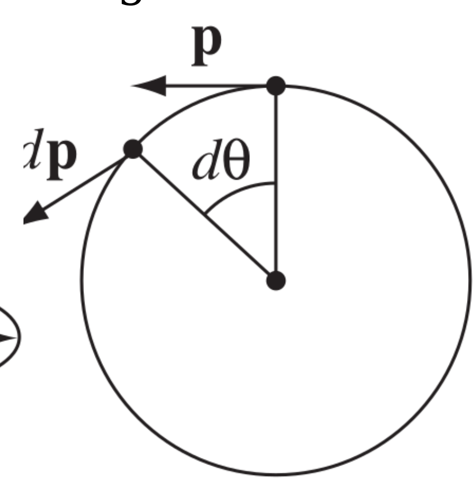
- The Lorentz force is an *ordinary* force—explain why this is so later and show how to construct the EM Minkowski force.



Example 12.11: The typical trajectory of a charged particle in a uniform magnetic field is **cyclotron motion**. The magnetic force pointing toward the center, $F = Q\,u\,B$

- In special relativity the centripetal force $\neq \dfrac{m\,u^2}{R}$, as in classical mechanics.





$$\mathrm{d}\,p = p\,\mathrm{d}\,\theta \quad \Rightarrow \quad F = \frac{\mathrm{d}\,p}{\mathrm{d}\,t} = p\,\frac{\mathrm{d}\,\theta}{\mathrm{d}\,t} = p\,\frac{u}{R} \quad \left(\text{classically} \;\; p = m\,u \quad \Rightarrow \quad F = \frac{m\,u^2}{R}\right)$$

$$\Rightarrow \quad Q\,u\,B = p\,\frac{u}{R} \quad \Rightarrow \quad p = Q\,B\,R$$

- In this form, the relativistic cyclotron formula is identical to the nonrelativistic one—the only difference is that p is now the *relativistic* momentum.

- In classical mechanics, the total momentum $\mathbf{P}$ of a collection of interacting particles can be expressed as the total mass M times the velocity of the center-of-mass: $\mathbf{P} = M\,\frac{\mathrm{d}\,\mathbf{R}_m}{\mathrm{d}\,t}$
- In relativity center-of-mass $\mathbf{R}_m = \frac{1}{M}\sum m_i\,\mathbf{r}_i$ $\rightarrow$ center-of-energy $\mathbf{R}_e = \frac{1}{E}\sum E_i\,\mathbf{r}_i$

  and $M \rightarrow \frac{E}{c^2} \quad \Rightarrow \quad \mathbf{P} = \frac{E}{c^2}\,\frac{\mathrm{d}\,\mathbf{R}_e}{\mathrm{d}\,t}$

- $\mathbf{P}$ now includes *all* forms of momentum, and E all forms of energy—not just mechanical, but also whatever may be stored in the fields.

- The momentum stored in the fields of a coaxial cable is not 0 (Ex 8.3), even though the cable itself is at rest. However, energy is being transported, from the battery to the resistor, and hence the center-of-energy is in motion.

- If the battery is at $z=0$, so the resistor is at $z=\ell$, then $\mathbf{R}_e = \dfrac{E_0 \mathbf{R}_0 + E_R \ell \hat{\mathbf{z}}}{E}$,

where E_R: energy in the resistor, E_0: rest of the energy, $\mathbf{R}_0$: center-of-energy of E_0,

$$\Rightarrow \quad \frac{d\mathbf{R}_e}{dt} = \frac{dE_R/dt}{E}\ell\hat{\mathbf{z}} = \frac{IV\ell}{E}\hat{\mathbf{z}} \quad \Rightarrow \quad \mathbf{P} = \frac{E}{c^2}\frac{d\mathbf{R}_e}{dt} = \frac{IV\ell}{c^2}\hat{\mathbf{z}}$$

momentum in the fields, as in Ex 8.3

- Imagine a shoe-box with a marble inside. The box is at rest, but the marble rolls around. Although the box is stationary, there is the momentum of the marble.

- In the case of the coaxial cable, no actual *object* is in motion, but energy flows around. In relativity *all* forms of energy in motion, not just rest energy (mass), constitute momentum.

- The EM field transports energy, and therefore contributes momentum, *even though the fields themselves are perfectly static*!

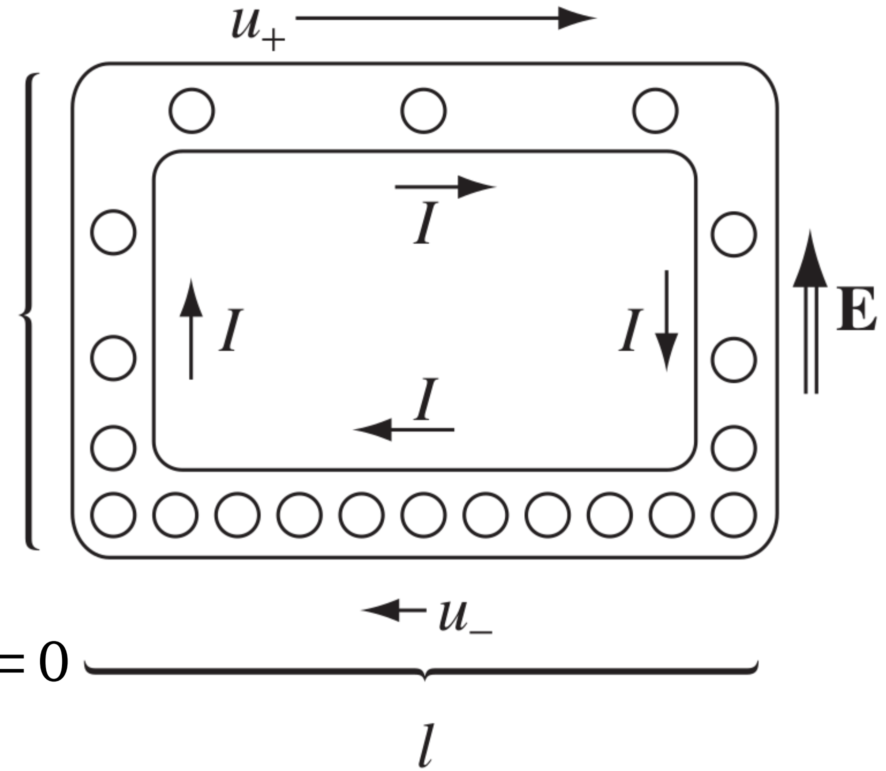
Example 12.13: For a magnetic dipole $\mathbf{m}$ with a rectangular loop of wire of a steady current I, let the current as a stream of noninteracting positive charges moving within the wire. When a uniform electric field $\mathbf{E}$ is applied, the charges accelerate/decelerate in the left/right segment. Find the total momentum of all the charges in the loop.

The momenta of the left and right segments cancel, so we need only consider the top and the bottom.

- N_+/N_- charges in the top/bottom segment with speed u_+/u_-(slower) to the right/left.

- The current is the same in all 4 segments

$$I = \lambda u = \frac{Q N_+}{\ell} u_+ = \frac{Q N_-}{\ell} u_- \quad \Rightarrow \quad N_\pm u_\pm = \frac{I \ell}{Q}$$



- *Classically,* $\mathbf{p} = M \mathbf{u}$

$$\Rightarrow \quad p_{\text{class}} = M N_+ u_+ - M N_- u_- = M \frac{I \ell}{Q} - M \frac{I \ell}{Q} = 0$$

- *Relativistically,*

$$\mathbf{p} = \gamma M \mathbf{u} \quad \Rightarrow \quad p = \gamma_+ M N_+ u_+ - \gamma_- M N_- u_- = \frac{M I \ell}{Q} (\gamma_+ - \gamma_-) > 0$$

- The gain in energy ($\gamma M c^2$), as a particle goes up the left segment, is equal to the work done by the electric force, $Q E w$

$$\Rightarrow \quad \gamma_+ - \gamma_- = \frac{Q E w}{M c^2} \quad \Rightarrow \quad p = \frac{I \ell E w}{c^2} \quad \Rightarrow \quad \mathbf{p} = \frac{\mathbf{m} \times \mathbf{E}}{c^2} \quad \Leftarrow \quad m = I A = I \ell w$$

- Thus a magnetic dipole at rest in an electric field carries linear momentum, *even though it is not moving*!

- This **hidden momentum** is strictly relativistic, and purely mechanical; it precisely cancels the electromagnetic momentum stored in the fields $\propto \mathbf{E} \times \mathbf{B}$.

Relativistic Electrodynamics

Magnetism as a Relativistic Phenomenon

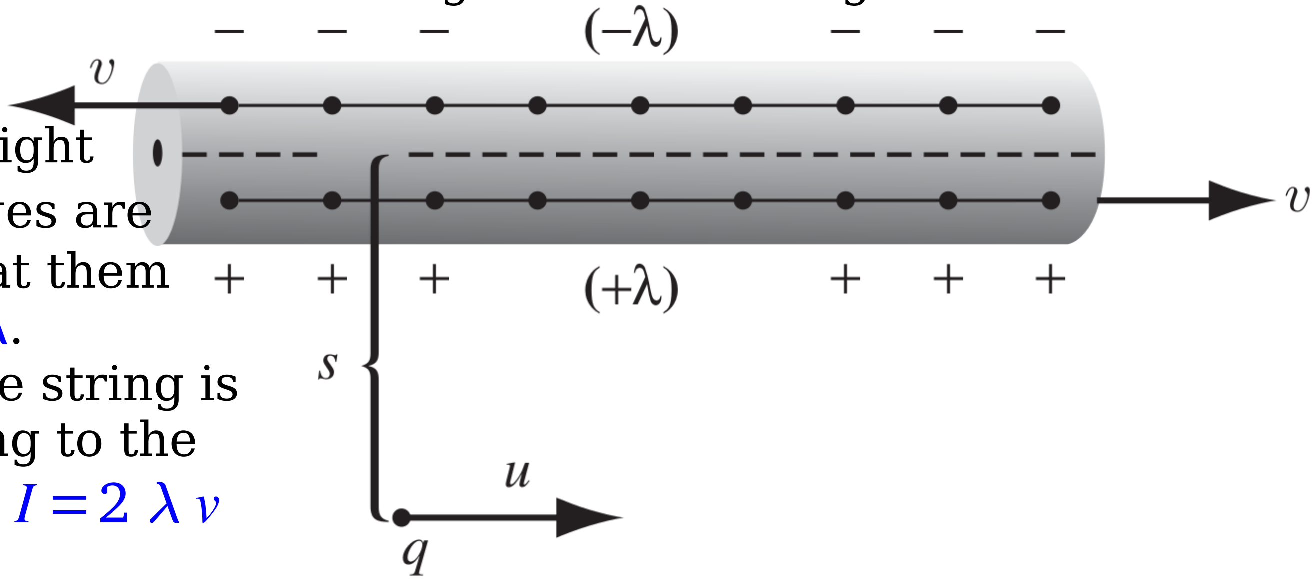
- Classical electrodynamics is *already* consistent with special relativity. Maxwell's eqns and the Lorentz force law can be applied legitimately in any inertial system.

- What one observer interprets as an electrical process another may regard as magnetic, but the actual particle motions they predict will be identical.

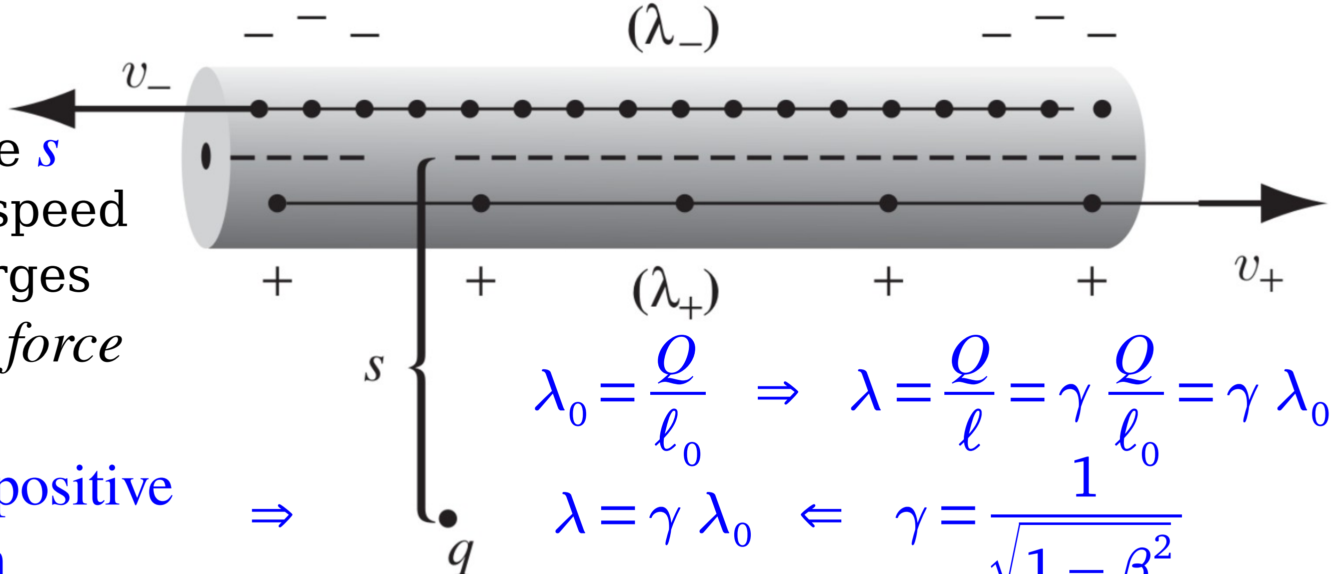
- For a complete and consistent formulation of relativistic electrodynamics, we will not change the rules of electrodynamics—rather, we will *express* these rules in a notation that exposes and illuminates their relativistic character.

- Show why there *had* to be such a thing as magnetism, given electrostatics and relativity, and how to calculate the magnetic force between a current-carrying wire and a moving charge without ever invoking the laws of magnetism.

- Let a string of positive charges moves along to the right at speed v. Assume the charges are close enough together to treat them as a continuous line charge λ. Superimposed on this positive string is a negative one, $-\lambda$ proceeding to the left at the same speed $v \quad \Rightarrow \quad I = 2\,\lambda\,v$



- A point charge q a distance s away travels to the right at speed $u \le v$. Because the 2 line charges cancel, there is *no electrical force on* q in this system S.



λ_0 : the charge density of the positive line in its own rest system $\Rightarrow$

$$\lambda_0 = \frac{Q}{\ell_0} \Rightarrow \lambda = \frac{Q}{\ell} = \gamma \frac{Q}{\ell_0} = \gamma \lambda_0$$

$$\lambda = \gamma \lambda_0 \Leftarrow \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

- Examine the same situation from $\bar{S}$, moving to the right with speed u. In $\bar{S}$, q is at rest. By the Einstein velocity addition rule, $v_\pm = \dfrac{v \mp u}{1 \mp \beta \beta_u} \Rightarrow \beta_\pm = \dfrac{\beta \mp \beta_u}{1 \mp \beta \beta_u}$

- $v_- > v_+ \Rightarrow$ the Lorentz contraction of the spacing between "$-$" charges is more severe than between "$+$" charges; so in $\bar{S}$, *the wire carries a net negative charge*!

$$\lambda_\pm = \pm \gamma_\pm \lambda_0 \Leftarrow \gamma_\pm = \frac{1}{\sqrt{1-\beta_\pm^2}} = \frac{1}{\sqrt{1-(\beta \mp \beta_u)^2 (1 \mp \beta \beta_u)^{-2}}}$$

$$= \frac{1 \mp \beta \beta_u}{\sqrt{(1 \mp \beta \beta_u)^2 - (\beta \mp \beta_u)^2}} = \frac{1 \mp \beta \beta_u}{\sqrt{(1-\beta^2)(1-\beta_u^2)}} = \gamma \gamma_u (1 \mp \beta \beta_u)$$

- The net line charge in $\bar{S}$: $\lambda_{tot} = \lambda_+ + \lambda_- = \lambda_0 (\gamma_+ - \gamma_-) = -2 \beta \gamma_u \beta_u \lambda$

- *Conclusion*: As a result of unequal Lorentz contraction of the "+" and "−" lines, a current-carrying wire that is electrically neutral in one inertial system will be charged in another.

- $\bar{E} = \dfrac{\lambda_{\text{tot}}}{2\,\pi\,\epsilon_0\,s}$ for a linear charge distribution $\Rightarrow$ $\bar{F} = q\,\bar{E} = -\dfrac{\lambda\,\beta}{\pi\,\epsilon_0\,s}\,q\,\gamma_u\,\beta_u$ in $\bar{S}$

- But if there's a force on q in $\bar{S}$, there must be one in S; we can calculate it by using the transformation rules for forces

$$q \text{ at rest in } \bar{S},\quad \bar{\mathbf{F}} \perp \mathbf{u} \quad \Rightarrow \quad F = \frac{\bar{F}}{\gamma_u} = -\frac{\lambda\,q\,\beta\,\beta_u}{\pi\,\epsilon_0\,s} \text{ in } S$$

- The charge is attracted toward the wire by a force that is purely electrical in $\bar{S}$, but distinctly *non*electrical in S (where the wire is neutral).

- Taken together electrostatics & relativity imply the existence of *magnetic* force.

$$F = -\frac{\lambda\,q\,\beta\,\beta_u}{\pi\,\epsilon_0\,s} = -q\,u\,\frac{\mu_0\,I}{2\,\pi\,s} = -q\,u\,B \quad \Leftarrow \quad c^2 = \frac{1}{\epsilon_0\,\mu_0}, \quad I = 2\,\lambda\,v$$

the force is what we would have obtained by using the Lorentz force law in S.

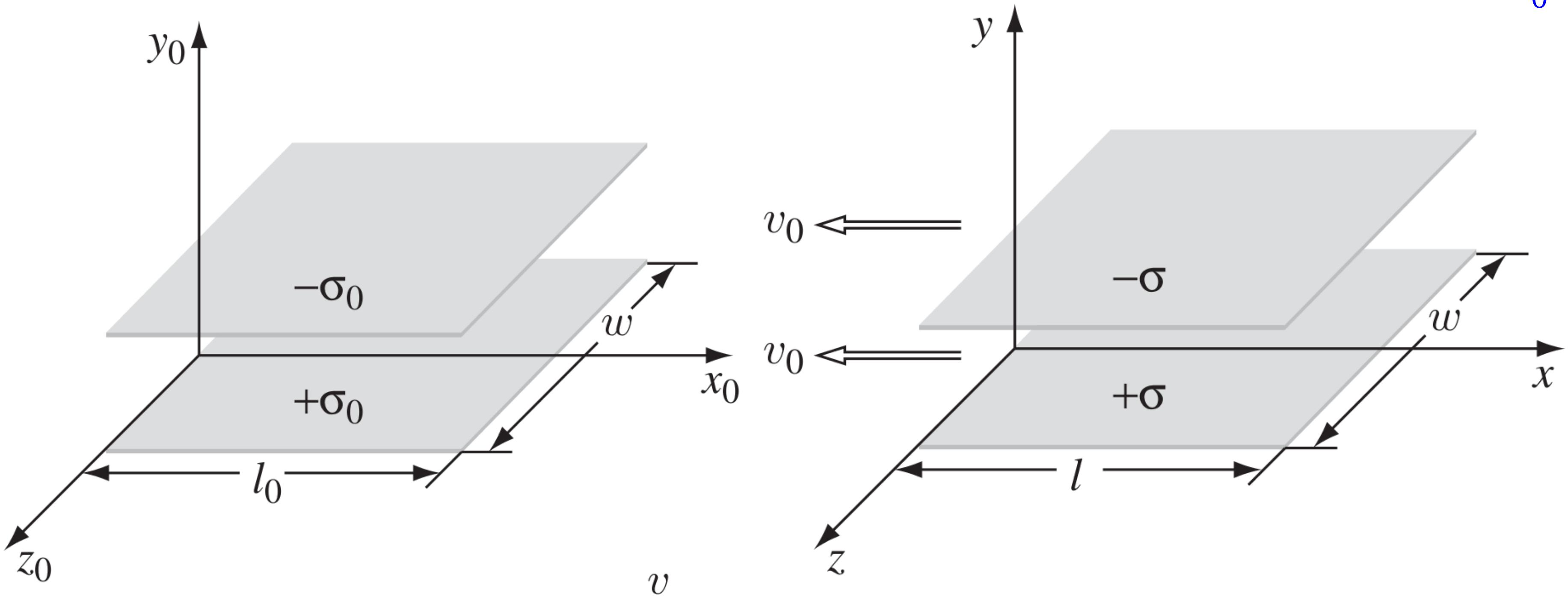
How the Fields Transform

- Assumption: *Charge is invariant*. The charge of a particle is a fixed number, independent of how fast it happens to be moving.

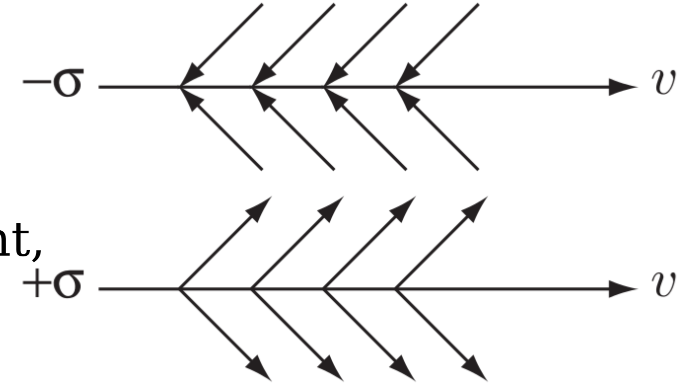
- Also assume that the transformation rules are the same no matter how the fields were produced—electric fields associated with changing magnetic fields transform the same way as those set up by stationary charges.

- In a field theory, the fields at a given point tell you *all there is to know* about that point; you do *not* have to append extra information regarding their source.

- the capacitor is at rest in S_0 and carries surface charges $\pm\sigma_0$ $\quad\Rightarrow\quad \mathbf{E}_0 = \dfrac{\sigma_0}{\epsilon_0}\,\hat{\mathbf{y}}$



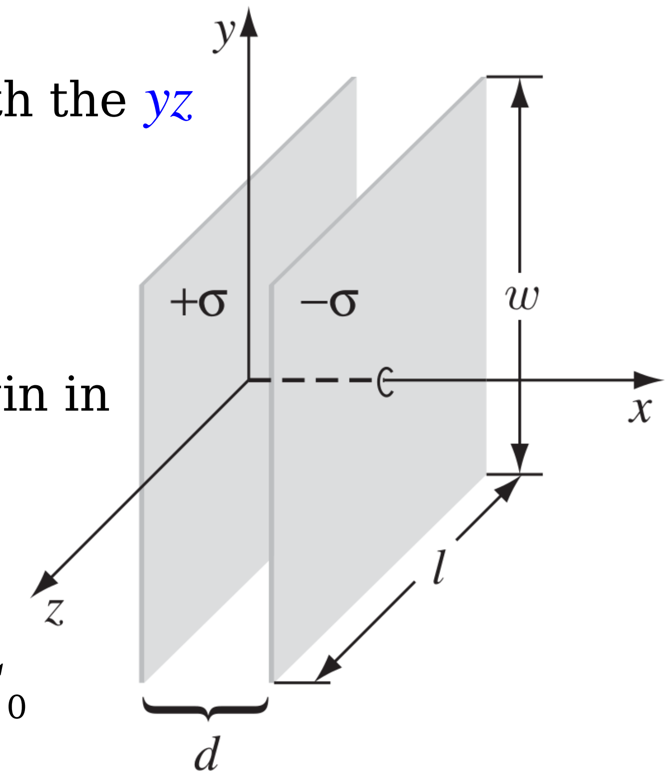
- From system S moving to the right at speed v_0, the plates are moving to the left, but the field still takes the form $\mathbf{E}=\frac{\sigma}{\epsilon_0}\hat{\mathbf{y}}$, and the surface charge σ is different, besides the other differences. And $\beta_0=\frac{v_0}{c}$.



- The total charge on each plate is invariant, but the length is Lorentz-contracted by a factor of $\gamma_0=\frac{1}{\sqrt{1-\beta_0^2}}$, so the charge/area is increased by a factor of γ_0:

$\sigma=\gamma_0\sigma_0 \quad \Rightarrow \quad \mathbf{E}^{\perp}=\gamma_0\mathbf{E}_0^{\perp}$ for components of $\mathbf{E}$ ⊥ the direction of motion of S.

- For ‖ components, consider the capacitor lined up with the yz plane, then the plate separation is Lorentz-contracted, whereas σ are the same in both frames $\Rightarrow \quad \mathbf{E}^{\parallel}=\mathbf{E}_0^{\parallel}$

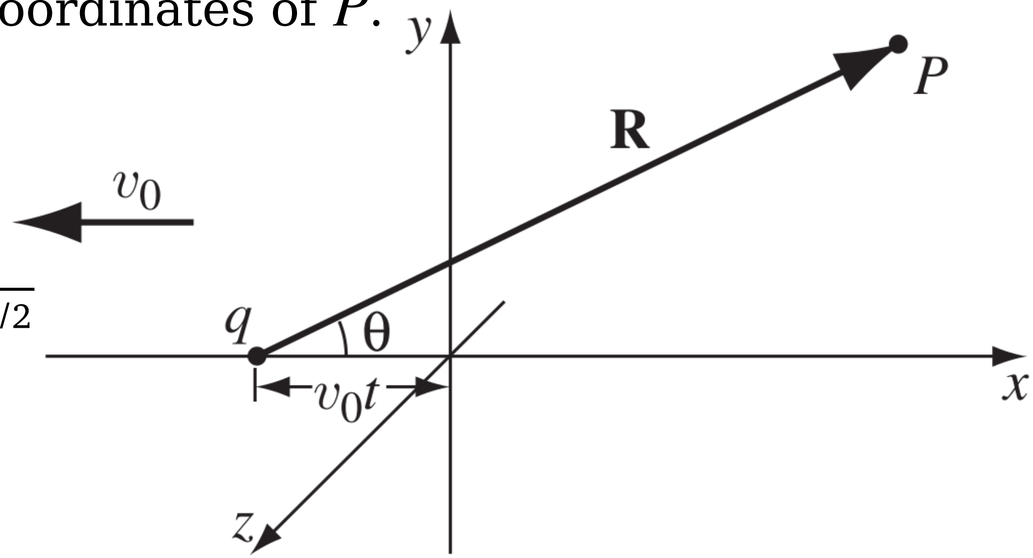


- <u>Example 12.14</u>: **Electric field of a point charge in uniform motion**. A point charge q is at rest at the origin in system S_0. Find the electric field of this same charge in system S, moving to the right at speed v_0 relative to S_0.

$$\mathbf{E}_0=\frac{q\,\hat{\mathbf{r}}_0}{4\pi\epsilon_0 r_0^2}\Rightarrow\begin{bmatrix}E_{0x}\\E_{0y}\\E_{0z}\end{bmatrix}=\frac{1}{4\pi\epsilon_0}\frac{q}{(x_0^2+y_0^2+z_0^2)^{3/2}}\begin{bmatrix}x_0\\y_0\\z_0\end{bmatrix} \text{ in } S_0$$

- From the above transformation $\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} E_{0x} \\ \gamma_0 E_{0y} \\ \gamma_0 E_{0z} \end{bmatrix} = \frac{1}{4\pi\epsilon_0} \frac{q}{(x_0^2 + y_0^2 + z_0^2)^{3/2}} \begin{bmatrix} x_0 \\ \gamma_0 y_0 \\ \gamma_0 z_0 \end{bmatrix}$

- These are still expressed in the S_0 coordinates (x_0, y_0, z_0) of the field point (P); we would prefer to write them in the S coordinates of P.



$$x_0 = \gamma_0 (x + v_0 t) = \gamma_0 R_x, \quad y_0 = y = R_y, \quad z_0 = z = R_z$$

$$\Rightarrow \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{\gamma_0 q \mathbf{R}}{(\gamma_0^2 R^2 \cos^2\theta + R^2 \sin^2\theta)^{3/2}}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q(1-\beta_0^2)}{(1-\beta_0^2 \sin^2\theta)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2}$$

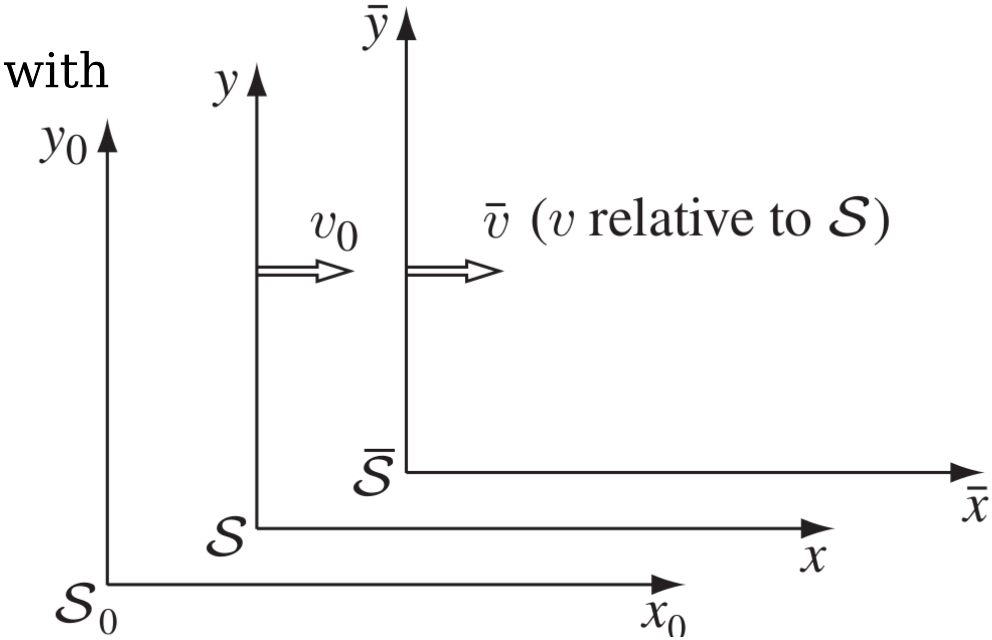
- This is the field of a charge in uniform motion; we got the same result in Ch 10 using the retarded potentials.

- This derivation is more efficient, and sheds some light on the fact that the field points away from the instantaneous (instead of retarded) position of the charge:

  E_x gets a factor of γ_0 from the Lorentz transformation of the *coordinates*;
  E_y & E_z pick up theirs from the transformation of the *field*.

- It's the balancing of these 2 γ_0's that leaves $\mathbf{E}$ parallel to $\mathbf{R}$.

- The above eqns are not the general transformation laws for no magnetic field.
- In the same case, In addition to the electric field $E_y = \dfrac{\sigma}{\epsilon_0}$, there is a magnetic field due to the surface currents $\boldsymbol{K}_\pm = \mp \sigma v_0 \hat{\mathbf{x}}$.
- By the right-hand rule, the magnetic field $\mathbf{B} = -\mu_0 \sigma v_0 \hat{\mathbf{z}}$ by Ampère's law.
- In a 3rd system, $\bar{S}$, traveling to the right with speed v relative to S, the fields would be



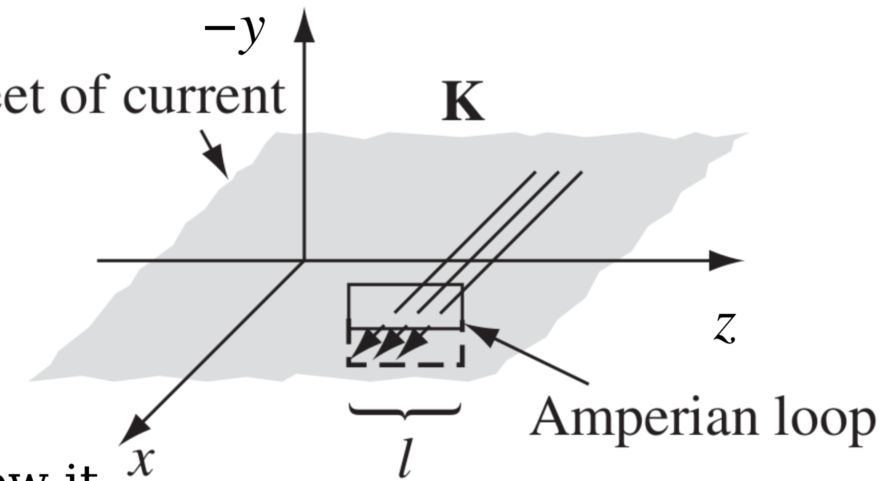
$$\bar{E}_y = \frac{\bar{\sigma}}{\epsilon_0}, \quad \bar{B}_z = -\mu_0 \bar{\sigma} \bar{v} \quad \Leftarrow \quad \bar{\sigma} = \bar{\gamma} \sigma_0$$

where $\bar{v} = \dfrac{v + v_0}{1 + \beta \beta_0}, \quad \bar{\gamma} = \dfrac{1}{\sqrt{1 - \beta_{\bar{v}}^2}}$

$$\Rightarrow \quad \bar{E}_y = \frac{\bar{\gamma}}{\gamma_0} \frac{\sigma}{\epsilon_0}, \quad \bar{B}_z = -\frac{\bar{\gamma}}{\gamma_0} \mu_0 \sigma \bar{v}$$

$$\frac{\bar{\gamma}}{\gamma_0} = \frac{\sqrt{1 - \beta_0^2}}{\sqrt{1 - \beta_{\bar{v}}^2}} = \frac{1 + \beta \beta_0}{\sqrt{1 - \beta^2}} = \gamma (1 + \beta \beta_0) \quad \Leftarrow \quad \beta_0 = \frac{v_0}{c}$$

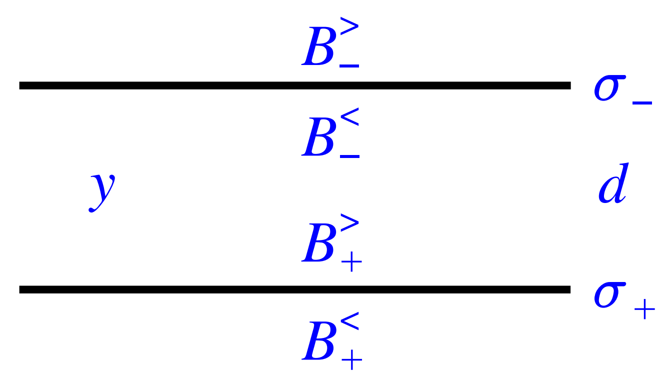
$$\Rightarrow \quad \begin{aligned} \bar{E}_y &= \gamma (1 + \beta \beta_0) \frac{\sigma}{\epsilon_0} = \gamma \left(E_y - \frac{\beta}{c \epsilon_0 \mu_0} B_z \right) = \gamma \left(E_y - c \beta B_z \right) \\ \bar{B}_z &= -\gamma (1 + \beta \beta_0) \mu_0 \sigma \frac{v + v_0}{1 + \beta \beta_0} = \gamma \left(B_z - \mu_0 \epsilon_0 v E_y \right) = \gamma \left(B_z - \frac{\beta}{c} E_y \right) \end{aligned}$$



<u>Example 5.8</u>: Find the magnetic field of an infinite uniform surface current $\boldsymbol{K} = K\,\hat{\mathbf{x}}$, flowing over the xz plane.

- $\mathbf{B}$ can only have a z component, and it points to the *left* above the plane and to the *right* below it.
- $\oint \mathbf{B}\cdot d\boldsymbol{\ell} = 2B\ell = \mu_0 I_{enc} = \mu_0 K \ell \quad \Leftarrow \quad$ Ampere's law

$$\Rightarrow \quad B = \frac{\mu_0}{2} K \quad \Rightarrow \quad \mathbf{B}^{\gtrless} = \pm \frac{\mu_0}{2} K\,\hat{\mathbf{z}} \quad \text{for} \quad y \gtrless 0$$

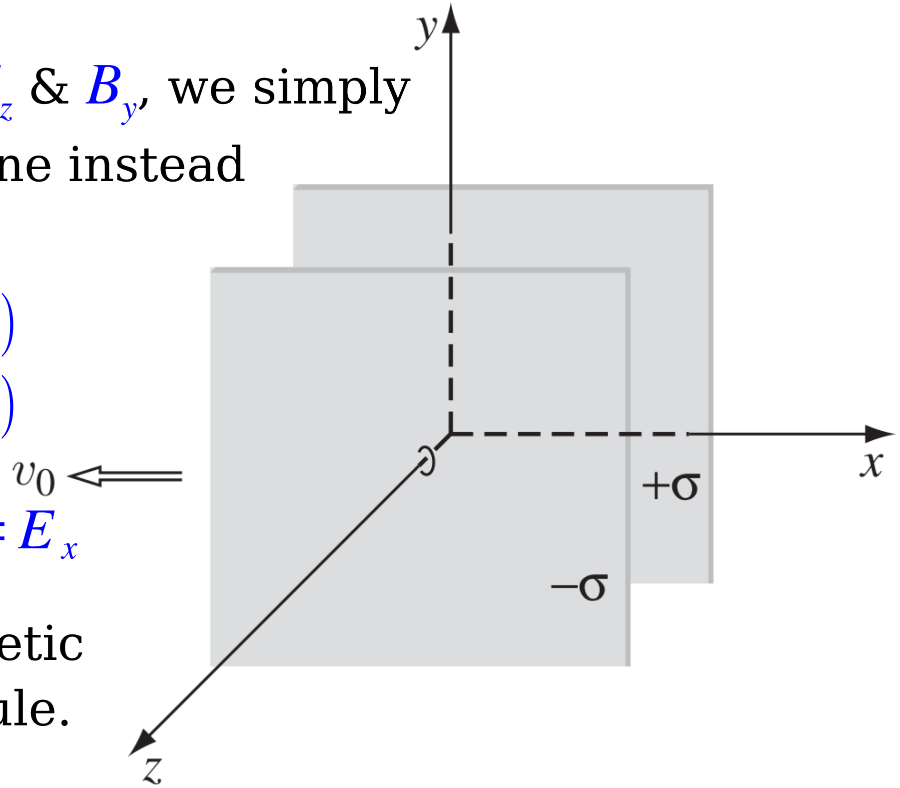


- Come back to the current problem: $\boldsymbol{K}_\pm = \mp \sigma v_0\,\hat{\mathbf{x}}$

$$\Rightarrow \quad \begin{aligned} \mathbf{B}_+^{\gtrless} &= \mp \frac{\mu_0}{2} \sigma v_0\,\hat{\mathbf{z}} \quad \text{for} \quad y \gtrless 0 \\ \mathbf{B}_-^{\gtrless} &= \pm \frac{\mu_0}{2} \sigma v_0\,\hat{\mathbf{z}} \quad \text{for} \quad y \gtrless d \end{aligned} \quad \Rightarrow \quad \mathbf{B} = \begin{bmatrix} -\mu_0 \sigma v_0\,\hat{\mathbf{z}} & \text{for} \;\; 0 < y < d \\ 0 & \text{elsewhere} \end{bmatrix}$$

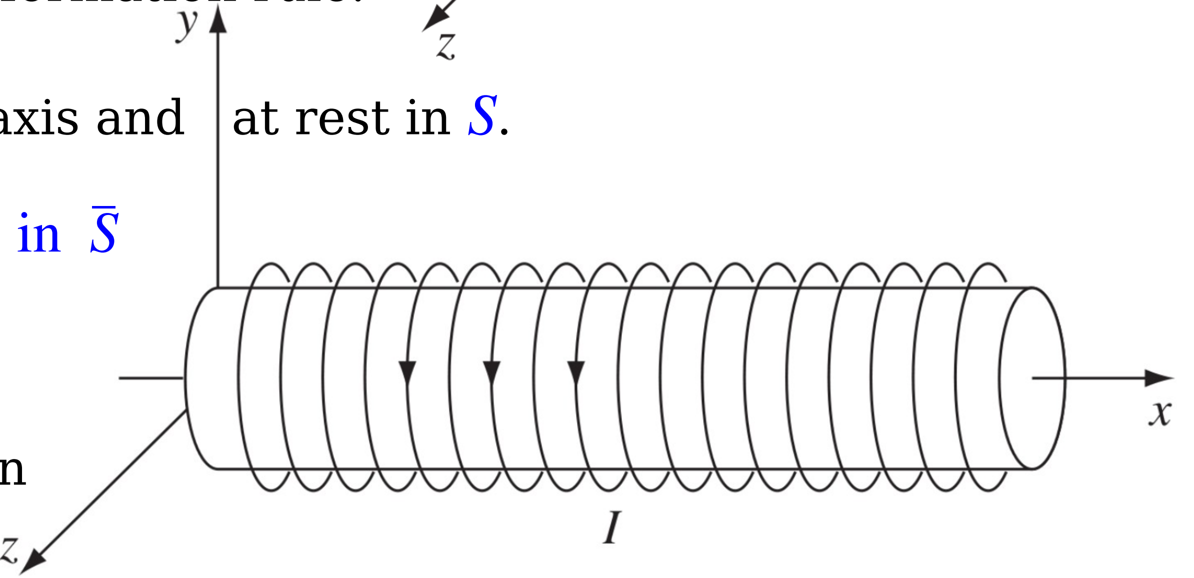
- This tells us how E_y & B_z transform—to do E_z & B_y, we simply align the same capacitor parallel to the xy plane instead of the xz plane

$$E_z = \frac{\sigma}{\epsilon_0} \qquad \text{in } S \quad \Rightarrow \quad \bar{E}_z = \gamma\left(E_z + \beta c B_y \right)$$
$$B_y = \mu_0 \sigma v_0 \qquad\qquad\qquad c\bar{B}_y = \gamma\left(c B_y + \beta E_z \right)$$



- As for the x components, we have seen $\bar{E}_x = E_x$
- In this case there is no accompanying magnetic field, we cannot deduce B_x's transformation rule.
- Imagine a long *solenoid* ∥ the x axis and at rest in S.

$$\bar{n} = \gamma n > n \quad \Leftarrow \quad \text{lenght contracts}$$
$$\bar{I} = \frac{I}{\gamma} < I \quad \Leftarrow \quad \text{time dilates} \qquad \text{in } \bar{S}$$
$$\bar{B}_x = \mu_0 \bar{n} \bar{I} = \mu_0 n I = B_x$$



- The component of $\mathbf{B}$ ∥ the motion is unchanged.
- $$\bar{E}_x = E_x, \quad \bar{E}_y = \gamma\left(E_y - \beta c B_z \right), \quad \bar{E}_z = \gamma\left(E_z + \beta c B_y \right)$$
$$\bar{B}_x = B_x, \quad c\bar{B}_y = \gamma\left(c B_y + \beta E_z \right), \quad c\bar{B}_z = \gamma\left(c B_z - \beta E_y \right)$$ (!)

- 2 special cases

1. If $\mathbf{B}$=0 in S, then $c\,\bar{\mathbf{B}} = \gamma\,\beta\left(E_z\,\hat{\mathbf{y}} - E_y\,\hat{\mathbf{z}}\right) = \beta\left(\bar{E}_z\,\hat{\mathbf{y}} - \bar{E}_y\,\hat{\mathbf{z}}\right)$ in $\bar{S}$

$$\Rightarrow \quad c\,\bar{\mathbf{B}} = -\boldsymbol{\beta} \times \bar{\mathbf{E}} \quad \Leftarrow \quad \mathbf{v} = v\,\hat{\mathbf{x}}, \quad \boldsymbol{\beta} = \frac{\mathbf{v}}{c}$$

2. If $\mathbf{E}$=0 in S, then $\bar{\mathbf{E}} = -\gamma\,\beta\left(c\,B_z\,\hat{\mathbf{y}} - c\,B_y\,\hat{\mathbf{z}}\right) = -\beta\left(c\,\bar{B}_z\,\hat{\mathbf{y}} - c\,\bar{B}_y\,\hat{\mathbf{z}}\right)$ in $\bar{S}$

$$\Rightarrow \quad \bar{\mathbf{E}} = \boldsymbol{\beta} \times \left(c\,\bar{\mathbf{B}}\right) \quad \Leftarrow \quad \mathbf{v} = v\,\hat{\mathbf{x}}, \quad \boldsymbol{\beta} = \frac{\mathbf{v}}{c}$$

- If either $\mathbf{E}$ or $\mathbf{B}$ is 0 (at a particular point) in one system, then in any other system the fields (at that point) are very simply related

<u>Example 12.15</u>: **Magnetic field of a point charge in uniform motion**. Find the magnetic field of a point charge q moving at constant velocity $\mathbf{v}$.

- In the particle's *rest* frame $\mathbf{B}$=0, so in a system moving with velocity $-\mathbf{v}$

$$\mathbf{B} = \frac{\boldsymbol{\beta}}{c} \times \mathbf{E} = \frac{\mu_0}{4\,\pi}\,\frac{q\,v\left(1-\beta^2\right)\sin\theta}{\left[1-\beta^2\sin^2\theta\right]^{3/2}}\,\frac{\hat{\boldsymbol{\phi}}}{R^2} \quad \text{using Ex 12.14}$$

$$\approx \frac{\mu_0}{4\,\pi}\,\frac{q\,\mathbf{v} \times \hat{\mathbf{R}}}{R^2} \quad \Leftarrow \quad v^2 \ll c^2$$

exactly what you get by naive application of the Biot-Savart law to a point charge

The Field Tensor

- **E** & **B** don't transform like the spatial parts of the 2 4-vectors—the components of **E** & **B** are stirred together when you go from one inertial system to another.

- An object with 6 components and transforming according to (!) is an **antisymmetric, second-rank tensor**.

- A 4-vector transforms by the rule $\bar{a}^{\mu} = \Lambda^{\mu}{}_{\nu}\, a^{\nu} \quad \Leftarrow \quad \mathbf{\Lambda} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- A (2^{nd}-rank) tensor is an object with 2 indices, which transforms with 2 factors of $\mathbf{\Lambda}$ (one for each index):

$$\bar{t}^{\mu\nu} = \Lambda^{\mu}{}_{\lambda}\, \Lambda^{\nu}{}_{\sigma}\, t^{\lambda\sigma} \quad \Leftarrow \quad t^{\mu\nu} = \begin{bmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{bmatrix} \quad 4 \times 4 = 16 \text{ components}$$

- The 16 elements need not all be different. A *symmetric* tensor has the property

$$t^{\mu\nu} = t^{\nu\mu} \quad \text{symmetric tensor, having 10 components}$$

$$\Rightarrow \quad t^{01} = t^{10}, \quad t^{02} = t^{20}, \quad t^{03} = t^{30}, \quad t^{12} = t^{21}, \quad t^{13} = t^{31}, \quad t^{23} = t^{32}$$

- An *anti*symmetric tensor obeys

$$\Rightarrow \quad t^{\mu\nu} = -t^{\nu\mu} \qquad \Rightarrow \quad t^{00} = t^{11} = t^{22} = t^{33} = 0$$

$$\Rightarrow \quad t^{\mu\nu} = \begin{bmatrix} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{bmatrix}$$ only 6 components

- Check the transformation rule for an antisymmetric tensor (6 components)

$$\bar{t}^{01} = \Lambda^0{}_\lambda \Lambda^1{}_\sigma t^{\lambda\sigma} = \Lambda^0{}_0 \Lambda^1{}_0 t^{00} + \Lambda^0{}_0 \Lambda^1{}_1 t^{01} + \Lambda^0{}_1 \Lambda^1{}_0 t^{10} + \Lambda^0{}_1 \Lambda^1{}_1 t^{11}$$

$$= \left(\Lambda^0{}_0 \Lambda^1{}_1 - \Lambda^0{}_1 \Lambda^1{}_0\right) t^{01} \quad \Leftarrow \quad t^{00} = t^{11} = 0, \quad t^{01} = -t^{10}$$

$$= \left(\gamma^2 - \gamma^2 \beta^2\right) t^{01} = t^{01}$$

$$\Rightarrow \quad \begin{array}{lll} \bar{t}^{01} = t^{01}, & \bar{t}^{02} = \gamma\left(t^{02} - \beta\, t^{12}\right), & \bar{t}^{03} = \gamma\left(t^{03} + \beta\, t^{31}\right) \\ \bar{t}^{23} = t^{23}, & \bar{t}^{31} = \gamma\left(t^{31} + \beta\, t^{03}\right), & \bar{t}^{12} = \gamma\left(t^{12} - \beta\, t^{02}\right) \end{array}$$

precisely the rules we obtained on physical grounds for the EM fields.

- we can construct the field tensor $\mathbb{F}$ by direct comparison:

$$\mathbb{F}^{01} \equiv \frac{E_x}{c}, \quad \mathbb{F}^{23} \equiv B_x$$

$$\mathbb{F}^{02} \equiv \frac{E_y}{c}, \quad \mathbb{F}^{31} \equiv B_y \quad \Rightarrow \quad \mathbb{F}^{\mu\nu} = \begin{bmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{bmatrix}$$

$$\mathbb{F}^{03} \equiv \frac{E_z}{c}, \quad \mathbb{F}^{12} \equiv B_z$$

- Swapping the roles of $\mathbf{E}$ & $\mathbf{B}$ and doing the same thing leads to **dual tensor** $\mathscr{F}^{\mu\nu}$

$$\mathscr{F}^{\mu\nu} = \frac{1}{2!}\epsilon^{\mu\nu}{}_{\lambda\sigma}\mathbb{F}^{\lambda\sigma} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -\frac{E_z}{c} & \frac{E_y}{c} \\ -B_y & \frac{E_z}{c} & 0 & -\frac{E_x}{c} \\ -B_z & -\frac{E_y}{c} & \frac{E_x}{c} & 0 \end{bmatrix} \Leftarrow$$

$$\epsilon^{0123} = 1\,, \quad \epsilon_{0123} = -1\,,$$

$$\epsilon_{\mu\nu\lambda\sigma} = \eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\lambda\gamma}\eta_{\sigma\delta}\epsilon^{\alpha\beta\gamma\delta}$$

$$\Rightarrow \quad \epsilon_{\mu\nu\lambda\sigma} = -\epsilon^{\mu\nu\lambda\sigma}$$

- $\mathscr{F}^{\mu\nu}$ can be obtained directly from $\mathbb{F}^{\mu\nu}$ by the substitution $\dfrac{\mathbf{E}}{c} \to \mathbf{B}\,, \quad \mathbf{B} \to -\dfrac{\mathbf{E}}{c}$.

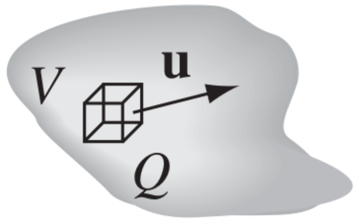
- $$\partial_\nu \mathscr{F}^{\mu\nu} = \frac{1}{2!}\epsilon^{\mu\nu\lambda\sigma}\partial_\nu \mathbb{F}_{\lambda\sigma} \quad \Leftarrow \quad \mathbb{F}_{\lambda\sigma} = \eta_{\lambda\mu}\eta_{\sigma\nu}\mathbb{F}^{\mu\nu}\,, \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

$$= \frac{1}{3!\,2!}\epsilon^{\mu\nu\lambda\sigma}\left(\partial_\nu \mathbb{F}_{\lambda\sigma} + \partial_\lambda \mathbb{F}_{\sigma\nu} + \partial_\sigma \mathbb{F}_{\nu\lambda} - \partial_\nu \mathbb{F}_{\sigma\lambda} - \partial_\sigma \mathbb{F}_{\lambda\nu} - \partial_\lambda \mathbb{F}_{\nu\sigma}\right)$$

$$= \frac{1}{3!}\epsilon^{\mu\nu\lambda\sigma}\left(\partial_\nu \mathbb{F}_{\lambda\sigma} + \partial_\lambda \mathbb{F}_{\sigma\nu} + \partial_\sigma \mathbb{F}_{\nu\lambda}\right)$$

$$\partial_\nu \mathscr{F}^{\mu\nu} = 0 \quad \Rightarrow \quad \partial_\nu \mathbb{F}_{\lambda\sigma} + \partial_\lambda \mathbb{F}_{\sigma\nu} + \partial_\sigma \mathbb{F}_{\nu\lambda} = 0 \quad \Rightarrow \quad \text{Problem 12.54}$$

Electrodynamics in Tensor Notation



- Determine how the *sources* of the fields, ρ and $\boldsymbol{J}$, transform.
- The charge density is $\rho = \dfrac{Q}{V}$, the current density is $\boldsymbol{J} = \rho\,\mathbf{u}$
- Express these quantities in terms of the **proper charge density** ρ_0, the density *in the rest system of the charge*:

$$\boldsymbol{\beta}_u \equiv \frac{\mathbf{u}}{c}, \quad \beta_u = \frac{u}{c}$$

$$\rho_0 = \frac{Q}{V_0} \;+\; V = \frac{V_0}{\gamma_u} \;\text{ Lorentz contraction } \Rightarrow \; \rho = \gamma_u\,\rho_0, \quad \boldsymbol{J} = \gamma_u\,\rho_0\,\mathbf{u} \;\Leftarrow\; \gamma_u \equiv \frac{1}{\sqrt{1-\beta_u^2}}$$

- Evidently charge density and current density go together to make a 4-vector:

$$\vec{J} = \rho_0\,\vec{U} \;\Rightarrow\; J^\mu = \rho_0\,U^\mu \;\Rightarrow\; J^\mu = \left(c\,\rho\,,\,J_x\,,\,J_y\,,\,J_z\right)$$ current density 4-vector

- The continuity eqn $\nabla\cdot\boldsymbol{J} = -\dfrac{\partial\rho}{\partial t}$ expressing the local conservation of charge,

takes on a nice compact form when written in terms of J^μ

$$\nabla\cdot\boldsymbol{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = \sum_{i=1}^{3}\frac{\partial J^i}{\partial x^i}, \quad \frac{\partial\rho}{\partial t} = \frac{1}{c}\frac{\partial J^0}{\partial t} = \frac{\partial J^0}{\partial x^0} \;\Rightarrow\; \partial_\mu J^\mu = \frac{\partial J^\mu}{\partial x^\mu} = 0$$

- $\dfrac{\partial J^\mu}{\partial x^\mu}$ is the 4d *divergence* of J^μ, so the continuity equation states that the

current density 4-vector is divergenceless.

- Maxwell's equations can be written as $\frac{\partial \mathbb{F}^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \frac{\partial \mathscr{F}^{\mu\nu}}{\partial x^\nu} = 0,$ 4 equations each.

- $\mu=0$: $$\frac{\partial \mathbb{F}^{0\nu}}{\partial x^\nu} = \frac{\partial \mathbb{F}^{00}}{\partial x^0} + \frac{\partial \mathbb{F}^{01}}{\partial x^1} + \frac{\partial \mathbb{F}^{02}}{\partial x^2} + \frac{\partial \mathbb{F}^{03}}{\partial x^3} = \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)$$
$$= \frac{1}{c}\nabla\cdot\mathbf{E} = \mu_0 J^0 = \mu_0 c\rho \quad \Rightarrow \quad \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{Gauss's law}$$

- $\mu=1$: $$\frac{\partial \mathbb{F}^{1\nu}}{\partial x^\nu} = \frac{\partial \mathbb{F}^{10}}{\partial x^0} + \frac{\partial \mathbb{F}^{11}}{\partial x^1} + \frac{\partial \mathbb{F}^{12}}{\partial x^2} + \frac{\partial \mathbb{F}^{13}}{\partial x^3} = -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}$$
$$= -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + (\nabla\times\mathbf{B})_x = \mu_0 J^1 = \mu_0 J_x \quad \text{combined with } \mu = 2, 3$$
$$\Rightarrow \quad \nabla\times\mathbf{B} = \mu_0 \boldsymbol{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t} \quad \text{Ampère-Maxwell law}$$

- $\mu=0$: $$\frac{\partial \mathscr{F}^{0\nu}}{\partial x^\nu} = \frac{\partial \mathscr{F}^{00}}{\partial x^0} + \frac{\partial \mathscr{F}^{01}}{\partial x^1} + \frac{\partial \mathscr{F}^{02}}{\partial x^2} + \frac{\partial \mathscr{F}^{03}}{\partial x^3} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \nabla\cdot\mathbf{B} = 0$$

- $\mu=1$: $$\frac{\partial \mathscr{F}^{1\nu}}{\partial x^\nu} = \frac{\partial \mathscr{F}^{10}}{\partial x^0} + \frac{\partial \mathscr{F}^{11}}{\partial x^1} + \frac{\partial \mathscr{F}^{12}}{\partial x^2} + \frac{\partial \mathscr{F}^{13}}{\partial x^3} = -\frac{1}{c}\frac{\partial B_x}{\partial t} - \frac{1}{c}\frac{\partial E_z}{\partial y} + \frac{1}{c}\frac{\partial E_y}{\partial z}$$
$$= -\frac{1}{c}\frac{\partial B_x}{\partial t} - \frac{(\nabla\times\mathbf{E})_x}{c} = 0 \quad \text{combined with } \mu = 2, 3 \quad \Rightarrow \quad \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{Faraday's law}$$

- In term of $\mathbb{F}^{\mu\nu}$ and the proper velocity U^{μ}, the Minkowski force on a charge q is

$$K^{\mu} = q\, U_{\nu}\, \mathbb{F}^{\mu\nu} \quad \Rightarrow \quad \vec{K} = q\, \mathbb{F} \cdot \vec{U}$$

- If $\mu=1$: $K^{1} = q\, U_{\nu}\, \mathbb{F}^{1\nu} = q\left(-U^{0}\,\mathbb{F}^{10} + U^{1}\,\mathbb{F}^{11} + U^{2}\,\mathbb{F}^{12} + U^{3}\,\mathbb{F}^{13}\right)$

$$= q\left(\left(-\gamma_u c\right)\frac{-E_x}{c} + \gamma_u u_y B_z + \gamma_u u_z\left(-B_y\right)\right)$$

$$= q\,\gamma_u\left(\mathbf{E} + \mathbf{u}\times\mathbf{B}\right)_x \quad \text{combined with} \quad \mu = 2{,}3$$

$$\Rightarrow \quad \mathbf{K} = q\,\gamma_u\left(\mathbf{E} + \mathbf{u}\times\mathbf{B}\right) = q\,\gamma_u\left[\mathbf{E} + \boldsymbol{\beta}_u \times\left(c\,\mathbf{B}\right)\right] \quad \Rightarrow \quad \mathbf{F} = q\left(\mathbf{E} + \mathbf{u}\times\mathbf{B}\right)$$

- Work out the $\mu=0$ part yourself.

Relativistic Potentials

- The electric and magnetic fields can be expressed in terms of a scalar potential Φ and a vector potential $\mathbf{A}$: $\mathbf{E} = -\nabla\Phi - \dfrac{\partial \mathbf{A}}{\partial t}$, $\quad \mathbf{B} = \nabla \times \mathbf{A}$

- Φ and $\mathbf{A}$ together constitute a 4-vector: $A^\mu = \left(\dfrac{\Phi}{c}, A_x, A_y, A_z \right)$

- In terms of this **4-vector potential**, the field tensor $\mathbb{F}^{\mu\nu} = \dfrac{\partial A^\nu}{\partial x_\mu} - \dfrac{\partial A^\mu}{\partial x_\nu}$

- For $\mu=0$, $\nu=1$: $\mathbb{F}^{01} = \dfrac{\partial A^1}{\partial x_0} - \dfrac{\partial A^0}{\partial x_1} = -\dfrac{\partial A_x}{\partial ct} - \dfrac{1}{c}\dfrac{\partial \Phi}{\partial x} = -\dfrac{1}{c}\left(\dfrac{\partial \mathbf{A}}{\partial t} + \nabla\Phi \right)_x = \dfrac{E_x}{c}$

$$\Rightarrow \quad \mathbb{F}^{02} = \frac{E_y}{c}, \quad \mathbb{F}^{03} = \frac{E_z}{c}$$

- For $\mu=1$, $\nu=2$: $\mathbb{F}^{12} = \dfrac{\partial A^2}{\partial x_1} - \dfrac{\partial A^1}{\partial x_2} = \dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y} = (\nabla \times \mathbf{A})_z = B_z$

$$\Rightarrow \quad \mathbb{F}^{23} = B_x, \quad \mathbb{F}^{31} = B_y$$

- The potential formulation automatically takes care of the homogeneous Maxwell equation $\dfrac{\partial \mathscr{F}^{\mu\nu}}{\partial x^\nu} = 0 \quad \Leftarrow \quad \epsilon^{\mu\nu\lambda\sigma}\partial_\nu\partial_\lambda A_\sigma = 0 \quad \Leftarrow \quad \partial_\nu\partial_\lambda = \partial_\lambda\partial_\nu$

- As for the inhomogeneous equation $\dfrac{\partial \mathbb{F}^{\mu\nu}}{\partial x^{\nu}} = \dfrac{\partial}{\partial x_{\mu}} \dfrac{\partial A^{\nu}}{\partial x^{\nu}} - \dfrac{\partial}{\partial x_{\nu}} \dfrac{\partial A^{\mu}}{\partial x^{\nu}} = \mu_0 J^{\mu} \quad (\&)$
- The potentials are not uniquely determined by the fields—you could add to A^{μ}

the gradient of any scalar function λ: $A^{\mu} \to A'^{\mu} = A^{\mu} + \dfrac{\partial \lambda}{\partial x_{\mu}}$ without changing $\mathbb{F}^{\mu\nu}$ —the **gauge invariance**.

- The Lorenz gauge condition $\nabla \cdot \mathbf{A} = -\dfrac{1}{c^2}\dfrac{\partial \Phi}{\partial t}$ can be written as $\dfrac{\partial A^{\mu}}{\partial x^{\mu}} = 0$
- In the Lorenz gauge, (&) reduces to

$$\boxed{\Box A^{\mu} = -\mu_0 J^{\mu}} \quad \Leftarrow \quad \text{d'Alembertian } \Box \equiv \frac{\partial}{\partial x_{\mu}} \frac{\partial}{\partial x^{\mu}} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

- The equation combines our previous results into a single 4-vector equation—it represents the most elegant formulation of Maxwell's equations.