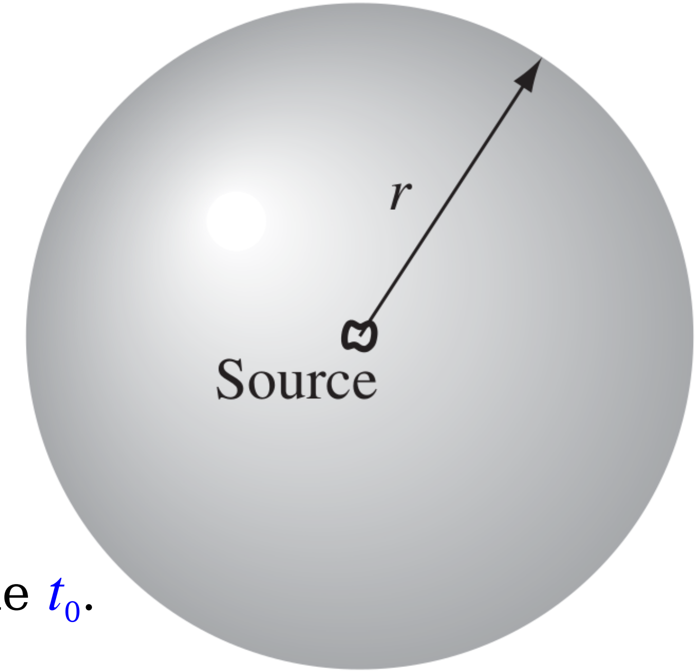
Chapter 11 Radiation

Dipole Radiation

What is Radiation?



- When charges *accelerate*, their fields can transport energy irreversibly out to infinity—a process we call **radiation**.

- Assume the source is localized near the origin; we would like to calculate the energy it is radiating at time t_0.

- Imagine a sphere, out at radius r. The power passing through its surface is the integral of the Poynting vector: $P(r,t) = \oint \mathbf{S} \cdot d\boldsymbol{a} = \frac{1}{\mu_0} \oint \mathbf{E} \times \mathbf{B} \cdot d\boldsymbol{a}$

- Because EM "news" travels at the speed of light, this energy actually left the source at the earlier time $t_0 = t - \frac{r}{c}$, so the power radiated is $P_{\text{rad}}(t_0) = \lim_{r \to \infty} P\left(r, t_0 + \frac{r}{c}\right)$

- This is energy (per unit time) that is carried away and never comes back.

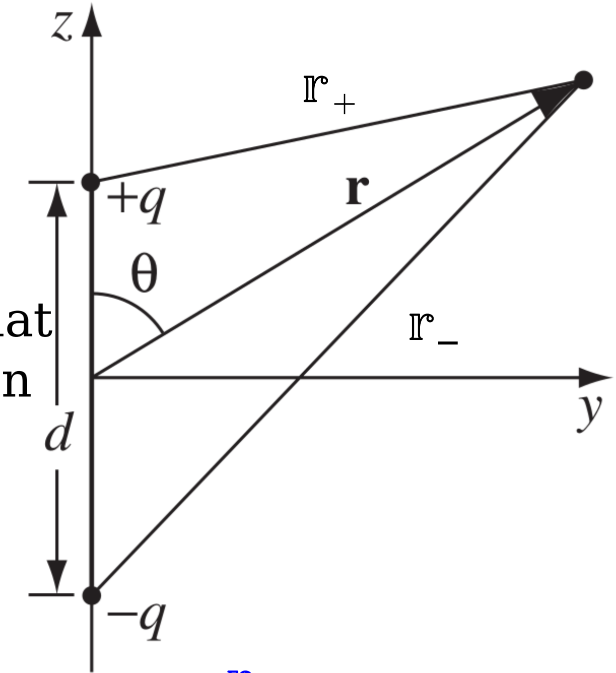
- The area of the sphere is $4\pi r^2$, so the Poynting vector must decrease (at large r) no faster than $\frac{1}{r^2}$. If it went $\frac{1}{r^3}$, then P would go $\frac{1}{r}$, and P_{rad} would be 0.

- According to Coulomb's law, electro*static* fields fall off like $\frac{1}{r^2}$ (or faster, if the charge is 0), and the Biot-Savart law says that magneto*static* fields go like $\frac{1}{r^2}$ (or faster), so $S \sim \frac{1}{r^4}$, for static configurations. So *static* sources do not radiate.

- Jefimenko's equations indicate that *time-dependent* fields include terms (involving $\dot{\rho}$ and $\dot{\mathbf{J}}$) that go like $\frac{1}{r}$; these are the terms that are responsible for EM radiation.

- The study of radiation involves picking out the parts of $\mathbf{E}$ & $\mathbf{B}$ that go like $\frac{1}{r}$ at large distances from the source, constructing from them the $\frac{1}{r^2}$ term in $\mathbf{S}$, integrating over a large spherical surface, and taking the limit as $r \to \infty$.

Electric Dipole Radiation (Hertzian dipole)



- 2 tiny metal spheres are separated by a distance d and connected by a fine wire; the charge on the upper sphere is $q(t)$, and the charge on the lower sphere is $-q(t)$. Let that we drive the charge back and forth through the wire, at an angular frequency ω:

$$q(t) = q_0 \cos \omega t \quad \Leftarrow \quad \mathbf{p}(t) = p_0 \cos \omega t \, \hat{\mathbf{z}} \quad \Leftarrow \quad p_0 \equiv q_0 d$$

electric dipole

$$\Rightarrow \quad \Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\left(\frac{q_0 \cos \omega t_{r+}}{\mathbb{r}_+} - \frac{q_0 \cos \omega t_{r-}}{\mathbb{r}_-}\right) \quad \Leftarrow \quad t_{r\pm} = t - \frac{\mathbb{r}_\pm}{c}$$

$$\text{where} \quad \mathbb{r}_\pm = \sqrt{r^2 \mp r\, d \cos\theta + \frac{d^2}{4}}$$

- To make this *physical* dipole into a *perfect* dipole, we want the separation distance to be extremely small:

approximation 1: $d \ll r$: to *1^st^ order* in d: $\quad \Rightarrow \quad \mathbb{r}_\pm \simeq r\left(1 \mp \frac{d}{2r}\cos\theta\right)$

$$\Rightarrow \quad \cos \omega t_{r\pm} = \cos \omega\left(t - \frac{\mathbb{r}_\pm}{c}\right) \simeq \cos \omega\left(t - \frac{r}{c} \pm \frac{d\cos\theta}{2c}\right)$$

$$= \cos\frac{\omega(ct - r)}{c}\cos\frac{\omega d\cos\theta}{2c} \mp \sin\frac{\omega(ct - r)}{c}\sin\frac{\omega d\cos\theta}{2c}$$

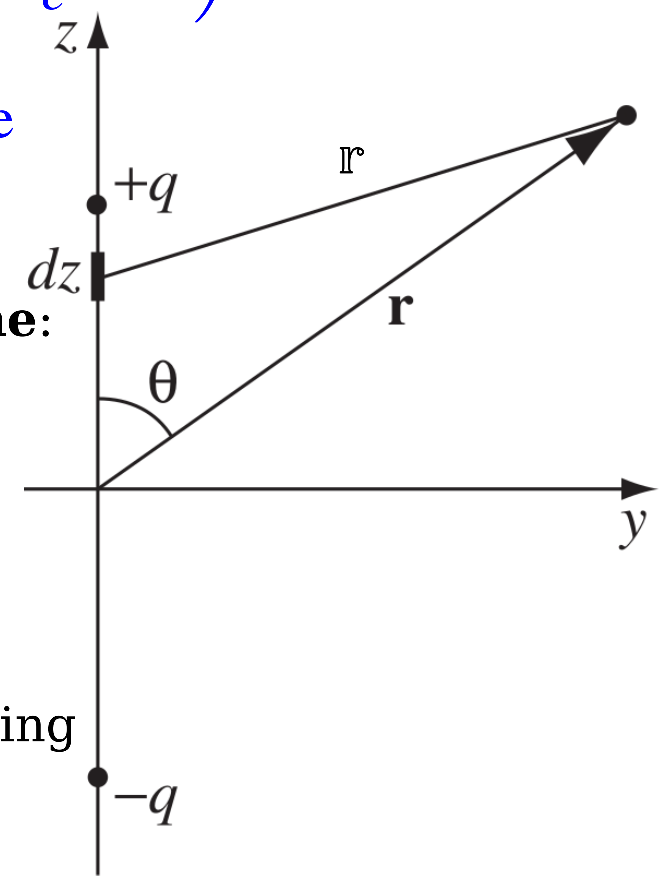
- In the perfect dipole limit we have, further,

Approximation 2: $d \ll \dfrac{c}{\omega} \;\Rightarrow\; d \ll \lambda \;\Leftarrow\; \lambda = \dfrac{2\,\pi\,c}{\omega}\,, \qquad \dfrac{1}{\mathscr{r}_\pm} \simeq \dfrac{1}{r}\left(1 \pm \dfrac{d}{2\,r}\cos\theta\right)$

$$\Rightarrow \quad \cos\omega\left(t - \frac{\mathscr{r}_\pm}{c}\right) \simeq \cos\frac{\omega\,(c\,t-r)}{c} \mp \frac{\omega\,d}{2\,c}\cos\theta\,\sin\frac{\omega\,(c\,t-r)}{c}$$

$$\Rightarrow \quad \Phi\,(r\,,\theta\,,t) \simeq \frac{p_0\cos\theta}{4\,\pi\,\epsilon_0\,r}\left(\frac{1}{r}\cos\frac{\omega\,(c\,t-r)}{c} - \frac{\omega}{c}\sin\frac{\omega\,(c\,t-r)}{c}\right) \quad \Leftarrow \quad p_0 = q_0\,d$$

$\omega \to 0 \quad \Rightarrow \quad \Phi \simeq \dfrac{p_0\cos\theta}{4\,\pi\,\epsilon_0\,r^2}$ potential of a stationary dipole



- We are interested in the fields that survive at *large distances from the source*, in the so-called **radiation zone**:

approximation 3: $r \gg \dfrac{c}{\omega} \;\Rightarrow\; r \gg \lambda \;\Rightarrow\; \dfrac{1}{r} \ll \dfrac{\omega}{c}$

$$\Rightarrow \quad \Phi\,(r\,,\theta\,,t) \simeq -\frac{p_0\,\omega}{4\,\pi\,\epsilon_0\,c}\,\frac{\cos\theta}{r}\,\sin\frac{\omega\,(c\,t-r)}{c}$$

- The *vector* potential is determined by the current flowing in the wire: $\boldsymbol{I}\,(t) = \dfrac{\mathrm{d}\,q}{\mathrm{d}\,t}\,\hat{\mathbf{z}} = -\,q_0\,\omega\sin\omega\,t\,\hat{\mathbf{z}}$

$$
\left[\begin{array}{l}
\dfrac{1}{\mathbb{r}_{\pm}} \simeq \dfrac{1}{r}\left(1 \pm \dfrac{d}{2\,r}\cos\theta\right) \\[2ex]
\cos\omega\, t_{r\,\pm} \simeq \cos\dfrac{\omega\,(c\,t-r)}{c} \mp \dfrac{\omega\, d}{2\,c}\cos\theta \sin\dfrac{\omega\,(c\,t-r)}{c} \quad \Leftarrow \quad \begin{array}{l} \cos\vartheta \simeq 1 \\ \sin\vartheta \simeq \vartheta \end{array} \quad \text{for} \ \ \vartheta \ll 1
\end{array}\right.
$$

$$
\begin{aligned}
\Rightarrow \quad \Phi\,(r\,,\theta\,,t) &= \frac{1}{4\,\pi\,\epsilon_0}\left(\frac{q_0\cos\omega\, t_{r\,+}}{\mathbb{r}_{+}} - \frac{q_0\cos\omega\, t_{r\,-}}{\mathbb{r}_{-}}\right) \\
&\simeq \frac{q_0}{4\,\pi\,\epsilon_0\, r}\left[\left(1+\frac{d}{2\,r}\cos\theta\right)\left(\cos\frac{\omega\,(c\,t-r)}{c} - \frac{\omega\, d}{2\,c}\cos\theta \sin\frac{\omega\,(c\,t-r)}{c}\right)\right. \\
&\qquad\qquad \left. - \left(1-\frac{d}{2\,r}\cos\theta\right)\left(\cos\frac{\omega\,(c\,t-r)}{c} + \frac{\omega\, d}{2\,c}\cos\theta \sin\frac{\omega\,(c\,t-r)}{c}\right)\right] \\
&= \frac{q_0}{4\,\pi\,\epsilon_0\, r}\left(\frac{d}{r}\cos\theta\cos\frac{\omega\,(c\,t-r)}{c} - \frac{\omega\, d}{c}\cos\theta \sin\frac{\omega\,(c\,t-r)}{c}\right) \\
&= \frac{p_0\cos\theta}{4\,\pi\,\epsilon_0\, r}\left(\frac{1}{r}\cos\frac{\omega\,(c\,t-r)}{c} - \frac{\omega}{c}\sin\frac{\omega\,(c\,t-r)}{c}\right) \quad \Leftarrow \quad p_0 = q_0\, d
\end{aligned}
$$

$$\Rightarrow \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{-d/2}^{+d/2} \frac{-q_0\,\omega}{\mathbb{r}} \sin\frac{\omega(c\,t - \mathbb{r})}{c}\,\hat{\mathbf{z}}\,\mathrm{d}\,z$$

$$\Rightarrow \quad \boxed{\mathbf{A}(r, \theta, t) \approx -\frac{\mu_0\,p_0\,\omega}{4\pi r} \sin\frac{\omega(c\,t - r)}{c}\,\hat{\mathbf{z}}} \quad \Leftarrow \quad \hat{\mathbf{z}} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$$

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\boldsymbol{\theta}} = \frac{p_0\,\omega}{4\pi\epsilon_0 c}\left[\frac{\cos\theta}{r^2}\left(\sin\frac{\omega(c\,t-r)}{c} + \frac{\omega\,r}{c}\cos\frac{\omega(c\,t-r)}{c}\right)\hat{\mathbf{r}}\right.$$

$$\left. + \frac{\sin\theta}{r^2}\sin\frac{\omega(c\,t-r)}{c}\,\hat{\boldsymbol{\theta}}\right] \simeq \frac{p_0\,\omega^2}{4\pi\epsilon_0 c^2}\frac{\cos\theta}{r}\cos\frac{\omega(c\,t-r)}{c}\,\hat{\mathbf{r}}$$

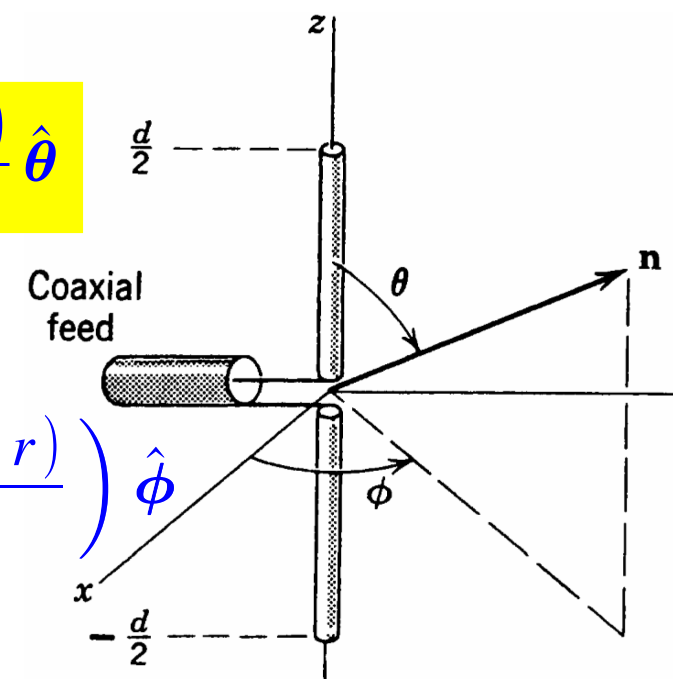
$$\frac{\partial\mathbf{A}}{\partial t} = -\frac{\mu_0\,p_0\,\omega^2}{4\pi}\left(\frac{\cos\theta}{r}\hat{\mathbf{r}} - \frac{\sin\theta}{r}\hat{\boldsymbol{\theta}}\right)\cos\frac{\omega(c\,t-r)}{c}$$

$$\Rightarrow \quad \boxed{\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} \simeq -\frac{\mu_0\,p_0\,\omega^2}{4\pi}\frac{\sin\theta}{r}\cos\frac{\omega(c\,t-r)}{c}\,\hat{\boldsymbol{\theta}}}$$

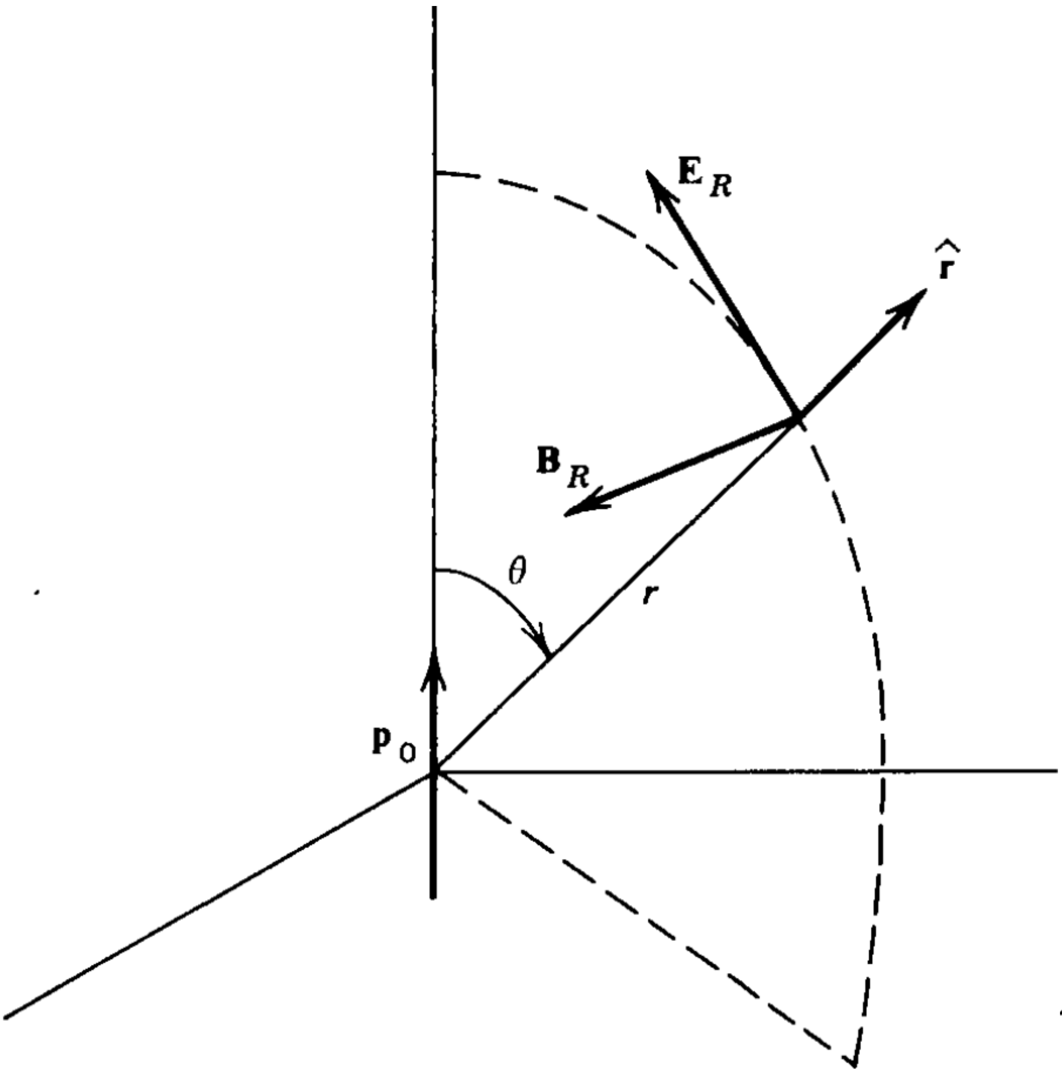
$$\nabla\times\mathbf{A} = \left(\frac{1}{r}\frac{\partial}{\partial r}(r\,A_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}}$$

$$= -\frac{\mu_0\,p_0\,\omega}{4\pi r}\left(\frac{\omega}{c}\sin\theta\cos\frac{\omega(c\,t-r)}{c} + \frac{\sin\theta}{r}\sin\frac{\omega(c\,t-r)}{c}\right)\hat{\boldsymbol{\phi}}$$

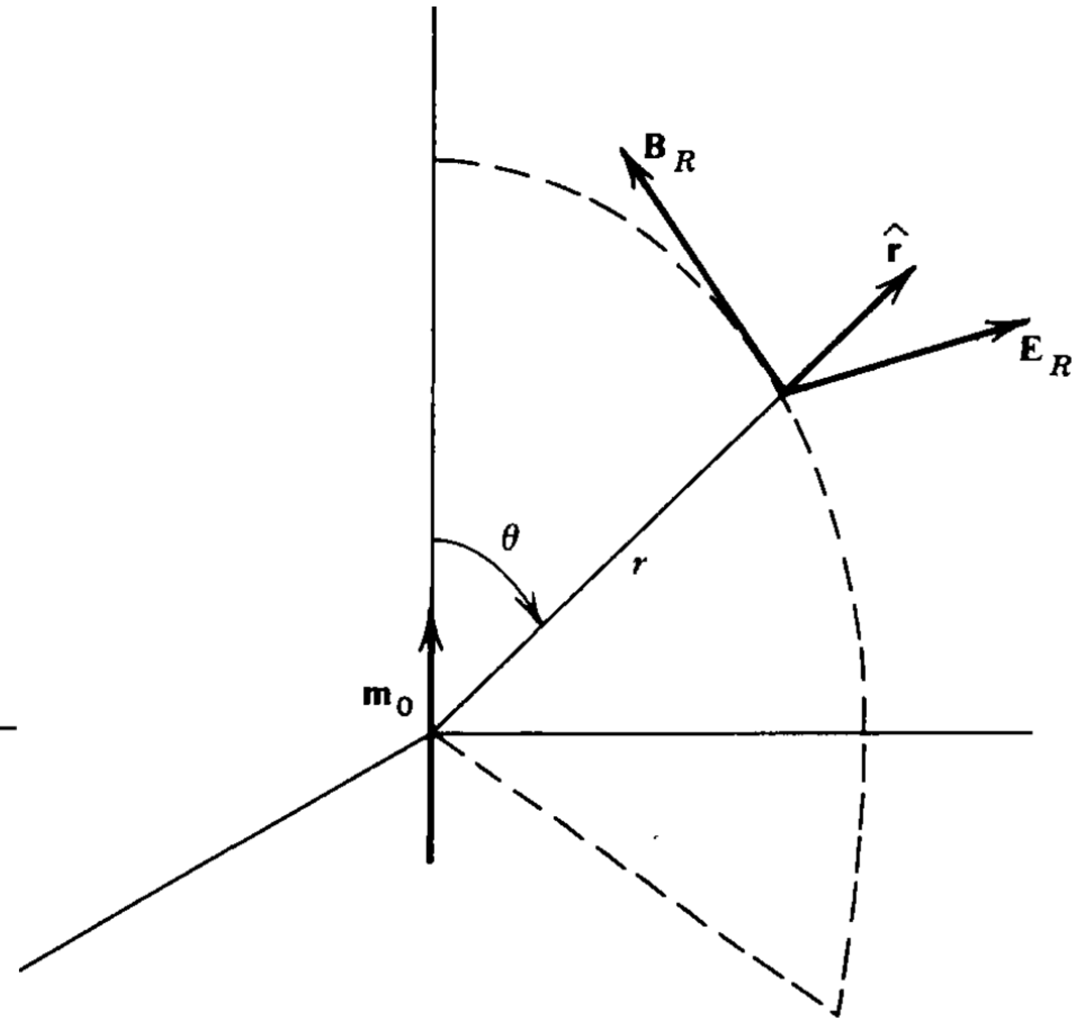
$$\Rightarrow \quad \boxed{\mathbf{B} = \nabla\times\mathbf{A} \simeq -\frac{\mu_0\,p_0\,\omega^2}{4\pi c}\frac{\sin\theta}{r}\cos\frac{\omega(c\,t-r)}{c}\,\hat{\boldsymbol{\phi}}}$$

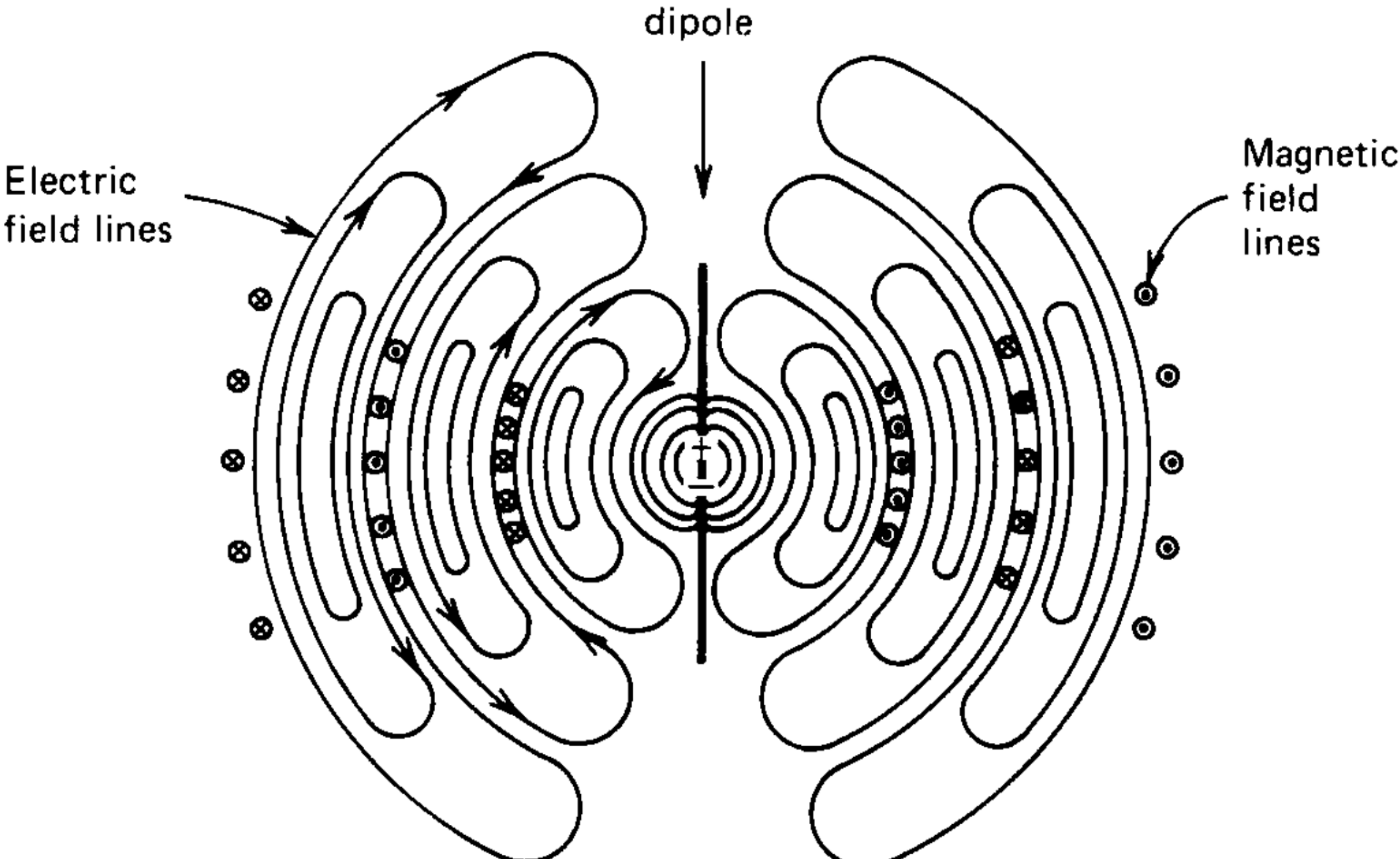


Electric dipole



magnetic dipole



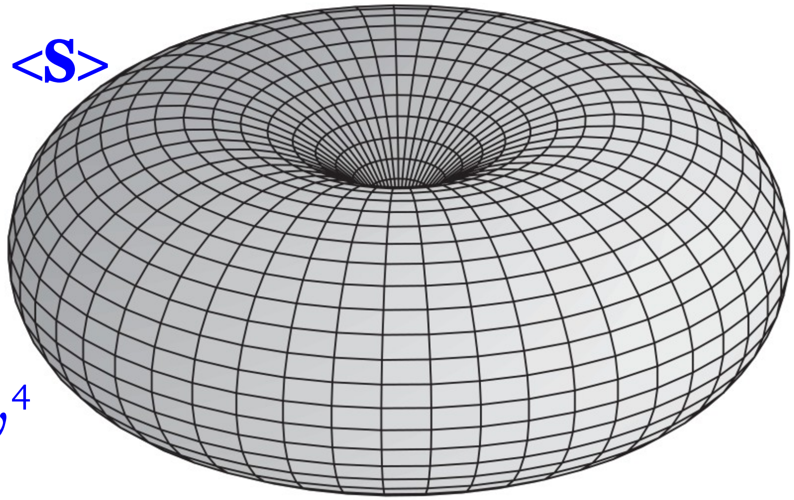

dipole
Electric field lines
Magnetic field lines

- Here $\mathbf{E}$ & $\mathbf{B}$ represent monochromatic waves of frequency ω traveling in the radial direction at the speed of light. The fields are in phase, mutually perpendicular, transverse; the ratio of their amplitudes is $\frac{E_0}{B_0}=c$, as expected.
- These are *spherical* waves, not plane waves, and their amplitude decreases like $\frac{1}{r}$ as they progress. For large r, they are approximately plane over small region.
- The energy radiated by an oscillating electric dipole is determined by the Poynting vector:

$$\mathbf{S}(\mathbf{r},t)=\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}=\frac{\mu_0}{c}\left[\frac{p_0\omega^2}{4\pi}\frac{\sin\theta}{r}\cos\frac{\omega(ct-r)}{c}\right]^2\hat{\mathbf{r}} \quad\Rightarrow\quad \langle\mathbf{S}\rangle=\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}}$$

- There is no radiation along the *axis* of the dipole (here $\sin\theta=0$); the intensity profile takes the form of a donut, with its maximum in the equatorial plane.

- The total power radiated is found by integrating **<S>** over a sphere of radius r:



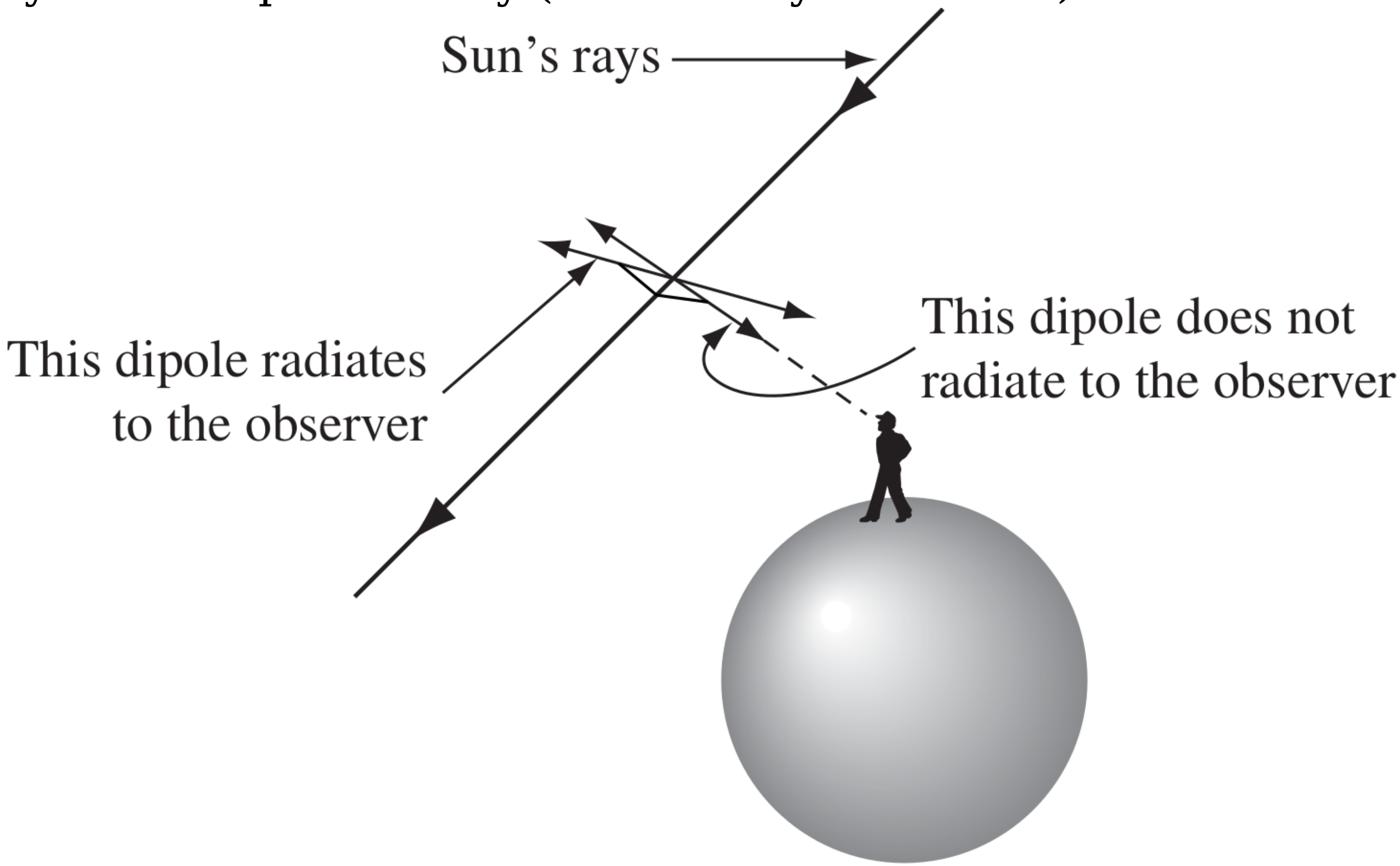
$$\langle P\rangle=\oint\langle\mathbf{S}\rangle\cdot d\boldsymbol{a}$$

$$=\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\int\frac{\sin^2\theta}{r^2}r^2\sin\theta\, d\theta\, d\phi=\frac{\mu_0 p_0^2}{12\pi c}\omega^4$$

Example 11.1: The strong frequency dependence of the power formula is what accounts for the blueness of the sky. Sunlight passing through the atmosphere stimulates atoms to oscillate as tiny dipoles.

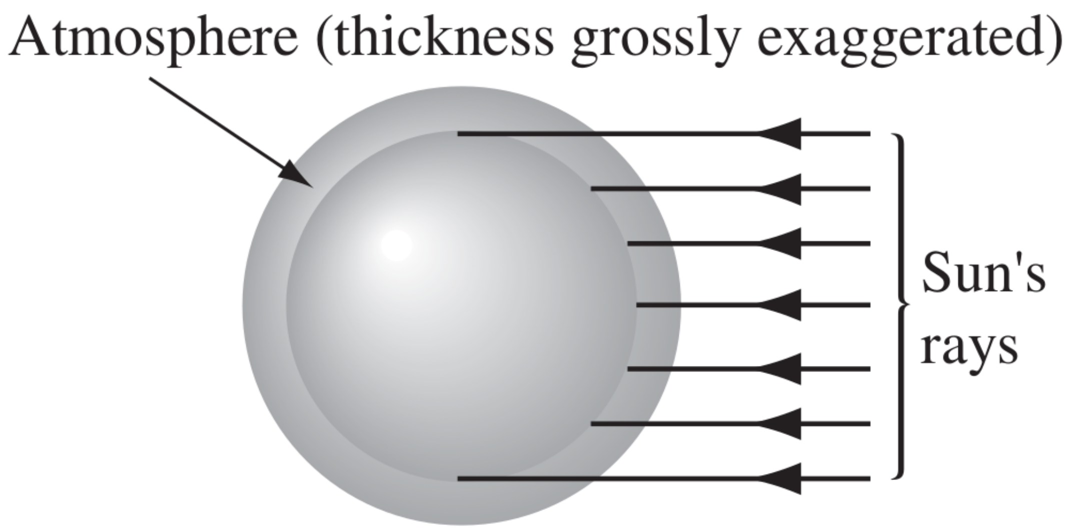
- The incident solar radiation covers a broad range of frequencies (white light), but the energy absorbed and re-radiated by the atmospheric dipoles is stronger at the higher frequencies because of the ω^4.

- It is more intense in the blue than in the red. It is this re-radiated light that you see when you look up in the sky (not directly at the sun).

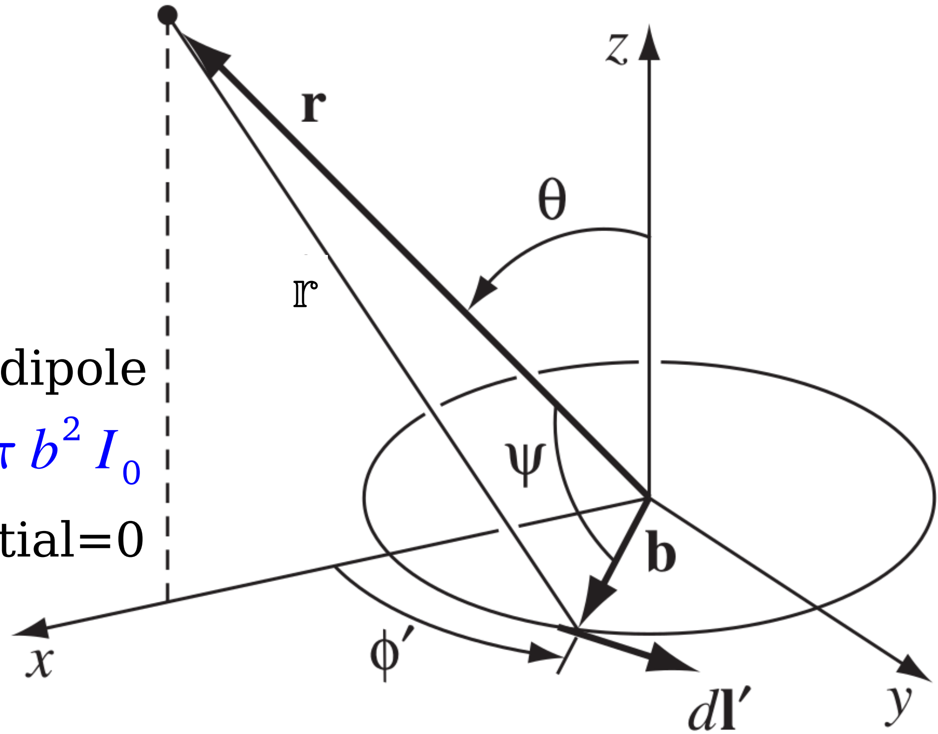


- In the celestial arc ⊥ the sun’s rays, where the blueness is most pronounced, the dipoles oscillating along the line of sight send no radiation to the observer; light received at this angle is polarized ⊥ the sun’s rays.

- The redness of sunset is the other side of the same coin: Sunlight coming in at a tangent to the earth’s surface must pass through a much longer stretch of atmosphere than sunlight coming from overhead. Accordingly, much of the blue has been removed by scattering, and what’s left is red.



Magnetic Dipole Radiation



- A wire loop of radius b with an alternating current: $I(t) = I_0 \cos\omega t$

- This is a model for an oscillating *magnetic* dipole

$$\mathbf{m}(t) = \pi b^2 I(t)\,\hat{\mathbf{z}} = m_0 \cos\omega t\,\hat{\mathbf{z}} \quad \Leftarrow \quad m_0 = \pi b^2 I_0$$

- The loop is uncharged, so the scalar potential=0

$$\Rightarrow \quad \mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{I_0}{\mathbb{r}} \cos\frac{\omega(ct-\mathbb{r})}{c}\,\mathrm{d}\boldsymbol{\ell}'$$

- For a point $\mathbf{r}$ directly above the x axis, $\mathbf{A}$ must aim in the y direction, since the x components from symmetrically placed points on either side of the x axis cancel.

$$\Rightarrow \quad \mathbf{A}(\mathbf{r},t) = \frac{\mu_0 I_0 b}{4\pi}\hat{\mathbf{y}}\int \frac{1}{\mathbb{r}}\cos\frac{\omega(ct-\mathbb{r})}{c}\cos\phi'\,\mathrm{d}\phi' \quad \Leftarrow \quad \cos\phi' \text{ for } \mathrm{d}\ell_y$$

$$\mathbf{r} = r\sin\theta\,\hat{\mathbf{x}} + r\cos\theta\,\hat{\mathbf{z}}, \quad \mathbf{b} = b\cos\phi'\,\hat{\mathbf{x}} + b\sin\phi'\,\hat{\mathbf{y}}$$

$$\Rightarrow \quad \mathbb{r} = |\mathbf{r}-\mathbf{b}| = \sqrt{r^2+b^2-2rb\cos\psi} = \sqrt{r^2+b^2-2rb\sin\theta\cos\phi'}$$

- For a “perfect” dipole, we want the loop to be extremely small:

approximation 1: $b \ll r \;\Rightarrow\; \mathbb{r} \simeq r - b\sin\theta\cos\phi' \;\Rightarrow\; \dfrac{1}{\mathbb{r}} \simeq \dfrac{1}{r}\left(1+\dfrac{b}{r}\sin\theta\cos\phi'\right)$

$$\Rightarrow \quad \cos\frac{\omega\left(c\,t-\mathbb{r}\right)}{c} \simeq \cos\omega\left(t-\frac{r}{c}+\frac{b}{c}\sin\theta\cos\phi'\right)$$

$$=\cos\frac{\omega\left(c\,t-r\right)}{c}\cos\frac{\omega\,b\sin\theta\cos\phi'}{c}-\sin\frac{\omega\left(c\,t-r\right)}{c}\sin\frac{\omega\,b\sin\theta\cos\phi'}{c}$$

- Assume the size of the dipole is small compared to the wavelength radiated:

approximation 2: $b \ll \frac{c}{\omega} \quad \Leftarrow \quad \cos\vartheta \simeq 1\,, \quad \sin\vartheta \simeq \vartheta \quad \text{for} \quad \vartheta \ll 1$

$$\Rightarrow \quad \cos\frac{\omega\left(c\,t-\mathbb{r}\right)}{c} \simeq \cos\frac{\omega\left(c\,t-r\right)}{c}-\frac{\omega\,b}{c}\sin\theta\cos\phi'\sin\frac{\omega\left(c\,t-r\right)}{c}$$

$$\Rightarrow \quad \mathbf{A}\left(\mathbf{r},t\right)=\frac{\mu_0 I_0 b}{4\pi r}\hat{\mathbf{y}}\int_0^{2\pi}\left[\cos\frac{\omega\left(c\,t-r\right)}{c}+b\sin\theta\cos\phi'\times\left(\frac{1}{r}\cos\frac{\omega\left(c\,t-r\right)}{c}-\frac{\omega}{c}\sin\frac{\omega\left(c\,t-r\right)}{c}\right)\right]\cos\phi'\,\mathrm{d}\phi'$$

$$\int_0^{2\pi}\cos\phi'\,\mathrm{d}\phi'=0\,, \quad \int_0^{2\pi}\cos^2\phi'\,\mathrm{d}\phi'=\pi \quad \text{vs} \quad \left\langle\cos^2\phi\right\rangle=\frac{1}{2\pi}\int_0^{2\pi}\cos^2\phi'\,\mathrm{d}\phi'=\frac{1}{2}$$

$$\Rightarrow \quad \mathbf{A}\left(r,\theta,t\right)=\frac{\mu_0 m_0}{4\pi}\frac{\sin\theta}{r}\left(\frac{1}{r}\cos\frac{\omega\left(c\,t-r\right)}{c}-\frac{\omega}{c}\sin\frac{\omega\left(c\,t-r\right)}{c}\right)\hat{\boldsymbol{\phi}}$$

static limit $\omega\to 0 \quad \Rightarrow \quad \mathbf{A}\left(r,\theta\right)=\frac{\mu_0}{4\pi}\frac{m_0\sin\theta}{r^2}\hat{\boldsymbol{\phi}}$ the potential of a magnetic dipole

- In the radiation zone,

approximation 3: $r \gg \frac{\omega}{c} \quad \Rightarrow \quad \mathbf{A}(r,\theta,t) = -\frac{\mu_0 m_0 \omega}{4\pi c}\frac{\sin\theta}{r}\sin\frac{\omega(ct-r)}{c}\,\hat{\boldsymbol{\phi}}$

$$\Rightarrow \quad \begin{aligned} \mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\frac{\sin\theta}{r}\cos\frac{\omega(ct-r)}{c}\,\hat{\boldsymbol{\phi}} \\ \mathbf{B} &= \nabla\times\mathbf{A} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\frac{\sin\theta}{r}\cos\frac{\omega(ct-r)}{c}\,\hat{\boldsymbol{\theta}} \end{aligned}$$

- These fields are in phase, mutually perpendicular, and transverse to the propagation direction ($\hat{\mathbf{r}}$), the ratio of their amplitudes is $\frac{E_0}{B_0} = c$, as expected.
- They are similar in structure to the fields of an oscillating electric dipole, only this time it is $\mathbf{B}/\mathbf{E}$ that points in the $\hat{\boldsymbol{\theta}}/\hat{\boldsymbol{\phi}}$ direction, whereas for electric dipoles it's the other way around.

- The energy flux for magnetic dipole radiation is

$$\mathbf{S}(\mathbf{r},t) = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0} = \frac{\mu_0}{c}\left(\frac{m_0\omega^2}{4\pi c}\frac{\sin\theta}{r}\cos\frac{\omega(ct-r)}{c}\right)^2\hat{\mathbf{r}}$$

$$\Rightarrow \quad \text{intensity } \langle\mathbf{S}\rangle = \frac{\mu_0 m_0^2\omega^4}{32\pi^2c^3}\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}} \quad \Rightarrow \quad \text{total radiated power } \langle P\rangle = \frac{\mu_0 m_0^2}{12\pi c^3}\omega^4$$

- The intensity profile has the shape of a donut, the power radiated goes like ω^4.

- There is one important difference between electric and magnetic dipole radiation: For configurations with comparable dimensions, the power radiated electrically is enormously greater.

- $$\frac{P_{\text{magnetic}}}{P_{\text{electric}}} = \frac{m_0^2}{p_0^2 c^2} = \frac{\omega^2 b^2}{c^2} \Leftarrow \quad m_0 = \pi b^2 I_0 , \quad p_0 = q_0 d , \quad I_0 = q_0 \omega , \quad d = \pi b$$

- $\dfrac{\omega b}{c}$ is the quantity we assumed was very small, not to mention being *squared*.

- Ordinarily one should expect electric dipole radiation to dominate. Only when the system is carefully contrived to exclude any electric contribution will the magnetic dipole radiation reveal itself.

<u>Example</u>: Find the radiation resistance for the magnetic dipole as described.

$$P = I^2 R \quad \Rightarrow \quad \langle P \rangle = \langle I^2 \rangle R_{\text{md}} \quad \Rightarrow \quad \frac{\mu_0 m_0^2}{12 \pi c^3} \omega^4 = \frac{1}{2} I_0^2 R_{\text{md}} \quad \Leftarrow \quad \begin{array}{l} I(t) = I_0 \cos \omega t \\ m_0 = \pi b^2 I_0 \end{array}$$

$$\Rightarrow \quad R_{\text{md}} = \frac{\mu_0 \pi}{6 c^3} b^4 \omega^4 = \frac{8 \pi^5}{3} \mu_0 c \left(\frac{b}{\lambda} \right)^4 \simeq 320 \pi^6 \left(\frac{b}{\lambda} \right)^4 \quad \Leftarrow \quad \omega = \frac{2 \pi c}{\lambda}$$

$$\text{For } d = 5 \text{ cm} , \quad \lambda = 1 \text{ km} \quad \Rightarrow \quad R_{\text{md}} \simeq 2 \times 10^{-12}\, \Omega \sim 10^{-6} R_{\text{ed}}$$

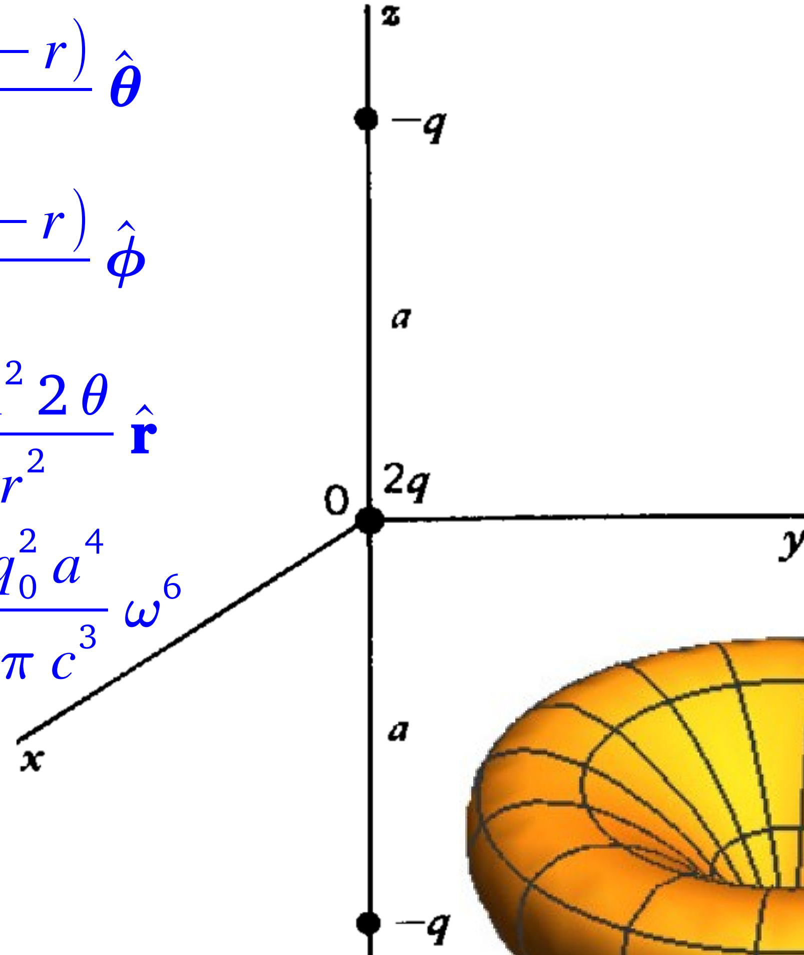
Linear Electric Quadrupole Radiation

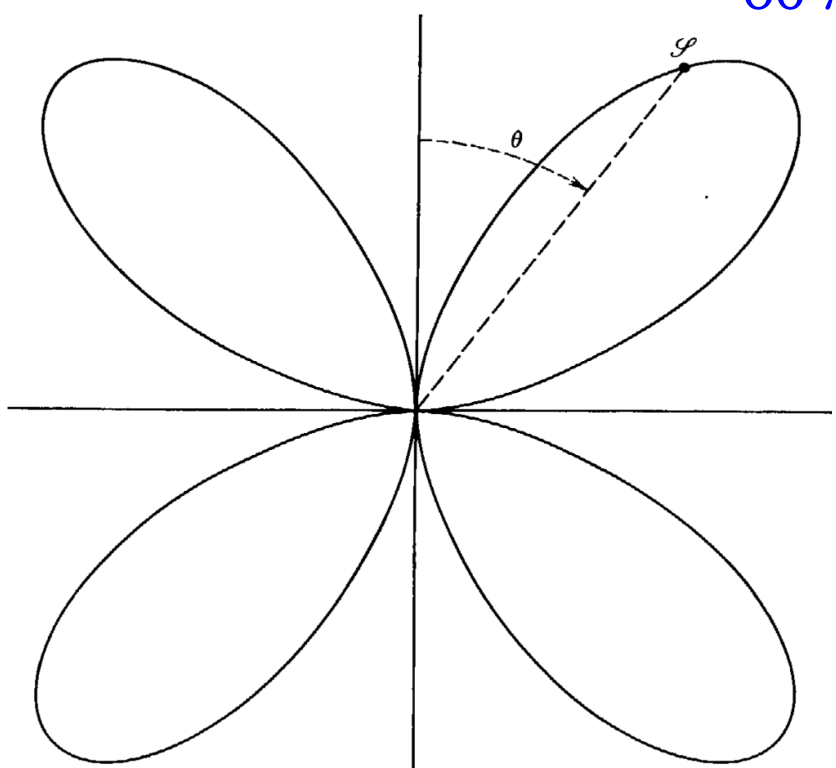
$$\mathbf{E}_r = \frac{\mu_0 q_0 a^2 \omega^3}{8\pi c} \frac{\sin 2\theta}{r} \sin \frac{\omega(c t - r)}{c} \hat{\boldsymbol{\theta}}$$

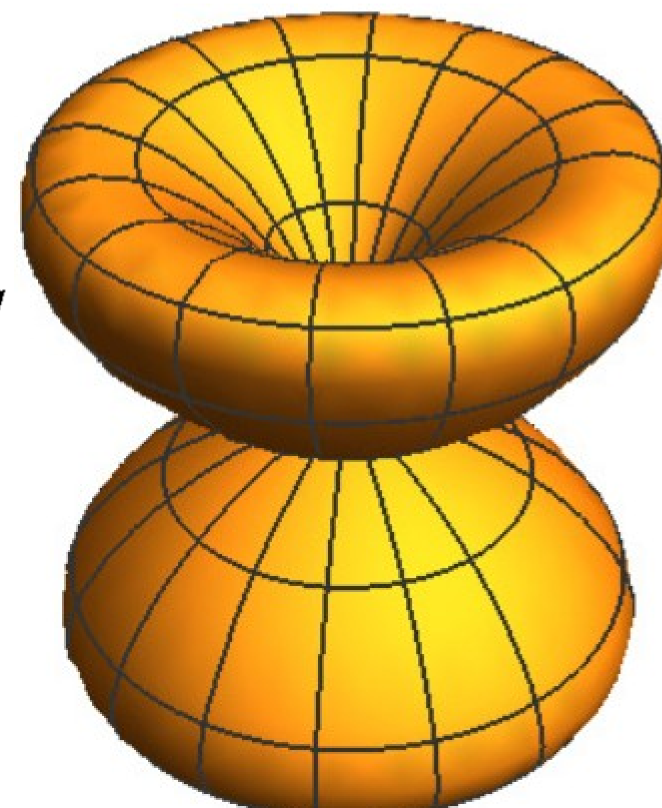
$$\mathbf{B}_r = \frac{\mu_0 q_0 a^2 \omega^3}{8\pi c^2} \frac{\sin 2\theta}{r} \sin \frac{\omega(c t - r)}{c} \hat{\boldsymbol{\phi}}$$

$$\Rightarrow \quad \text{intensity } \langle \mathbf{S} \rangle = \frac{\mu_0 q_0^2 a^4 \omega^6}{128\pi^2 c^3} \frac{\sin^2 2\theta}{r^2} \hat{\mathbf{r}}$$

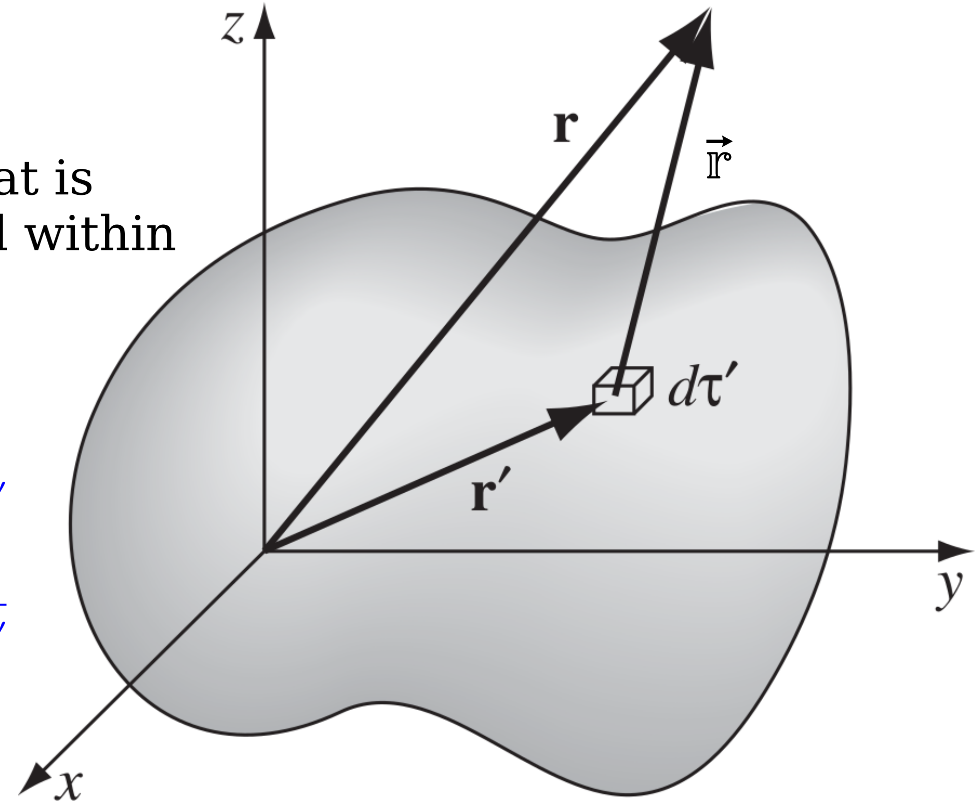
$$\Rightarrow \quad \text{total radiated power } \langle P \rangle = \frac{\mu_0 q_0^2 a^4}{60\pi c^3} \omega^6$$







Radiation from an Arbitrary Source



- A configuration of charge and current that is entirely arbitrary, except that it is localized within some finite volume near the origin.

- The retarded scalar potential is

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{\mathscr{r}} \rho\left(\mathbf{r}', t - \frac{\mathscr{r}}{c}\right) d\tau'$$

$$\text{where } \mathscr{r} = \sqrt{r^2 + r'^2 - 2\,\mathbf{r}\cdot\mathbf{r}'}$$

- Assume that $\mathbf{r}$ is far away, in comparison to the dimensions of the source:

approximation 1: $r' \ll r \;\Rightarrow\; \mathscr{r} \simeq r\left(1 - \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r}\right) \;\Rightarrow\; \frac{1}{\mathscr{r}} \simeq \frac{1}{r}\left(1 + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r}\right)$

$$\Rightarrow \rho\left(\mathbf{r}', t - \frac{\mathscr{r}}{c}\right) \simeq \rho\left(\mathbf{r}', t_0 + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right) = \rho(\mathbf{r}', t_0) + \dot{\rho}(\mathbf{r}', t_0)\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c} + \cdots \Leftarrow t_0 = t - \frac{r}{c}$$

- We can afford to drop them, provided

approximation 2: $r' \ll \frac{c}{|\ddot{\rho}/\dot{\rho}|}, \;\; \frac{c}{|\dddot{\rho}/\dot{\rho}|^{1/2}}, \cdots \;\propto\; \text{wavelength}$

- For an oscillating system, each of these ratios is $\frac{c}{\omega}$, and we recover the old approximation 2.

- As a *procedural* matter approximations 1 and 2 amount to *keeping only the 1st-order terms in* r'.

- Discarding the higher-order terms

$$\Phi(\mathbf{r},t) \simeq \frac{1}{4\pi\epsilon_0 r}\left(\int \rho(\mathbf{r}',t_0)\,d\tau' + \frac{\hat{\mathbf{r}}}{r}\cdot\int \mathbf{r}'\rho(\mathbf{r}',t_0)\,d\tau' + \frac{\hat{\mathbf{r}}}{c}\cdot\frac{d}{dt}\int \mathbf{r}'\rho(\mathbf{r}',t_0)\,d\tau'\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{p}(t_0)}{r^2} + \frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)}{rc}\right)$$

- Because charge is conserved, Q is independent of time. The other 2 integrals represent the electric dipole moment at time t_0.

- In the static case, the 1st 2 terms are the monopole and dipole contributions to the multipole expansion for Φ; the 3rd term would not be present.

- $$\boldsymbol{J} = \sum_k J^k \hat{\mathbf{x}}_k = \sum_k J^k \partial_k \mathbf{r} = \sum_k \left[\partial_k (J^k \mathbf{r}) - \mathbf{r}\,\partial_k J^k\right] = \sum_i \nabla\cdot(x^i \boldsymbol{J})\,\hat{\mathbf{x}}_i - (\nabla\cdot\boldsymbol{J})\,\mathbf{r}$$

$$\Rightarrow \int \boldsymbol{J}(\mathbf{r}',t_0)\,d\tau' = \sum_i \hat{\mathbf{x}}_i \int \nabla'\cdot(x'^i \boldsymbol{J})\,d\tau' - \int \mathbf{r}'(\nabla'\cdot\boldsymbol{J})\,d\tau'$$

$$= \sum_i \hat{\mathbf{x}}_i \oint x'^i \boldsymbol{J}\cdot d\boldsymbol{a}' + \int \mathbf{r}'\frac{\partial\rho}{\partial t_0}\,d\tau' = \frac{d}{dt}\int \rho\,\mathbf{r}'\,d\tau' = \dot{\mathbf{p}} \quad \Leftarrow \quad \frac{\partial\rho}{\partial t} + \nabla\boldsymbol{J} = 0$$

[Problem 5.7]

$$\Rightarrow \quad \mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{1}{\mathbb{r}}\boldsymbol{J}\left(\mathbf{r}', t-\frac{\mathbb{r}}{c}\right)d\tau' \simeq \frac{\mu_0}{4\pi r}\int \boldsymbol{J}(\mathbf{r}', t_0)\,d\tau' = \frac{\mu_0}{4\pi}\frac{\dot{\mathbf{p}}(t_0)}{r}$$

- It was unnecessary to carry the approximation of $\mathbb{r}$ beyond the 0$^{\text{th}}$-order($\mathbb{r}\cong r$): $\mathbf{p}$ is *already* 1$^{\text{st}}$ order in r', any refinements are corrections of 2$^{\text{nd}}$ order (or higher).

- In the radiation we keep only those terms that go like $\dfrac{1}{r}$:

approximation 3: discard $\dfrac{1}{r^2}$ terms in $\mathbf{E}$ and $\mathbf{B}$

- For instance, the Coulomb field, $\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r^2}\hat{\mathbf{r}}$, from the 1$^{\text{st}}$ term in $\Phi(\mathbf{r}, t)$, does not contribute to the EM radiation.

- The radiation comes from the terms in which we differentiate the argument t_0,

$$\nabla t_0 = -\frac{\nabla r}{c} = -\frac{\hat{\mathbf{r}}}{c} \quad \Leftarrow \quad t_0 = t - \frac{r}{c}$$

$$\nabla\Phi \simeq \nabla\left(\frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)}{r\,c}\right) \simeq \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}(t_0)}{r\,c}\nabla t_0 = -\frac{1}{4\pi\epsilon_0 c^2}\frac{\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}(t_0)}{r}\hat{\mathbf{r}}$$

$$\Rightarrow \quad \frac{\partial\mathbf{A}}{\partial t} \simeq \frac{\mu_0}{4\pi}\frac{\ddot{\mathbf{p}}(t_0)}{r}$$

$$\nabla\times\mathbf{A} \simeq \frac{\mu_0}{4\pi r}\nabla\times\dot{\mathbf{p}}(t_0) = \frac{\mu_0}{4\pi r}\nabla t_0\times\ddot{\mathbf{p}}(t_0) = -\frac{\mu_0}{4\pi r c}\hat{\mathbf{r}}\times\ddot{\mathbf{p}}(t_0)$$

$$\Rightarrow \boxed{\begin{aligned}\mathbf{E}(\mathbf{r},t) &\simeq \frac{\mu_0}{4\pi r}\left[(\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}})\,\hat{\mathbf{r}}-\ddot{\mathbf{p}}\right]=\frac{\mu_0}{4\pi r}\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\ddot{\mathbf{p}}) \\ \mathbf{B}(\mathbf{r},t) &\simeq -\frac{\mu_0}{4\pi r c}\hat{\mathbf{r}}\times\ddot{\mathbf{p}}=\frac{\hat{\mathbf{r}}}{c}\times\mathbf{E}(\mathbf{r},t)\end{aligned}} \Leftarrow \quad \ddot{\mathbf{p}}=\ddot{\mathbf{p}}(t_0)=\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)$$

- Use spherical polar coordinates, with the z axis in the direction of $\ddot{\mathbf{p}}(t_0)=\ddot{p}\ \hat{\mathbf{z}}$

$$\mathbf{E}(r,\theta,t)\simeq\frac{\mu_0\,\ddot{p}(t_0)}{4\pi}\frac{\sin\theta}{r}\hat{\boldsymbol{\theta}}$$

$$\mathbf{B}(r,\theta,t)\simeq\frac{\mu_0\,\ddot{p}(t_0)}{4\pi c}\frac{\sin\theta}{r}\hat{\boldsymbol{\phi}}$$

$$\Rightarrow \quad \text{Poynting vector } \mathbf{S}(\mathbf{r},t)\simeq\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}=\frac{\mu_0\,\ddot{p}^2(t_0)}{16\pi^2 c}\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}}$$

$$\Rightarrow \quad P(r,t)=\oint\mathbf{S}(\mathbf{r},t)\cdot d\boldsymbol{a}=\frac{\mu_0\,\ddot{p}^2(t_0)}{6\pi c}=P_{\text{rad}}(t_0)\ \text{ total radiated power}$$

- $\mathbf{E}$ and $\mathbf{B}$ are mutually perpendicular, transverse to the direction of propagation ($\hat{\mathbf{r}}$), and in the ratio $\dfrac{E}{B}=c$, as always.

Example 11.2: (a) In the case of an oscillating electric dipole,

$$\mathbf{p}(t) = p_0 \cos \omega t \, \hat{\mathbf{z}} \qquad \mathbf{E} \simeq -\frac{\mu_0 p_0 \omega^2}{4\pi} \frac{\sin\theta}{r} \cos \frac{\omega (c t - r)}{c} \hat{\boldsymbol{\theta}}$$

$$\Rightarrow \qquad \Leftarrow \text{ same as the earlier}$$

$$\ddot{\mathbf{p}}(t) = -\omega^2 p_0 \cos \omega t \, \hat{\mathbf{z}} \qquad \mathbf{B} \simeq -\frac{\mu_0 p_0 \omega^2}{4\pi c} \frac{\sin\theta}{r} \cos \frac{\omega (c t - r)}{c} \hat{\boldsymbol{\phi}}$$

(b) For a single point charge q, the dipole moment is

$$\mathbf{p}(t) = q\, \mathbf{d}(t) \;\Rightarrow\; \ddot{\mathbf{p}}(t) = q\, \mathbf{a}(t) \;\Rightarrow\; \text{Larmor formula } P = \frac{\mu_0 q^2 a^2}{6\pi c} \propto a^2$$

- In this section for a multipole expansion of the retarded potentials, use the lowest order in r' that is capable of producing EM radiation (fields being like $\frac{1}{r}$). This turns out to be the electric dipole term.

- Because charge is conserved, an electric *monopole* does not radiate.

- If charge were *not* conserved, the 1st term in Φ would read $\Phi_{\text{mono}} = \frac{1}{4\pi\epsilon_0} \frac{Q(t_0)}{r} \Rightarrow \mathbf{E}_{\text{mono}} = \frac{1}{4\pi\epsilon_0 c} \frac{\dot{Q}(t_0)}{r} \hat{\mathbf{r}}$

- You might think that a charged sphere whose radius oscillates in and out would radiate, but it *doesn't*—the field outside, according to Gauss's law, is exactly $\frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}$, regardless of the fluctuations in size.

- In the acoustical analog, monopoles *do* radiate: transverse vs longitudinal.

- If the electric dipole moment or its 2nd time derivative should happen to vanish, then there is no electric dipole radiation, and one must look to the next term: the one of 2nd order in r'.

- This term can be separated into 2 parts, one of which is related to the *magnetic* dipole moment of the source, the other to its electric *quadrupole* moment.

- If the magnetic dipole and electric quadrupole contributions vanish, the r'^3 term must be considered, etc.

Selected problems: 4, 10, 18, 22, 26

Point Charges

Power Radiated by a Point Charge

- The fields of a point charge q in arbitrary motion

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0 \mathfrak{r}^2}\frac{c\left(1-\beta^2\right)\left(\hat{\mathfrak{r}}-\boldsymbol{\beta}\right)+\vec{\mathfrak{r}}\times\left[\left(\hat{\mathfrak{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]}{c\left(1-\hat{\mathfrak{r}}\cdot\boldsymbol{\beta}\right)^3} \quad (\$)$$

$$\mathbf{B}(\mathbf{r},t)=\frac{\hat{\mathfrak{r}}}{c}\times\mathbf{E}(\mathbf{r},t)$$

The 1st term in ($) is the **velocity field**, and the 2nd one is the **acceleration field**.

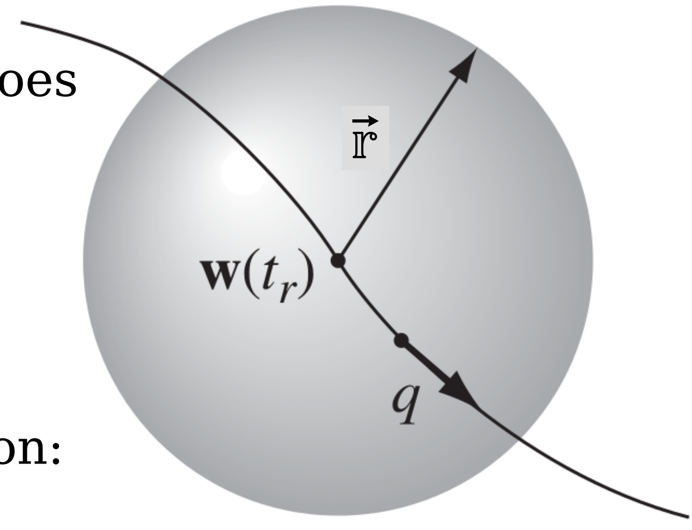
- Poynting vector $\mathbf{S}(\mathbf{r},t)\simeq\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}=\frac{\mathbf{E}\times\left(\hat{\mathfrak{r}}\times\mathbf{E}\right)}{\mu_0 c}=\frac{E^2\,\hat{\mathfrak{r}}-\left(\hat{\mathfrak{r}}\cdot\mathbf{E}\right)\mathbf{E}}{\mu_0 c}$

- Not all of this energy flux constitutes *radiation*; some of it is just field energy carried along by the particle as it moves.

- The *radiated* energy is the stuff that, in effect, *detaches* itself from the charge and propagates off to infinity.

- To calculate the total power *radiated* by the particle at time t_r, consider a huge sphere of radius $\mathfrak{r}$, centered at the position of the particle (at t_r), wait the appropriate interval $c(t-t_r)=\mathfrak{r}$ for the radiation to reach the sphere, and at that moment integrate the Poynting vector over the surface.

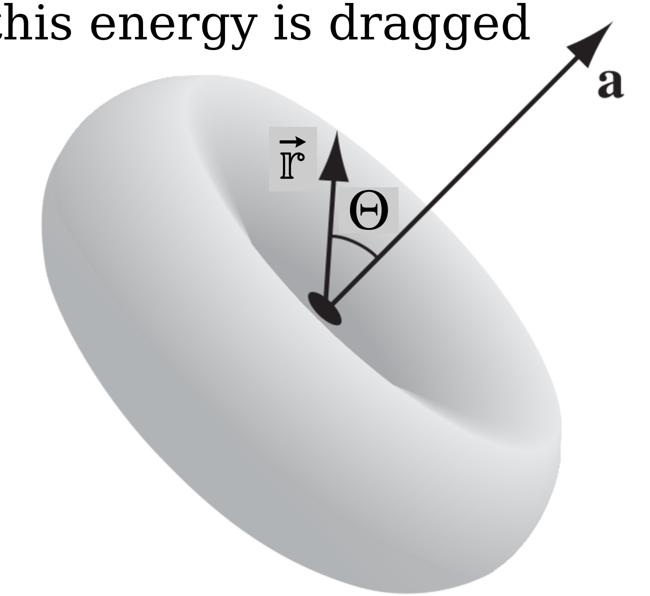
- Tthe area of the sphere $\propto \mathscr{r}^2$, so any term in $\mathbf{S}$ that goes like $\dfrac{1}{\mathscr{r}^2}$ will yield a finite answer, but terms like $\dfrac{1}{\mathscr{r}^3}$ or $\dfrac{1}{\mathscr{r}^4}$ will contribute nothing in the limit $r \to \infty$.



- So only the *acceleration* fields represent true radiation:

$$\text{Radiation fields}\quad \mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0 \mathscr{r}} \frac{\hat{\mathscr{r}} \times [(\hat{\mathscr{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{c\,(1 - \hat{\mathscr{r}}\cdot\boldsymbol{\beta})^3}$$

- The velocity fields carry energy, as the charge moves this energy is dragged along—but it's not *radiation*.



$$\mathbf{E}_{\text{rad}} \perp \hat{\mathscr{r}} \quad\Rightarrow\quad \mathbf{S}_{\text{rad}} \simeq \frac{E_{\text{rad}}^2\,\hat{\mathscr{r}} - (\hat{\mathscr{r}}\cdot\mathbf{E}_{\text{rad}})\,\mathbf{E}_{\text{rad}}}{\mu_0 c} = \frac{E_{\text{rad}}^2}{\mu_0 c}\,\hat{\mathscr{r}}$$

$$\text{Let}\ \ \mathbf{v}(t_r) = 0\ \ \text{at rest} \ \Rightarrow\ \boldsymbol{\beta}(t_r) = 0$$

$$\Rightarrow\quad \mathbf{E}_{\text{rad}} = \frac{q\,\hat{\mathscr{r}} \times (\hat{\mathscr{r}} \times \dot{\boldsymbol{\beta}})}{4\pi\epsilon_0 c\,\mathscr{r}} = \frac{\mu_0 q c}{4\pi\mathscr{r}}\left[(\hat{\mathscr{r}}\cdot\dot{\boldsymbol{\beta}})\,\hat{\mathscr{r}} - \dot{\boldsymbol{\beta}}\right]$$

$$\Rightarrow\quad \mathbf{S}_{\text{rad}} \simeq \frac{1}{\mu_0 c}\left(\frac{\mu_0 q c}{4\pi\mathscr{r}}\right)^2 \left[\dot{\beta}^2 - (\hat{\mathscr{r}}\cdot\dot{\boldsymbol{\beta}})^2\right]\hat{\mathscr{r}} = \frac{\mu_0 c\, q^2\, \dot{\beta}^2}{16\pi^2\mathscr{r}^2}\sin^2\Theta\,\hat{\mathscr{r}} \quad (\#) \ \Leftarrow\ \cos\Theta = \hat{\mathscr{r}}\cdot\hat{\dot{\boldsymbol{\beta}}}$$

- No power is radiated in the forward or backward direction—rather, it is emitted in a donut about the direction of instantaneous acceleration.

$$\Downarrow \quad \dot{\boldsymbol{\beta}} = \frac{\mathbf{a}}{c}, \quad \dot{\beta} = \frac{a}{c}$$

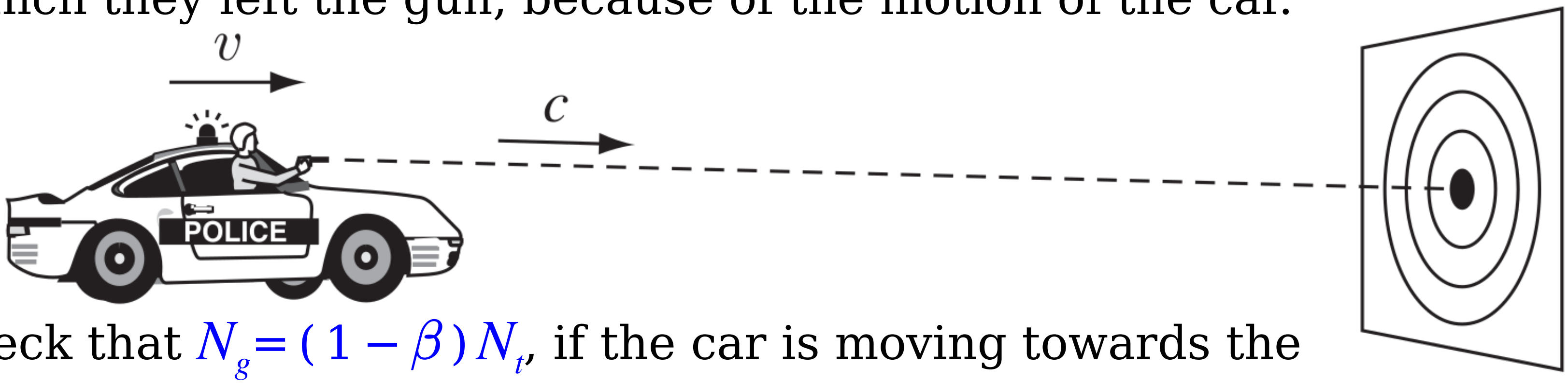
- The total power radiated is

$$P = \oint \mathbf{S}_{\mathrm{rad}} \cdot \mathrm{d}\,\boldsymbol{a} = \frac{\mu_0 c\, q^2 \dot{\beta}^2}{16\,\pi^2} \int \frac{\sin^2\theta}{\mathbb{r}^2}\, \mathbb{r}^2 \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\phi \;\Rightarrow\; P = \frac{\mu_0 q^2 a^2}{6\,\pi\, c}$$

Larmor formula (again)

- Although we derived them on the assumption that $v=0$, the result actually holds to good approximation as long as $v \ll c$.

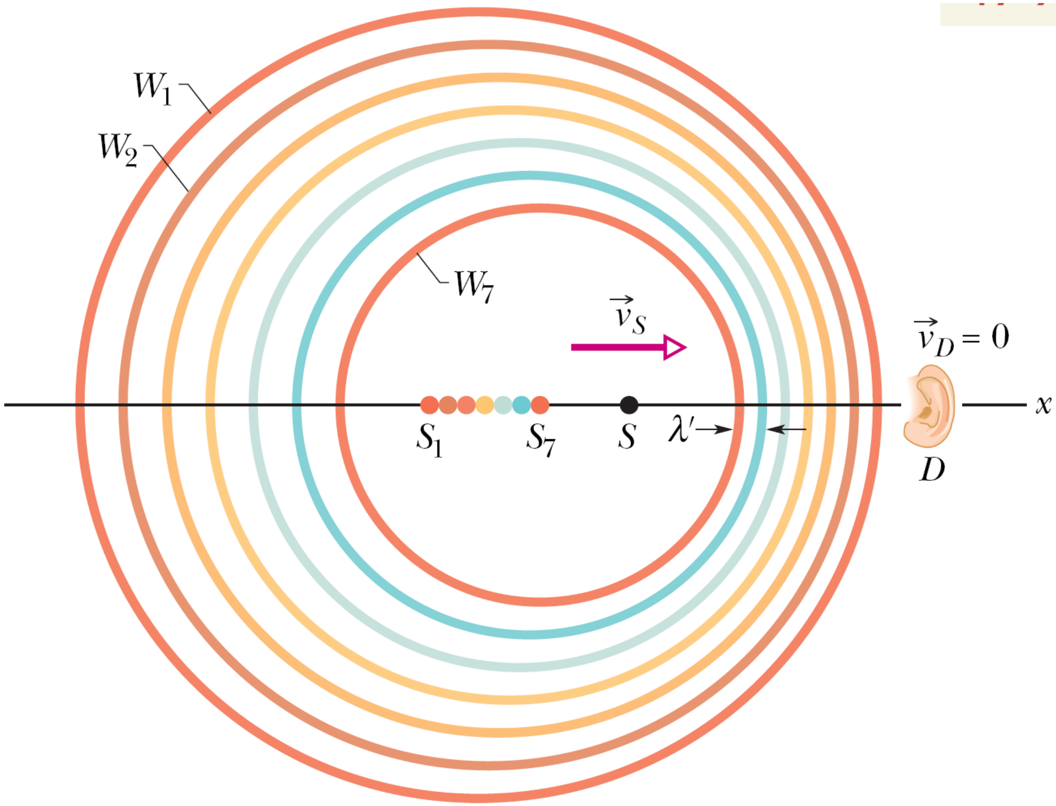
- Suppose someone is firing a stream of bullets out the window of a moving car. The rate N_t at which the bullets strike a stationary target is not the same as the rate N_g at which they left the gun, because of the motion of the car.

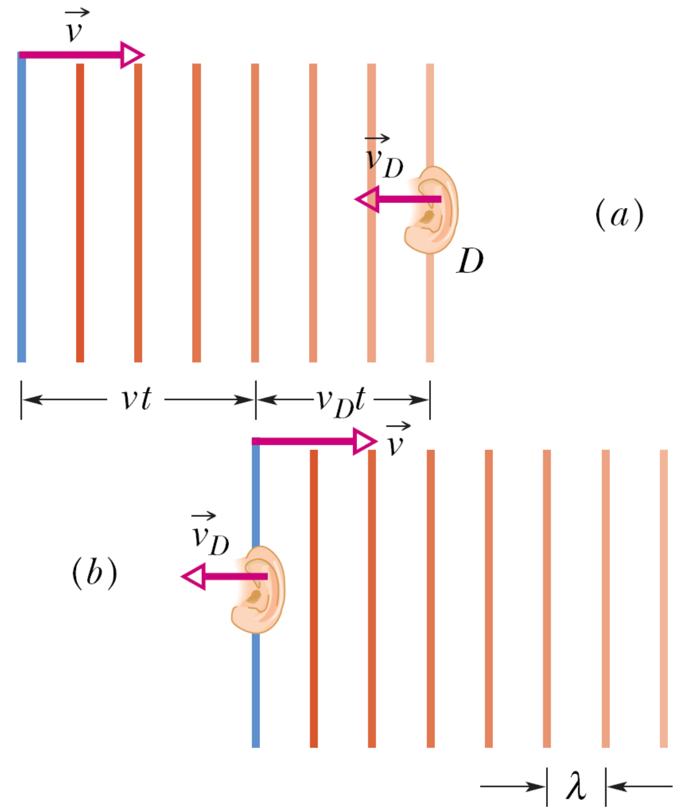


- We can check that $N_g = (1-\beta)\,N_t$, if the car is moving towards the target, and $N_g = \left(1 - \hat{\mathbb{r}} \cdot \boldsymbol{\beta}\right) N_t$ for arbitrary directions, where $\boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c}$, $\beta = |\boldsymbol{\beta}|$.

- If $\frac{\mathrm{d}W}{\mathrm{d}t}$ is the rate at which energy passes through the sphere at radius $\mathbb{r}$, the rate of energy left the charge $\frac{\mathrm{d}W}{\mathrm{d}t_r} = \frac{\mathrm{d}W/\mathrm{d}t}{\partial t_r/\partial t} = \left(1 - \hat{\mathbb{r}} \cdot \boldsymbol{\beta}\right) \frac{\mathrm{d}W}{\mathrm{d}t} \;\Leftarrow\; \frac{\partial t_r}{\partial t} = \frac{1}{1 - \hat{\mathbb{r}} \cdot \boldsymbol{\beta}}$

Source Moving, Detector Stationary





- Let source S move toward D at speed v_S

$$\lambda' = v\,T - v_S\,T \quad \Rightarrow \quad f' = \frac{v}{\lambda'} = \frac{v}{v\,T - v_S\,T} = \frac{v}{v/f - v_S/f} = f\,\frac{v}{v - v_S} \geqslant f$$

$$\lambda_t = c\,T_g - v\,T_g \quad \Rightarrow \quad f_t = \frac{c}{\lambda_t} = \frac{c}{c\,T_g - v\,T_g} = \frac{f_g}{1-\beta} \quad \Leftarrow \quad f = \frac{1}{T}$$

$$\Rightarrow \quad N_g = (1-\beta)\,N_t$$

$$t_r = t - \frac{\mathbb{r}}{c} \quad \Leftarrow \quad \vec{\mathbb{r}} = \mathbf{r} - \vec{\mathbb{x}}\,(t_r)\,, \quad \mathbb{r} = |\vec{\mathbb{r}}| = \sqrt{r^2 + \mathbb{x}^2 - 2\,\mathbf{r}\cdot\vec{\mathbb{x}}}\,, \quad \hat{\mathbb{r}} = \frac{\vec{\mathbb{r}}}{\mathbb{r}}$$

$$\Rightarrow \quad \frac{\partial\, \mathbb{r}}{\partial\, t_r} = \frac{(\vec{\mathbb{x}} - \mathbf{r})\cdot\mathbf{v}}{\mathbb{r}} = \frac{-\,\vec{\mathbb{r}}\cdot\mathbf{v}}{\mathbb{r}} = -\,c\,\hat{\mathbb{r}}\cdot\boldsymbol{\beta} \quad \Leftarrow \quad \mathbf{v} = \frac{\mathrm{d}\,\vec{\mathbb{x}}}{\mathrm{d}\,t_r}\,, \quad \boldsymbol{\beta} = \frac{\mathbf{v}}{c}$$

$$t = t_r + \frac{\mathbb{r}}{c} \quad \Rightarrow \quad \frac{\partial\, t}{\partial\, t_r} = 1 + \frac{1}{c}\,\frac{\partial\, \mathbb{r}}{\partial\, t_r} = 1 - \hat{\mathbb{r}}\cdot\boldsymbol{\beta} \quad \Rightarrow \quad \frac{\partial\, t_r}{\partial\, t} = \frac{1}{1 - \hat{\mathbb{r}}\cdot\boldsymbol{\beta}}$$

$$\hat{\mathbb{r}} \times [(\hat{\mathbb{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}] = \hat{\mathbb{r}}\cdot\dot{\boldsymbol{\beta}}\,(\hat{\mathbb{r}} - \boldsymbol{\beta}) - (1 - \hat{\mathbb{r}}\cdot\boldsymbol{\beta})\,\dot{\boldsymbol{\beta}}$$

$$\Rightarrow \quad |\hat{\mathbb{r}} \times [(\hat{\mathbb{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2 = (\hat{\mathbb{r}}\cdot\dot{\boldsymbol{\beta}})^2\,(1 - 2\,\hat{\mathbb{r}}\cdot\boldsymbol{\beta} + \beta^2) + (1 - \hat{\mathbb{r}}\cdot\boldsymbol{\beta})^2\,\dot{\beta}^2$$
$$-\,2\,\hat{\mathbb{r}}\cdot\dot{\boldsymbol{\beta}}\,(\hat{\mathbb{r}}\cdot\dot{\boldsymbol{\beta}} - \boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})\,(1 - \hat{\mathbb{r}}\cdot\boldsymbol{\beta})$$

$$= (\hat{\mathbb{r}}\cdot\dot{\boldsymbol{\beta}})^2\,(\beta^2 - 1) + 2\,(\hat{\mathbb{r}}\cdot\dot{\boldsymbol{\beta}})\,(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})\,(1 - \hat{\mathbb{r}}\cdot\boldsymbol{\beta}) + (1 - \hat{\mathbb{r}}\cdot\boldsymbol{\beta})^2\,\dot{\beta}^2$$

$$= \dot{\beta}^2\,[(\beta^2 - 1)\cos^2\Theta + 2\,\beta\cos\psi\cos\Theta\,(1 - \hat{\mathbb{r}}\cdot\boldsymbol{\beta}) + (1 - \hat{\mathbb{r}}\cdot\boldsymbol{\beta})^2]$$

where $\hat{\mathbb{r}}\cdot\dot{\boldsymbol{\beta}} = \dot{\beta}\cos\Theta\,, \quad \boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}} = \beta\,\dot{\beta}\cos\psi$

- So the ratio of the energy rate is precisely the ratio of N_g to N_t; it's a purely geometrical factor.

- The power radiated by the particle into a patch of area $\mathfrak{r}^2 \sin\theta \, d\theta \, d\phi = \mathfrak{r}^2 \, d\Omega$, where $d\Omega = \sin\theta \, d\theta \, d\phi$ is the **solid angle**, on the sphere is given by

$$\frac{dP}{d\Omega} = (1 - \hat{\mathfrak{r}}\cdot\boldsymbol{\beta}) \frac{E_{\text{rad}}^2 \mathfrak{r}^2}{\mu_0 c} = \frac{q^2}{16\pi^2 \epsilon_0} \frac{|\hat{\mathfrak{r}} \times [(\hat{\mathfrak{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{c(1 - \hat{\mathfrak{r}}\cdot\boldsymbol{\beta})^5} \quad \Leftarrow \quad \dot{\boldsymbol{\beta}} = \frac{\mathbf{a}}{c} \quad \Rightarrow \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$\Rightarrow \quad P = \frac{\mu_0 c q^2 \gamma^6}{6\pi} \left(\dot{\beta}^2 - |\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}|^2\right) = \frac{\mu_0 q^2 \gamma^6}{6\pi c} \left(a^2 - |\boldsymbol{\beta} \times \mathbf{a}|^2\right)$$

Liénard's generalization of the Larmor formula

- The factor γ^6 means that the radiated power increases enormously as the particle velocity approaches the speed of light.

Example 11.3: Let **v** & **a** are instantaneously collinear (at t_r) as in straight-line motion. Find the angular distribution of the radiation & the total power emitted.

- $\mathbf{v} \| \mathbf{a} \| \hat{\mathbf{z}} \quad \Rightarrow \quad \boldsymbol{\beta} \| \dot{\boldsymbol{\beta}} \| \hat{\mathbf{z}} \quad \Rightarrow \quad (\hat{\mathfrak{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} = \hat{\mathfrak{r}} \times \dot{\boldsymbol{\beta}} \quad \Rightarrow \quad \dfrac{dP}{d\Omega} = \dfrac{q^2}{16\pi^2\epsilon_0} \dfrac{|\hat{\mathfrak{r}} \times (\hat{\mathfrak{r}} \times \dot{\boldsymbol{\beta}})|^2}{c(1 - \hat{\mathfrak{r}}\cdot\boldsymbol{\beta})^5}$

$$\hat{\mathfrak{r}} \times (\hat{\mathfrak{r}} \times \dot{\boldsymbol{\beta}}) = (\hat{\mathfrak{r}} \cdot \dot{\boldsymbol{\beta}}) \hat{\mathfrak{r}} - \dot{\boldsymbol{\beta}} \quad \Rightarrow \quad |\hat{\mathfrak{r}} \times (\hat{\mathfrak{r}} \times \dot{\boldsymbol{\beta}})|^2 = \dot{\beta}^2 - (\hat{\mathfrak{r}} \cdot \dot{\boldsymbol{\beta}})^2$$

$$\Rightarrow \quad \frac{dP}{d\Omega} = \frac{\mu_0 q^2}{16\pi^2 \epsilon_0} \frac{\dot{\beta}^2 \sin^2\theta}{c(1 - \beta\cos\theta)^5} \quad \Leftarrow \quad \beta \equiv \frac{v}{c}, \quad \boldsymbol{\beta} = \beta\,\hat{\mathbf{z}}, \quad \dot{\boldsymbol{\beta}} = \dot{\beta}\,\hat{\mathbf{z}}$$

$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{q^2}{16\pi^2 \epsilon_0 c} \int \frac{|\hat{\mathbb{r}} \times [(\hat{\mathbb{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \hat{\mathbb{r}} \cdot \boldsymbol{\beta})^5} \sin\theta \, d\theta \, d\phi$$

Choose $\hat{\boldsymbol{\beta}} = \hat{\mathbf{z}}$, $\hat{\dot{\boldsymbol{\beta}}} = \sin\psi\,\hat{\mathbf{x}} + \cos\psi\,\hat{\mathbf{z}}$, ψ fixed

$$\hat{\mathbb{r}} = \sin\theta\,(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}) + \cos\theta\,\hat{\mathbf{z}}$$

$$\Rightarrow \quad \hat{\mathbb{r}} \cdot \hat{\boldsymbol{\beta}} = \cos\theta, \quad \hat{\boldsymbol{\beta}} \cdot \hat{\dot{\boldsymbol{\beta}}} = \cos\psi$$

$$\hat{\mathbb{r}} \cdot \hat{\dot{\boldsymbol{\beta}}} = \cos\Theta = \cos\psi\cos\theta + \sin\psi\sin\theta\cos\phi$$

$$\Rightarrow \quad (\hat{\mathbb{r}} \cdot \hat{\dot{\boldsymbol{\beta}}})^2 = \cos^2\Theta = \cos^2\psi\cos^2\theta + \sin 2\psi\cos\theta\sin\theta\cos\phi + \sin^2\psi\sin^2\theta\cos^2\phi$$

$$\Rightarrow \quad |\hat{\mathbb{r}} \times [(\hat{\mathbb{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2 = \dot{\beta}^2 [(\beta^2 - 1)\cos^2\Theta + 2\beta\cos\psi\cos\Theta\,(1 - \hat{\mathbb{r}} \cdot \boldsymbol{\beta}) + (1 - \hat{\mathbb{r}} \cdot \boldsymbol{\beta})^2]$$

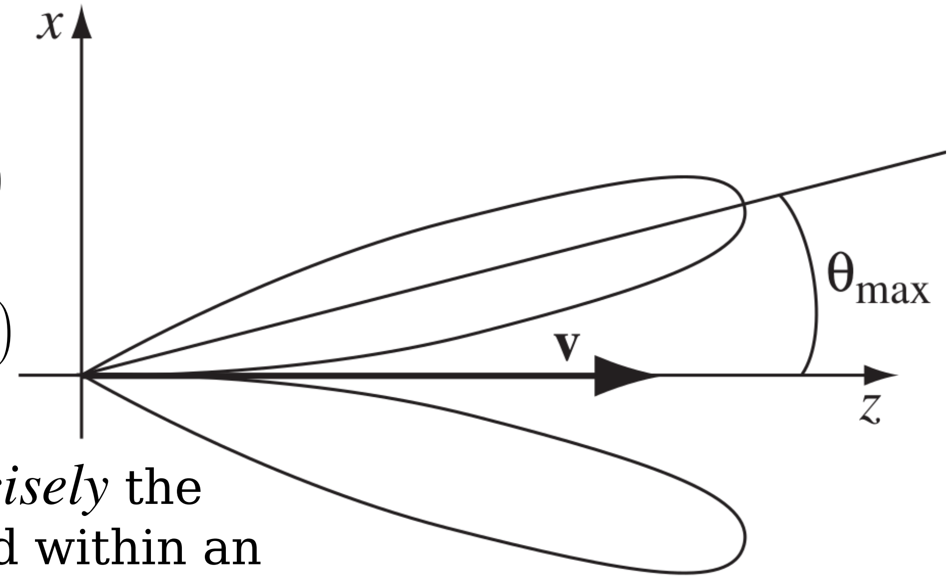
$$\Rightarrow \quad \left[\begin{array}{l} \dot{\beta}^2 (\beta^2 - 1) \int \frac{\cos^2\Theta}{(1 - \beta\cos\theta)^5} \sin\theta\,d\theta\,d\phi = -\frac{4\pi}{3}\gamma^4\dot{\beta}^2 - 8\pi\gamma^6(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2 \\ 2\beta\dot{\beta}^2\cos\psi \int \frac{\cos\Theta}{(1 - \beta\cos\theta)^4} \sin\theta\,d\theta\,d\phi = \frac{32\pi}{3}\gamma^6(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2 \\ \dot{\beta}^2 \int \frac{1}{(1 - \beta\cos\theta)^3} \sin\theta\,d\theta\,d\phi = 4\pi\gamma^4\dot{\beta}^2 \end{array}\right.$$

$$\Rightarrow \quad P = \frac{q^2}{16\pi^2\epsilon_0 c}\left(-\frac{4\pi}{3}\gamma^4\dot{\beta}^2 - 8\pi\gamma^6(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2 + \frac{32\pi}{3}\gamma^6(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2 + 4\pi\gamma^4\dot{\beta}^2\right)$$

$$= \frac{q^2}{16\pi^2\epsilon_0 c}\left(\frac{8\pi}{3}\gamma^4\dot{\beta}^2 + \frac{8\pi}{3}\gamma^6(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2\right) = \frac{\mu_0 c\, q^2\gamma^6}{6\pi}\left(\dot{\beta}^2 - |\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}|^2\right)$$

$$\Rightarrow \quad \frac{dP}{d\Omega} = \mathbb{r}^2 \times (\#) \quad \text{for } v = 0 \ (\beta = 0)$$

$$\frac{dP}{d\Omega} \propto \frac{1}{(1-\beta\cos\theta)^5} \quad \text{for } v \to c \ (\beta \to 1)$$



- Although there is still no radiation in *precisely* the forward direction, most of it is concentrated within an increasingly narrow cone about the forward direction.

- $$P = \int \frac{dP}{d\Omega}\, d\Omega = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \sin\theta\, d\theta\, d\phi$$

$$= \frac{\mu_0 q^2 a^2}{8\pi c} \int_{-1}^{1} \frac{(1-x^2)}{(1-\beta x)^5}\, dx = \frac{\mu_0 q^2 a^2}{8\pi c} \frac{4}{3(1-\beta^2)^3} = \frac{\mu_0 q^2 a^2 \gamma^6}{6\pi c}$$

consistent with the Liénard formula, for the case of collinear **v** and **a**.

- The angular distribution of the radiation is the same whether the particle is *accelerating* or *decelerating*; it only depends on a^2, and is concentrated in the forward direction (with respect to the velocity) in either case.

- When a high speed electron hits a metal target it rapidly decelerates, giving off what is called **bremsstrahlung**, or “braking radiation.”

Problem 11.15: Find θ_{max} at which the maximum radiation is emitted in Ex. 11.3.

- $$\frac{d}{d\theta}\frac{dP}{d\Omega}=0 \;\Rightarrow\; 0=\frac{d}{d\theta}\frac{\sin\theta^2}{(1-\beta\cos\theta)^5}=\frac{\sin\theta\,(2\cos\theta-2\beta\cos^2\theta-5\beta\sin^2\theta)}{(1-\beta\cos\theta)^6}$$

$$\Rightarrow\; 3\beta\cos^2\theta+2\cos\theta-5\beta=0 \;\Rightarrow\; \cos\theta=\frac{\pm\sqrt{15\beta^2+1}-1}{3\beta}$$ choose + sign to fit $\beta\to 0$

$$\Rightarrow\; \theta_{max}=\cos^{-1}\frac{\sqrt{15\beta^2+1}-1}{3\beta}$$ and $\theta_{min}=0,\pi$ for $\sin\theta_{min}=0 \Rightarrow \frac{dP}{d\Omega}_{|min}=0$

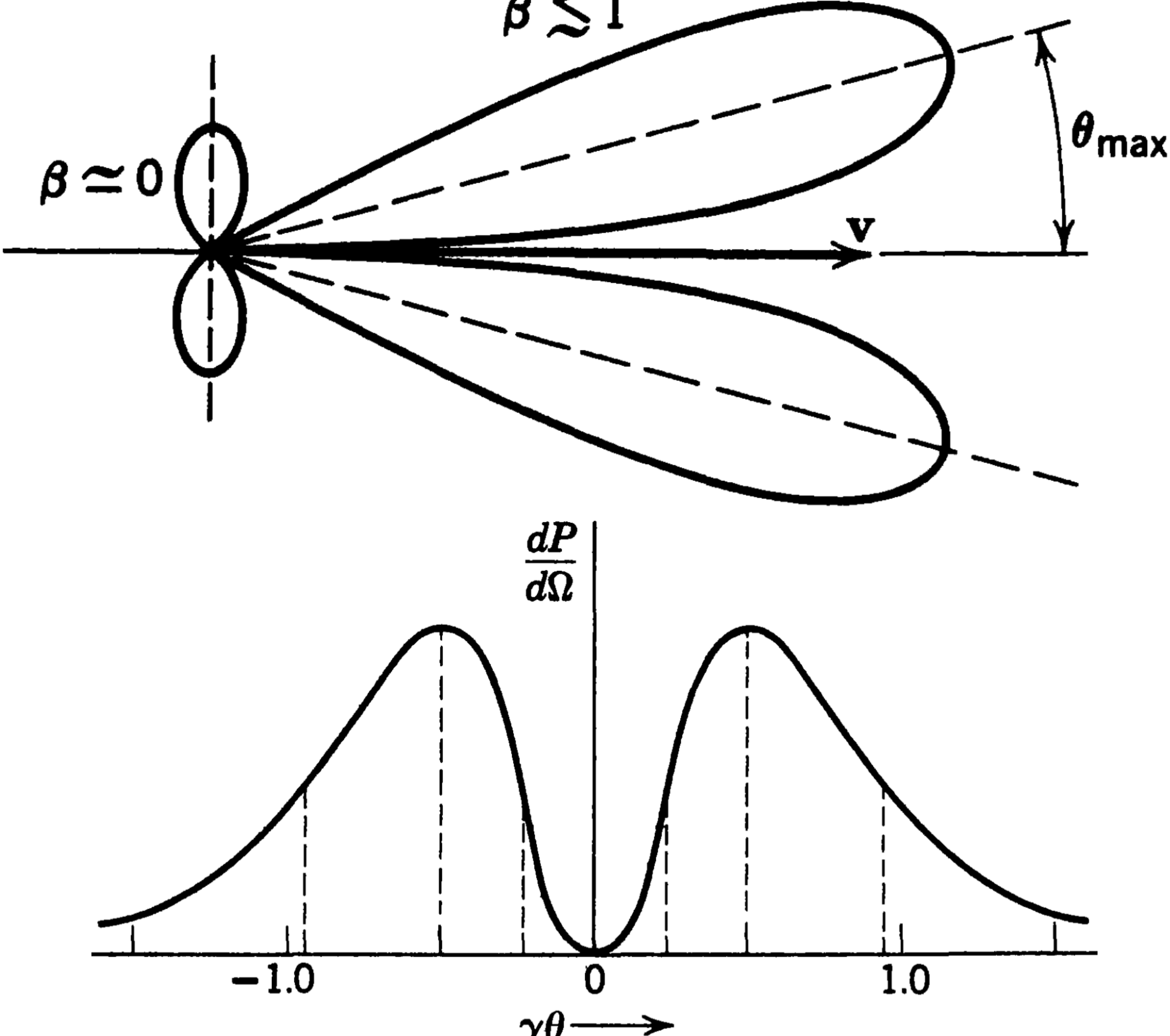
For ultra-relativistic speeds $\Rightarrow \beta\to 1 \Rightarrow \beta=1-\delta,\ \delta\ll 1 \Rightarrow \gamma=\frac{1}{\sqrt{1-\beta^2}}\simeq\frac{1}{\sqrt{2\delta}}$

$$\Rightarrow\; \frac{\sqrt{15\beta^2+1}-1}{3\beta}=\frac{\sqrt{15(1-\delta)^2+1}-1}{3(1-\delta)}\simeq\frac{\sqrt{16-30\delta}-1}{3(1-\delta)}\simeq\frac{1+\delta}{3}\left(3-\frac{15}{4}\delta\right)$$

$$\simeq 1-\frac{\delta}{4} \;\Rightarrow\; \cos\theta_{max}\simeq 1-\frac{\theta_{max}^2}{2}\simeq 1-\frac{\delta}{4} \;\Rightarrow\; \theta_{max}\simeq\sqrt{\frac{\delta}{2}}\simeq\sqrt{\frac{1-\beta}{2}}\simeq\frac{1}{2\gamma}$$

$$\Rightarrow\; \frac{dP}{d\Omega}_{|max}=\frac{\mu_0 q^2}{16\pi^2\epsilon_0 c}\frac{\dot\beta^2\sin\theta_{max}^2}{(1-\beta\cos\theta_{max})^5}\simeq\frac{\mu_0 q^2\dot\beta^2}{16\pi^2\epsilon_0 c}\frac{\theta_{max}^2}{[1-(1-\delta)(1-\delta/4)]^5}$$

$$\simeq\frac{\mu_0 q^2\dot\beta^2}{16\pi^2\epsilon_0 c}\frac{\delta/2}{[1-(1-\delta)(1-\delta/4)]^5}\simeq\frac{\mu_0 q^2\dot\beta^2}{16\pi^2\epsilon_0 c}\frac{\delta/2}{(5\delta/4)^5}\simeq\frac{\mu_0 q^2\dot\beta^2}{2\pi^2\epsilon_0 c}\left(\frac{4}{5}\right)^5\gamma^8$$


$\beta \lesssim 1$
$\beta \simeq 0$
θ_{max}
$\mathbf{v}$
$\frac{dP}{d\Omega}$
-1.0
0
1.0
$\gamma\theta \longrightarrow$

Radiation Reaction

- An accelerating charge radiates. This radiation carries off energy, which comes at the expense of the particle's kinetic energy.

- Under the influence of a given force, a charged particle accelerates less than a neutral one of the same mass.

- The radiation exerts a force ($\mathbf{F}_{\text{rad}}$) back on the charge—a *recoil* force, like that of a bullet on a gun—the **radiation reaction** force (from conservation of energy).

- For a nonrelativistic particle ($v \ll c$), the total power radiated is given by the Larmor formula: $P = \dfrac{\mu_0 q^2 a^2}{6\pi c}$

- Conservation of energy asks that this is also the rate at which the particle *loses* energy, under the influence of the radiation reaction force: (only correct averagely) $\mathbf{F}_{\text{rad}} \cdot \mathbf{v} = -\dfrac{\mu_0 q^2 a^2}{6\pi c}$

- In the calculation of the radiated power the *velocity* fields played no part, since they fall off too rapidly as a function of $\mathscr{r}$ to make any contribution.

- The velocity fields *do* carry energy—they just don't transport it out to infinity.

- As the particle accelerates/decelerates, energy is exchanged between it and the velocity fields, as energy is also radiated away by the acceleration fields.

- The earlier consideration accounts only for the latter. If we want to know the recoil force by the fields on the charge, we should consider the total power lost at any instant, not just the portion that eventually escapes in the form of radiation.

- The energy lost by the particle in any given time interval must equal the energy carried away by the radiation plus the energy pumped into the velocity fields.

- If we consider only intervals over which the system returns to its initial state, eg, the periodic motion, then the energy in the velocity fields is the same at both ends, and the only net loss is in the form of radiation.

- $\int_{t_1}^{t_2} \mathbf{F}_{\text{rad}} \cdot \mathbf{v} \, \mathrm{d}t = -\frac{\mu_0 q^2}{6\pi c} \int_{t_1}^{t_2} a^2 \, \mathrm{d}t \quad \Leftarrow \quad \mathbf{v}(t_1) = \mathbf{v}(t_2), \quad \mathbf{a}(t_1) = \mathbf{a}(t_2)$

$$\int_{t_1}^{t_2} a^2 \, \mathrm{d}t = \int_{t_1}^{t_2} \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \, \mathrm{d}t = \mathbf{v} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\mathrm{d}^2\mathbf{v}}{\mathrm{d}t^2} \cdot \mathbf{v} \, \mathrm{d}t$$

$$\Rightarrow \int_{t_1}^{t_2} \left(\mathbf{F}_{\text{rad}} - \frac{\mu_0 q^2}{6\pi c} \dot{\mathbf{a}} \right) \cdot \mathbf{v} \, \mathrm{d}t = 0 \quad \Rightarrow \quad \boxed{\mathbf{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c} \dot{\mathbf{a}}} \quad \text{Abraham-Lorentz formula}$$

- The expression tells nothing about the component of $\mathbf{F}_{\text{rad}} \perp \mathbf{v}$, it only tells the time average of the parallel component—the average over special time intervals.

- However, it represents the *simplest* form the radiation reaction force could take, consistent with conservation of energy.

- The Abraham-Lorentz formula has disturbing implications. Suppose a particle is subject to no *external* forces; then Newton's 2nd law says

$$F_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\dot{a} = m a \quad \Rightarrow \quad a(t) = a_0 e^{\frac{t}{\tau}} \quad \Leftarrow \quad \tau \equiv \frac{\mu_0 q^2}{6\pi m c} \quad \Rightarrow \quad \tau_{\text{electron}} = 6\times 10^{-24}\ \text{s}$$

The acceleration *increases* exponentially with time unless $a_0=0$!

- The systematic exclusion of such **runaway solutions** has a more unpleasant consequence: If you *do* apply an external force, the particle starts to respond *before the force acts*! [Problem 11.19]

- This **acausal preacceleration** jumps the gun by only a short time τ; however, it is unacceptable that the theory should countenance it at all.

<u>Example 11.4</u>: Calculate the **radiation damping** of a charged particle attached to a spring of natural frequency ω_0, driven at frequency ω.

- The equation of motion is

$$m\ddot{x} = F_{\text{spring}} + F_{\text{rad}} + F_{\text{drive}} = -m\omega_0^2 x + m\tau\dddot{x} + F_{\text{drive}}(\omega) \quad \Rightarrow \quad x(t) = x_0\cos(\omega t + \delta)$$

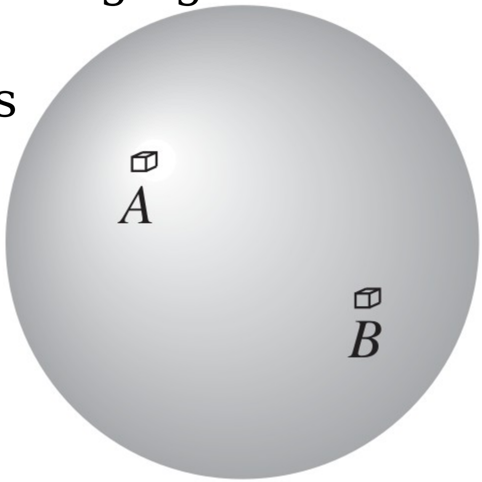
$$\Rightarrow \quad \dddot{x} = -\omega^2\dot{x} \quad \Rightarrow \quad m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = F_{\text{drive}} \quad \Leftarrow \quad \text{damping factor}\ \ \gamma = \omega^2\tau$$

$$\Rightarrow \quad F_{\text{damping}} = -\gamma m\dot{x} = -m\omega^2\tau\dot{x}$$

The Mechanism Responsible for the Radiation Reaction

- The fields of a point charge blow up right at the particle, so it's hard to see how one can calculate the radiation reaction force they exert.

- Avoid this problem by considering an extended charge distribution with the field is finite everywhere; then take the limit as the size of the charge goes to 0.



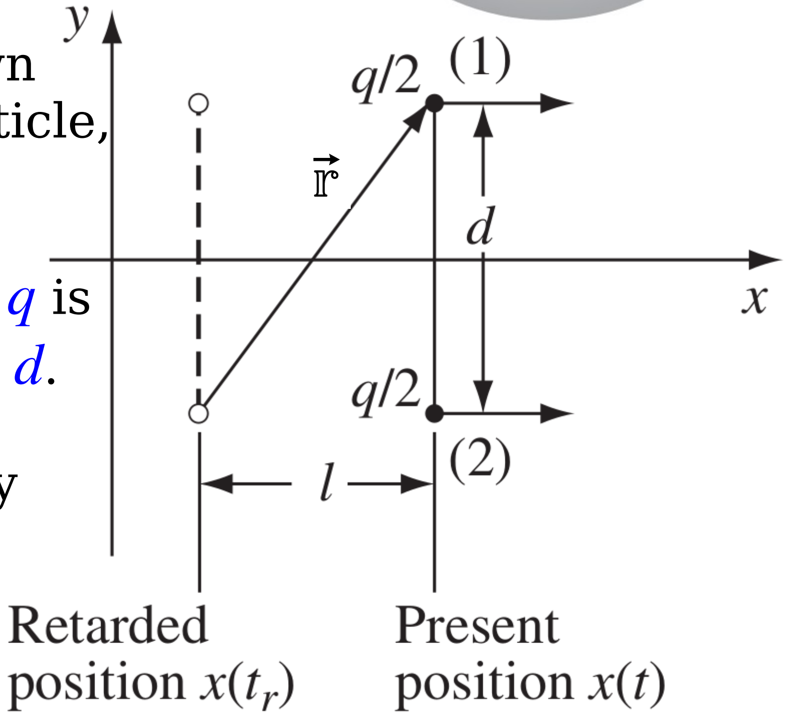
- In general, the EM force of one part (A) on another part (B) is *not* equal and opposite to the force of B on A.

- If the distribution is divided up into infinitesimal chunks, and the imbalances are added up for all such pairs, the result is a *net force of the charge* on itself.

- It is this **self-force**, resulting from the breakdown of Newton's 3rd law within the structure of the particle, that accounts for the radiation reaction.

- Consider a "dumbbell" in which the total charge q is divided into 2 halves separated by a fixed distance d.

- Although it's an unlikely model for an elementary particle: in the point limit ($d \to 0$) any model must yield the Abraham-Lorentz formula, to the extent that conservation of energy dictates that answer.



- Assume the dumbbell moves in the x direction, and is instantaneously at rest [$\boldsymbol{\beta}(t_r=0)=0$] at the retarded time. The electric field at 1 due to 2 is

$$\mathbf{E}_1 = \frac{q/2}{4\pi\epsilon_0 \mathfrak{r}^2} \frac{c\left(1-\beta^2\right)\left(\hat{\mathfrak{r}}-\boldsymbol{\beta}\right)+\vec{\mathfrak{r}}\times\left[\left(\hat{\mathfrak{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]}{c\left(1-\hat{\mathfrak{r}}\cdot\boldsymbol{\beta}\right)^3} = \frac{q}{8\pi\epsilon_0} \frac{\left(c+\vec{\mathfrak{r}}\cdot\dot{\boldsymbol{\beta}}\right)\vec{\mathfrak{r}}-\mathfrak{r}^2\dot{\boldsymbol{\beta}}}{c\,\mathfrak{r}^3}$$

$$\vec{\mathfrak{r}} = \ell\,\hat{\mathbf{x}} + d\,\hat{\mathbf{y}} \quad \Rightarrow \quad \mathfrak{r} = \sqrt{\ell^2+d^2}\,, \quad \vec{\mathfrak{r}}\cdot\dot{\boldsymbol{\beta}} = \ell\,\dot{\beta}$$

- Only interested in the x component of $\mathbf{E}_1$, since the y components will cancel when we add the forces on the 2 ends.

- $$E_{1x} = \frac{q}{8\pi\epsilon_0 c} \frac{\ell c - \dot{\beta} d^2}{\left(\ell^2+d^2\right)^{3/2}} \quad \Rightarrow \quad E_{2x} = E_{1x} \text{ by symmetry}$$

$$\Rightarrow \mathbf{F}_{\text{self}} = \frac{q}{2}\left(\mathbf{E}_1+\mathbf{E}_2\right) = \frac{q^2}{8\pi\epsilon_0 c^2} \frac{\ell c^2 - a d^2}{\left(\ell^2+d^2\right)^{3/2}}\hat{\mathbf{x}} \simeq \frac{q^2}{8\pi\epsilon_0 d^3}\left(\ell - \frac{a}{c^2}d^2\right)\hat{\mathbf{x}} \text{ for } \ell \ll d$$

$$x(t) = x(t_r) + \dot{x}(t_r)(t-t_r) + \frac{1}{2}\ddot{x}(t_r)(x-t_r)^2 + \frac{1}{3!}\dddot{x}(t_r)(t-t_r)^3 + \cdots$$

$$\Rightarrow \quad \ell = x(t) - x(t_r) = \frac{1}{2}a T^2 + \frac{1}{6}\dot{a} T^3 + \cdots \quad \Leftarrow \quad T \equiv t - t_r\,, \quad \dot{x}(t_r) = 0$$

$$(cT)^2 = \ell^2 + d^2 \quad \Rightarrow \quad d = \sqrt{c^2T^2 - \ell^2} = cT\sqrt{1-\left(\frac{aT}{2c}+\frac{\dot{a}T^2}{6c}+\cdots\right)^2} = cT - \frac{a^2T^3}{8c} + \cdots$$

- Need to solve the equation for T as a function of d. There is a systematic procedure for doing this, known as **reversion of series**,

$$0^{\text{th}}\text{-order:}\quad d \simeq c\,T \quad\Rightarrow\quad T \simeq \frac{d}{c}$$

$$1^{\text{st}}\text{-order:}\quad d \simeq c\,T - \frac{a^2}{8\,c}\frac{d^3}{c^3} \quad\Rightarrow\quad T \simeq \frac{d}{c} + \frac{a^2\,d^3}{8\,c^5}$$

$$\Rightarrow\quad T = \frac{d}{c} + \frac{a^2}{8\,c^5}\,d^3 + O\left(d^4\right)$$

$$\Rightarrow\quad \ell = \frac{a}{2\,c^2}\,d^2 + \frac{\dot{a}}{6\,c^3}\,d^3 + O\left(d^4\right) \quad\Rightarrow\quad \mathbf{F}_{\text{self}} \simeq \frac{q^2}{4\,\pi\,\epsilon_0}\left(-\frac{a\left(t_r\right)}{4\,c^2\,d} + \frac{\dot{a}\left(t_r\right)}{12\,c^3} + O\left(d\right)\right)\hat{\mathbf{x}}$$

$$\Rightarrow\quad a\left(t_r\right) = a\left(t\right) + \dot{a}\left(t\right)\left(t_r - t\right) + \cdots = a\left(t\right) - \dot{a}\left(t\right)T + \cdots = a\left(t\right) - \dot{a}\left(t\right)\frac{d}{c} + \cdots$$

$$\Rightarrow\quad \mathbf{F}_{\text{self}} = \frac{q^2}{4\,\pi\,\epsilon_0}\left(-\frac{a\left(t\right)}{4\,c^2\,d} + \frac{\dot{a}\left(t\right)}{3\,c^3} + O\left(d\right)\right)\hat{\mathbf{x}}$$

- The 1^{st} term $\propto a(t)$; if we pull it over to the other side of Newton's 2^{nd} law, it simply adds to the dumbbell's mass.

- In effect, the total inertia of the charged dumbbell $m = 2\,m_0 + \dfrac{1}{4\,\pi\,\epsilon_0}\dfrac{q^2}{4\,d\,c^2}$

- In special relativity, it is not surprising that the electrical repulsion of the charges should enhance the mass of the dumbbell.

- For the potential energy of this configuration is $\frac{1}{4\pi\epsilon_0}\frac{(q/2)^2}{d}$ and according to Einstein's formula $E = mc^2$, this energy contributes to the inertia of the object.
- The 2nd term is the radiation reaction: $F_{\text{rad}}^{\text{int}} = \frac{\mu_0 q^2 \dot{a}}{12\pi c}$
- The term survives in the "point dumbbell" limit $d \to 0$. But it differs from the Abraham-Lorentz formula by a factor of 2.

- But this is only the self-force associated with the *interaction* between 1 and 2. There remains the force of each end on itself. When the latter is included, the result is $F_{\text{rad}} = \frac{\mu_0 q^2 \dot{a}}{6\pi c}$, reproducing the Abraham-Lorentz formula exactly.

$$F_{\text{rad}}(q) = F_{\text{rad}}^{\text{int}}(q) + 2F_{\text{rad}}\left(\frac{q}{2}\right) = F_{\text{rad}}^{\text{int}}(q) + 2\cdot\frac{1}{2^2}\cdot F_{\text{rad}}(q) \quad \Leftarrow \quad F_{\text{rad}}(q) \propto q^2$$

$$\Rightarrow \quad F_{\text{rad}}(q) = 2F_{\text{rad}}^{\text{int}}(q) = \frac{\mu_0 q^2 \dot{a}}{6\pi c}$$

- *Conclusion: The radiation reaction is due to the force of the charge on itself*—or the net force exerted by the fields generated by different parts of the charge distribution acting on one another.