Chapter 9 Electromagnetic Waves

Waves in One Dimension

The Wave Equation

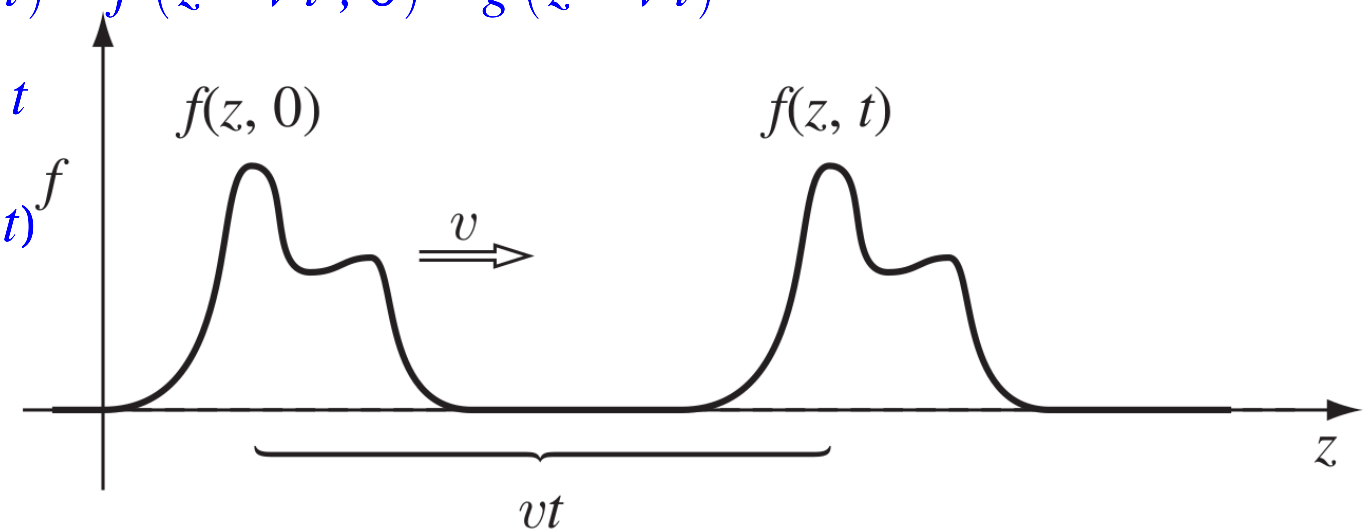
- A wave is a *disturbance of a continuous medium that propagates with a fixed shape at constant velocity*.

- In the presence of absorption, the wave will diminish in size as it moves; if the medium is dispersive, different frequencies travel at different speeds; in 2-dim or 3-dim, as the wave spreads out, its amplitude will decrease; and *standing* waves don't propagate at all.

- $f(z, t)$ represents the displacement of the string at the point z, at time t. Given the *initial* shape of the string, $g(z) \equiv f(z, 0)$, the displacement at z, at t, is the same as the displacement a distance vt to the left (ie. at $z-vt$), back at $t=0$:

$$f(z, t) = f(z - vt, 0) = g(z - vt)$$

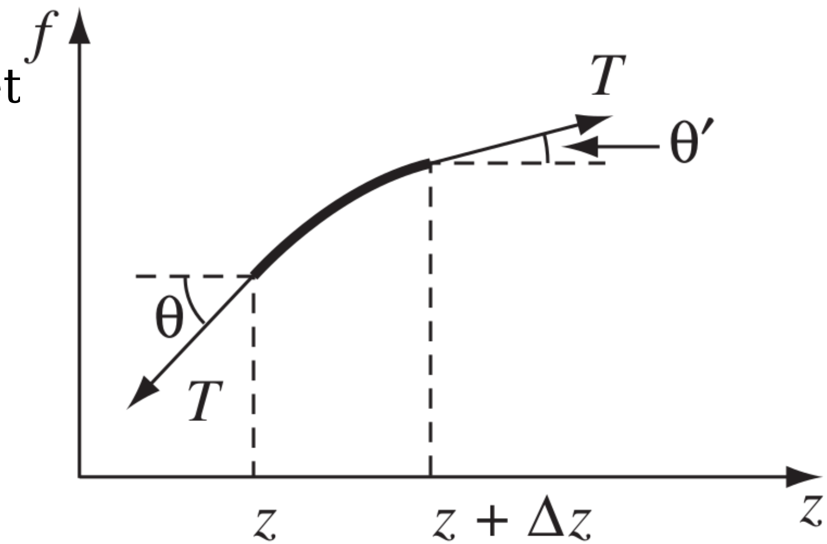
- $f(z, t)$ depended on z and t only in the very special combination $z-vt$; and $f(z, t)$ represents a wave of fixed shape traveling in the z direction at speed v.



- If A and b are constants $f_1(z,t) = A\, e^{-b(z-vt)^2}$, $f_2(z,t) = A \sin[b(z-vt)]$, $f_3(z,t) = \dfrac{A}{b(z-vt)^2+1}$ all represent waves, but $f_4(z,t) = A\, e^{-b(bz^2+vt)}$, $f_5(z,t) = A \sin(bz)\cos(bvt)^3$ do not.

- A stretched string supporting wave motion it follows from Newton's 2nd law.

- If a string is displaced from equilibrium, the net transverse force on the segment, for $\theta, \theta' \ll 1$,



$$\Delta F = T\sin\theta' - T\sin\theta \simeq T\left(\tan\theta' - \tan\theta\right)$$

$$= T\left(\frac{\partial f}{\partial z}\bigg|_{z+\Delta z} - \frac{\partial f}{\partial z}\bigg|_{z}\right) \simeq T\,\frac{\partial^2 f}{\partial z^2}\,\Delta z$$

$$\Delta F = \mu\,\Delta z\,\frac{\partial^2 f}{\partial t^2} \quad \Leftarrow \quad \mu:\ \text{linear mass density}$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial z^2} = \frac{\mu}{T}\,\frac{\partial^2 f}{\partial t^2} \quad \Rightarrow \quad \boxed{\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\,\frac{\partial^2 f}{\partial t^2}} \quad \text{wave equation} \quad \Leftarrow \quad v = \sqrt{\frac{T}{\mu}}$$

- The wave eqn admits as solutions all functions of the form $f(z,t) = g(z-vt)$ that is, all functions that depend on the variables z & t in the special combination $u \equiv z - vt$.

$$\Rightarrow \quad \frac{\partial f}{\partial z}=\frac{\mathrm{d}g}{\mathrm{d}u}\frac{\partial u}{\partial z}=\frac{\mathrm{d}g}{\mathrm{d}u}, \quad \frac{\partial f}{\partial t}=\frac{\mathrm{d}g}{\mathrm{d}u}\frac{\partial u}{\partial t}=-v\frac{\mathrm{d}g}{\mathrm{d}u}$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial z^2}=\frac{\partial}{\partial z}\frac{\mathrm{d}g}{\mathrm{d}u}=\frac{\mathrm{d}^2 g}{\mathrm{d}u^2}\frac{\partial u}{\partial z}=\frac{\mathrm{d}^2 g}{\mathrm{d}u^2}, \quad \frac{\partial^2 f}{\partial t^2}=-v\frac{\partial}{\partial t}\frac{\mathrm{d}g}{\mathrm{d}u}=-v\frac{\mathrm{d}^2 g}{\mathrm{d}u^2}\frac{\partial u}{\partial t}=v^2\frac{\mathrm{d}^2 g}{\mathrm{d}u^2}$$

$$\Rightarrow \quad \frac{\mathrm{d}^2 g}{\mathrm{d}u^2}=\frac{\partial^2 f}{\partial z^2}=\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

- $g(u)$ can be *any* (differentiable) *function whatever*. If the disturbance propagates without changing its shape, then it satisfies the wave equation.

- Functions of the form $g(z-vt)$ are not the only solutions. We can generate another class of solutions by changing the sign of the velocity: $f(z,t)=h(z+vt)$. This represents a wave propagating in the negative z direction.

- The most general solution to the wave equation is the sum of a wave to the right and a wave to the left: $f(z,t)=g(z-vt)+h(z+vt)$

- Since the wave eqn is **linear**: The sum of any 2 solutions is itself a solution.

- Like the simple harmonic oscillator eqn, the wave eqn is ubiquitous in physics. If something is vibrating, the oscillator eqn is responsible, and if something is waving, the wave eqn is bound to be involved.

Sinusoidal Waves

(i) Terminology: The sinusoidal wave form: $f(z,t) = A\cos[k(z - vt) + \delta]$

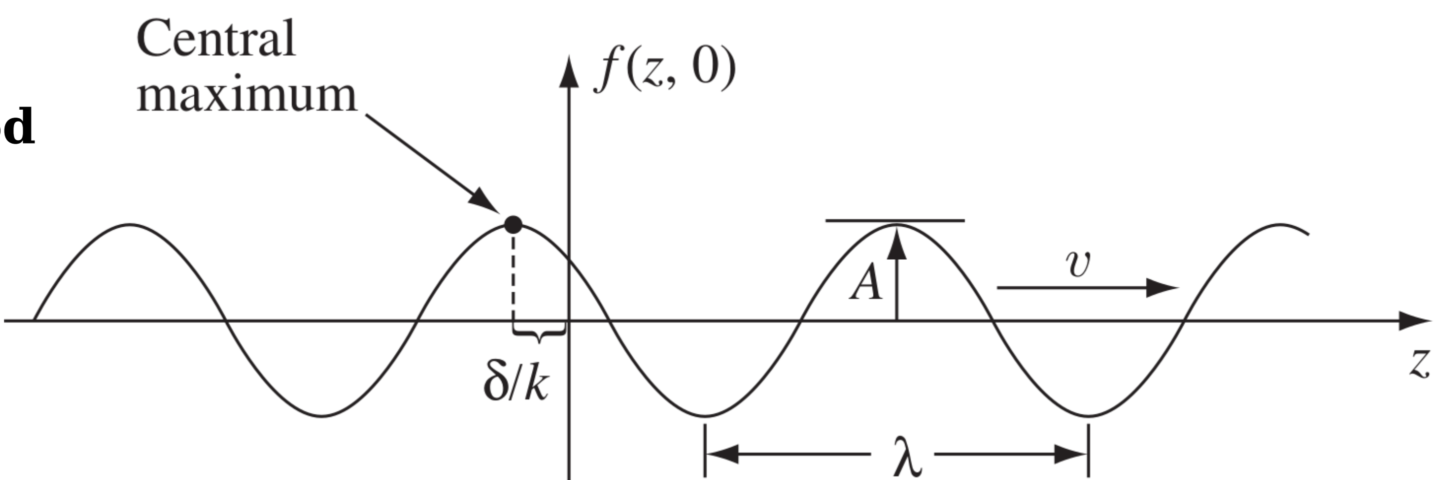
- A is the **amplitude** of the wave (positive, the maximum displacement). The argument of the cosine is called the **phase**, and δ is the **phase constant** (can add $2n\pi$ to δ without changing $f(z,t)$; One usually uses a value in $0 \le \delta < 2\pi$).

- At $z = vt - \frac{\delta}{k}$, the phase is 0, ie, central maximum. If $\delta = 0$, the central maximum passes the origin at $t = 0$; $\frac{\delta}{k}$ is the distance by which the central max is delayed.

- k is the **wave number**; it is related to the **wavelength** λ by $\lambda = \frac{2\pi}{k}$, for when z advances by $\frac{2\pi}{k}$, the cosine executes one complete cycle.

- As time passes, the wave train proceeds to right, at speed v. At any fixed point z, the string vibrates up and down, undergoing one full cycle in a **period**

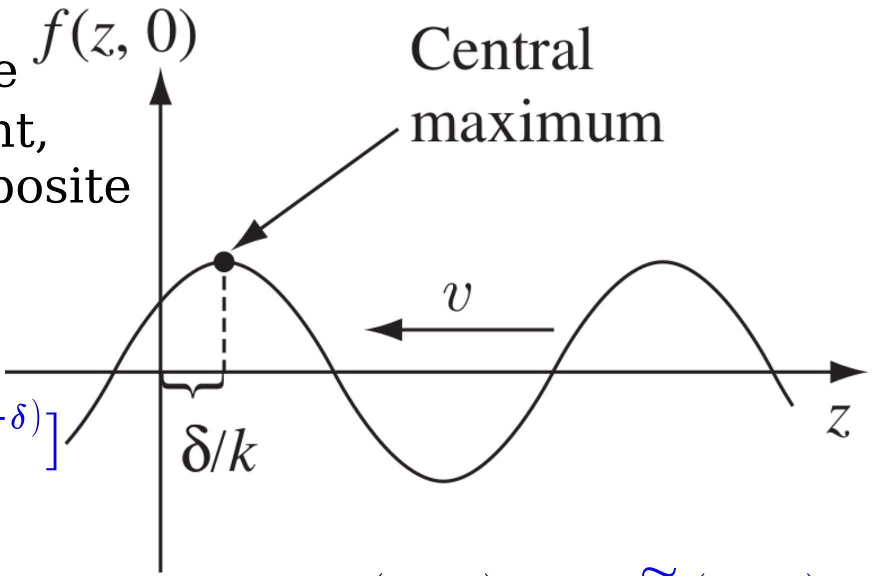
$$T = \frac{2\pi}{kv}$$



- The frequency ν (number of oscillations/time) is $\nu = \frac{1}{T} = \frac{k\,v}{2\pi} = \frac{v}{\lambda}$
- A more convenient unit is the **angular frequency** ω, it represents the number of radians swept out per unit time:

$$\omega = 2\pi\nu = k\,v \quad \Rightarrow \quad f(z,t) = A\cos(kz - \omega t + \delta)$$

- A sinusoidal oscillation of wave number k and (angular) frequency ω traveling to the *left* $f(z,t) = A\cos(kz + \omega t - \delta) = A\cos(-kz - \omega t + \delta)$
- *We could simply switch the sign of k* to produce a wave with the same amplitude, phase constant, frequency, and wavelength, traveling in the opposite direction.



(ii) Complex notation: In **Euler's formula**,

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \Rightarrow \quad f(z,t) = \Re\left[A\,e^{i(kz - \omega t + \delta)}\right]$$

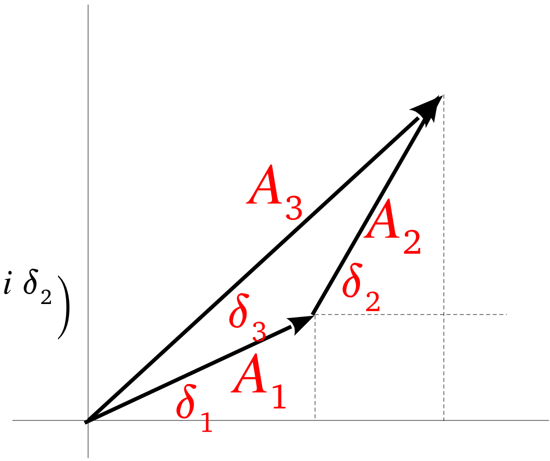
- The **complex wave function**

$$\tilde{f}(z,t) \equiv \tilde{A}\,e^{i(kz-\omega t)} \quad \Leftarrow \quad \tilde{A} \equiv A\,e^{i\delta} \text{ complex amplitude} \Rightarrow \quad f(z,t) = \Re\left[\tilde{f}(z,t)\right]$$

- The *advantage* of the complex notation is that exponentials are much easier to manipulate than sines and cosines.

Example 9.1: show $f_3 = \Re[\tilde{f}_3] = \Re[\tilde{f}_1] + \Re[\tilde{f}_2] = f_1 + f_2 \quad \Leftarrow \quad \tilde{f}_3 = \tilde{f}_1 + \tilde{f}_2$

Let them have the same frequency and wave number,



$$\tilde{f}_j = \tilde{A}_j e^{i(kz-\omega t)} = A_j e^{i\delta_j} e^{i(kz-\omega t)}, \quad j = 1, 2, 3$$

$$\tilde{f}_3 = \tilde{f}_1 + \tilde{f}_2 \Rightarrow \tilde{A}_3 = \tilde{A}_1 + \tilde{A}_2 \Rightarrow A_3 e^{i\delta_3} = A_1 e^{i\delta_1} + A_2 e^{i\delta_2}$$

$$\Rightarrow \quad A_3^2 = \left(A_3 e^{i\delta_3}\right)\left(A_3 e^{-i\delta_3}\right) = \left(A_1 e^{i\delta_1} + A_2 e^{i\delta_2}\right)\left(A_1 e^{-i\delta_1} + A_2 e^{-i\delta_2}\right)$$

$$= A_1^2 + A_2^2 + 2 A_1 A_2 \cos(\delta_2 - \delta_1)$$

$$\Rightarrow \quad A_3 = \sqrt{A_1^2 + A_2^2 + 2 A_1 A_2 \cos(\delta_2 - \delta_1)}$$

$$A_3 e^{i\delta_3} = A_3(\cos\delta_3 + i\sin\delta_3) = A_1(\cos\delta_1 + i\sin\delta_1) + A_2(\cos\delta_2 + i\sin\delta_2)$$

$$= (A_1\cos\delta_1 + A_2\cos\delta_2) + i(A_1\sin\delta_1 + A_2\sin\delta_2)$$

$$\Rightarrow \quad \tan\delta_3 = \frac{A_3\sin\delta_3}{A_3\cos\delta_3} = \frac{A_1\sin\delta_1 + A_2\sin\delta_2}{A_1\cos\delta_1 + A_2\cos\delta_2}$$

$$f_j = A_j\cos(kz - \omega t + \delta_j), \; j = 1, 2, 3 \quad \text{and} \quad f_3 = f_1 + f_2$$

$$\Rightarrow \quad A_3\left[\cos\delta_3\cos(kz-\omega t) - \sin\delta_3\sin(kz-\omega t)\right]$$

$$= (A_1\cos\delta_1 + A_2\cos\delta_2)\cos(kz-\omega t) - (A_1\sin\delta_1 + A_2\sin\delta_2)\sin(kz-\omega t)$$

$$\Rightarrow \quad A_3 = \sqrt{A_1^2 + A_2^2 + 2 A_1 A_2 \cos(\delta_2 - \delta_1)} \;\; \& \;\; \tan\delta_3 = \frac{A_1\sin\delta_1 + A_2\sin\delta_2}{A_1\cos\delta_1 + A_2\cos\delta_2} \quad \text{the same}$$

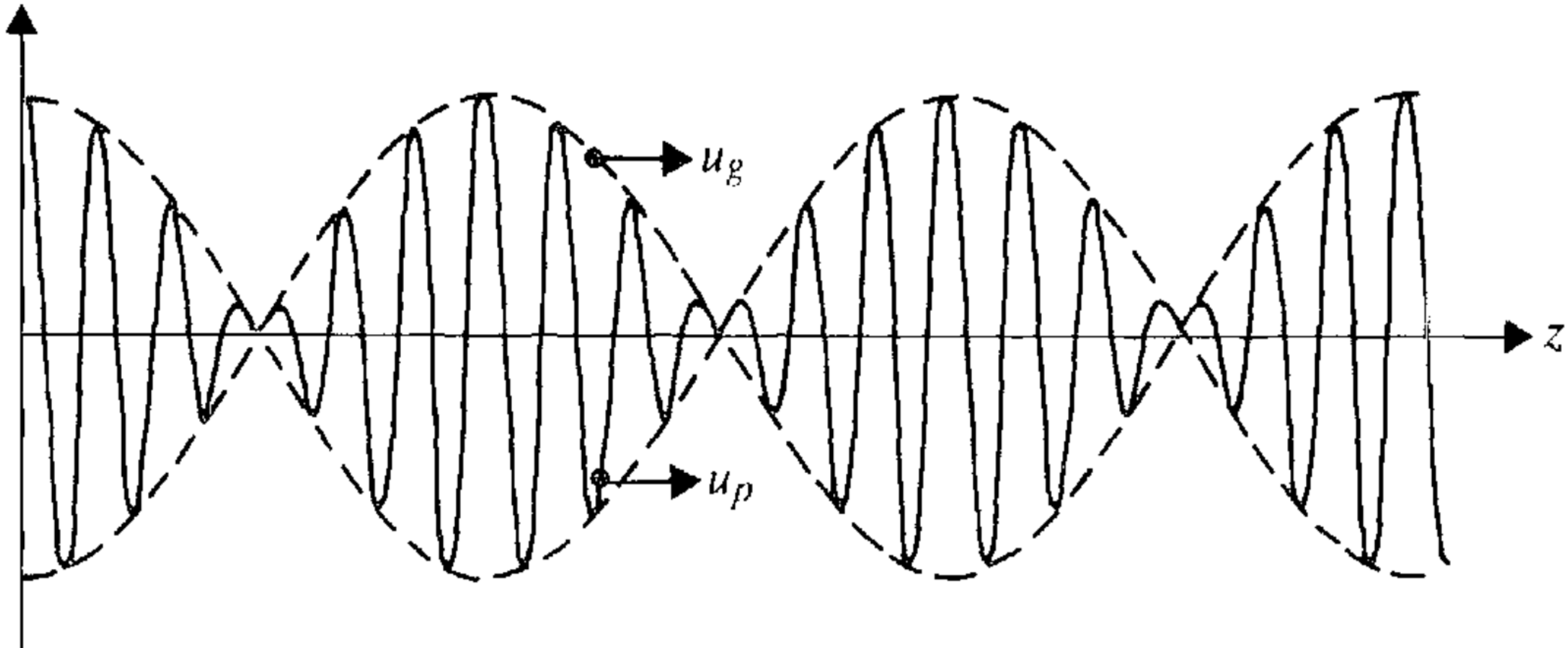
(iii) Linear combinations of sinusoidal waves: *Any* wave can be expressed as a linear combination of sinusoidal ones: $\tilde{f}(z,t) = \int_{-\infty}^{+\infty} \tilde{A}(k)\, e^{i(kz-\omega t)}\, \mathrm{d}k$

- The formula for $\tilde{A}(k)$, in terms of the initial conditions $f(z,0)$ and $\dot{f}(z,0)$, can be obtained from the theory of Fourier transforms.

- So any wave can be written as a linear combination of sinusoidal waves, and if you know how sinusoidal waves behave, you know how any wave behaves.

Phase velocity and group velocity

- $k\,x - \omega\,t = \text{constant phase} \quad \Rightarrow$ phase velocity $v_p \equiv \dfrac{\mathrm{d}\,x}{\mathrm{d}\,t} = \dfrac{\omega}{k}$
- In some cases, waves of different frequencies propagates with different phase velocities, since information-bearing signals consist of a band of frequencies.
- Waves of the component frequencies travel with different phase velocities, causing a distortion in the signal wave shape, called **dispersion**.
- Such signal normally has a small spread of frequencies (side bands) around a high carrier frequency. Such a signal comprises a "group" of frequencies and forms a wave packet.



- A **group velocity** is the velocity of propagation of the wave-packet envelope (of a group of frequencies).

- Consider a wave packet that consists of 2 traveling waves with equal amplitude and slightly different angular frequencies $\omega_0 \pm \Delta\omega$ ($\Delta\omega \ll \omega_0$) and wave numbers $k_0 \pm \Delta k$ ($\Delta k \ll k_0$), then the combined wave is

$$f(x,t) = f_0 \cos\left[(k_0 + \Delta k)x - (\omega_0 + \Delta\omega)t\right] + f_0 \cos\left[(k_0 - \Delta k)x - (\omega_0 - \Delta\omega)t\right]$$
$$= 2 f_0 \cos(x\,\Delta k - t\,\Delta\omega)\cos(k_0 x - \omega_0 t)$$

- This expression represents a rapidly oscillating wave wit an angular frequency ω_0 and an amplitude that varies slowly with an angular frequency $\Delta\omega$.

- The wave inside the envelope propagates with a phase velocity $v_p \equiv \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\omega_0}{k_0}$

- The velocity of the envelope (ie, group velocity) can be determined by

$$x\,\Delta k - t\,\Delta\omega = \text{constant phase} \quad \Rightarrow \quad \text{group velocity} \;\; v_g \equiv \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\Delta\omega}{\Delta k} = \frac{1}{\Delta k / \Delta\omega}$$

$$\text{As } \Delta\omega \to 0 \quad \Rightarrow \quad v_g \equiv \frac{1}{\mathrm{d}k/\mathrm{d}\omega}$$

- In a normal dispersion, $v_p \geq v_g$; in an anomalous dispersion $v_p \leq v_g$.

Boundary Conditions: Reflection and Transmission

- What happens to a wave depends a lot on how the string is *attached*—ie, on the specific boundary conditions to which the wave is subject.

- If the string is simply tied onto a 2nd string. The tension T is the same for both, but the mass per unit length μ presumably is not, and hence the wave velocities v_1 and v_2 are different ($v=\sqrt{\dfrac{T}{\mu}}$).

- Let the knot occurs at $z=0$. The **incident** wave: $\tilde{f}_I(z,t)=\tilde{A}_I e^{i(k_1 z-\omega t)}, \quad z<0$

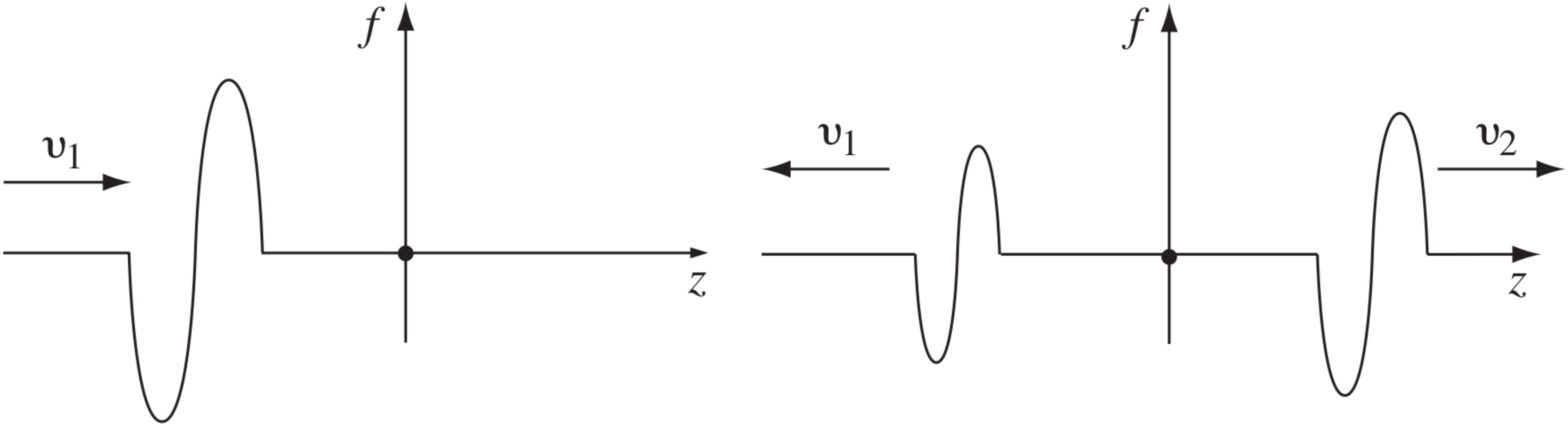
- The **reflected** wave travels back along string 1: $\tilde{f}_R(z,t)=\tilde{A}_R e^{i(-k_1 z-\omega t)}, \quad z<0$

- The **transmitted** wave continues onto the right: $\tilde{f}_T(z,t)=\tilde{A}_T e^{i(k_2 z-\omega t)}, \quad z>0$ in string 2.

- The incident wave $f_I(z,t)$ is a sinusoidal oscillation that extends all the way to $-\infty$, and doing so for all of history. Same as f_R and f_T (to $+\infty$).

- *All parts of the system are oscillating at the same frequency* ω.

- Since the wave velocities are different in the 2 strings, the wavelengths and wave numbers are also different: $\dfrac{\lambda_1}{\lambda_2}=\dfrac{k_2}{k_1}=\dfrac{v_1}{v_2}$



(a) Incident pulse (b) Reflected and transmitted pulses

- With incident and reflected waves of infinite extent traveling on the same piece of string, it's going to be hard to tell them apart.

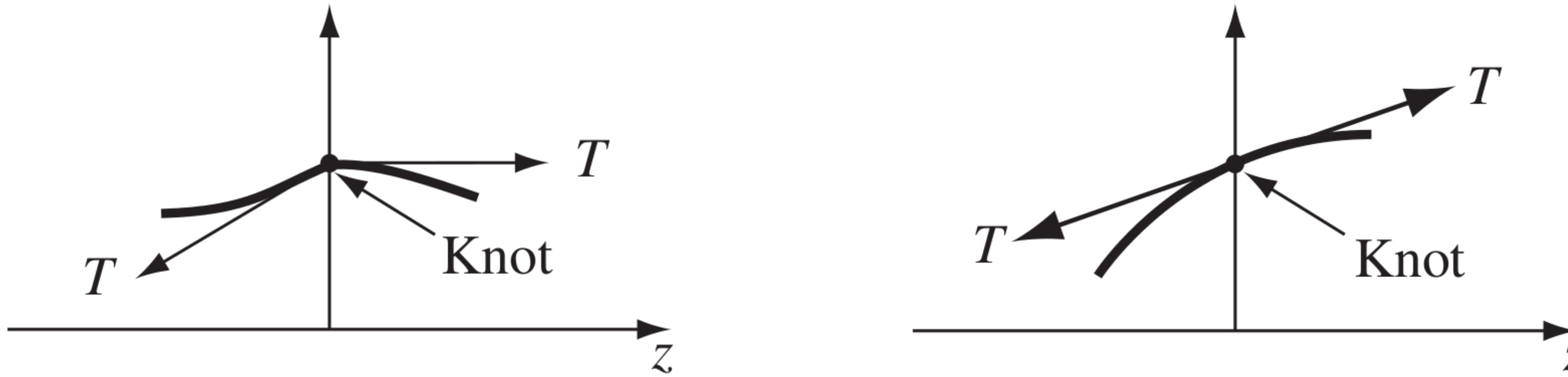
- No *finite* pulse is truly sinusoidal. They can be built up as *linear combinations* of sinusoidal functions, but only by putting together a whole range of frequencies and wavelengths.

- For a sinusoidal incident wave, the net disturbance of the string is:

$$\tilde{f}(z,t)=\begin{cases}\tilde{A}_I e^{i(k_1 z-\omega t)}+\tilde{A}_R e^{i(-k_1 z-\omega t)}, & z<0\\ \tilde{A}_T e^{i(k_2 z-\omega t)}, & z>0\end{cases}$$

- At the join ($z=0$), the displacement just slightly to the left ($z=0^-$) must equal the one slightly to the right ($z=0^+$). So the *real* wave $f(z,t)$ is *continuous* at $z=0$:

$$f(0^-,t)=f(0^+,t)$$



(a) Discontinuous slope; force on knot (b) Continuous slope; no force on knot

- If the knot is of negligible mass, the *derivative* of f must *also* be continuous:

$$\left.\frac{\partial f}{\partial z}\right|_{0^-} = \left.\frac{\partial f}{\partial z}\right|_{0^+}$$

Otherwise there would be a net force on the knot, and therefore an infinite acceleration.

- The complex wave function obeys the same rules:

$$\tilde{f}(0^-, t) = \tilde{f}(0^+, t)$$

$$\left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^-} = \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^+}$$

- These boundary conditions determine the outgoing amplitudes in terms of the incoming one:

$$\begin{aligned} \tilde{A}_I + \tilde{A}_R &= \tilde{A}_T \\ k_1(\tilde{A}_I - \tilde{A}_R) &= k_2 \tilde{A}_T \end{aligned} \quad \Rightarrow \quad \tilde{A}_R = \frac{k_1 - k_2}{k_1 + k_2}\tilde{A}_I = \frac{v_2 - v_1}{v_1 + v_2}\tilde{A}_I, \quad \tilde{A}_T = \frac{2k_1}{k_1 + k_2}\tilde{A}_I = \frac{2v_2}{v_1 + v_2}\tilde{A}_I$$

$$\Rightarrow \quad A_R e^{i\delta_R} = \frac{v_2 - v_1}{v_1 + v_2} A_I e^{i\delta_I}, \quad A_T e^{i\delta_T} = \frac{2v_2}{v_1 + v_2} A_I e^{i\delta_I} \quad \text{for real wave}$$

- If the 2nd string is lighter than the 1st ($\mu_2 < \mu_1 \Rightarrow v_2 > v_1$), all 3 waves have the same phase angle ($\delta_R = \delta_T = \delta_I$), and the outgoing amplitudes are

$$A_R = \frac{v_2 - v_1}{v_1 + v_2} A_I , \quad A_T = \frac{2 v_2}{v_1 + v_2} A_I$$

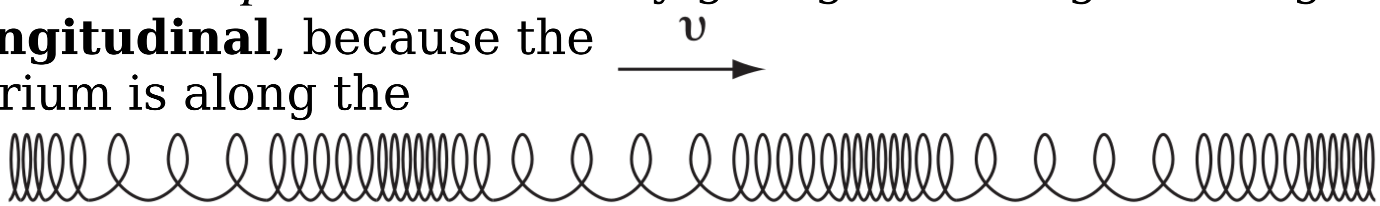
- If the 2nd string is heavier than the 1st ($v_2 < v_1$), the reflected wave is out of phase by 180° ($\delta_R + \pi = \delta_T = \delta_I$). Since $\cos(-k_1 z - \omega t + \delta_I - \pi) = -\cos(-k_1 z - \omega t + \delta_I)$ the reflected wave is "upside down," and $A_R = \frac{v_1 - v_2}{v_1 + v_2} A_I , \quad A_T = \frac{2 v_2}{v_1 + v_2} A_I$

- If the 2nd string is *infinitely* massive—or if the 1st string is simply *nailed down* at the end—then $A_R = A_I$, $A_T = 0$, *no* transmitted wave—*all* of it reflects back.

Polarization

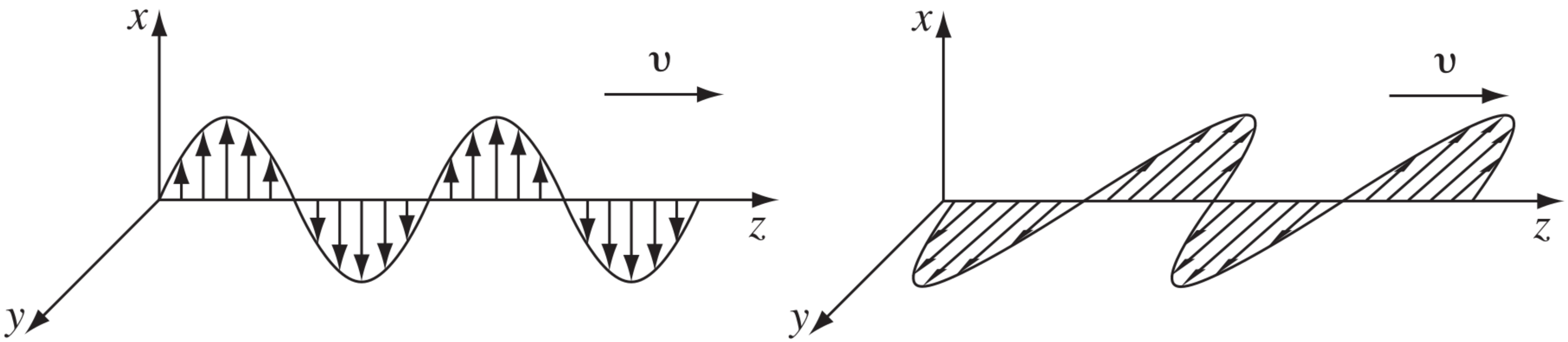
- The waves that travel down a string when you shake it are called **transverse**, because the displacement $\perp$ the direction of propagation.

- It is also possible to stimulate *compression* waves by giving the string little tugs. These waves are called **longitudinal**, because the displacement from equilibrium is along the direction of propagation.



- Sound waves, compression in air, are longitudinal; EM waves are transverse.

- There are 2 dimensions $\perp$ any given line of propagation. Accordingly transverse waves occur in 2 independent states of **polarization**.



(a) Vertical polarization (b) Horizontal polarization

- You can shake the string up-and-down, ie, "vertical" polarization

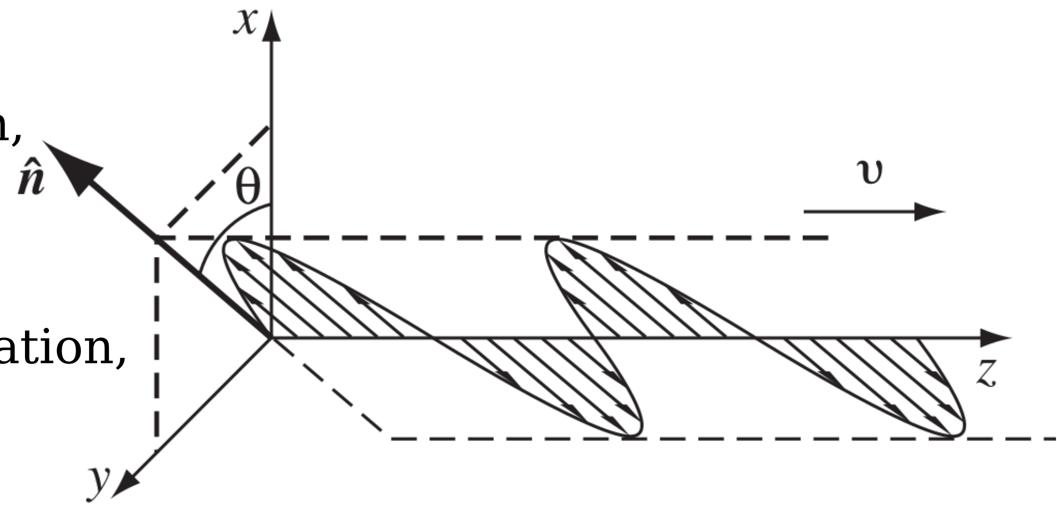
$$\tilde{\boldsymbol{f}}_v(z,t) = \tilde{A}\, e^{i(kz-\omega t)}\, \hat{\mathbf{x}}$$

or left-and-right, ie,"horizontal" polarization,

$$\tilde{\boldsymbol{f}}_h(z,t) = \tilde{A}\, e^{i(kz-\omega t)}\, \hat{\mathbf{y}}$$

or along any other direction in the *xy* plane

$$\tilde{\boldsymbol{f}}(z,t) = \tilde{A}\, e^{i(kz-\omega t)}\, \hat{\mathbf{n}}$$



(c) Polarization vector

- The **polarization vector** $\hat{\mathbf{n}}$ defines the plane of vibration. Because the waves are transverse, $\hat{\mathbf{n}}$ ⊥ the direction of propagation: $\hat{\mathbf{n}}\cdot\hat{\mathbf{z}} = 0$

- In terms of the **polarization angle** θ, $\hat{\mathbf{n}} = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}$

- The wave along any direction in the *xy* plane can be considered a superposition of 2 waves—one horizontally polarized, the other vertically:

$$\tilde{\boldsymbol{f}}(z,t) = (\tilde{A}\cos\theta)\, e^{i(kz-\omega t)}\,\hat{\mathbf{x}} + (\tilde{A}\sin\theta)\, e^{i(kz-\omega t)}\,\hat{\mathbf{y}}$$

Electromagnetic Waves in Vacuum

The Wave Equation for **E** and **B**

- In regions of space without charge or current, Maxwell's equations read

$$\nabla\cdot\mathbf{E}=0\,,\quad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\,,\quad \nabla\cdot\mathbf{B}=0\,,\quad \nabla\times\mathbf{B}=\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

- They constitute a set of coupled, 1st-order, partial differential eqns for **E** and **B**.

- They can be decoupled by

$$\nabla\times(\nabla\times\mathbf{E})=\nabla(\cancel{\nabla\cdot\mathbf{E}})-\nabla^2\mathbf{E}$$
$$=\nabla\times\left(-\frac{\partial\mathbf{B}}{\partial t}\right)=-\frac{\partial}{\partial t}(\nabla\times\mathbf{B})=-\mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\mathbf{E}$$
$$\nabla\times(\nabla\times\mathbf{B})=\nabla(\cancel{\nabla\cdot\mathbf{B}})-\nabla^2\mathbf{B}$$
$$=\nabla\times\left(\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)=\mu_0\epsilon_0\frac{\partial}{\partial t}(\nabla\times\mathbf{E})=-\mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\mathbf{B}$$
$$\Rightarrow\quad \nabla^2\mathbf{E}=\mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}\,,\quad \nabla^2\mathbf{B}=\mu_0\epsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2}$$

- We now have separate equations for **E** and **B**, but they are of 2nd-order.

- In vacuum, each Cartesian component of $\mathbf{E}$ and $\mathbf{B}$ satisfies the **3d wave eqn**,

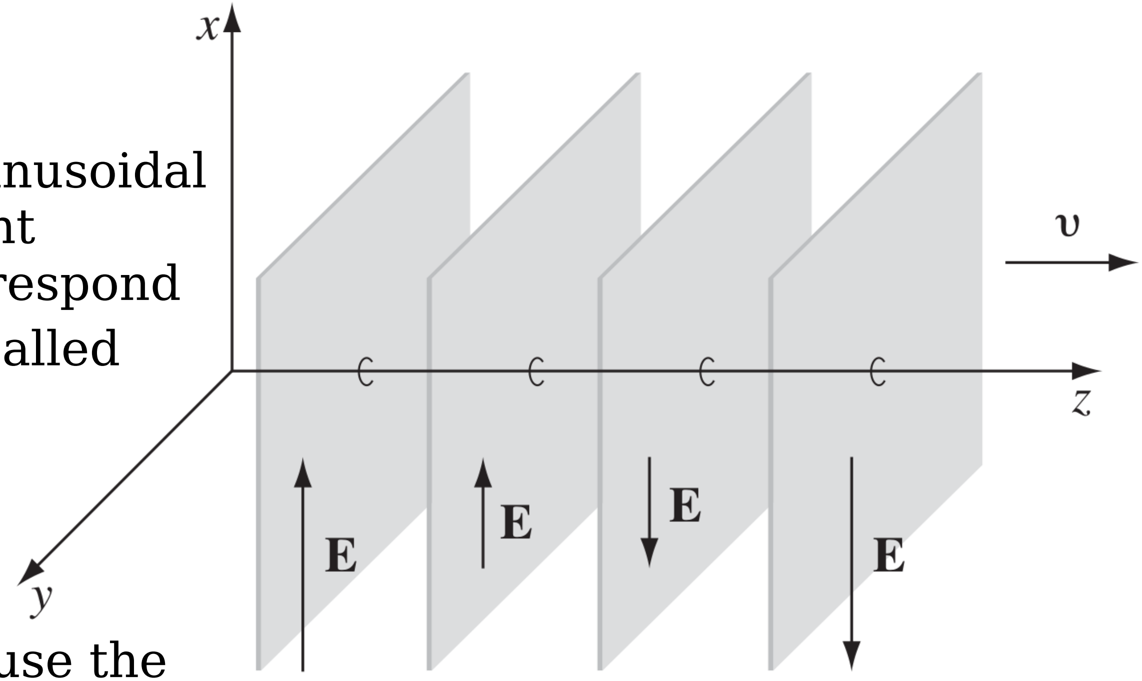
$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

- So Maxwell's equations imply that empty space supports the propagation of EM waves, traveling at a speed $v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ m/s} = c$
- The implication is astounding: Light is an EM wave.

- ϵ_0 & μ_0 came into the theory as constants in Coulomb's law and the Biot-Savart law. You measure them in experiments having nothing whatever to do with light. But according to Maxwell's theory, you can calculate c from these 2 numbers.

- The crucial role played by Maxwell's contribution ($\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$) to Ampère's law; without it, the wave equation would not emerge, there would be no EM theory of light.

Monochromatic Plane Waves



- We now confine our attention to sinusoidal waves of frequency ω. Since different frequencies in the visible range correspond to different *colors*, such waves are called **monochromatic**.

- If the waves are traveling in the z direction and no x or y dependence; these are called **plane waves**, because the fields are uniform over every plane ⊥ the direction of propagation.

- $\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(kz-\omega t)}$, where $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ are the (complex) amplitudes.

- The physical fields are the real parts of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$, and $\omega = ck$.

- Whereas every solution to Maxwell's equations (in empty space) must obey the wave equation, the converse is *not* true; Maxwell's equations impose extra constraints on $\tilde{\mathbf{E}}_0, \tilde{\mathbf{B}}_0$.

- $\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \tilde{E}_{0z} = \tilde{B}_{0z} = 0$. That is, EM *waves are transverse*: the electric and magnetic fields ⊥ the direction of propagation.

$$\mathbf{E} = \mathbf{E}(\boldsymbol{k}\cdot\mathbf{r} - \omega t) = \mathbf{E}(\varphi) \quad \Leftarrow \quad \varphi \equiv \boldsymbol{k}\cdot\mathbf{r} - \omega t\,, \quad \mathbf{r} = \sum_i x_i \hat{\mathbf{x}}_i$$

$$\Rightarrow \quad \nabla\cdot\mathbf{E} = \sum_i \frac{\partial E_i}{\partial x_i} = \sum_i \frac{\mathrm{d} E_i}{\mathrm{d}\varphi}\frac{\partial \varphi}{\partial x_i} = \sum_i k_i \frac{\mathrm{d} E_i}{\mathrm{d}\varphi} = \boldsymbol{k}\cdot\frac{\mathrm{d}\mathbf{E}}{\mathrm{d}\varphi} = \frac{\mathrm{d}}{\mathrm{d}\varphi}(\boldsymbol{k}\cdot\mathbf{E}) = 0$$

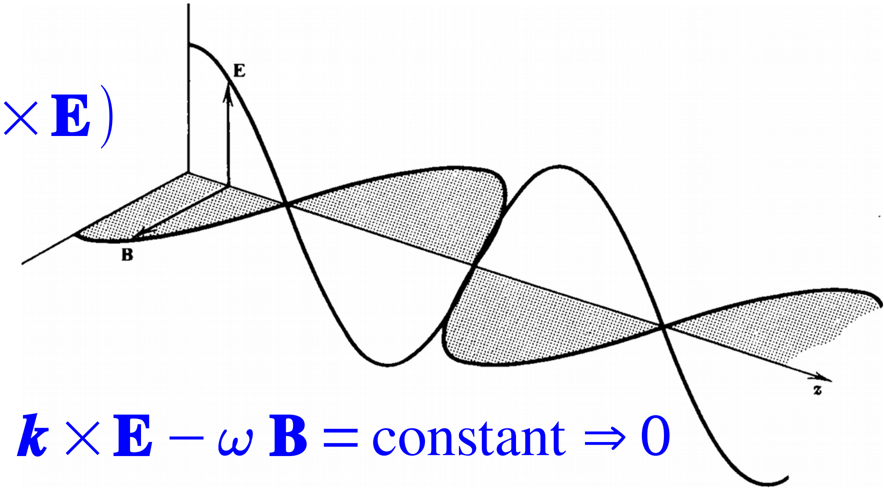
$$\Rightarrow \quad \boldsymbol{k}\cdot\mathbf{E} = \text{constant} \Rightarrow 0 \quad \Rightarrow \quad \hat{\boldsymbol{k}} \perp \mathbf{E} \quad \Leftarrow \text{ wave, not constant (static)}$$

$$\text{Similiarly, } \hat{\boldsymbol{k}} \perp \mathbf{B}$$

Thus $\mathbf{E}$ and $\mathbf{B}$, when plane waves, are always transverse to the motion of the wave.

$$\nabla\times\mathbf{E} = \sum_i \nabla E_i \times \hat{\mathbf{x}}_i = \sum_{i,j}\left(\frac{\partial E_i}{\partial x_j}\hat{\mathbf{x}}_j \times \hat{\mathbf{x}}_i\right) \quad \Leftarrow \quad \nabla\times(f\,\mathbf{A}) = \nabla f \times \mathbf{A} + f\,\nabla\times\mathbf{A}$$

$$= \sum_i \left(\frac{\mathrm{d} E_i}{\mathrm{d}\varphi}\boldsymbol{k}\times\hat{\mathbf{x}}_i\right) = \boldsymbol{k}\times\frac{\mathrm{d}\mathbf{E}}{\mathrm{d}\varphi} = \frac{\mathrm{d}}{\mathrm{d}\varphi}(\boldsymbol{k}\times\mathbf{E})$$

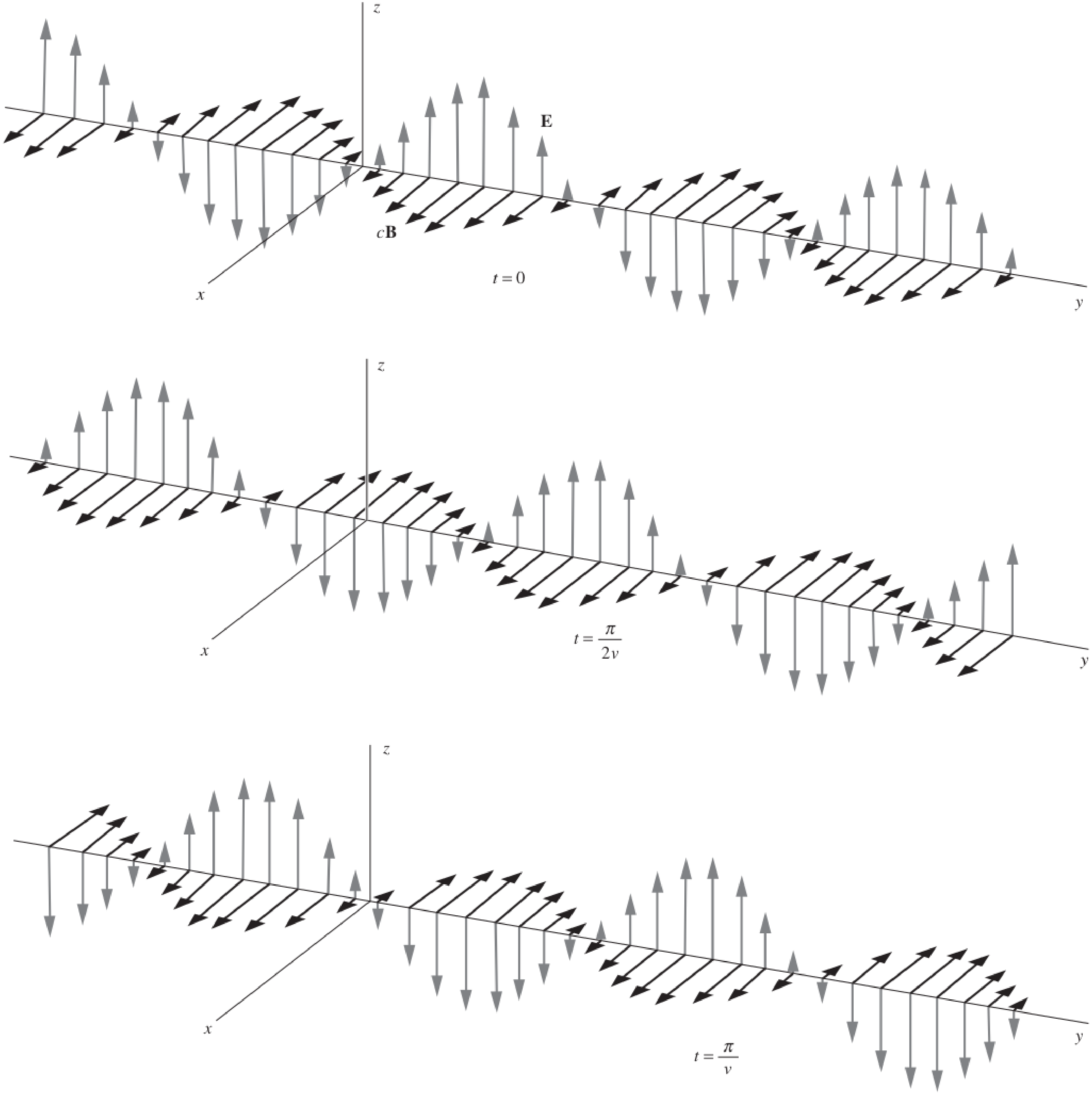


$$-\frac{\partial \mathbf{B}}{\partial t} = -\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\varphi}\frac{\partial\varphi}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}\varphi}(\omega\,\mathbf{B})$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\varphi}(\boldsymbol{k}\times\mathbf{E} - \omega\,\mathbf{B}) = 0 \quad \Rightarrow \quad \boldsymbol{k}\times\mathbf{E} - \omega\,\mathbf{B} = \text{constant} \Rightarrow 0$$

$$\Rightarrow \quad \hat{\boldsymbol{k}}\times\mathbf{E} = \frac{\omega}{k}\mathbf{B} = v_p\,\mathbf{B} \quad \Rightarrow \quad \mathbf{E}\perp\mathbf{B}\,, \quad \frac{E}{B} = v_p \quad \Rightarrow \quad \boldsymbol{k}\perp\mathbf{E}\perp\mathbf{B}$$

The Electromagnetic Spectrum		
Frequency (Hz)	Type	Wavelength (m)
10^{22}		10^{-13}
10^{21}	gamma rays	10^{-12}
10^{20}		10^{-11}
10^{19}		10^{-10}
10^{18}	x-rays	10^{-9}
10^{17}		10^{-8}
10^{16}	ultraviolet	10^{-7}
10^{15}	visible	10^{-6}
10^{14}	infrared	10^{-5}
10^{13}		10^{-4}
10^{12}		10^{-3}
10^{11}		10^{-2}
10^{10}	microwave	10^{-1}
10^{9}		1
10^{8}	TV, FM	10
10^{7}		10^{2}
10^{6}	AM	10^{3}
10^{5}		10^{4}
10^{4}	RF	10^{5}
10^{3}		10^{6}



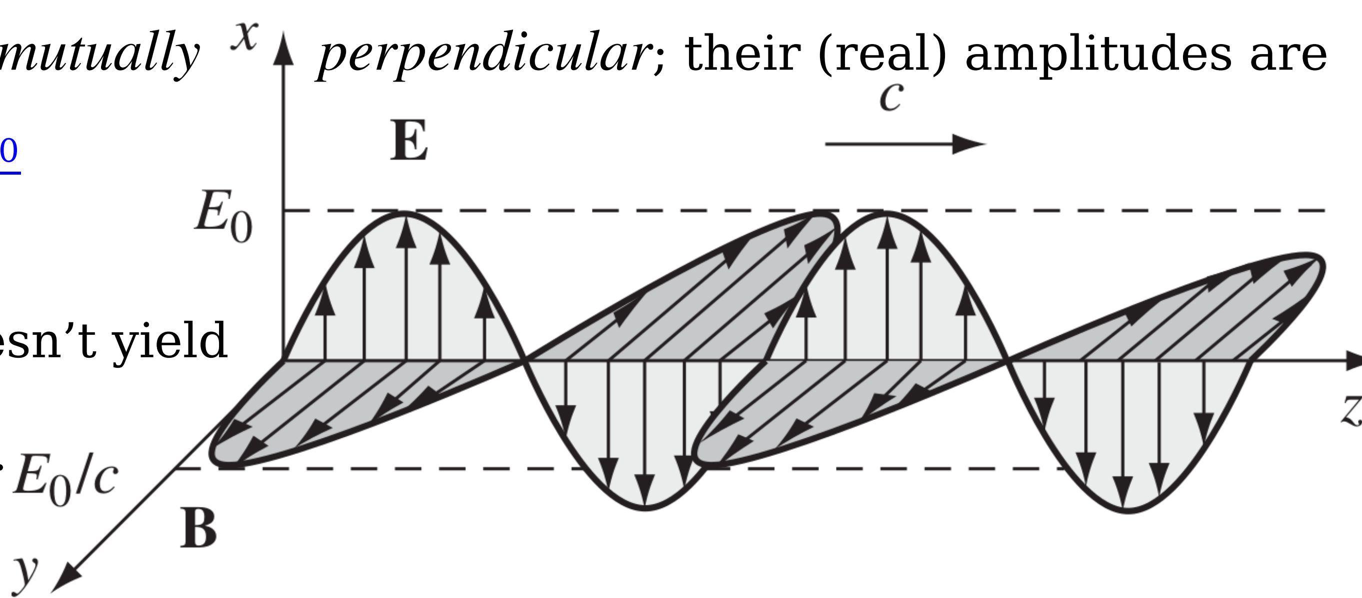
z
E
cB
x
t = 0
y
z
x
$t = \frac{\pi}{2\nu}$
y
z
x
$t = \frac{\pi}{\nu}$
y

The Visible Range		
Frequency (Hz)	Color	Wavelength (m)
1.0×10^{15}	near ultraviolet	3.0×10^{-7}
7.5×10^{14}	shortest visible blue	4.0×10^{-7}
6.5×10^{14}	blue	4.6×10^{-7}
5.6×10^{14}	green	5.4×10^{-7}
5.1×10^{14}	yellow	5.9×10^{-7}
4.9×10^{14}	orange	6.1×10^{-7}
3.9×10^{14}	longest visible red	7.6×10^{-7}
3.0×10^{14}	near infrared	1.0×10^{-6}

- Faraday's law, $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$, implies a relation between the electric and magnetic amplitudes, to wit: $-k\,\tilde{E}_{0y} = \omega\,\tilde{B}_{0x}, \quad k\,\tilde{E}_{0x} = \omega\,\tilde{B}_{0y} \quad \Rightarrow \quad \tilde{\mathbf{B}}_0 = \dfrac{k}{\omega}\,\hat{\mathbf{z}} \times \tilde{\mathbf{E}}_0$

- $\mathbf{E}$ & $\mathbf{B}$ are *in phase* and *mutually perpendicular*; their (real) amplitudes are related by $B_0 = \dfrac{k}{\omega} E_0 = \dfrac{E_0}{c}$

- $\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$ doesn't yield an independent condition.



$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)} = \left(\tilde{E}_{0x}\hat{\mathbf{x}} + \tilde{E}_{0y}\hat{\mathbf{y}}\right) e^{i(kz-\omega t)} = \tilde{E}_x \hat{\mathbf{x}} + \tilde{E}_y \hat{\mathbf{y}}$$

$$\tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(kz-\omega t)} = \left(\tilde{B}_{0x}\hat{\mathbf{x}} + \tilde{B}_{0y}\hat{\mathbf{y}}\right) e^{i(kz-\omega t)} = \tilde{B}_x \hat{\mathbf{x}} + \tilde{B}_y \hat{\mathbf{y}}$$

$$\Rightarrow \quad \begin{array}{l} \tilde{E}_x = \tilde{E}_{0x} e^{i(kz-\omega t)}, \quad \tilde{E}_y = \tilde{E}_{0y} e^{i(kz-\omega t)} \\ \tilde{B}_x = \tilde{B}_{0x} e^{i(kz-\omega t)}, \quad \tilde{B}_y = \tilde{B}_{0y} e^{i(kz-\omega t)} \end{array}$$

$$\Rightarrow \quad \nabla \times \tilde{\mathbf{E}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ \tilde{E}_x & \tilde{E}_y & 0 \end{vmatrix} = -\partial_z \tilde{E}_y \hat{\mathbf{x}} + \partial_z \tilde{E}_x \hat{\mathbf{y}} + \left(\partial_x \tilde{E}_y - \partial_y \tilde{E}_x\right)\hat{\mathbf{z}}$$

$$= +i\left(-k\tilde{E}_y \hat{\mathbf{x}} + k\tilde{E}_x \hat{\mathbf{y}}\right)$$

$$\frac{\partial \tilde{\mathbf{B}}}{\partial t} = -i\left(\omega \tilde{B}_x \hat{\mathbf{x}} + \omega \tilde{B}_x \hat{\mathbf{y}}\right)$$

$$\nabla \times \tilde{\mathbf{E}} = -\frac{\partial \tilde{\mathbf{B}}}{\partial t} \quad \Rightarrow \quad -k\tilde{E}_y = \omega \tilde{B}_x, \quad k\tilde{E}_x = \omega \tilde{B}_y \quad \Rightarrow \quad \tilde{\mathbf{B}} = \frac{k}{\omega}\hat{\mathbf{z}} \times \tilde{\mathbf{E}} = \frac{\hat{\mathbf{z}} \times \tilde{\mathbf{E}}}{c}$$

Example 9.2: If **E** points in the x direction, then **B** points in the y direction,

$$\tilde{\mathbf{E}}(z,t)=\tilde{E}_0 e^{i(kz-\omega t)}\,\hat{\mathbf{x}},\quad \tilde{\mathbf{B}}(z,t)=\frac{\tilde{E}_0}{c}e^{i(kz-\omega t)}\,\hat{\mathbf{y}}$$. Take the real part

$$\mathbf{E}(z,t)=E_0\cos(kz-\omega t+\delta)\,\hat{\mathbf{x}},\quad \mathbf{B}(z,t)=\frac{E_0}{c}\cos(kz-\omega t+\delta)\,\hat{\mathbf{y}}$$

the paradigm for a monochromatic plane wave.

- The wave is said to be polarized in the x direction (by using the direction of **E**).

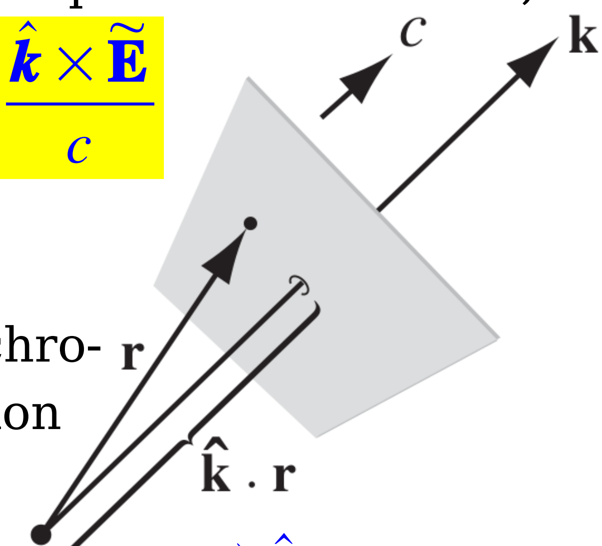
- Generalize to monochromatic plane waves traveling in an arbitrary direction by introducing the **propagation** (or **wave**) **vector**, $\boldsymbol{k}$, pointing in the direction of propagation, whose magnitude is the wave number k.

- With $\boldsymbol{k}\cdot\mathbf{r}$ as the appropriate generalization of kz, $\hat{\mathbf{n}}$ as the polarization vector,

$$\tilde{\mathbf{E}}(\mathbf{r},t)=\tilde{E}_0 e^{i(\boldsymbol{k}\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{n}},\quad \tilde{\mathbf{B}}(\mathbf{r},t)=\frac{\tilde{E}_0}{c}e^{i(\boldsymbol{k}\cdot\mathbf{r}-\omega t)}\,\hat{\boldsymbol{k}}\times\hat{\mathbf{n}}=\frac{\hat{\boldsymbol{k}}\times\tilde{\mathbf{E}}}{c}$$

- Because **E** is transverse, $\hat{\mathbf{n}}\cdot\hat{\boldsymbol{k}}=0$. (So is **B**.)

- The actual (real) electric and magnetic fields in a monochromatic plane wave with propagation vector $\boldsymbol{k}$ and polarization $\hat{\mathbf{n}}$ are



$$\mathbf{E}(\mathbf{r},t)=E_0\cos(\boldsymbol{k}\cdot\mathbf{r}-\omega t+\delta)\,\hat{\mathbf{n}},\quad \mathbf{B}(\mathbf{r},t)=\frac{E_0}{c}\cos(\boldsymbol{k}\cdot\mathbf{r}-\omega t+\delta)\,\hat{\boldsymbol{k}}\times\hat{\mathbf{n}}$$

- In some cases the direction of $\mathbf{E}$ of a plane wave at a given point may change with time.

- Consider the superposition of 2 linearly polarized waves: one polarized in the x-direction, and the other in the y-direction and lagging/leading $\pi/2$ in time phase:

$$\mathbf{E}(z,t)=\mathbf{E}_1(z,t)+\mathbf{E}_2(z,t)=E_{10}\cos(kz-\omega t)\,\hat{\mathbf{x}}+E_{20}\cos(kz-\omega t\pm\pi/2)\,\hat{\mathbf{y}}$$
$$=E_{10}\cos(\omega t-kz)\,\hat{\mathbf{x}}\pm E_{20}\sin(\omega t-kz)\,\hat{\mathbf{y}}$$

$$\Rightarrow\quad \frac{E_1^2}{E_{10}^2}+\frac{E_2^2}{E_{20}^2}=\cos^2(\omega t-kz)+\sin^2(\omega t-kz)=1\quad\Rightarrow$$

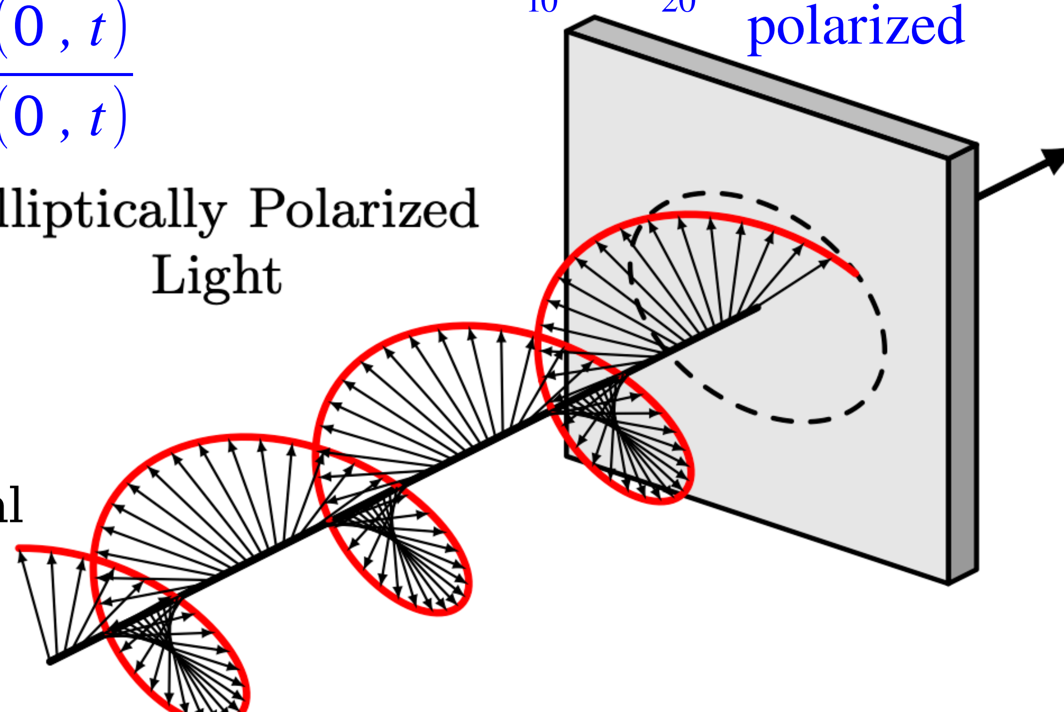
$E_{10}\neq E_{20}$ elliptically polarized

$E_{10}=E_{20}$ circularly polarized

- When $E_{10}=E_{20}\quad\Rightarrow\quad \omega t=\tan^{-1}\dfrac{E_2(0,t)}{E_1(0,t)}$

- If $\omega \gtrless 0$, ie, counterclockwise/clockwise, it is called **positive/negative (right-/left-hand) circularly polarized wave**.

- In general, $\mathbf{E}_1$ & $\mathbf{E}_2$ can have unequal amplitudes and differ with arbitrary phase. Their sum $\mathbf{E}$ will be elliptically polarized with tilt principal axes.



Elliptically Polarized Light

- A linearly polarized plane wave can be resolved into a right-hand (circularly) polarized wave and a left-hand (circularly) polarized wave of equal amplitude.

$$\mathbf{E}(z,t) = E_0 \cos(kz - \omega t)\,\hat{\mathbf{x}}$$

$$= \frac{E_0}{2}\left[\cos(\omega t - kz)\,\hat{\mathbf{x}} + \sin(\omega t - kz)\,\hat{\mathbf{y}}\right]$$

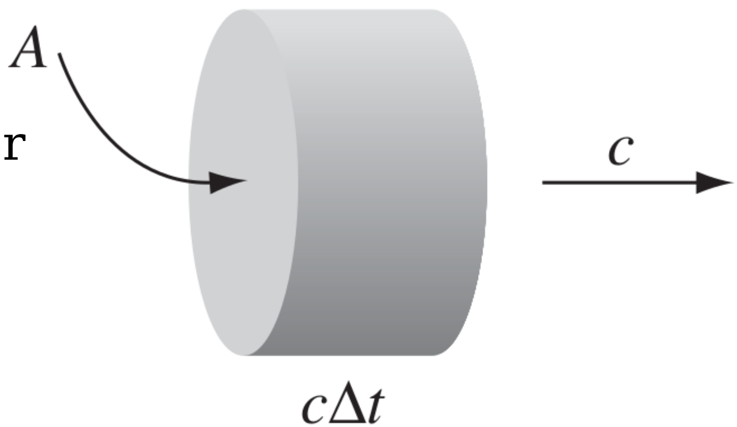
$$+ \frac{E_0}{2}\left[\cos(\omega t - kz)\,\hat{\mathbf{x}} - \sin(\omega t - kz)\,\hat{\mathbf{y}}\right]$$

Energy and Momentum in Electromagnetic Waves

- The energy per unit volume in electromagnetic fields is $u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right)$
- In the case of a monochromatic plane wave $B^2 = \frac{E^2}{c^2} = \mu_0 \epsilon_0 E^2$, so the *electric and magnetic contributions are equal*: $u = \epsilon_0 E^2 = \epsilon_0 E_0^2 \cos^2(k z - \omega t + \delta)$
- As the wave travels, it carries this energy along with it. The energy flux density (energy/area/time) transported by the fields is the Poynting vector: $\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}$
- For monochromatic plane waves propagating in the z direction,

$$\mathbf{S} = c\, \epsilon_0 E_0^2 \cos^2(k z - \omega t + \delta)\, \hat{\mathbf{z}} = c\, u\, \hat{\mathbf{z}}$$

- So $\mathbf{S}$ is the energy density (u) times the velocity of the waves ($c\, \hat{\mathbf{z}}$)—as it *should*.
- For in a time Δt, a length $c\, \Delta t$ passes through area A, carrying with it an energy $u\, A\, c\, \Delta t$. The energy per unit time, per unit area, transported by the wave is therefore $u\, c$.



- EM fields not only carry energy, they also carry momentum. The momentum density stored in the fields is $\mathbf{g} = \frac{\mathbf{S}}{c^2}$
- For monochromatic plane waves $\mathbf{g} = \frac{\epsilon_0}{c} E_0^2 \cos^2(kz - \omega t + \delta)\,\hat{\mathbf{z}} = \frac{u}{c}\,\hat{\mathbf{z}}$
- In the case of *light*, the wavelength is short ($\sim 5\times10^{-7}$m), and the period brief ($\sim 10^{-15}$s), that any macroscopic measurement will encompass many cycles.
- Therefore we're not interested in the fluctuating cosine-squared term in the energy and momentum densities; all we want is the *average* value.
- The average of cosine-squared over a complete cycle is 1/2, so

$$\langle u \rangle = \frac{\epsilon_0}{2} E_0^2, \quad \langle \mathbf{S} \rangle = \frac{c\,\epsilon_0}{2} E_0^2\,\hat{\mathbf{z}}, \quad \langle \mathbf{g} \rangle = \frac{\epsilon_0}{2c} E_0^2\,\hat{\mathbf{z}}$$

The brackets <> denote the (time) average over at least one complete cycle.

- The average power/area transported by an EM wave is called the **intensity**:

$$I \equiv \langle S \rangle = \frac{c\,\epsilon_0}{2} E_0^2$$

- When light falls (at normal incidence) on a perfect absorber, it delivers its momentum to the surface.

- In Δt, the momentum transfer is $\Delta \mathbf{p} = \langle \mathbf{g} \rangle A\, c\, \Delta t$, so the **radiation pressure** (average force/area) is $P = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{\epsilon_0}{2} E_0^2 = \frac{I}{c}$
- On a perfect *reflector* the pressure is *twice* as great, because the momentum switches direction, instead of simply being absorbed.

- Explanation: The electric field drives charges in the x direction, and the magnetic field then exerts on them a force $q\,\mathbf{v}\times\mathbf{B}$ in the z direction. The net force on all the charges in the surface produces the pressure.

Example: Fields of a Laser Beam: Consider a He-Ne laser beam of 100 W/mm^2.

For its average energy density in such a beam,

$$\langle u \rangle = \frac{\langle S \rangle}{c} = \frac{100\ \mathrm{J/s}/\left(10^{-6}\ \mathrm{m}^2\right)}{3\times 10^{8}\ \mathrm{m/s}} = 0.33\ \mathrm{J/m^3} \quad \Rightarrow \quad E_{\mathrm{rms}}^2 = \langle E^2 \rangle = \frac{E_0^2}{2} = \frac{\langle u \rangle}{\epsilon_0}$$

$$\Rightarrow \quad E_{\mathrm{rms}} = \sqrt{\frac{\langle u \rangle}{\epsilon_0}} = \sqrt{\frac{0.33\ \mathrm{J/m^3}}{8.8\times 10^{-12}\ \mathrm{C/m\,V}}} = 1.94\times 10^{5}\ \mathrm{V/m}$$

$$\Rightarrow \quad B_{\mathrm{rms}} = \frac{E_{\mathrm{rms}}}{c} = 6.5\times 10^{-4}\ \mathrm{T}\ (= 6.5\ \mathrm{G})$$

Electromagnetic Waves in Matter

Propagation in Linear Media

- In Inside matter, but no *free* charge or *free* current, Maxwell's eqns become

$$\nabla\cdot\mathbf{D}=0\,,\quad \nabla\times\mathbf{E}=-\frac{\partial\,\mathbf{B}}{\partial\,t}\,,\quad \nabla\cdot\mathbf{B}=0\,,\quad \nabla\times\mathbf{H}=\frac{\partial\,\mathbf{D}}{\partial\,t}$$

- If the medium is *linear*, $\mathbf{D}=\epsilon\,\mathbf{E}\,,\quad \mathbf{H}=\frac{1}{\mu}\mathbf{B}$, and *homogeneous* (ϵ and μ do not vary from point to point), the equations reduce to

$$\nabla\cdot\mathbf{E}=0\,,\quad \nabla\times\mathbf{E}=-\frac{\partial\,\mathbf{B}}{\partial\,t}\,,\quad \nabla\cdot\mathbf{B}=0\,,\quad \nabla\times\mathbf{B}=\mu\,\epsilon\,\frac{\partial\,\mathbf{E}}{\partial\,t}$$

which differ from the vacuum analogs only in the replacement of $\mu_0\,\epsilon_0$ by $\mu\,\epsilon$.

- EM waves propagate through a linear homogeneous medium at a speed

$$v=\frac{1}{\sqrt{\mu\,\epsilon}}=\frac{c}{n}\quad\Leftarrow\quad n\equiv\sqrt{\frac{\epsilon\,\mu}{\epsilon_0\,\mu_0}}\quad \text{index of refraction}$$

- For most materials, μ is very close to μ_0, so $n\simeq\sqrt{\epsilon_r}\quad\Leftarrow\quad \epsilon_r:$ dielectric constant

- Since $\epsilon_r\geq 1$, light travels *more slowly* through matter.

- All of our previous results carry over, with the simple transcription $\epsilon_0\rightarrow\epsilon$, $\mu_0\rightarrow\mu$, and hence $c\rightarrow v$.

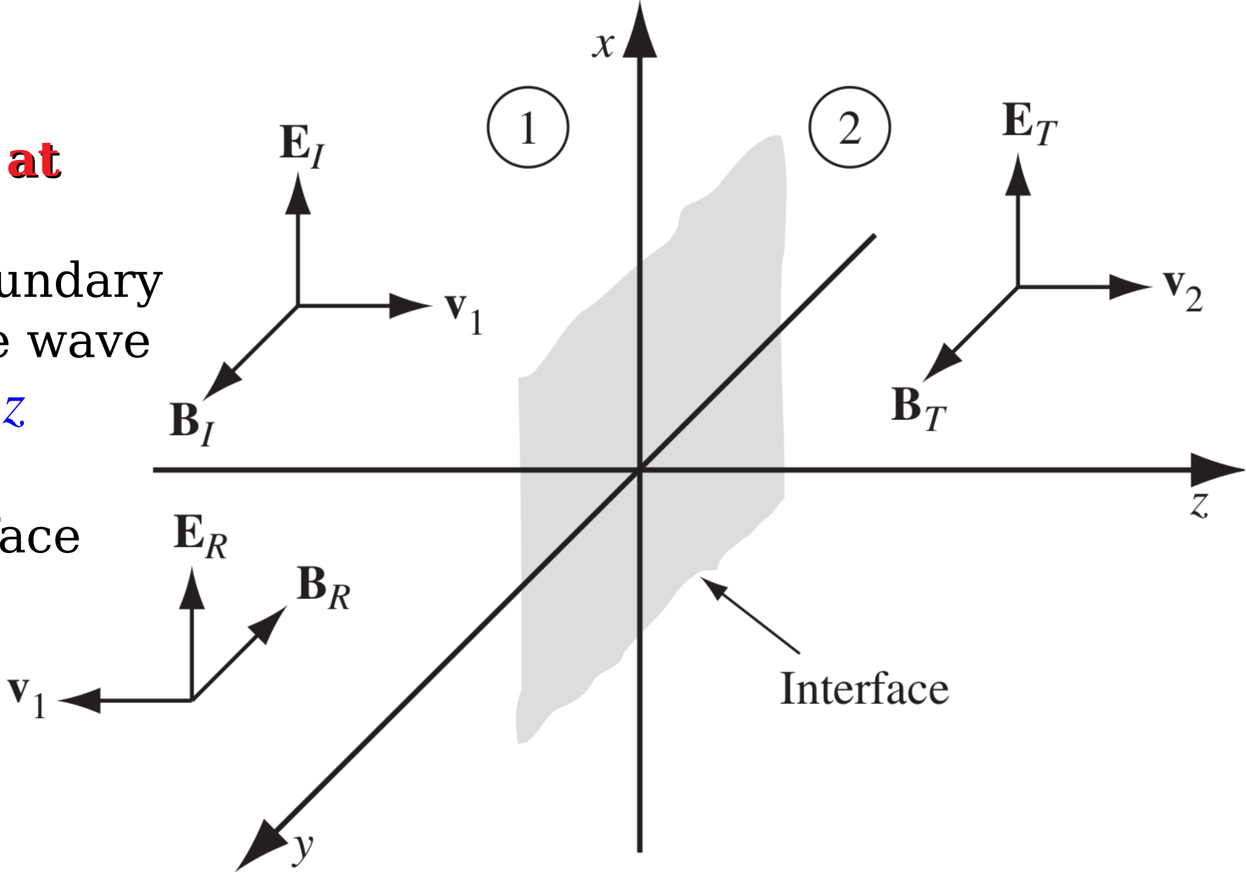
- The energy density is $u = \frac{\epsilon E^2}{2} + \frac{B^2}{2\mu}$, and the Poynting vector is $\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu}$
- For monochromatic plane waves, the frequency and wave number are related by $\omega = k v$, also $B = \frac{E}{v}$, and the intensity is $I = \frac{\epsilon v}{2} E_0^2$
- What happens when a wave passes from one transparent medium into another —air to water, say, or glass to plastic? It's related to the boundary conditions.
- As in the case of waves on a string, there is a reflected wave and a transmitted wave. The details depend on the electrodynamic boundary conditions,

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}, \quad \mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel}, \quad B_1^{\perp} = B_2^{\perp}, \quad \frac{\mathbf{B}_1^{\parallel}}{\mu_1} = \frac{\mathbf{B}_2^{\parallel}}{\mu_2}$$

- These equations relate the electric and magnetic fields just to the left and just to the right of the interface between 2 linear media.

Reflection and Transmission at Normal Incidence

- Let the xy plane forms the boundary between 2 linear media. A plane wave of frequency ω, traveling in the z direction and polarized in the x direction, approaches the interface from the left:



$$\tilde{\mathbf{E}}_I(z,t)=\tilde{E}_{0I}\,e^{i(k_1 z-\omega t)}\,\hat{\mathbf{x}}$$

$$\tilde{\mathbf{B}}_I(z,t)=\frac{\tilde{E}_{0I}}{v_1}\,e^{i(k_1 z-\omega t)}\,\hat{\mathbf{y}}$$

- The reflected wave

$$\tilde{\mathbf{E}}_R(z,t)=\tilde{E}_{0R}\,e^{i(-k_1 z-\omega t)}\,\hat{\mathbf{x}}$$

$$\tilde{\mathbf{B}}_R(z,t)=-\frac{\tilde{E}_{0R}}{v_1}\,e^{i(-k_1 z-\omega t)}\,\hat{\mathbf{y}}$$

back to the left in medium 1.

- The transmitted wave

$$\tilde{\mathbf{E}}_T(z,t)=\tilde{E}_{0T}\,e^{i(k_2 z-\omega t)}\,\hat{\mathbf{x}}$$

$$\tilde{\mathbf{B}}_T(z,t)=\frac{\tilde{E}_{0T}}{v_2}\,e^{i(k_2 z-\omega t)}\,\hat{\mathbf{y}}$$

on to the right in medium 2.

- The minus sign in $\tilde{\mathbf{B}}_R$, as required by $v\tilde{\mathbf{B}}=\hat{\boldsymbol{k}}\times\tilde{\mathbf{E}}$ — or by the fact that the Poynting vector aims in the direction of propagation.

- At $z=0$, the combined fields on the left, $\tilde{\mathbf{E}}_I+\tilde{\mathbf{E}}_R$ and $\tilde{\mathbf{B}}_I+\tilde{\mathbf{B}}_R$, must join the field on the right, $\tilde{\mathbf{E}}_T$ and $\tilde{\mathbf{B}}_T$, in accordance with the boundary conditions.

- In this case there are no components ⊥ the surface, so

$$\left[\begin{array}{l} \tilde{E}_{0I}+\tilde{E}_{0R}=\tilde{E}_{0T} \\ \dfrac{\tilde{E}_{0I}-\tilde{E}_{0R}}{\mu_1 v_1}=\dfrac{\tilde{E}_{0T}}{\mu_2 v_2} \quad \Rightarrow \quad \tilde{E}_{0I}-\tilde{E}_{0R}=\beta\,\tilde{E}_{0T} \quad \Leftarrow \quad \beta=\dfrac{\mu_1 v_1}{\mu_2 v_2}=\dfrac{\mu_1 n_2}{\mu_2 n_1} \end{array}\right.$$

$$\Rightarrow \quad \tilde{E}_{0R}=\frac{1-\beta}{1+\beta}\,\tilde{E}_{0I}, \quad \tilde{E}_{0T}=\frac{2}{1+\beta}\,\tilde{E}_{0I}$$

- These results are similar to the ones for waves on a string.

$$\text{If } \mu_1\simeq\mu_2 \quad \Rightarrow \quad \beta\simeq\frac{v_1}{v_2} \Rightarrow \quad \tilde{E}_{0R}\simeq\frac{v_2-v_1}{v_1+v_2}\,\tilde{E}_{0I}, \quad \tilde{E}_{0T}\simeq\frac{2\,v_2}{v_1+v_2}\,\tilde{E}_{0I} \quad \text{as in string}$$

- In that case, the reflected wave is *in phase* if $v_2>v_1$ and *out of phase* if $v_2<v_1$.

- The real amplitudes are related by

$$E_{0R}=\left|\frac{v_1-v_2}{v_1+v_2}\right|E_{0I}=\left|\frac{n_2-n_1}{n_1+n_2}\right|E_{0I}, \quad E_{0T}=\frac{2\,v_2}{v_1+v_2}\,E_{0I}=\frac{2\,n_1}{n_1+n_2}\,E_{0I}$$

- The ratio of the reflected intensity to the incident intensity is (for $\mu_1 = \mu_2 = \mu_0$)

$$\text{Reflection coefficient} \quad R = \frac{I_R}{I_I} = \frac{E_{0R}^2}{E_{0I}^2} = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2$$

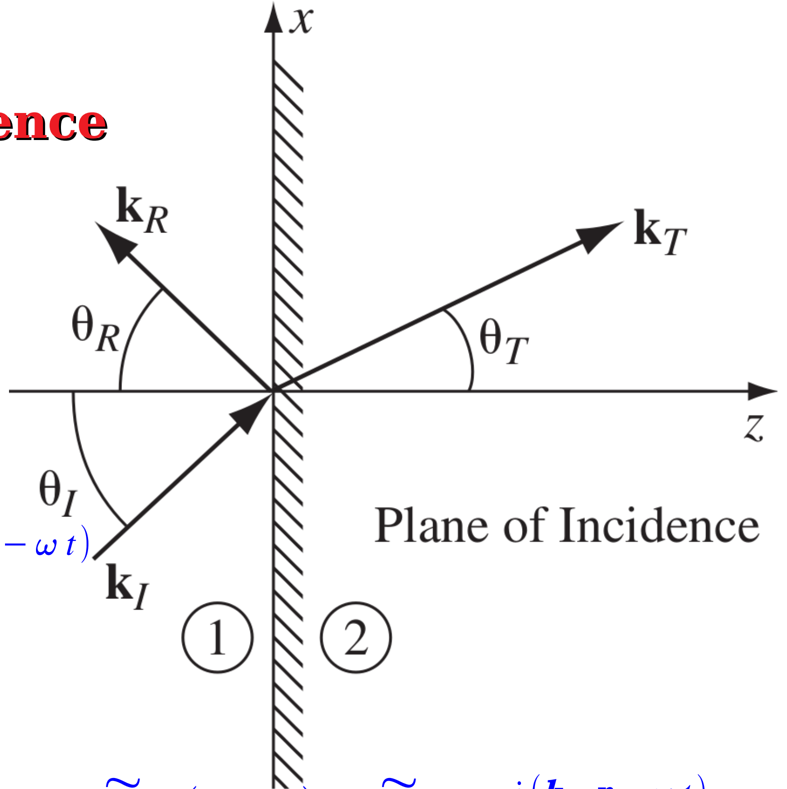
- The ratio of the transmitted intensity to the incident intensity is

$$\text{Transmission coefficient} \quad T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} \frac{E_{0T}^2}{E_{0I}^2} = \frac{4 n_1 n_2}{(n_1 + n_2)^2}$$

- R and T measure the fraction of the incident energy that is reflected and transmitted. $R+T=1$, as conservation of energy requires.

- When light passes from air ($n_1=1$) into glass ($n_2=1.5$), $R=0.04$ and $T=0.96$. So most of the light is transmitted.

Reflection and Transmission at Oblique Incidence



- The more general case of *oblique* incidence, in which the incoming wave meets the boundary at an arbitrary angle θ_I.

- Normal incidence is a special case of oblique incidence, with $\theta_I=0$.

- Monochromatic plane wave

$$\tilde{\mathbf{E}}_I(\mathbf{r},t)=\tilde{\mathbf{E}}_{0I}\,e^{i(\boldsymbol{k}_I\cdot\mathbf{r}-\omega t)}$$

$$\tilde{\mathbf{B}}_I(\mathbf{r},t)=\frac{\hat{\boldsymbol{k}}_I\times\tilde{\mathbf{E}}_I}{v_1}$$

- Reflected wave

$$\tilde{\mathbf{E}}_R(\mathbf{r},t)=\tilde{\mathbf{E}}_{0R}\,e^{i(\boldsymbol{k}_R\cdot\mathbf{r}-\omega t)}$$

$$\tilde{\mathbf{B}}_R(\mathbf{r},t)=\frac{\hat{\boldsymbol{k}}_R\times\tilde{\mathbf{E}}_R}{v_1}$$

, transmitted wave

$$\tilde{\mathbf{E}}_T(\mathbf{r},t)=\tilde{\mathbf{E}}_{0T}\,e^{i(\boldsymbol{k}_T\cdot\mathbf{r}-\omega t)}$$

$$\tilde{\mathbf{B}}_T(\mathbf{r},t)=\frac{\hat{\boldsymbol{k}}_T\times\tilde{\mathbf{E}}_T}{v_2}$$

- All 3 waves have the same *frequency* ω—that is determined once and for all at the source: $v=\dfrac{\omega}{k} \;\Rightarrow\; k_I v_1=k_R v_1=k_T v_2=\omega \;\Rightarrow\; k_I=k_R=\dfrac{v_2}{v_1}k_T=\dfrac{n_1}{n_2}k_T$

- The combined fields in medium 1, $\tilde{\mathbf{E}}_I+\tilde{\mathbf{E}}_R$ and $\tilde{\mathbf{B}}_I+\tilde{\mathbf{B}}_R$, must be joined to the fields $\tilde{\mathbf{E}}_T$ and $\tilde{\mathbf{B}}_T$ in medium 2, using the boundary conditions

$$[\quad]\,e^{i(\boldsymbol{k}_I\cdot\mathbf{r}-\omega t)}+[\quad]\,e^{i(\boldsymbol{k}_R\cdot\mathbf{r}-\omega t)}=[\quad]\,e^{i(\boldsymbol{k}_T\cdot\mathbf{r}-\omega t)}\quad \text{at } z=0$$

- *Because the boundary conditions must hold at all points on the plane, and for all times, these exponential factors must be equal* (when $z=0$)

$$\Rightarrow \quad \boldsymbol{k}_I \cdot \mathbf{r} = \boldsymbol{k}_R \cdot \mathbf{r} = \boldsymbol{k}_T \cdot \mathbf{r} \quad \text{at } z=0 \quad \Rightarrow \quad k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$$

$$\Rightarrow \quad x\,k_{Ix} + y\,k_{Iy} = x\,k_{Rx} + y\,k_{Ry} = x\,k_{Tx} + y\,k_{Ty} \quad \Rightarrow \quad \begin{array}{ll} k_{Iy} = k_{Ry} = k_{Ty}, & x=0 \\ k_{Ix} = k_{Rx} = k_{Tx}, & y=0 \end{array}$$

θ_I is the **angle of incidence**, θ_R is the **angle of reflection**, θ_T is the **angle of refraction** (or the angle of transmission), measured with respect to the normal.

- Orient our axes so that $\boldsymbol{k}_I$ lies in the xz plane (ie, $k_{Iy}=0$); so too will $\boldsymbol{k}_R$ and $\boldsymbol{k}_T$.

1st Law: The incident, reflected, and transmitted wave vectors form a plane (called the **plane of incidence**), which also includes the normal to the surface (here, the z axis).

2nd Law: The angle of incidence = the angle of reflection, $\theta_I = \theta_R$ law of reflection

3rd Law: $\dfrac{\sin\theta_T}{\sin\theta_I} = \dfrac{n_1}{n_2}$ law of refraction—Snell's law

- These are the 3 fundamental laws of geometrical optics. Little *electrodynamics* went into them: no specific boundary conditions involved. Therefore, any *other* waves can be expected to obey the same "optical" laws when they pass from one medium into another.

Total Reflection

- For Snell's law with $n_1 > n_2$, ie, when the wave in medium 1 is incident on a less dense medium 2, then In that case, $\theta_T > \theta_I$.

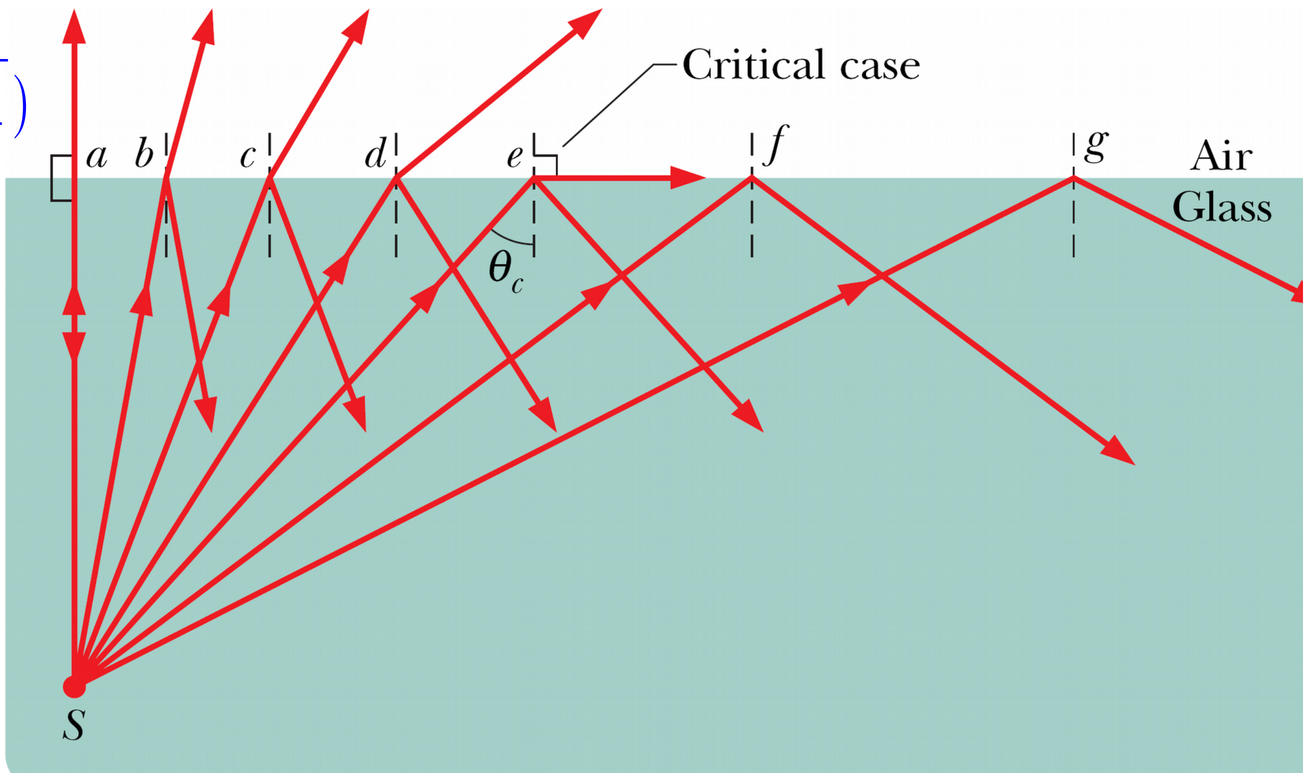
- When $\theta_T = \pi/2$, at which angle the refracted wave glazes along the interface; a further increase in θ_I would result in no refracted wave, and the incident wave is then said to be *totally reflected*. The angle of incidence in this situation is called the **critical angle** θ_c: $\sin\theta_c = \frac{n_2}{n_1}\sin\frac{\pi}{2} \Rightarrow \theta_c = \sin^{-1}\frac{n_2}{n_1}$

- If $\theta_I > \theta_c \Rightarrow \sin\theta_T = \frac{n_1}{n_2}\sin\theta_I \equiv \gamma > 1 \Rightarrow \cos\theta_T = \pm\sqrt{1-\sin^2\theta_T} \Rightarrow -i\sqrt{\gamma^2-1}$

$$\Rightarrow \theta_T = \frac{\pi}{2} + i\ln\left(\gamma + \sqrt{\gamma^2-1}\right)$$

- $n_2 < n_1$ tells us that total internal reflection cannot occur when the incident light is in the medium of lower index of refraction.

- This effect is heavily applied to optical fibers.



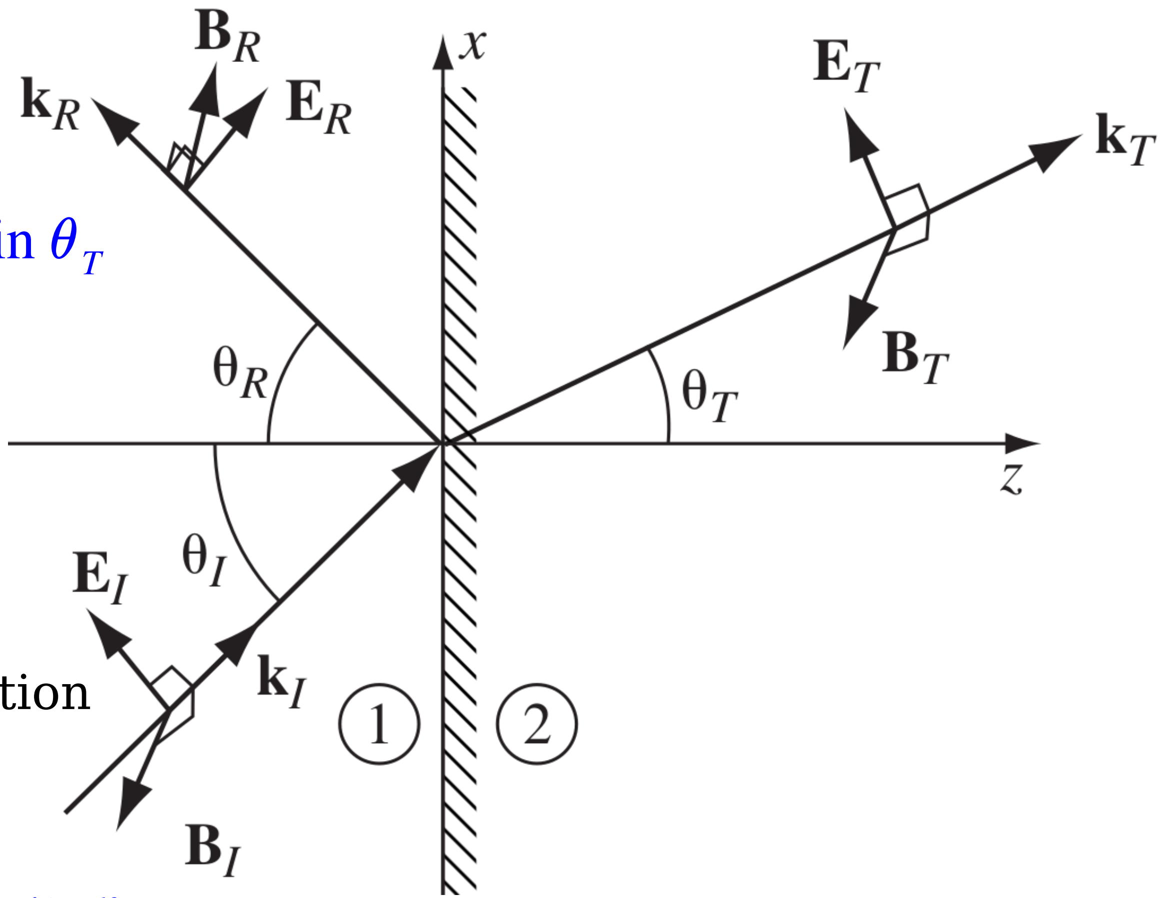
- After taking care of the exponential factors, the boundary conditions becomes

$$\epsilon_1 (\tilde{\mathbf{E}}_{0I} + \tilde{\mathbf{E}}_{0R})_\perp = \epsilon_2 (\tilde{\mathbf{E}}_{0T})_\perp , \quad (\tilde{\mathbf{B}}_{0I} + \tilde{\mathbf{B}}_{0R})_\perp = (\tilde{\mathbf{B}}_{0T})_\perp , \quad \perp : \text{ normal to the plane}$$

$$(\tilde{\mathbf{E}}_{0I} + \tilde{\mathbf{E}}_{0R})_\parallel = (\tilde{\mathbf{E}}_{0T})_\parallel , \quad \frac{(\tilde{\mathbf{B}}_{0I} + \tilde{\mathbf{B}}_{0R})_\parallel}{\mu_1} = \frac{(\tilde{\mathbf{B}}_{0T})_\parallel}{\mu_2} , \quad \parallel : \text{ parallel to the plane}$$

$$v_p \tilde{\mathbf{B}}_0 = \hat{\boldsymbol{k}} \times \tilde{\mathbf{E}}_0$$

- Suppose the polarization of the incident wave ∥ the plane of incidence (the xz plane); it follows that the reflected and transmitted waves are also polarized in this plane (Prob. 9.15).



- The boundary conditions give

$$-\tilde{E}_{0I} \sin\theta_I + \tilde{E}_{0R} \sin\theta_R = -\frac{\epsilon_2}{\epsilon_1} \tilde{E}_{0T} \sin\theta_T$$

$$0 = 0$$

$$\tilde{E}_{0I} \cos\theta_I + \tilde{E}_{0R} \cos\theta_R = \tilde{E}_{0T} \cos\theta_T$$

$$\frac{\tilde{E}_{0I} - \tilde{E}_{0R}}{\mu_1 v_1} = \frac{\tilde{E}_{0T}}{\mu_2 v_2}$$

- By the laws of reflection & refraction

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \alpha \tilde{E}_{0T} \quad \Leftarrow \quad \alpha \equiv \frac{\cos\theta_T}{\cos\theta_I}$$

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \beta \tilde{E}_{0T} \quad \Leftarrow \quad \beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$$

$$\Rightarrow \quad \tilde{E}_{0R} = \frac{\alpha - \beta}{\alpha + \beta}\,\tilde{E}_{0I}, \quad \tilde{E}_{0T} = \frac{2}{\alpha + \beta}\,\tilde{E}_{0I}$$

Fresnel's equations
polarization in the plane of incidence

- The transmitted wave is always *in phase* with the incident one; the reflected wave is either in phase ("right side up"), if $\alpha > \beta$, or 180° out of phase ("upside down"), if $\alpha < \beta$.

- In the case of normal incidence ($\theta_I = 0$), $\alpha = 1$,

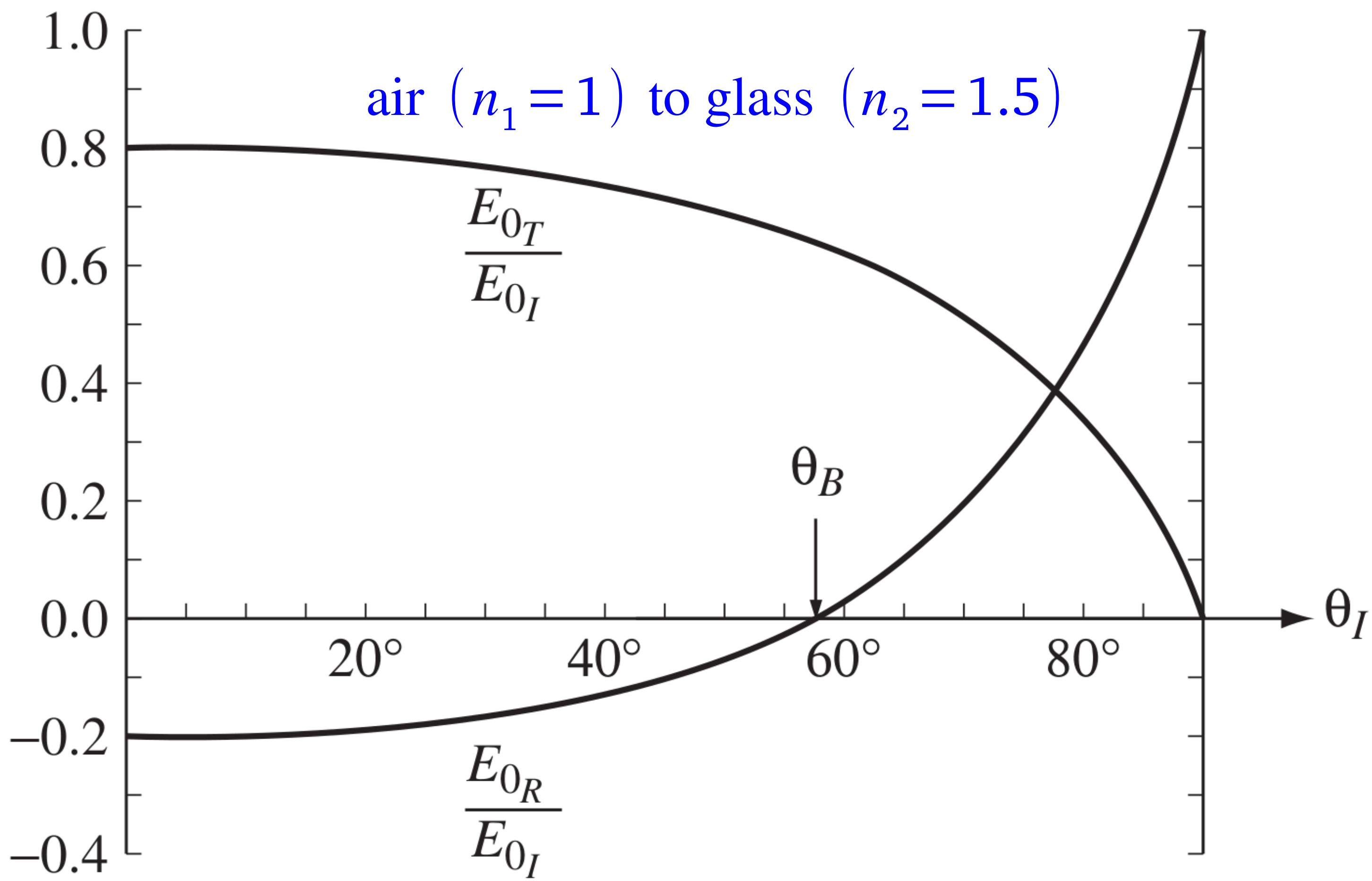
$$\Rightarrow \quad \tilde{E}_{0R} = \frac{1 - \beta}{1 + \beta}\,\tilde{E}_{0I}, \quad \tilde{E}_{0T} = \frac{2}{1 + \beta}\,\tilde{E}_{0I}, \text{ as before}$$

- The amplitudes of the transmitted and reflected waves depend on the angle of incidence, because α is a function of θ_I: $\alpha = \frac{\sqrt{1 - \sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1 - \left(\frac{n_1}{n_2}\sin\theta_I\right)^2}}{\cos\theta_I}$

- At grazing incidence ($\theta_I = 90^\circ$), α diverges, and the wave is totally reflected.

- There is an intermediate angle, θ_B (called **Brewster's angle**), at which the reflected wave is completely extinguished. This occurs when

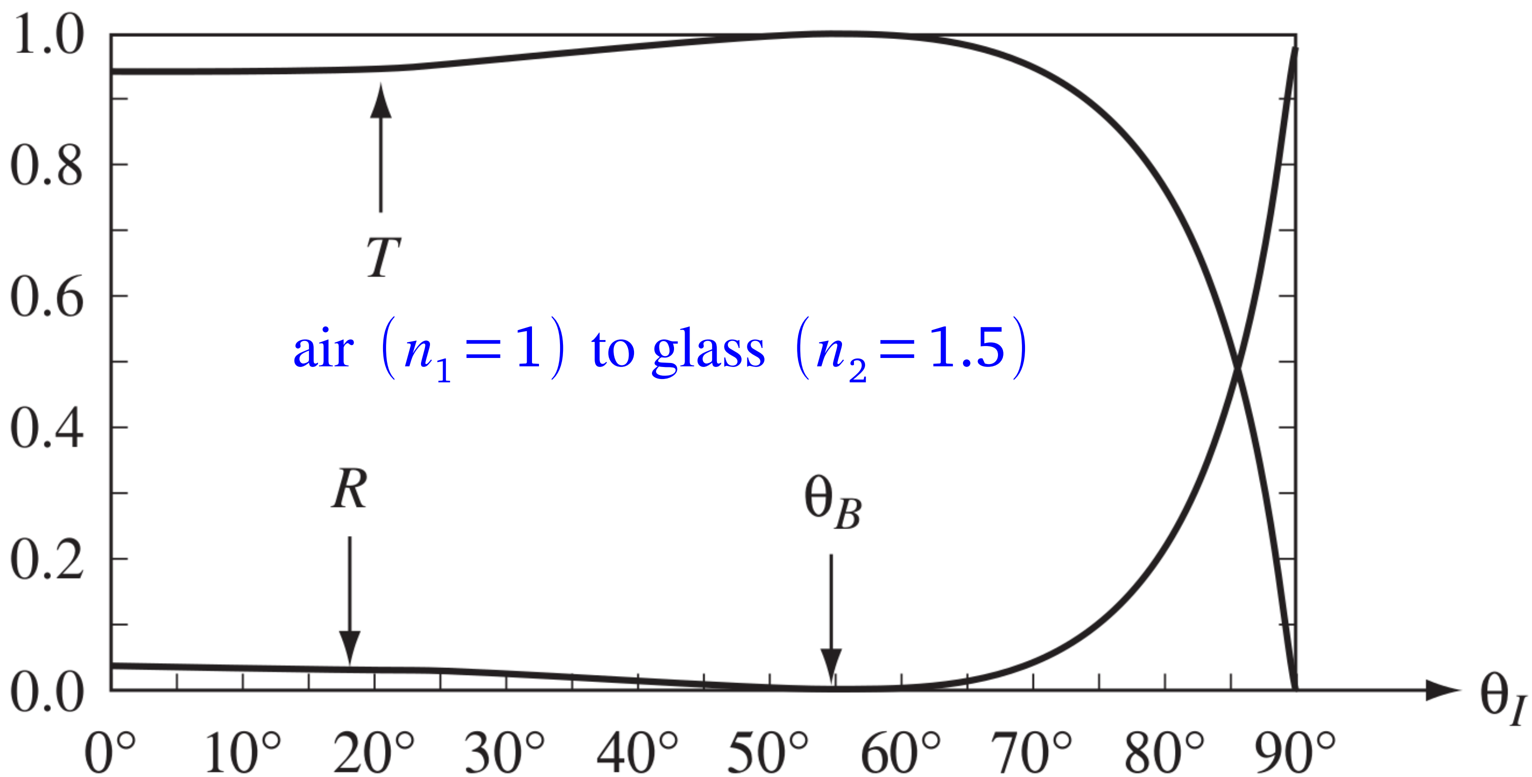
$$\alpha = \beta \quad \Rightarrow \quad \alpha^2 = \beta^2 \quad \Rightarrow \quad \sin^2\theta_B = \frac{1 - \beta^2}{\left(\frac{n_1}{n_2}\right)^2 - \beta^2}$$



- For the typical case $\mu_1 \simeq \mu_2 \;\Rightarrow\; \beta \simeq \frac{n_2}{n_1} \;\Rightarrow\; \sin^2\theta_B = \frac{\beta^2}{1+\beta^2} \;\Rightarrow\; \tan\theta_B \simeq \frac{n_2}{n_1}$
- The power per unit area striking the interface is $\mathbf{S}\cdot\hat{\mathbf{z}}$. Thus the incident

intensity $I_I = \frac{\epsilon_1 v_1}{2} E_{0I}^2 \cos\theta_I$, the reflected intensities $I_R = \frac{\epsilon_1 v_1}{2} E_{0R}^2 \cos\theta_R$

the transmitted intensities $I_T = \frac{\epsilon_2 v_2}{2} E_{0T}^2 \cos\theta_T$



- The reflection and transmission coefficients for waves polarized ∥ the plane of incidence are

$$R \equiv \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2, \quad T \equiv \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0R}}{E_{0I}}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \frac{4\alpha\beta}{(\alpha+\beta)^2}$$

- R is the fraction of the incident energy that is reflected—it goes to 0 at Brewster's angle θ_B; T is the fraction transmitted—it goes to 1 at θ_B.

- $R+T=1$, as required by conservation of energy: the energy per unit time reaching a particular patch of area on the surface is equal to the energy per unit time leaving the patch.

- Now consider the case of polarization ⊥ the plane of incidence.

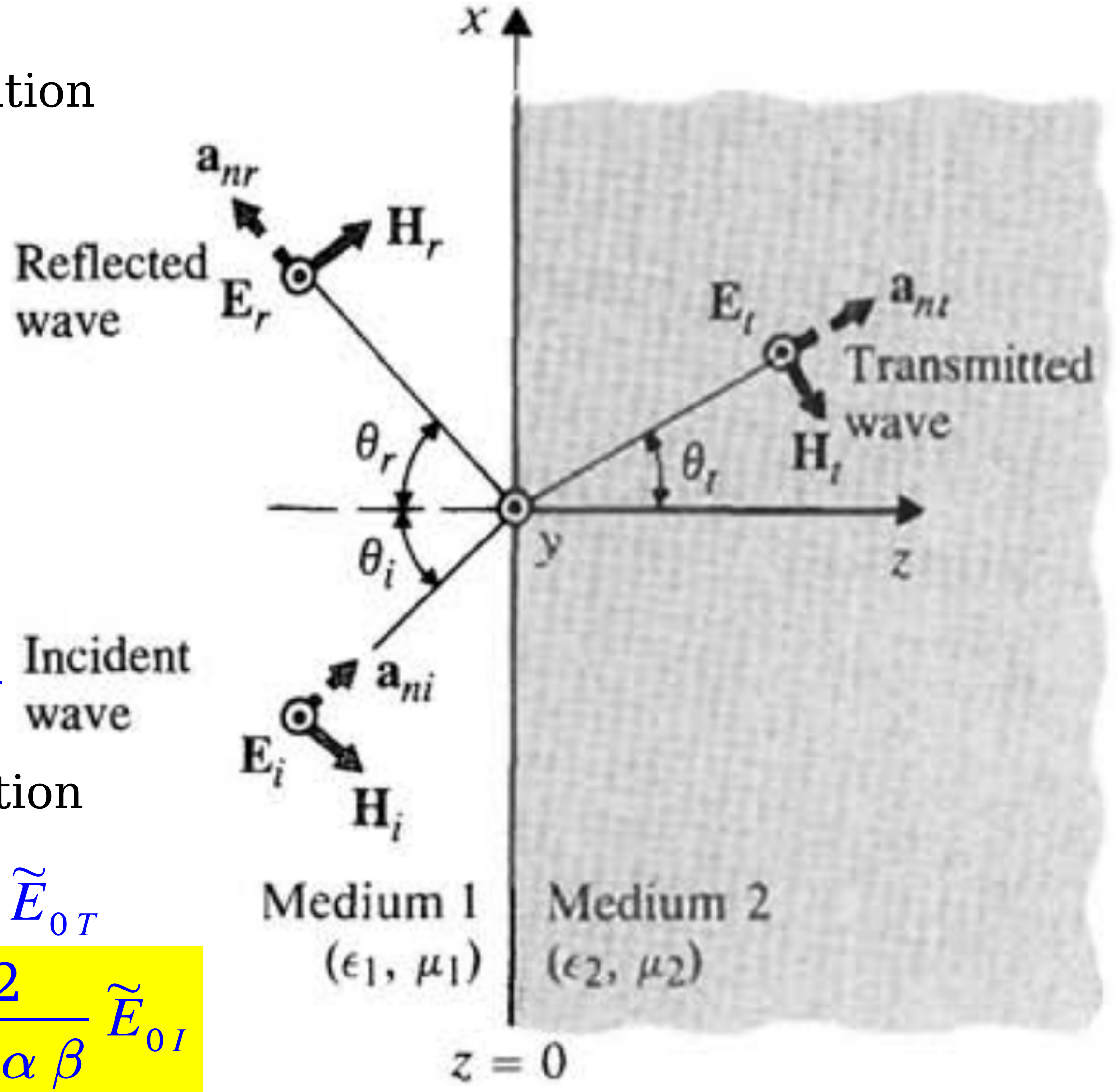
- The boundary conditions give

$$0=0$$

$$\frac{\tilde{E}_{0I}}{v_1}\sin\theta_I+\frac{\tilde{E}_{0R}}{v_1}\sin\theta_R=\frac{\tilde{E}_{0T}}{v_2}\sin\theta_T$$

$$\tilde{E}_{0I}+\tilde{E}_{0R}=\tilde{E}_{0T}$$

$$\frac{\tilde{E}_{0I}\cos\theta_I}{\mu_1 v_1}-\frac{\tilde{E}_{0R}\cos\theta_R}{\mu_1 v_1}=\frac{\tilde{E}_{0T}\cos\theta_T}{\mu_2 v_2}$$



- By the laws of reflection & refraction

$$\tilde{E}_{0I}+\tilde{E}_{0R}=\tilde{E}_{0T}, \quad \tilde{E}_{0I}-\tilde{E}_{0R}=\alpha\beta\,\tilde{E}_{0T}$$

$$\Rightarrow \quad \tilde{E}_{0R}=\frac{1-\alpha\beta}{1+\alpha\beta}\tilde{E}_{0I}, \quad \tilde{E}_{0T}=\frac{2}{1+\alpha\beta}\tilde{E}_{0I}$$

Fresnel's equations, polarization ⊥ the plane of incidence

- The transmitted wave is always *in phase* with the incident one; the reflected wave is either in phase if $\alpha\beta<1$, or 180° out of phase if $\alpha\beta>1$.

- In the case of normal incidence ($\theta_I=0$), $\alpha=1$, $\Rightarrow \quad \tilde{E}_{0R}=\frac{1-\beta}{1+\beta}\tilde{E}_{0I}, \quad \tilde{E}_{0T}=\frac{2}{1+\beta}\tilde{E}_{0I}$, as before

- The amplitudes of the transmitted and reflected waves depend on the angle of incidence, because α is a function of θ_I:

$$\alpha\,\beta = \beta\,\frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \frac{\mu_1}{\mu_2}\,\frac{\sqrt{\dfrac{n_2^2}{n_1^2}-\sin^2\theta_I}}{\cos\theta_I}$$

- *Is there a Brewster's angle?* $\tilde{E}_{0R}=0$ means $\alpha\,\beta=1 \;\Rightarrow\; \dfrac{n_2^2}{n_1^2} = \sin^2\theta_B + \dfrac{\mu_2^2}{\mu_1^2}\cos^2\theta_B$

$$\Rightarrow\quad \cos^2\theta_B = \frac{(n_2/n_1)^2-1}{(\mu_2/\mu_1)^2-1} \le 1 \quad\Rightarrow\quad \frac{\mu_2}{\mu_1} \ge \frac{n_2}{n_1} \ge 1 \;\text{ or }\; \frac{\mu_2}{\mu_1} \le \frac{n_2}{n_1} \le 1$$

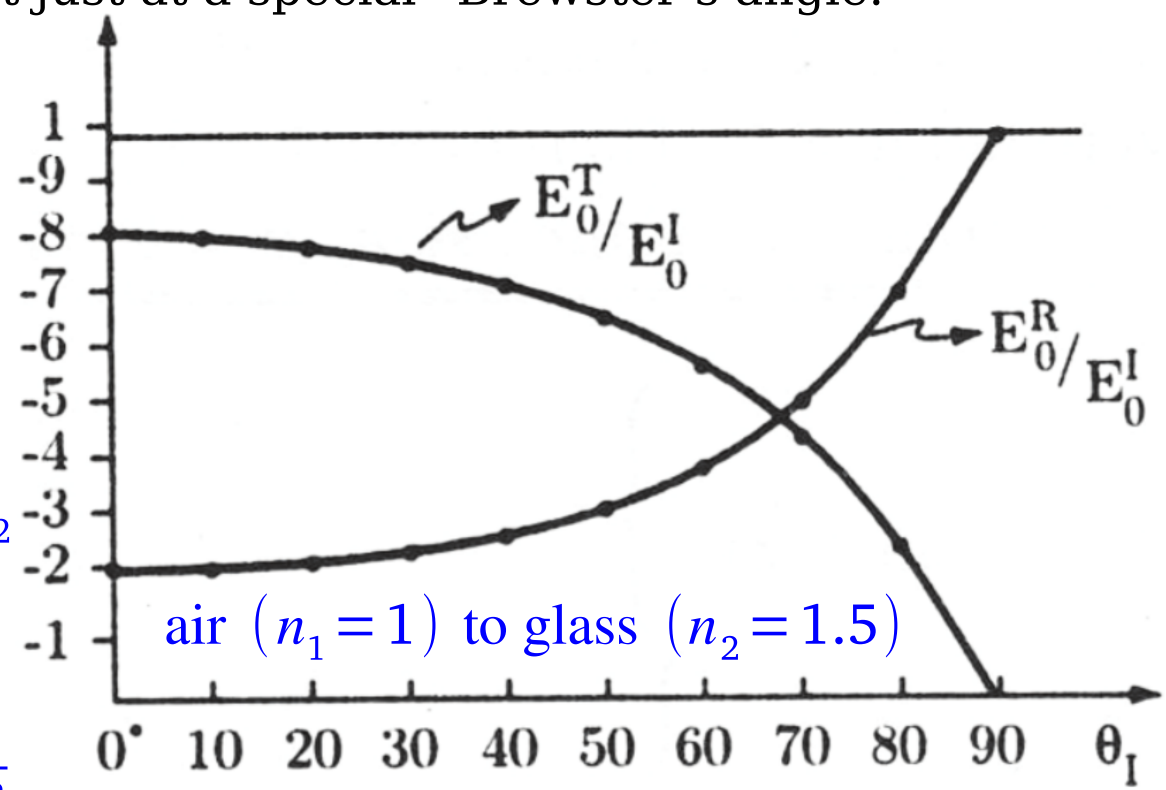
- For $\mu_2 \approx \mu_1 \Rightarrow n_2 \approx n_1$, only true for indistinguishable media, so no reflection. But that becomes true at any angle, not just at a special "Brewster's angle."

- If μ_2 were substantially different from μ_1, and the conditions above are satisfied, it would be possible to get a Brewster's angle, but the media would be very peculiar.

- The reflection and transmission coefficients

$$R \equiv \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{1-\alpha\,\beta}{1+\alpha\,\beta}\right)^2$$

$$T \equiv \frac{\epsilon_2\,v_2}{\epsilon_1\,v_1}\left(\frac{E_{0R}}{E_{0I}}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \frac{4\,\alpha\,\beta}{(1+\alpha\,\beta)^2}$$



$R+T=1$

Absorption and Dispersion

Electromagnetic Waves in Conductors

- The free charge density ρ_f and the free current density $\boldsymbol{J}_f$ being 0 is perfectly reasonable when you're talking about wave propagation through a vacuum or through insulating materials such as glass or (pure) water.

- But in the case of conductors we do not independently control the flow of charge, and in general $\boldsymbol{J}_f$ is certainly *not* 0.

- According to Ohm's law, the (free) current density in a conductor is proportional to the electric field: $\boldsymbol{J} = \boldsymbol{J}_{\text{induced}} + \boldsymbol{J}_f \quad \Leftarrow \quad \boldsymbol{J} = \sigma \mathbf{E}$

- Thus Maxwell's equations for linear media assume the form

$$\nabla \cdot \mathbf{E} = \frac{\rho_f}{\epsilon}, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu \sigma \mathbf{E} + \mu \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

- The continuity equation for free charge

$$\nabla \cdot \boldsymbol{J}_f = -\frac{\partial \rho_f}{\partial t} \quad \Rightarrow \quad \frac{\partial \rho_f}{\partial t} = -\sigma \nabla \cdot \mathbf{E} = -\frac{\sigma}{\epsilon} \rho_f \quad \Rightarrow \quad \rho_f(t) = \rho_f(0)\, e^{-\frac{\sigma}{\epsilon} t}$$

any initial free charge $\rho_f(0)$ dissipates in a characteristic time $\tau \equiv \dfrac{\epsilon}{\sigma}$.

- So if you put some free charge on a conductor, it will flow out to the edges. The time constant τ affords a measure of how "good" a conductor is.

- For a "perfect" conductor, $\sigma=\infty$ and $\tau=0$; for a "good" conductor, τ is much less than the other relevant times in the problem (in oscillatory systems, $\tau \ll \frac{1}{\omega}$); for a "poor" conductor, τ is greater than the characteristic times ($\tau \gg \frac{1}{\omega}$).

- Not interested in this transient behavior. From then on, we focus on $\rho_f=0$, $\boldsymbol{J}_f=0$,

$$\nabla\cdot\mathbf{E}=0\,,\quad \nabla\cdot\mathbf{B}=0\,,\quad \nabla\times\mathbf{E}=-\frac{\partial\,\mathbf{B}}{\partial\,t}\,,\quad \nabla\times\mathbf{B}=\mu\,\sigma\,\mathbf{E}+\mu\,\epsilon\,\frac{\partial\,\mathbf{E}}{\partial\,t}$$

- These differ from the corresponding equations for *non*conducting media only in the last term—which is absent when $\sigma=0$.

- Manage to obtain modified wave equations for $\mathbf{E}$ / $\mathbf{B}$:

$$\nabla^2\mathbf{E}=\mu\,\epsilon\,\frac{\partial^2\mathbf{E}}{\partial\,t^2}+\mu\,\sigma\,\frac{\partial\,\mathbf{E}}{\partial\,t}$$

$$\nabla^2\mathbf{B}=\mu\,\epsilon\,\frac{\partial^2\mathbf{B}}{\partial\,t^2}+\mu\,\sigma\,\frac{\partial\,\mathbf{B}}{\partial\,t}$$

- These equations still admit plane-wave solutions,

$$\tilde{\mathbf{E}}(z,t)=\tilde{\mathbf{E}}_0\,e^{i(\tilde{k}z-\omega t)}\,,\quad \tilde{\mathbf{B}}(z,t)=\tilde{\mathbf{B}}_0\,e^{i(\tilde{k}z-\omega t)}\quad\Rightarrow\quad \tilde{k}^2=\mu\,\epsilon\,\omega^2+i\,\mu\,\sigma\,\omega\,\in\,\mathbb{C}$$

$$\Rightarrow\quad \tilde{k}=k+i\,\xi\quad\Leftarrow\quad k\equiv\sqrt{\frac{\mu\,\omega}{2}\left(\sqrt{\epsilon^2\,\omega^2+\sigma^2}+\epsilon\,\omega\right)}\,,\quad \xi\equiv\sqrt{\frac{\mu\,\omega}{2}\left(\sqrt{\epsilon^2\,\omega^2+\sigma^2}-\epsilon\,\omega\right)}$$

- The imaginary part of $\tilde{k}$ results in an attenuation of the wave

$$\tilde{\mathbf{E}}(z,t)=\tilde{\mathbf{E}}_0\,e^{-\xi z}\,e^{i(kz-\omega t)}\,,\quad \tilde{\mathbf{B}}(z,t)=\tilde{\mathbf{B}}_0\,e^{-\xi z}\,e^{i(kz-\omega t)}$$

$$\nabla\cdot\mathbf{E}=0\,,\quad \nabla\cdot\mathbf{B}=0\,,\quad \nabla\times\mathbf{E}=-\frac{\partial\,\mathbf{B}}{\partial\,t}\,,\quad \nabla\times\mathbf{B}=\mu\,\sigma\,\mathbf{E}+\mu\,\epsilon\,\frac{\partial\,\mathbf{E}}{\partial\,t}$$

$$\nabla\times\left(\nabla\times\mathbf{E}=-\frac{\partial\,\mathbf{B}}{\partial\,t}\right)\;\Rightarrow\;\nabla\left(\nabla\cdot\mathbf{E}\right)-\nabla^2\,\mathbf{E}=-\nabla\times\frac{\partial\,\mathbf{B}}{\partial\,t}=-\frac{\partial}{\partial\,t}\left(\nabla\times\mathbf{B}\right)$$

$$\Rightarrow\;-\nabla^2\,\mathbf{E}=-\frac{\partial}{\partial\,t}\left(\mu\,\sigma\,\mathbf{E}+\mu\,\epsilon\,\frac{\partial\,\mathbf{E}}{\partial\,t}\right)\;\Rightarrow\;\nabla^2\,\mathbf{E}=\mu\,\epsilon\,\frac{\partial^2\,\mathbf{E}}{\partial\,t^2}+\mu\,\sigma\,\frac{\partial\,\mathbf{E}}{\partial\,t}$$

The equation for $\mathbf{B}$ is similar to the one for $\mathbf{E}$.

$$\tilde{k}^2=\left(k+i\,\xi\right)^2=k^2+2\,k\,\xi\,i-\xi^2=\mu\,\epsilon\,\omega^2+i\,\mu\,\sigma\,\omega\;\Rightarrow\;\begin{array}{c}k^2-\xi^2=\mu\,\epsilon\,\omega^2\\ 2\,k\,\xi=\mu\,\sigma\,\omega\end{array}$$

$$\Rightarrow\;\xi=\frac{\mu\,\sigma\,\omega}{2\,k}\;\Rightarrow\;k^2-\left(\frac{\mu\,\sigma\,\omega}{2\,k}\right)^2=\mu\,\epsilon\,\omega^2\;\Rightarrow\;k^4-\mu\,\epsilon\,\omega^2\,k^2-\frac{\mu^2\,\sigma^2\,\omega^2}{4}=0$$

$$k\in\mathbb{R}\Rightarrow\;k^2=\frac{\mu\,\epsilon\,\omega^2+\sqrt{\mu^2\,\epsilon^2\,\omega^4+\mu^2\,\sigma^2\,\omega^2}}{2}=\frac{\mu\,\omega}{2}\left(\sqrt{\epsilon^2\,\omega^2+\sigma^2}+\epsilon\,\omega\right)$$

$$\Rightarrow\;k\equiv\pm\sqrt{\frac{\mu\,\omega}{2}\left(\sqrt{\epsilon^2\,\omega^2+\sigma^2}+\epsilon\,\omega\right)}\;\Rightarrow\;\xi\equiv\pm\sqrt{\frac{\mu\,\omega}{2}\left(\sqrt{\epsilon^2\,\omega^2+\sigma^2}-\epsilon\,\omega\right)}$$

Choosing $+$ or $-$ depends on $e^{-\xi\,z}<1$. Choose $+$ if $e^{-\xi\,z}\to e^{-\xi\,|z|}$.

- The distance taken to reduce the amplitude by a factor of $\frac{1}{e}$ (about $\frac{1}{3}$) is the **skin depth**: $d \equiv \frac{1}{\xi}$, a measure of how far the wave penetrates into the conductor.
- The real part of $\tilde{k}$ determines the wavelength, the propagation speed, and the index of refraction, in the usual way: $\lambda = \frac{2\pi}{k}, \quad v = \frac{\omega}{k}, \quad n = \frac{ck}{\omega}$
- The attenuated plane waves satisfy the modified wave eqn for any $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$. But Maxwell's equations impose further constraints, which serve to determine the relative amplitudes, phases, and polarizations of $\mathbf{E}$ and $\mathbf{B}$.

- $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$ rule out any z components: the fields are *transverse*. We orient our axes so that $\mathbf{E}$ is polarized along the x direction:

$$\tilde{\mathbf{E}}(z,t) = \tilde{E}_0 e^{-\xi z} e^{i(kz-\omega t)} \hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}(z,t) = \frac{\tilde{k}}{\omega} \tilde{E}_0 e^{-\xi z} e^{i(kz-\omega t)} \hat{\mathbf{y}} \quad \Leftarrow \quad \nabla \times \tilde{\mathbf{E}} = -\frac{\partial \tilde{\mathbf{B}}}{\partial t}$$

- Define $\tilde{k} = K e^{i\phi} \quad \Leftarrow \quad K \equiv |\tilde{k}| = \sqrt{k^2 + \xi^2} = \sqrt{\mu\omega} \sqrt[4]{\epsilon^2 \omega^2 + \sigma^2}, \quad \phi \equiv \tan^{-1} \frac{\xi}{k}$

$$\Rightarrow \text{ the complex amplitudes } \tilde{E}_0 = E_0 e^{i\delta_E}, \quad \tilde{B}_0 = B_0 e^{i\delta_B} = \frac{K e^{i\phi}}{\omega} E_0 e^{i\delta_E}$$

- The electric and magnetic fields are no longer in phase: $\delta_B - \delta_E = \phi$; the magnetic field lags behind the electric field.

$$\tilde{k} = k + i\,\xi \quad \Leftarrow \quad k \equiv \sqrt{\frac{\mu\,\omega}{2}\left(\sqrt{\epsilon^2\,\omega^2 + \sigma^2} + \epsilon\,\omega\right)}, \quad \xi \equiv \sqrt{\frac{\mu\,\omega}{2}\left(\sqrt{\epsilon^2\,\omega^2 + \sigma^2} - \epsilon\,\omega\right)}$$

$$\Rightarrow \quad 2\,k\,\xi = \mu\,\sigma\,\omega \quad \Rightarrow \quad v_p = \frac{\omega}{k} = \frac{2\,\xi}{\mu\,\sigma}$$

$$\lambda = \frac{2\,\pi}{k} = \frac{2\,\pi}{\sigma}\sqrt{\frac{2}{\mu\,\omega}}\sqrt{\sqrt{\epsilon^2\,\omega^2 + \sigma^2} - \epsilon\,\omega} = \frac{4\,\pi\,\xi}{\mu\,\sigma\,\omega}$$

$$d \equiv \frac{1}{\xi} = \frac{1}{\sigma}\sqrt{\frac{2}{\mu\,\omega}}\sqrt{\sqrt{\epsilon^2\,\omega^2 + \sigma^2} + \epsilon\,\omega} = \frac{2\,k}{\mu\,\sigma\,\omega} = \frac{4\,\pi}{\mu\,\sigma\,\omega\,\lambda} = \frac{2}{\mu\,\sigma\,v_p}$$

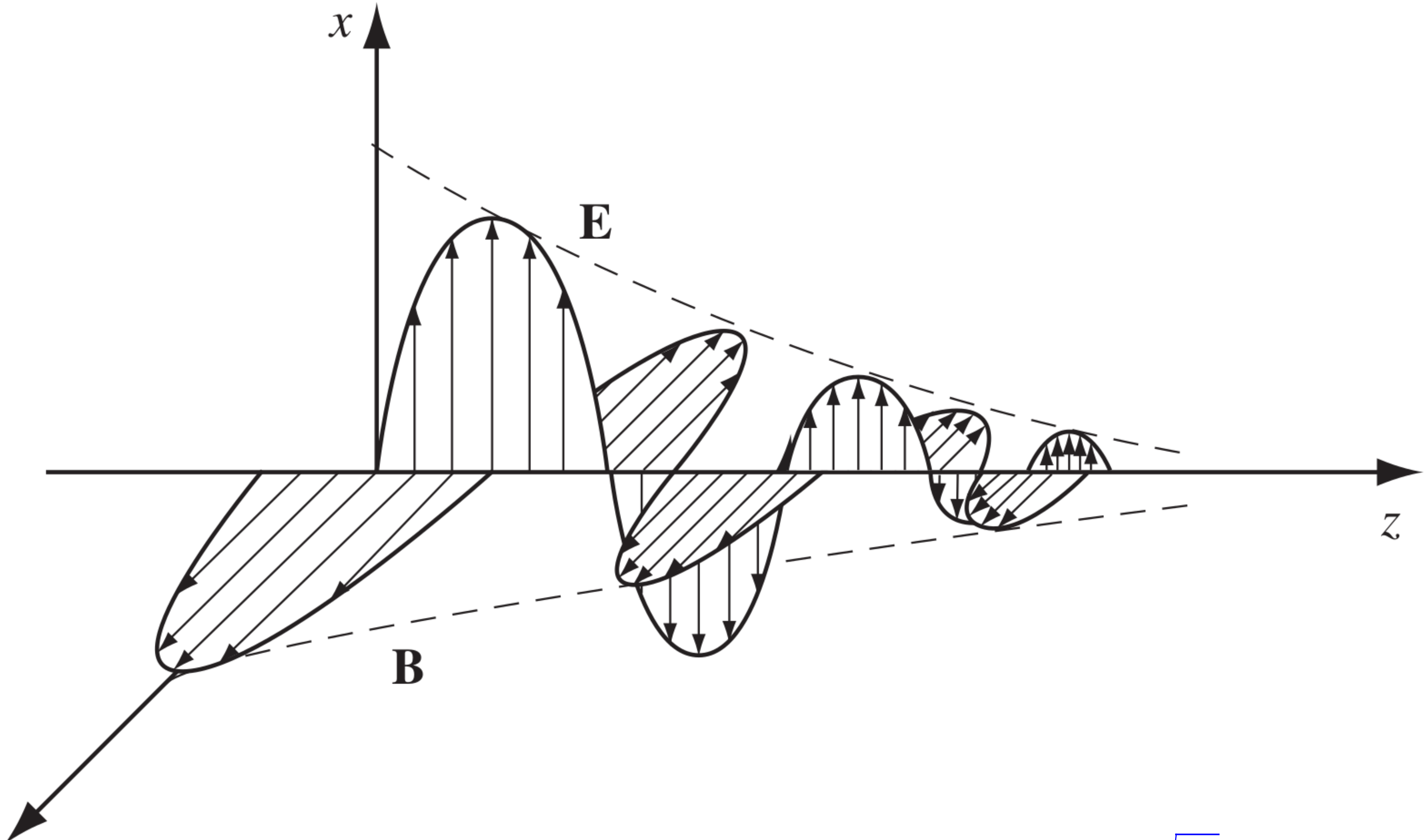
$$K \equiv |\tilde{k}| = \sqrt{k^2 + \xi^2} = \sqrt{\mu\,\omega}\,\sqrt[4]{\epsilon^2\,\omega^2 + \sigma^2}$$

$$\tilde{k} = K\,e^{i\,\phi} \quad \Leftarrow \quad \phi \equiv \tan^{-1}\frac{\xi}{k} = \tan^{-1}\frac{\sqrt{\epsilon^2\,\omega^2 + \sigma^2} - \epsilon\,\omega}{\sigma} \;\to\; \frac{\pi}{4} \text{ as } \sigma \to \infty$$

$$\to \; 0 \text{ as } \sigma \to 0$$

$$\tilde{v}_p \equiv \frac{\omega}{\tilde{k}} = \frac{\omega}{K}\,e^{-i\,\phi} = \frac{\omega}{K^2}\left(k - i\,\xi\right) = \frac{k\,v_p}{K^2}\left(k - i\,\xi\right)$$

$$\tilde{n} \equiv \frac{c}{\tilde{v}_p} = \frac{c\,\tilde{k}}{\omega} = \frac{c\,k}{\omega} + i\,\frac{c\,\xi}{\omega} = \frac{c}{v_p}\left(1 + i\,\frac{\xi}{k}\right) = n\left(1 + i\,\frac{\sqrt{\epsilon^2\,\omega^2 + \sigma^2} - \epsilon\,\omega}{\sigma}\right)$$



- The (real) amplitudes of **E** and **B** are related by $\frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\frac{\mu}{\omega}} \sqrt[4]{\epsilon^2 \omega^2 + \sigma^2}$
- The (real) electric and magnetic fields are,

$$\mathbf{E}(z,t) = E_0 e^{-\xi z} \cos(kz - \omega t + \delta_E) \hat{\mathbf{x}}$$

$$\mathbf{B}(z,t) = B_0 e^{-\xi z} \cos(kz - \omega t + \delta_E + \phi) \hat{\mathbf{y}} = \frac{K}{\omega} E_0 e^{-\xi z} \cos(kz - \omega t + \delta_E + \phi) \hat{\mathbf{y}}$$

Insulator ($\epsilon\omega \gg \sigma$):

$$k \simeq \omega\sqrt{\epsilon\mu}\left(1+\frac{\sigma^2}{8\epsilon^2\omega^2}\right),\quad \xi \simeq \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}},\quad K \simeq \omega\sqrt{\epsilon\mu}\left(1+\frac{\sigma^2}{4\epsilon^2\omega^2}\right),\quad \tan\phi \simeq \frac{\sigma}{2\epsilon\omega}$$

$$v_p \simeq \frac{1}{\sqrt{\epsilon\mu}}\left(1-\frac{\sigma^2}{8\epsilon^2\omega^2}\right),\quad \lambda \simeq \frac{2\pi}{\omega\sqrt{\epsilon\mu}}\left(1-\frac{\sigma^2}{8\epsilon^2\omega^2}\right),\quad \frac{B_0}{E_0} \simeq \sqrt{\epsilon\mu}\left(1+\frac{\sigma^2}{8\epsilon^2\omega^2}\right)$$

Good conductor ($\epsilon\omega \ll \sigma$):

$$k \simeq \sqrt{\frac{\mu\omega\sigma}{2}}\left(1+\frac{\epsilon\omega}{2\sigma}\right),\quad \xi \simeq \sqrt{\frac{\mu\omega\sigma}{2}}\left(1-\frac{\epsilon\omega}{2\sigma}\right),\quad K \simeq \sqrt{\mu\omega\sigma},\quad \tan\phi \simeq 1-\frac{\epsilon\omega}{\sigma}$$

$$v_p \simeq \sqrt{\frac{2\omega}{\mu\sigma}}\left(1-\frac{\epsilon\omega}{2\sigma}\right),\quad \lambda \simeq 2\pi\sqrt{\frac{2}{\mu\omega\sigma}}\left(1-\frac{\epsilon\omega}{2\sigma}\right),\quad \frac{B_0}{E_0} \simeq \sqrt{\frac{\mu\sigma}{\omega}} \gg \left.\frac{B_0}{E_0}\right|_{\text{insulator}}$$

- For $\sigma \to \infty$,

$$k \simeq \xi \simeq \sqrt{\frac{\mu\omega\sigma}{2}},\quad \tan\phi \simeq 1 \quad \Rightarrow \quad \phi = \frac{\pi}{4},\quad v_p \simeq \sqrt{\frac{2\omega}{\mu\sigma}} \simeq \sqrt{\frac{2\epsilon\omega}{\sigma}}\, v_{p,\,\text{insulator}} \to 0$$

$$\lambda \simeq 2\pi\sqrt{\frac{2}{\mu\omega\sigma}} = 2\pi\delta$$

- For copper, $\sigma = 6\times 10^7/(\text{Ohm-meter}) \;\Rightarrow\; \frac{\epsilon\omega}{\sigma} \simeq 10^{-18}\nu \;\Leftarrow\; \epsilon \simeq \epsilon_0$

$$\mu \simeq \mu_0 \;\Rightarrow\; \delta \approx \frac{6.5\times 10^{-2}\ \text{meter}}{\sqrt{\nu}} \to 10^{-6}\ \text{meter for}\ \nu \sim 10^9\ \text{Hz} \quad \text{microwave}$$

Material	σ (S/m)	$f = 60$ (Hz)	1 (MHz)	1 (GHz)
Silver	6.17×10^7	8.27 (mm)	0.064 (mm)	0.0020 (mm)
Copper	5.80×10^7	8.53	0.066	0.0021
Gold	4.10×10^7	10.14	0.079	0.0025
Aluminum	3.54×10^7	10.92	0.084	0.0027
Iron ($\mu_r \cong 10^3$)	1.00×10^7	0.65	0.005	0.00016
Seawater	4	32 (m)	0.25 (m)	†

Example: The electric field of a linearly polarized uniform plane wave propagating in the $+z$-direction in seawater is $\mathbf{E}(z=0) = \hat{\mathbf{x}}\, 100 \cos(10^7 \pi t)$ V/m. The constitutive parameters of seawater are ϵ_r=72, μ_r=1, and σ=4 S/m, S=Ω^{-1}.

$$\omega = 10^7 \pi \quad \Rightarrow \quad \frac{\sigma}{\omega \epsilon} = \frac{\sigma}{\omega \epsilon_r \epsilon_0} = \frac{4}{10^7 \pi \times 72 \times 8.85 \times 10^{-12}} \approx 200 \gg 1 \quad \text{good conductor}$$

Wave number k

Attenuation constant ξ

$$k, \xi \approx \sqrt{\frac{\mu \omega \sigma}{2}} = \sqrt{5\pi \times 10^6 \times 4\pi \times 10^{-7} \times 4} = 8.89\,/\mathrm{m}$$

phase velocity $v_p = \dfrac{\omega}{k} = \dfrac{10^7 \pi}{8.85} = 3.53 \times 10^6$ m/s, wavelength $\lambda = \dfrac{2\pi}{k} = 0.707$ m

skin depth $d = \dfrac{1}{\xi} = \dfrac{1}{8.87} = 0.112$ m

Distance z_1 at which the amplitude of wave decreases to 1% of its value at z=0:

$$e^{-\xi z_1} = 0.01 \quad \Rightarrow \quad e^{\xi z_1} = 100 \quad \Rightarrow \quad z_1 = \frac{\ln 100}{\xi} = \frac{4.605}{8.89} = 0.518 \text{ m}$$

The expression of $\mathbf{E}$ (V/m) & $\mathbf{B}$ (T):

$$\mathbf{E}(z, t) = E_0 e^{-\xi z} \cos(k z - \omega t) \hat{\mathbf{x}} = 100 e^{-8.89 z} \cos(8.89 z - 10^7 \pi t) \hat{\mathbf{x}} \quad \Leftarrow \quad \delta_E = 0$$

$$K \approx \sqrt{\mu \omega \sigma} = \sqrt{\mu_r \mu_0 \omega \sigma} = 4\pi / \text{m}$$

$$\Rightarrow \quad \mathbf{B}(z, t) = B_0 e^{-\xi z} \cos(k z - \omega t + \phi) \hat{\mathbf{y}} = \frac{K}{\omega} E_0 e^{-\xi z} \cos(k z - \omega t + \pi/4) \hat{\mathbf{y}}$$

$$= 4 \times 10^{-5} e^{-8.89 z} \cos\left(8.89 z - 10^7 \pi t + \frac{\pi}{4} \right) \hat{\mathbf{y}}$$

At z=0.8m:

$$\mathbf{E}(0.8 \text{ m}, t) = 100 e^{-8.89 z} \cos(8.89 z - 10^7 \pi t) \hat{\mathbf{x}} = 0.0815 \cos(10^7 \pi t - 7.11) \hat{\mathbf{x}}$$

$$\mathbf{B}(0.8 \text{ m}, t) = 4 \times 10^{-5} e^{-8.89 z} \cos\left(8.89 z - 10^7 \pi t + \frac{\pi}{4} \right) \hat{\mathbf{y}}$$

$$= 3.26 \times 10^{-8} \cos(10^7 \pi t - 7.9) \hat{\mathbf{y}}$$

Reflection at a Conducting Surface

- The boundary conditions used to analyze reflection/refraction at an interface between 2 dielectrics do not hold in the presence of free charges and currents.

- The more general relations

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f \,, \quad B_1^{\perp} = B_2^{\perp} \,, \quad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0 \,, \quad \frac{\mathbf{B}_1^{\parallel}}{\mu_1} - \frac{\mathbf{B}_2^{\parallel}}{\mu_2} = \boldsymbol{K}_f \times \hat{\mathbf{n}}$$

σ_f : free surface charge

$\boldsymbol{K}_f$: free surface current

$\hat{\mathbf{n}}$: normal vector to the surface from medium 2 to 1

- For ohmic conductors ($\boldsymbol{J}_E = \sigma \mathbf{E}$) there can be no free surface current, since this would require an infinite electric field at the boundary, $\boldsymbol{K}_f = 0$.

- Let the xy plane forms the boundary between a nonconducting linear medium 1 and a conductor 2. A monochromatic plane wave, traveling in the z direction and polarized in the x direction, approaches from the left

$$\tilde{\mathbf{E}}_I(z,t) = \tilde{E}_{0I}\, e^{i(k_1 z - \omega t)}\, \hat{\mathbf{x}} \,, \qquad \tilde{\mathbf{B}}_I(z,t) = \frac{\tilde{E}_{0I}}{v_1}\, e^{i(k_1 z - \omega t)}\, \hat{\mathbf{y}}$$

the reflected wave $\quad \tilde{\mathbf{E}}_R(z,t) = \tilde{E}_{0R}\, e^{i(-k_1 z - \omega t)}\, \hat{\mathbf{x}} \,, \quad \tilde{\mathbf{B}}_R(z,t) = -\frac{\tilde{E}_{0R}}{v_1}\, e^{i(-k_1 z - \omega t)}\, \hat{\mathbf{y}}$

the transmitted wave $\quad \tilde{\mathbf{E}}_T(z,t) = \tilde{E}_{0T}\, e^{i(\tilde{k}_2 z - \omega t)}\, \hat{\mathbf{x}} \,, \quad \tilde{\mathbf{B}}_T(z,t) = \frac{\tilde{k}_2}{\omega}\, \tilde{E}_{0T}\, e^{i(\tilde{k}_2 z - \omega t)}\, \hat{\mathbf{y}}$

which is attenuated as it penetrates into the conductor.

- At $z=0$, the combined wave in medium 1 must join the wave in medium 2, pursuant to the boundary conditions.

$$E_1^{\perp}=E_2^{\perp}=0\,(\text{transverse})\Rightarrow\quad \sigma_f=0 \quad\Bigg|\quad B_1^{\perp}=B_2^{\perp}=0\,(\text{transverse})\Rightarrow\quad 0=0$$

$$\mathbf{E}_1^{\parallel}=\mathbf{E}_2^{\parallel} \qquad\Rightarrow\quad \tilde{E}_{0I}+\tilde{E}_{0R}=\tilde{E}_{0T}$$

$$\mathbf{K}_f=0\quad\Rightarrow\quad \frac{\tilde{E}_{0I}-\tilde{E}_{0R}}{\mu_1 v_1}=\frac{\tilde{k}_2}{\mu_2\omega}\tilde{E}_{0T}\quad\Rightarrow\quad \tilde{E}_{0I}-\tilde{E}_{0R}=\tilde{\beta}\,\tilde{E}_{0T}\quad\Leftarrow\quad \tilde{\beta}\equiv\frac{\mu_1 v_1}{\mu_2\omega}\tilde{k}_2$$

$$\Rightarrow\quad \tilde{E}_{0R}=\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\tilde{E}_{0I},\qquad \tilde{E}_{0T}=\frac{2}{1+\tilde{\beta}}\tilde{E}_{0I}$$

- These results are formally identical to the ones that apply at the boundary between *non*conductors, but notice $\tilde{\beta}$ is now a complex number.

- For a *perfect* conductor $\sigma\to\infty \quad\Rightarrow\quad k_2\to\infty \quad\Rightarrow\quad \tilde{\beta}\to\infty \quad\Rightarrow\quad \begin{array}{l}\tilde{E}_{0R}=-\tilde{E}_{0I}\\ \tilde{E}_{0T}=\ 0\end{array}$

- In the case the wave is totally reflected, with a 180° phase shift, standing wave.

- That's why excellent conductors make good mirrors. You can paint a thin coating of silver onto the back of a pane of glass—the glass has nothing to do with the *reflection*; it's just there to support the silver.

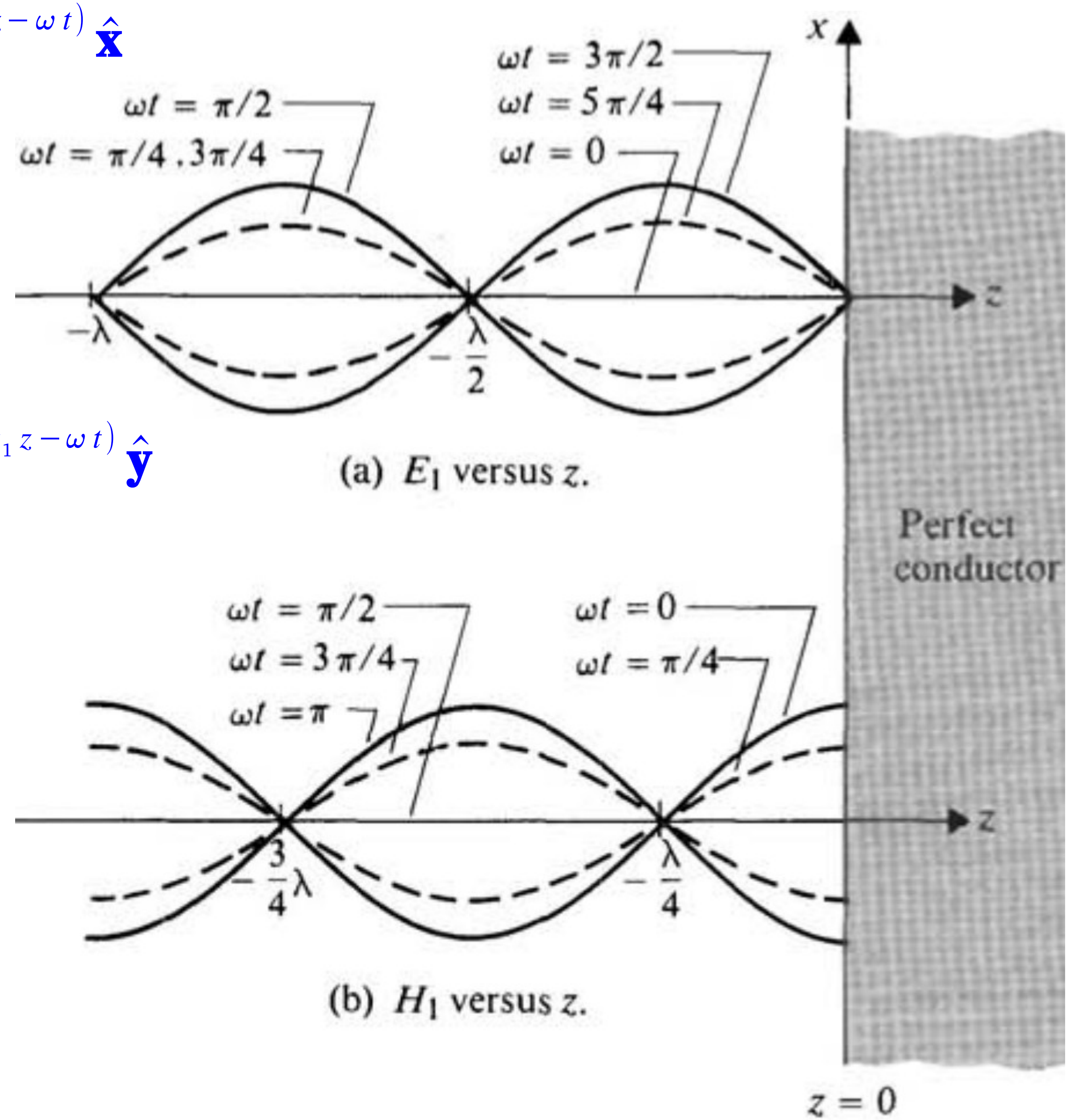
- Since the skin depth in silver at optical frequencies is less than 100Å, you don't need a very thick layer.

- For a *perfect* conductor $\sigma \to \infty \;\Rightarrow\; k_2 \to \infty \;\Rightarrow\; \tilde{\beta} \to \infty \;\Rightarrow\;$ $\tilde{E}_{0R} = -\tilde{E}_{0I}$, $\tilde{E}_{0T} = 0$

$$\tilde{\mathbf{E}}_1 \equiv \tilde{\mathbf{E}}_I + \tilde{\mathbf{E}}_R$$
$$= \tilde{E}_{0I}\, e^{i(k_1 z - \omega t)}\,\hat{\mathbf{x}} + \tilde{E}_{0R}\, e^{i(-k_1 z - \omega t)}\,\hat{\mathbf{x}}$$
$$= \tilde{E}_{0I}\left(e^{ik_1 z} - e^{-k_1 z}\right) e^{-i\omega t}\,\hat{\mathbf{x}}$$
$$= 2i\,\tilde{E}_{0I}\, e^{-i\omega t} \sin k_1 z\,\hat{\mathbf{x}}$$

$$\tilde{\mathbf{B}}_1 \equiv \tilde{\mathbf{B}}_I + \tilde{\mathbf{B}}_R$$
$$= \frac{\tilde{E}_{0I}}{v_1} e^{i(k_1 z - \omega t)}\,\hat{\mathbf{y}} - \frac{\tilde{E}_{0R}}{v_1} e^{i(-k_1 z - \omega t)}\,\hat{\mathbf{y}}$$
$$= \frac{\tilde{E}_{0I}}{v_1}\left(e^{ik_1 z} + e^{-k_1 z}\right) e^{-i\omega t}\,\hat{\mathbf{y}}$$
$$= 2\frac{\tilde{E}_{0I}}{v_1} e^{-i\omega t} \cos k_1 z\,\hat{\mathbf{y}}$$

standing wave



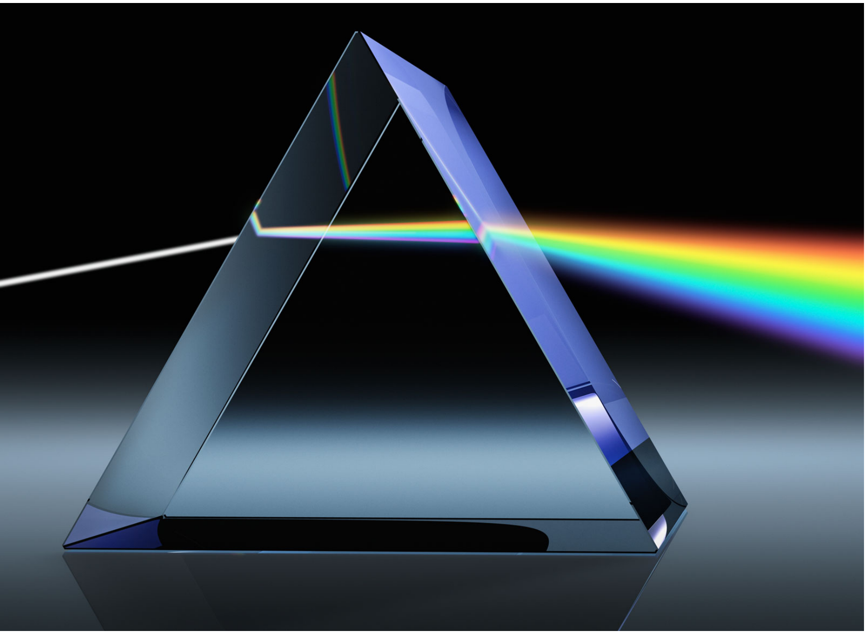
The Frequency Dependence of Permittivity

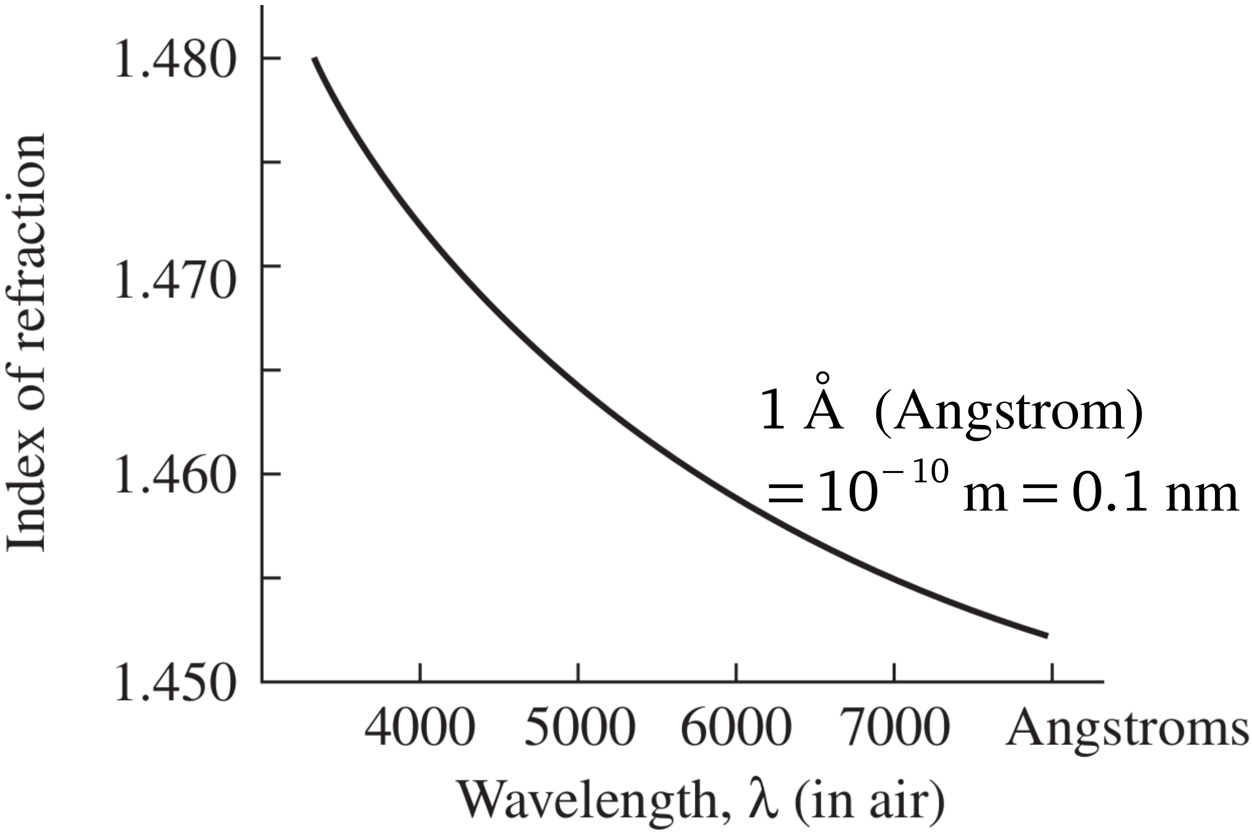
- The propagation of EM waves through matter is governed by 3 properties of the material: the permittivity ϵ, the permeability μ, and the conductivity σ.

- Each of these parameters depends on the frequency of the waves. It is well known from optics that $n \simeq \sqrt{\epsilon_r}$ is a function of wavelength.

- A prism or a raindrop bends blue light more sharply than red, and spreads white light out into a rainbow of colors—**dispersion**.

- Whenever the speed of a wave depends on its frequency, the supporting medium is called **dispersive**.





- Because waves of different frequency travel at different speeds in a dispersive medium, a wave form that incorporates a range of frequencies will change shape as it propagates.

- A sharply peaked wave typically flattens out, and whereas each sinusoidal component travels at the ordinary **wave** (or **phase**) **velocity**, $v_p = \frac{\omega}{k}$

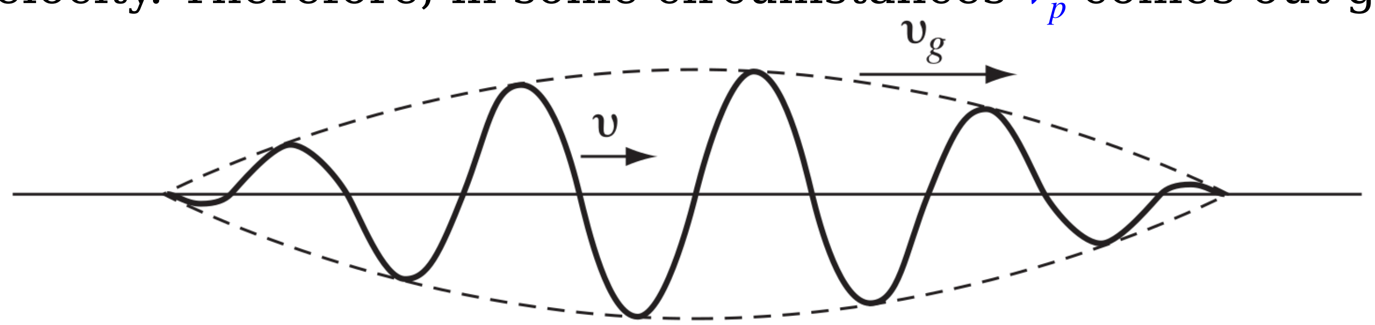
- The packet as a whole (the "envelope") moves at the so-called **group velocity**

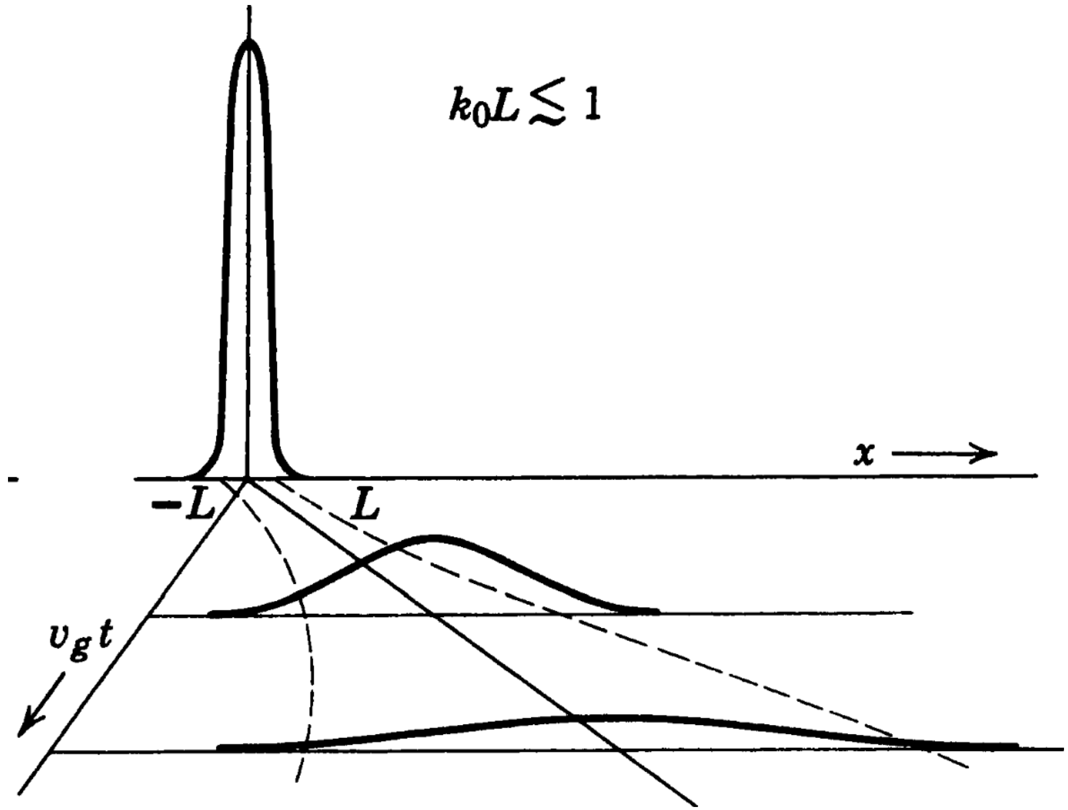
$$v_g = \frac{d\,\omega}{d\,k}$$

- While the disturbance by dropping rock into water as a whole spreads out in a circle at v_g, the ripples making it up travel twice as fast ($v_p=2v_g$ in this case).

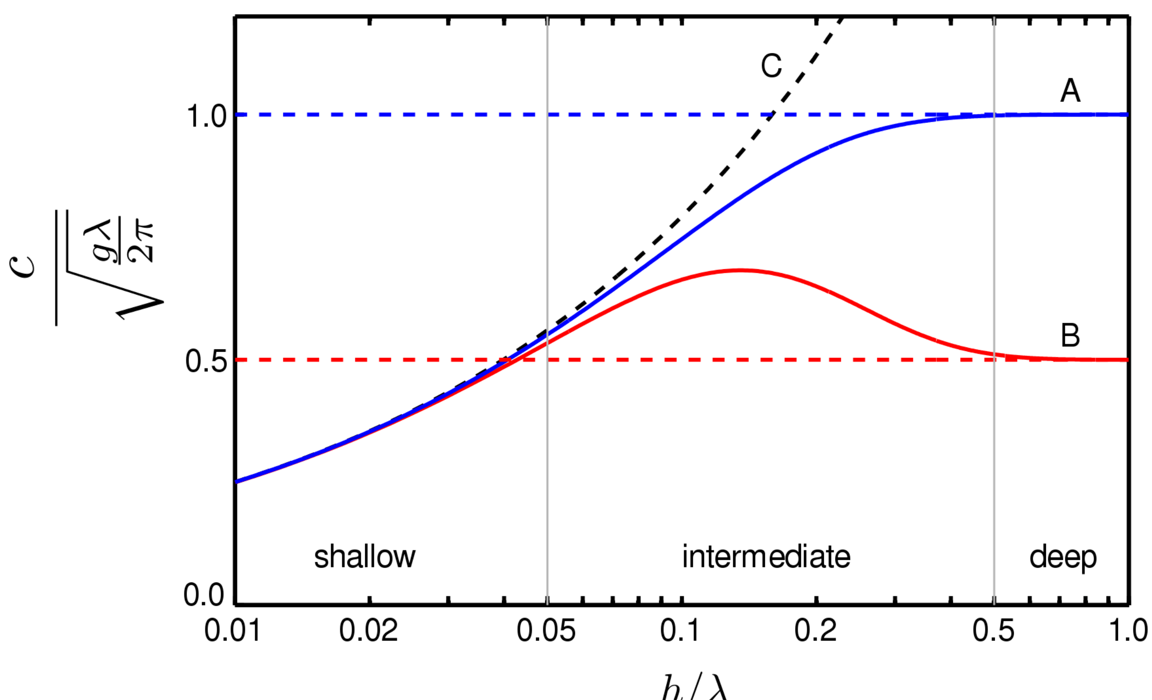
- They appear at the back end of the packet, growing as they move forward to the center, then shrinking again and fading away at the front.

- The *energy* carried by a wave packet in a dispersive medium does not travel at the phase velocity. Therefore, in some circumstances v_p comes out greater than c.





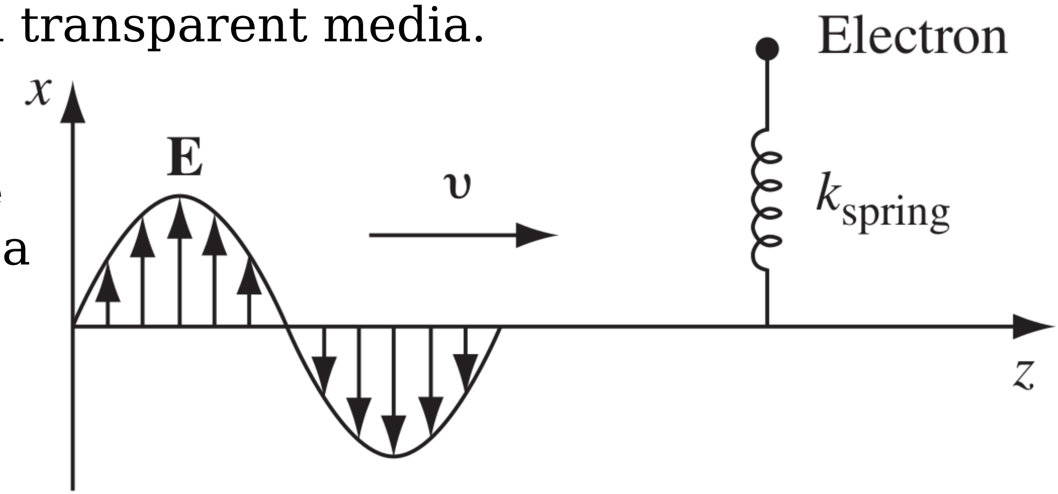
$k_0 L \lesssim 1$
x
$-L$
L
$v_g t$



C
A
B
1.0
0.5
0.0
shallow
intermediate
deep
0.01
0.02
0.05
0.1
0.2
0.5
1.0
$\frac{c}{\sqrt{\frac{g\lambda}{2\pi}}}$
h/λ

- Try to account for the frequency dependence of in dielectrics by using a simplified model for the behavior of the electrons.

- The classical model of atomic-scale phenomena is an approximation to the truth; nevertheless, it does yield qualitatively satisfactory results, and it provides a plausible mechanism for dispersion in transparent media.



- The electrons in a nonconductor are bound molecules. Here we shall picture each electron as attached to the end of a spring, with force constant k_{spring}:

$$F_{\text{binding}} = -k_{\text{spring}}\, x = -m\,\omega_0^2\, x$$

$$\omega_0 = \sqrt{\frac{k_{\text{spring}}}{m}} \quad \text{natural oscillation frequency}$$

- *Any* binding force can be approximated this way for sufficiently small displacements from equilibrium, since $U(x) = U(0) + U'(0)\,x + \frac{1}{2} U''(0)\,x^2 + \cdots$

Set $U(0) = 0$, and $-F = U'(0) = 0$, $k_{\text{spring}} = U''(0)$

- There will be some damping force on the electron: $F_{\text{damping}} = -m\gamma \frac{\mathrm{d}x}{\mathrm{d}t}$

- One possible damping is the EM radiation.

- In the presence of an EM wave of frequency ω, polarized in the x direction, the electron is subject to a driving force $F_{\text{driving}} = qE = qE_0 \cos\omega t \quad \Leftarrow \quad B = \frac{E}{c}$ negligible

- Putting all this into Newton's 2nd law gives

$$m\frac{d^2x}{dt^2} = F_{\text{tot}} = F_{\text{binding}} + F_{\text{damping}} + F_{\text{driving}} \quad \Rightarrow \quad m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} + m\omega_0^2 x = qE_0\cos\omega t$$

- Our model describes the electron as a damped harmonic oscillator, driven at frequency ω. (The much more massive nucleus remains at rest.)

- Regard the equation as the real part of a *complex* equation:

$$\frac{d^2\tilde{x}}{dt^2} + \gamma\frac{d\tilde{x}}{dt} + \omega_0^2\tilde{x} = \frac{q}{m}E_0 e^{-i\omega t} \quad \Rightarrow \quad \tilde{x} = \tilde{x}_0 e^{-i\omega t} \quad \Leftarrow \quad \tilde{x}_0 = \frac{q}{m}\frac{E_0}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

$$\Rightarrow \quad \tilde{p}(t) = q\tilde{x}(t) = \frac{q^2}{m}\frac{E_0 e^{-i\omega t}}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

complex dipole moment
use the real part

- The imaginary term in the denominator means that p is out of phase with E—lagging behind by an angle $\tan^{-1}\frac{\gamma\omega}{\omega_0^2 - \omega^2}$, very small when $\omega \ll \omega_0$ and rises to π when $\omega \gg \omega_0$.

- In general, differently situated electrons within a given molecule experience different natural frequencies and damping coefficients.

- If there are f_j electrons with frequency ω_j and damping γ_j in each molecule, and N molecules per unit volume, the polarization $\mathbf{P}$ is given by the real part of

$$\tilde{\mathbf{P}}(t) = N\frac{q^2}{m}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\tilde{\mathbf{E}} = \epsilon_0\tilde{\chi}_e\tilde{\mathbf{E}} \quad \Leftarrow \quad \tilde{\chi}_e : \text{complex susceptibility}$$

- We used to use $\mathbf{P}=\epsilon_0\chi_e\mathbf{E}$. In the present case, $\mathbf{P}$ is not proportional to $\mathbf{E}$ (not a linear medium) because of the difference in phase.

- All of the manipulations carry over, on the understanding that the physical polarization is the real part of $\tilde{\mathbf{P}}$, just as the physical field is the real part of $\tilde{\mathbf{E}}$.

- $\tilde{\mathbf{D}} = \tilde{\epsilon}\,\tilde{\mathbf{E}} \quad \Leftarrow \quad \tilde{\epsilon} = \epsilon_0(1+\tilde{\chi}_e)$ complex permittivity

$$\Rightarrow \quad \tilde{\epsilon}_r = \frac{\tilde{\epsilon}}{\epsilon_0} = 1 + \frac{Nq^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega} \quad \text{complex dielectric constant} \qquad (\$)$$

- Ordinarily, the imaginary term is negligible; however, when ω is very close to one of the resonant frequencies ω_j it plays an important role.

- In a dispersive medium, the wave eqn for a given frequency $\nabla^2\tilde{\mathbf{E}} = \tilde{\epsilon}\mu_0\dfrac{\partial^2\tilde{\mathbf{E}}}{\partial t^2}$.

It admits plane wave solutions, $\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)}$, with the complex wave number $\tilde{k} \equiv \sqrt{\tilde{\epsilon}\mu_0}\,\omega = k + i\xi \quad \Rightarrow \quad \tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-\xi z} e^{i(kz-\omega t)}$

- The wave is *attenuated* because the intensity $\propto E^2 \propto e^{-\alpha z} = e^{-2\xi z}$, where $\alpha \equiv 2\xi$ is called the **absorption coefficient**.

- The wave velocity $v_p = \dfrac{\omega}{k}$, and the index of refraction $n = \dfrac{c\,k}{\omega}$

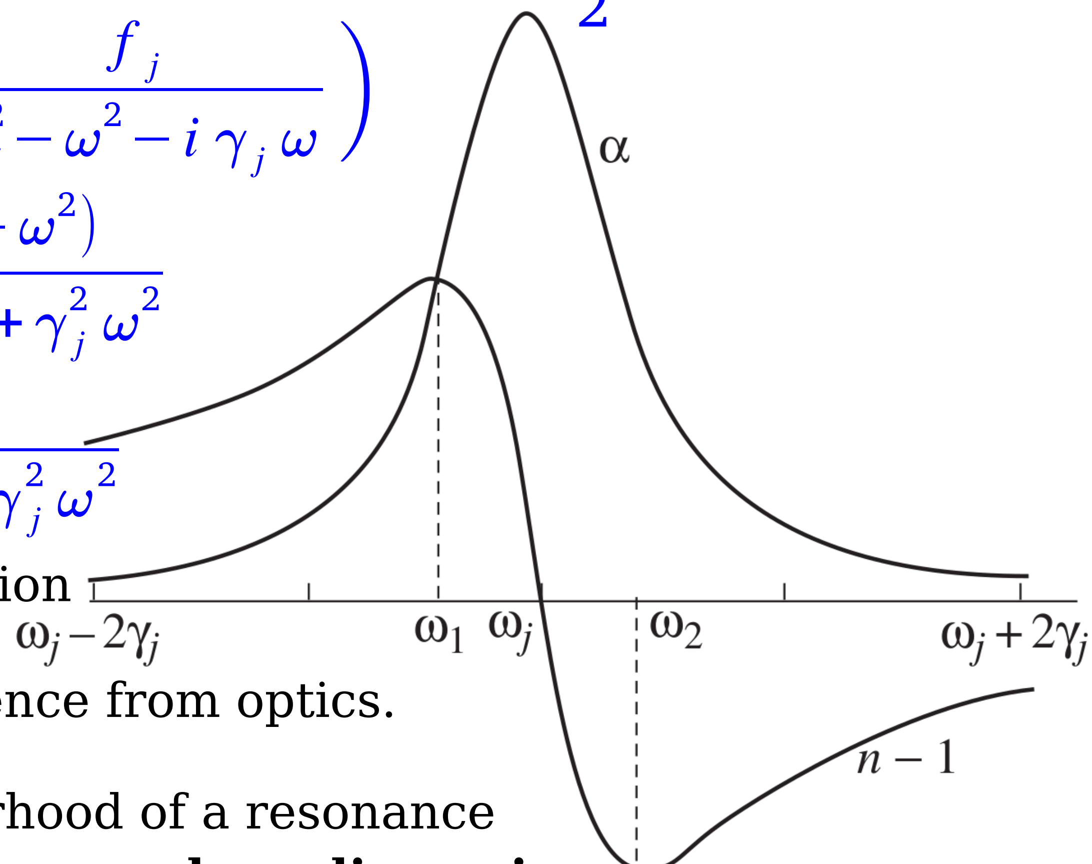
- Here k and ξ have nothing to do with conductivity; they are determined by the parameters of our damped harmonic oscillator.

- For gases, the 2nd term in ($) is small, by using $\sqrt{1+x} \simeq 1 + \dfrac{x}{2}$ for $x \ll 1$

$$\Rightarrow \quad \tilde{k} = \frac{\omega}{c}\sqrt{\tilde{\epsilon}_r} \simeq \frac{\omega}{c}\left(1 + \frac{N q^2}{2 m \epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega}\right)$$

$$\Rightarrow \quad n = \frac{c\,k}{\omega} \simeq 1 + \frac{N q^2}{2 m \epsilon_0}\sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2}$$

$$\alpha = 2\xi \simeq \frac{N q^2 \omega^2}{m \epsilon_0 c}\sum_j \frac{f_j \gamma_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2}$$



- *Most* of the time the index of refraction rises gradually with increasing frequency, consistent with our experience from optics.

- However, in the immediate neighborhood of a resonance the index of refraction *drops* sharply—**anomalous dispersion**.

- The region of anomalous dispersion ($\omega_1 < \omega < \omega_2$) coincides with the region of maximum absorption; the material may be opaque in this frequency range.

- The is because the electrons are driven at their “favorite” frequency; the amplitude of their oscillation is relatively large, and thus a large amount of energy is dissipated by the damping mechanism.

- It shows $n < 1$ above the resonance, suggesting that the wave speed exceeds c. But it is ok since energy does not travel at the wave velocity.

- We also need to consider the contributions of other terms in the sum, which add a relatively constant “background” that, in some cases, keeps $n > 1$ on both sides of the resonance.

- The group velocity can also exceed c in the vicinity of a resonance in this model. But no causality is violated.

- Staying away from the resonances, the damping can be ignored, and the formula for the index of refraction simplifies: $n = 1 + \frac{N q^2}{2 m \epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2}$

- For transparent materials, the nearest significant resonances typically lie in the ultraviolet, so $\omega < \omega_j \;\Rightarrow\; \frac{1}{\omega_j^2 - \omega^2} = \frac{1}{\omega_j^2}\left(1 - \frac{\omega^2}{\omega_j^2}\right)^{-1} \simeq \frac{1}{\omega_j^2}\left(1 + \frac{\omega^2}{\omega_j^2}\right)$

$$\Rightarrow \quad n \simeq 1 + \frac{N q^2}{2 m \epsilon_0} \sum_j \frac{f_j}{\omega_j^2} + \omega^2 \frac{N q^2}{2 m \epsilon_0} \sum_j \frac{f_j}{\omega_j^4}$$

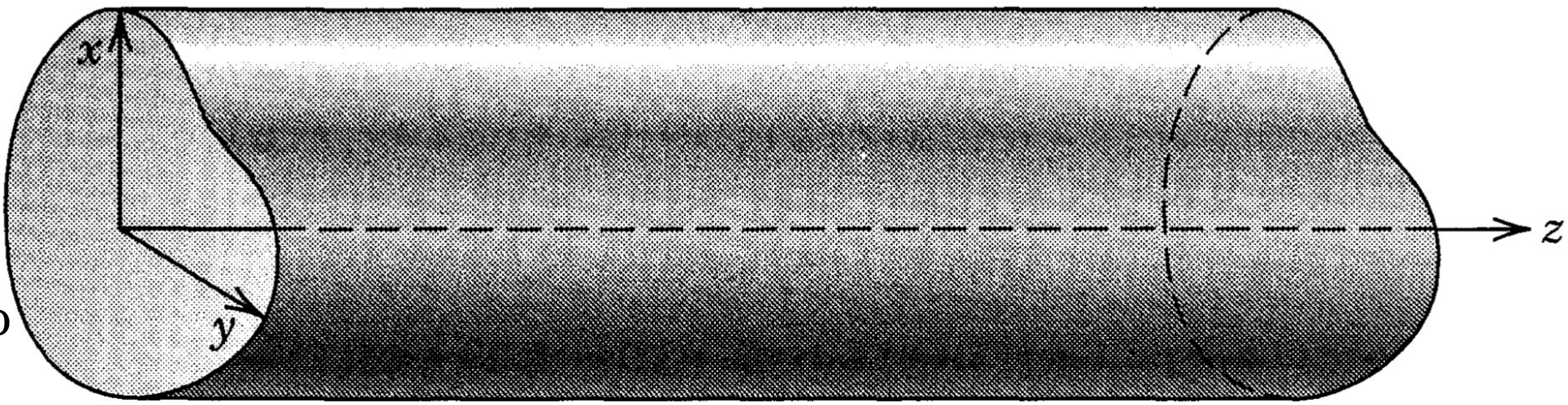
$$\Rightarrow \quad n = 1 + A \left(1 + \frac{B}{\lambda^2} \right) \quad \text{Cauchy's formula} \quad \Leftarrow \quad \lambda = 2 \pi \frac{c}{\omega}$$

A the coefficient of refraction , B : the coefficient of dispersion

- Cauchy's equation applies reasonably well to most gases, in the optical region.

Selected Problems: 9, 13, 18, 21, 26, 29, 35, 39

Guided Waves



Wave Guides

- Consider EM waves confined to the interior of a hollow pipe—**wave guide**.

- Assume the wave guide is a perfect conductor, so $\mathbf{E}=\mathbf{B}=0$ inside the material, and hence the boundary conditions at the inner wall are $\mathbf{E}^{\parallel}=0\,,\quad B^{\perp}=0$

- Charges and currents will be induced on the surface in such a way as to enforce these constraints.

- For monochromatic waves propagating down the tube, $\mathbf{E}$ & $\mathbf{B}$ have the form

$$\tilde{\mathbf{E}}(x,y,z,t)=\tilde{\mathbf{E}}_0(x,y)\,e^{i(kz-\omega t)},\quad \tilde{\mathbf{B}}(x,y,z,t)=\tilde{\mathbf{B}}_0(x,y)\,e^{i(kz-\omega t)}$$

- The electric & magnetic fields must satisfy Maxwell's equations, in the interior of the wave guide: $\nabla\cdot\mathbf{E}=0\,,\quad \nabla\cdot\mathbf{B}=0\,,\quad \nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}\,,\quad \nabla\times\mathbf{B}=\dfrac{1}{c^2}\dfrac{\partial\mathbf{E}}{\partial t}$

- The problem is to find functions $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ such that the fields obey the Maxwell's equations, and subject to the boundary conditions.

- *Confined* waves are *not* (in general) transverse; in order to fit the boundary conditions we have to include longitudinal components (E_z and B_z):

$$\tilde{\mathbf{E}}_0(x,y)=\mathcal{E}_x\,\hat{\mathbf{x}}+\mathcal{E}_y\,\hat{\mathbf{y}}+\mathcal{E}_z\,\hat{\mathbf{z}}\,,\quad \tilde{\mathbf{B}}_0(x,y)=\mathcal{B}_x\,\hat{\mathbf{x}}+\mathcal{B}_y\,\hat{\mathbf{y}}+\mathcal{B}_z\,\hat{\mathbf{z}}\,,\quad c\,\tilde{\mathbf{B}}\neq\hat{\boldsymbol{k}}\times\tilde{\mathbf{E}}\ \text{ in general}$$

- Putting this into Maxwell's equations, $c\,k\neq\omega$ here

Use $\nabla\times\mathbf{E}=-\dfrac{\partial\,\mathbf{B}}{\partial\,t}$, $\nabla\times\mathbf{B}=\dfrac{1}{c^2}\dfrac{\partial\,\mathbf{E}}{\partial\,t}$ $\Leftarrow$ $E_i=\mathcal{E}_i\,e^{i(kz-\omega t)}$ $\quad B_i=\mathcal{B}_i\,e^{i(kz-\omega t)}$

$$\Rightarrow\ \begin{array}{l}\dfrac{\partial\,\mathcal{E}_y}{\partial\,x}-\dfrac{\partial\,\mathcal{E}_x}{\partial\,y}=\quad i\,\omega\,\mathcal{B}_z\,,\quad \dfrac{\partial\,\mathcal{E}_z}{\partial\,y}-i\,k\,\mathcal{E}_y=\quad i\,\omega\,\mathcal{B}_x\,,\quad i\,k\,\mathcal{E}_x-\dfrac{\partial\,\mathcal{E}_z}{\partial\,x}=\quad i\,\omega\,\mathcal{B}_y\\ \dfrac{\partial\,\mathcal{B}_y}{\partial\,x}-\dfrac{\partial\,\mathcal{B}_x}{\partial\,y}=-i\,\dfrac{\omega}{c^2}\,\mathcal{E}_z\,,\quad \dfrac{\partial\,\mathcal{B}_z}{\partial\,y}-i\,k\,\mathcal{B}_y=-i\,\dfrac{\omega}{c^2}\,\mathcal{E}_x\,,\quad i\,k\,\mathcal{B}_x-\dfrac{\partial\,\mathcal{B}_z}{\partial\,x}=-i\,\dfrac{\omega}{c^2}\,\mathcal{E}_y\end{array}$$

$$\Rightarrow\ \begin{array}{l}\mathcal{E}_x=\dfrac{i\,c^2}{\omega^2-c^2k^2}\left(k\,\dfrac{\partial\,\mathcal{E}_z}{\partial\,x}+\ \omega\,\dfrac{\partial\,\mathcal{B}_z}{\partial\,y}\right)\,,\quad \mathcal{E}_y=\dfrac{i\,c^2}{\omega^2-c^2k^2}\left(k\,\dfrac{\partial\,\mathcal{E}_z}{\partial\,y}-\omega\,\dfrac{\partial\,\mathcal{B}_z}{\partial\,x}\right)\\ \mathcal{B}_x=\dfrac{i\,c^2}{\omega^2-c^2k^2}\left(k\,\dfrac{\partial\,\mathcal{B}_z}{\partial\,x}-\dfrac{\omega}{c^2}\,\dfrac{\partial\,\mathcal{E}_z}{\partial\,y}\right)\,,\quad \mathcal{B}_y=\dfrac{i\,c^2}{\omega^2-c^2k^2}\left(k\,\dfrac{\partial\,\mathcal{B}_z}{\partial\,y}+\dfrac{\omega}{c^2}\,\dfrac{\partial\,\mathcal{E}_z}{\partial\,x}\right)\end{array}$$

Use $\nabla\cdot\mathbf{E}=0\,,\quad\nabla\cdot\mathbf{B}=0$

$$\Rightarrow\ \left(\frac{\partial^2}{\partial\,x^2}+\frac{\partial^2}{\partial\,y^2}+\frac{\omega^2}{c^2}-k^2\right)\mathcal{E}_z=0\,,\quad \left(\frac{\partial^2}{\partial\,x^2}+\frac{\partial^2}{\partial\,y^2}+\frac{\omega^2}{c^2}-k^2\right)\mathcal{B}_z=0\quad\text{decoupled}$$

$$i\ k\,\mathcal{E}_x - i\,\omega\,\mathcal{B}_y = \frac{\partial\,\mathcal{E}_z}{\partial\,x} \qquad \Rightarrow \qquad i\ k^2\,\mathcal{E}_x - i\,k\,\omega\,\mathcal{B}_y = k\,\frac{\partial\,\mathcal{E}_z}{\partial\,x}$$

$$-i\,\frac{\omega}{c^2}\,\mathcal{E}_x + i\,k\,\mathcal{B}_y = \frac{\partial\,\mathcal{B}_z}{\partial\,y} \qquad\qquad -i\,\frac{\omega^2}{c^2}\,\mathcal{E}_x + i\,k\,\omega\,\mathcal{B}_y = \omega\,\frac{\partial\,\mathcal{B}_z}{\partial\,y}$$

$$\Rightarrow \quad i\left(k^2 - \frac{\omega^2}{c^2}\right)\mathcal{E}_x = k\,\frac{\partial\,\mathcal{E}_z}{\partial\,x} + \omega\,\frac{\partial\,\mathcal{B}_z}{\partial\,y} \quad \Rightarrow \quad \mathcal{E}_x = \frac{i\,c^2}{\omega^2 - c^2\,k^2}\left(k\,\frac{\partial\,\mathcal{E}_z}{\partial\,x} + \omega\,\frac{\partial\,\mathcal{B}_z}{\partial\,y}\right)$$

Similarly, $\mathcal{E}_y = \dfrac{i\,c^2}{\omega^2 - c^2\,k^2}\left(k\,\dfrac{\partial\,\mathcal{E}_z}{\partial\,y} - \omega\,\dfrac{\partial\,\mathcal{B}_z}{\partial\,x}\right)$

$$\mathbf{E}\,(x\,,\,y\,,\,z) = \boldsymbol{\mathcal{E}}\,(x\,,\,y)\,e^{i\,(k\,z - \omega\,t)}$$

$$0 = \nabla\cdot\mathbf{E} = \frac{\partial\,E_x}{\partial\,x} + \frac{\partial\,E_y}{\partial\,y} + \frac{\partial\,E_z}{\partial\,z} = e^{i\,(k\,z-\omega\,t)}\left(\frac{\partial\,\mathcal{E}_x}{\partial\,x} + \frac{\partial\,\mathcal{E}_y}{\partial\,y} + i\,k\,\mathcal{E}_z\right)$$

$$\Rightarrow \quad 0 = \frac{\partial\,\mathcal{E}_x}{\partial\,x} + \frac{\partial\,\mathcal{E}_y}{\partial\,y} + i\,k\,\mathcal{E}_z$$

$$= \frac{i\,c^2}{\omega^2 - c^2\,k^2}\left(k\,\frac{\partial^2\,\mathcal{E}_z}{\partial\,x^2} + \omega\,\cancel{\frac{\partial^2\,\mathcal{B}_z}{\partial\,x\,\partial\,y}}\right) + \frac{i\,c^2}{\omega^2 - c^2\,k^2}\left(k\,\frac{\partial^2\,\mathcal{E}_z}{\partial\,y^2} - \omega\,\cancel{\frac{\partial^2\,\mathcal{B}_z}{\partial\,y\,\partial\,x}}\right) + i\,k\,\mathcal{E}_z$$

$$\Rightarrow \quad \left(\frac{\partial^2}{\partial\,x^2} + \frac{\partial^2}{\partial\,y^2} + \frac{\omega^2}{c^2} - k^2\right)\mathcal{E}_z = 0$$

- It is convenient to classify the propagating waves in a uniform waveguide into 3 types according to whether E_z or B_z exists:

  1. *Transverse electromagnetic* (**TEM**) *waves*. These are waves with $E_z = B_z = 0$.
  2. *Transverse magnetic* (**TM**) *waves*: These are waves with $B_z = 0$ but $E_z \neq 0$.
  3. *Transverse electric* (**TE**) *waves*: These are waves with $E_z = 0$ but $B_z \neq 0$.

- It turns out that TEM waves cannot occur in a hollow wave guide.

Proof:

$$\left[\begin{array}{l} E_z = 0 \;\Rightarrow\; \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} = 0 \;\Leftarrow\; \nabla \cdot \tilde{\mathbf{E}} = 0 \\ B_z = 0 \;\Rightarrow\; \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = 0 \;\Leftarrow\; \nabla \times \tilde{\mathbf{E}} = -\dfrac{\partial \tilde{\mathbf{B}}}{\partial t} \end{array}\right. \;\Rightarrow\; \begin{array}{l} \nabla_t \equiv \hat{\mathbf{x}} \dfrac{\partial}{\partial x} + \hat{\mathbf{y}} \dfrac{\partial}{\partial y} \\ \nabla_t \cdot \tilde{\mathbf{E}}_0 = \nabla_t \times \tilde{\mathbf{E}}_0 = 0 \end{array}$$

$$\Rightarrow\; \begin{array}{l} \tilde{\mathbf{E}}_0 = -\nabla_t \Phi \\ \tilde{\mathbf{E}}_{0,\,\text{boundary}} = 0 \end{array} \;\Rightarrow\; \begin{array}{l} \nabla_t^2 \Phi = 0 \\ \left.\dfrac{\partial \Phi}{\partial n}\right|_{\text{boundary}} = 0 \end{array} \;\Rightarrow\; \Phi = \text{constant} \;\Rightarrow\; \tilde{\mathbf{E}}_0 = 0$$

- This argument applies only to a completely *empty* pipe—if you run a separate conductor down the middle, the potential at *its* surface need not be the same as on the outer wall, eg, the coaxial cable and the parallel-wire transmission line, and hence a nontrivial potential is possible.

The Coaxial Transmission Line

- A coaxial transmission line does admit modes with $E_z=0$ and $B_z=0$, ie, the TEM waves.

- In this case Maxwell's equations yield $k=\frac{\omega}{c} \Rightarrow v=c$, and are nondispersive

$$\Rightarrow c\,B_y=E_x\,,\quad c\,B_x=-E_y\,,\quad \mathbf{E}\perp\mathbf{B}\,,\quad \nabla\cdot\mathbf{E}=0\,,\quad \nabla\cdot\mathbf{B}=0$$

$$\Rightarrow \begin{array}{l} \frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}=0\,,\quad \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=0 \\ \frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}=0\,,\quad \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=0 \end{array}$$

- These are precisely the eqns of *electrostatics* and *magnetostatics*, for empty space, in 2D; the solution with cylindrical symmetry can be borrowed from the case of an infinite line charge and an infinite straight current

$$\mathbf{E}_0(s,\phi)=\frac{A}{s}\hat{\mathbf{s}}\,,\quad \mathbf{B}_0(s,\phi)=\frac{A}{c\,s}\hat{\boldsymbol{\phi}} \quad \Leftarrow \quad A:\ \text{some constant}$$

$$\Rightarrow \text{ the real part } \begin{array}{l} \mathbf{E}(s,\phi,z,t)=\frac{A}{s}\cos(k\,z-\omega\,t)\,\hat{\mathbf{s}} \\ \mathbf{B}(s,\phi,z,t)=\frac{A}{c\,s}\cos(k\,z-\omega\,t)\,\hat{\boldsymbol{\phi}} \end{array}$$

Transverse Magnetic (TM) Waves

- TM waves do not have a component of the magnetic field in the direction of propagation, $B_z=0$. The behavior of TM waves can be analyzed, subject to the boundary conditions of the guide, by solving $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\omega^2}{c^2}-k^2\right)E_z=0$

$$\Rightarrow \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \frac{i\,c^2 k}{\omega^2 - c^2 k^2}\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} E_z, \quad \begin{bmatrix} B_x \\ B_y \end{bmatrix} = \frac{i\,\omega}{\omega^2 - c^2 k^2}\begin{bmatrix} -\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \end{bmatrix} E_z$$

$$\Rightarrow \mathbf{E}_t \equiv E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}} = \frac{i\,c^2 k}{\omega^2 - c^2 k^2}\nabla_t E_z, \quad \mathbf{B} = \frac{\omega}{c^2 k}\hat{\mathbf{z}}\times\mathbf{E} \quad \Leftarrow \quad \nabla_t \equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y}$$

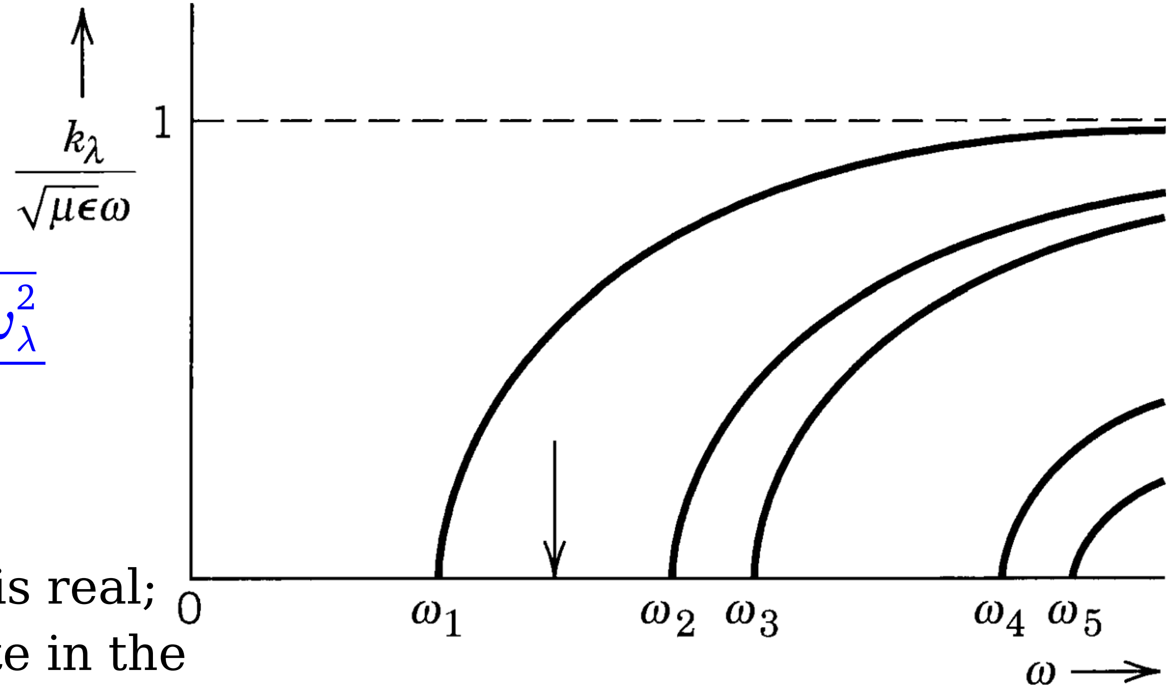
- It is easy to see that the constant $\omega^2-c^2k^2$ must be nonnegative because E_z must be oscillatory to satisfy boundary conditions on the sides of the cylinder.

- There will be a spectrum of eigenvalues $\omega^2-c^2k_\lambda{}^2$ and corresponding solutions $E_{z\lambda}$, $\lambda= 1, 2, 3, \ldots$, which form an orthogonal set. These different solutions are called the *modes of the guide*.

- For a given frequency ω, the wave number k_λ is determined for each value of λ:

$$\omega_\lambda = \sqrt{\omega^2 - c^2 k_\lambda^2} \quad \Rightarrow \quad k_\lambda = \frac{\sqrt{\omega^2 - \omega_\lambda^2}}{c}$$

cutoff frequency



- For $\omega > \omega_\lambda$, the wave number k_λ is real; waves of the λ mode can propagate in the guide. For $\omega < \omega_\lambda$, k_λ is imaginary; such modes cannot propagate and are called *cutoff modes* or *evanescent modes*.

- It is often convenient to choose the dimensions of the guide so that at the operating frequency only the lowest mode can occur.

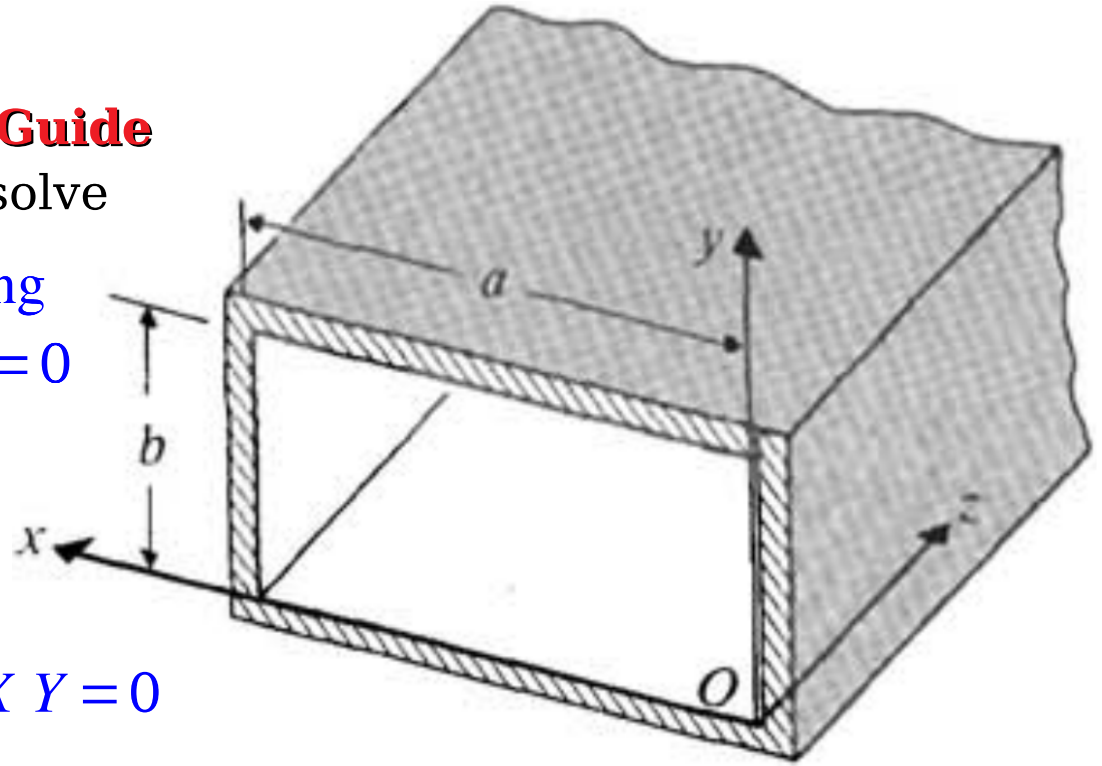
- Since the wave number $k_\lambda < \frac{\omega}{c}$, the wavelength in the guide is always greater than the free-space wavelength. Thus the phase velocity $v_p = \frac{\omega}{k_\lambda} = \frac{\omega\, c}{\sqrt{\omega^2 - \omega_\lambda^2}} > c$

- The phase velocity becomes infinite exactly at the cutoff frequency.

TM Waves in a Rectangular Wave Guide

- The TM wave ($B_z=0$) problem is to solve

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\omega^2}{c^2}-k^2\right)\mathcal{E}_z=0 \quad \text{using} \quad E^{\parallel}=0$$



- Do it by separation of variables

$$\mathcal{E}_z=X(x)\,Y(y)$$

$$\Rightarrow \quad Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}+\left(\frac{\omega^2}{c^2}-k^2\right)XY=0$$

$$\Rightarrow \quad \frac{1}{X}\frac{d^2X}{dx^2}=-k_x^2, \quad \frac{1}{Y}\frac{d^2Y}{dy^2}=-k_y^2 \quad \Leftarrow \quad k_x^2+k_y^2=\frac{\omega^2}{c^2}-k^2=\frac{\omega^2}{c^2}-k_z^2$$

$$\Rightarrow \quad X(x)=A\sin k_x x+B\cos k_x x$$

$$\mathcal{E}_{z,\,\text{boundary}}=0 \quad \Rightarrow \quad X(0)=X(a)=0 \quad \Rightarrow \quad B=0, \quad k_x=\frac{m\pi}{a}, \quad m=0,\ 1,\ 2,\cdots$$

$$\text{Similarly} \quad k_y=\frac{n\pi}{b}, \quad n=0,\ 1,\ 2,\cdots \quad \Rightarrow \quad k_z^2=\frac{\omega^2}{c^2}-\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

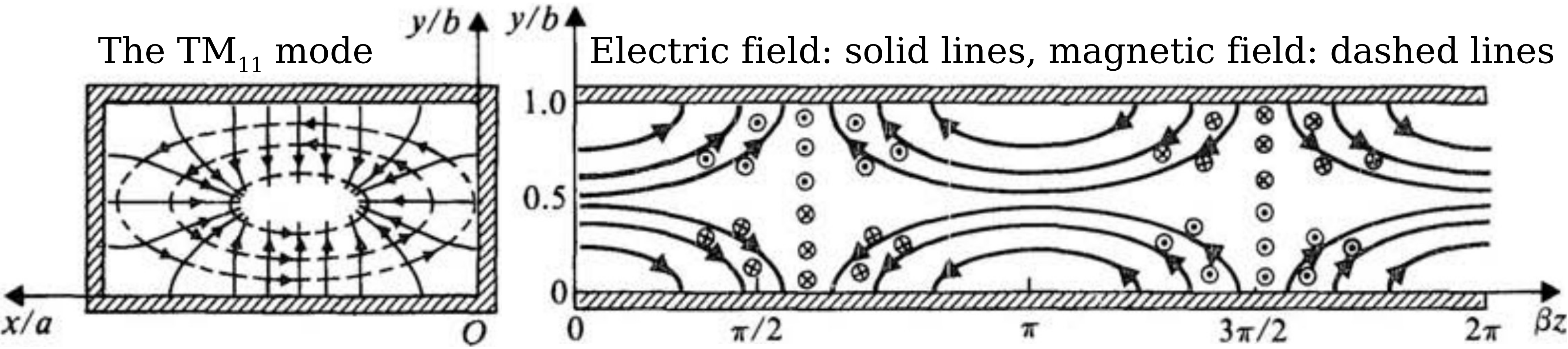
$$\Rightarrow \quad \mathcal{E}_{z\,mn}=E_{0\,mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad \Rightarrow \quad \mathcal{E}_z=\sum_{m,n}E_{0\,mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

- The other field components are

$$\begin{bmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{bmatrix} = \sum_{m,n} \frac{i c^2 k_z}{\omega^2 - c^2 k_z^2} E_{0mn} \begin{bmatrix} \frac{m\pi}{a} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \\ \frac{n\pi}{b} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \end{bmatrix}, \quad \mathbf{B} = \frac{\omega}{c^2 k_z} \hat{\mathbf{z}} \times \mathbf{E}$$

- The solution is called the TM_{mn} mode, assuming $a \ge b$. Neither one of the indices can be 0.
- If $\omega < c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \equiv \omega_{mn}$, the wave number k_z is imaginary, and instead of a traveling wave we have exponentially attenuated fields. For this reason, ω_{mn} is called the **cutoff frequency** for the mode in question.
- The **cutoff wavelength**: $\lambda_{mn} = \frac{2\pi c}{\omega_{mn}} = \frac{2ab}{\sqrt{b^2 m^2 + a^2 n^2}}$
- The *lowest* cutoff frequency for a given wave guide occurs for the mode TM_{11}: $\omega_{11} = c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$. Frequencies less than this will not propagate at all.
- The wave number can be written in terms of the cutoff frequency:

$$k_z = \frac{\sqrt{\omega^2 - \omega_{mn}^2}}{c} \quad \Rightarrow \text{ phase velocity } \; v_p = \frac{\omega}{k_z} = \frac{\omega c}{\sqrt{\omega^2 - \omega_{mn}^2}} > c$$

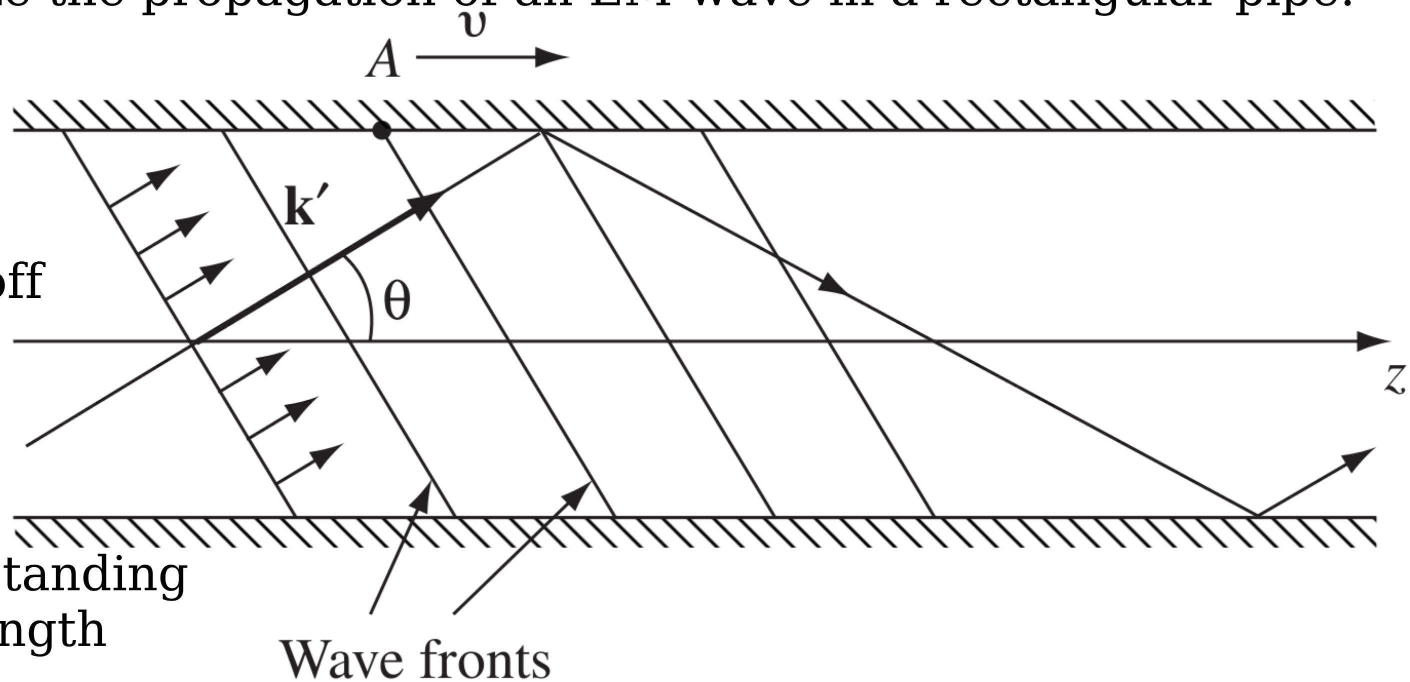


- However, the energy carried by the wave travels at the *group* velocity:

$$v_g = \frac{d\,\omega}{d\,k_z} = \frac{1}{d\,k_z / d\,\omega} = \frac{c}{\omega}\sqrt{\omega^2 - \omega_{mn}^2} < c$$

- Another way to visualize the propagation of an EM wave in a rectangular pipe.

- Consider an ordinary plane wave, traveling at an angle θ to the z axis, and reflecting perfectly off each conducting surface.



- In the x & y directions, the (multiply reflected) waves interfere to form standing wave patterns, of wavelength

$$\lambda_x = \frac{2a}{m} \text{ and } \lambda_y = \frac{2b}{n} \left(k_x = \frac{2\pi}{\lambda_x} = \frac{m\pi}{a},\ k_y = \frac{2\pi}{\lambda_y} = \frac{n\pi}{b} \right).$$

- In the z direction there remains a traveling wave, with wave number $k_z = k$,

$$\boldsymbol{k}' = \frac{m\pi}{a}\hat{\mathbf{x}} + \frac{n\pi}{b}\hat{\mathbf{y}} + k_z\hat{\mathbf{z}} \quad \Rightarrow \quad \omega = c\,|\boldsymbol{k}'| = c\sqrt{k_z^2 + \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)} = \sqrt{c^2 k_z^2 + \omega_{mn}^2}$$

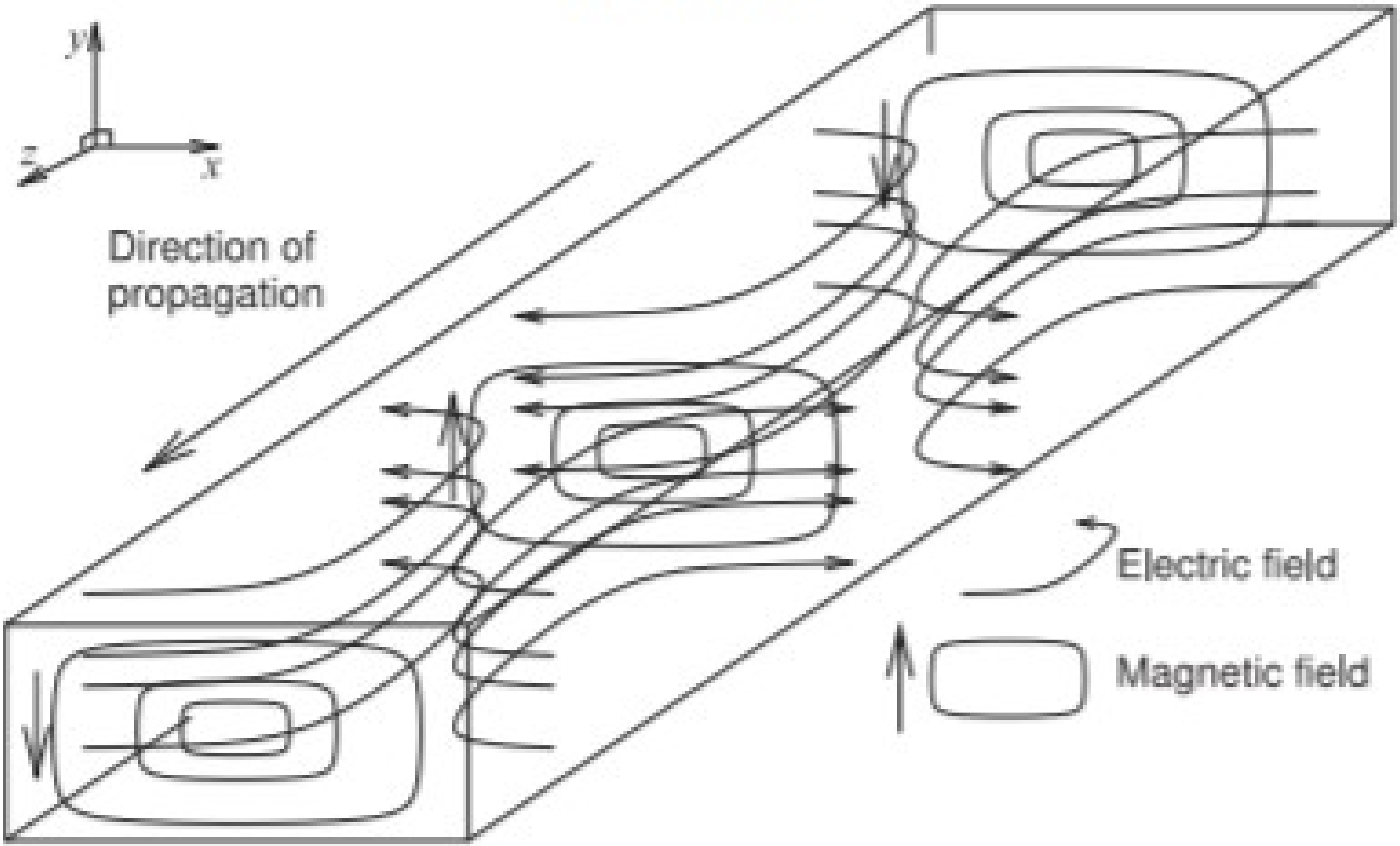
- Only certain angles will lead to one of the allowed standing wave patterns:

$$\cos\theta = \frac{k_z}{|\boldsymbol{k}'|} = \frac{\sqrt{\omega^2 - \omega_{mn}^2}}{\omega}$$

- The plane wave travels at speed c, but because it is going at an angle θ to the z axis, its net velocity down the wave guide is $v_g = c\cos\theta = \frac{c}{\omega}\sqrt{\omega^2 - \omega_{mn}^2}$

- The *wave* (*phase*) velocity is the speed of the wave fronts down the pipe. Like the intersection of a line of breakers with the beach, they can move much faster than the waves themselves—in fact $v_p = \frac{c}{\cos\theta} = \frac{\omega c}{\sqrt{\omega^2 - \omega_{mn}^2}}$

TM$_{11}$ Mode



y
z
x
Direction of propagation
Electric field
Magnetic field

Transverse Electric (TE) Waves

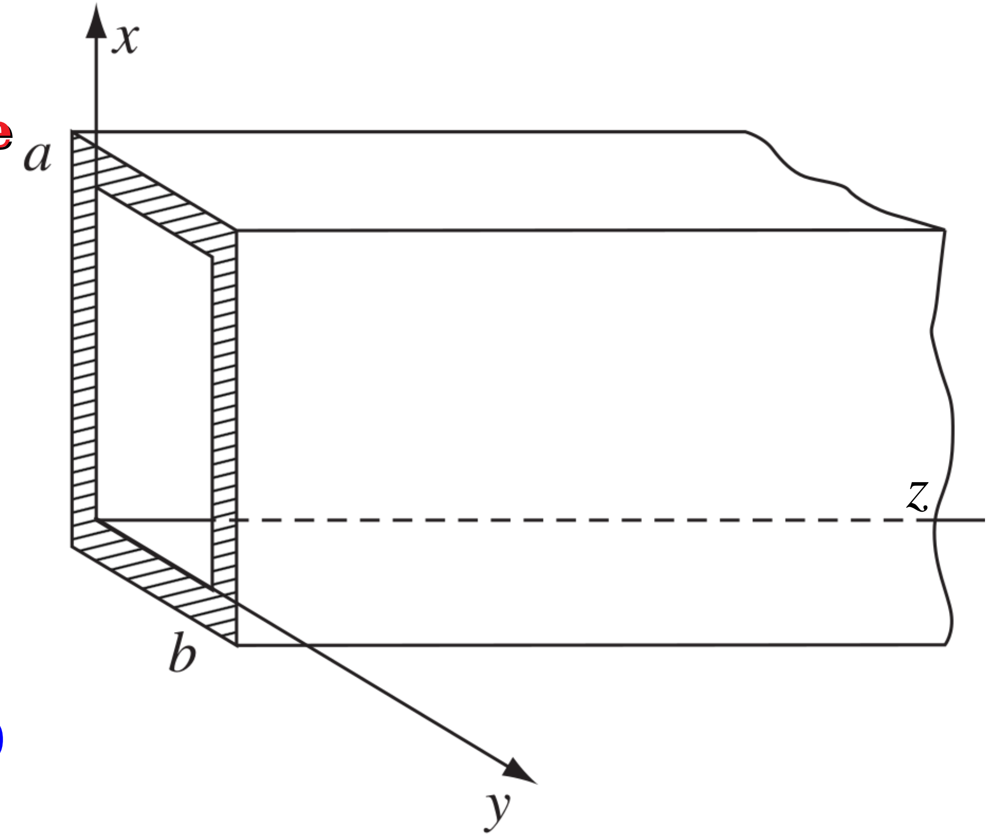
- TE waves do not have a component of the electric field in the direction of propagation, $E_z=0$. The behavior of TE waves can be analyzed, subject to the boundary conditions of the guide, by solving $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\omega^2}{c^2}-k^2\right)B_z=0$

$$\Rightarrow \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \frac{i\,c^2\,\omega}{\omega^2-c^2k^2}\begin{bmatrix} \frac{\partial}{\partial y} \\ -\frac{\partial}{\partial x} \end{bmatrix} B_z\,, \quad \begin{bmatrix} B_x \\ B_y \end{bmatrix} = \frac{i\,c^2\,k}{\omega^2-c^2k^2}\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} B_z$$

$$\Rightarrow \mathbf{B}_t \equiv B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} = \frac{i\,c^2\,k}{\omega^2-c^2k^2}\nabla_t B_z\,, \quad \mathbf{E} = -\frac{\omega}{k}\,\hat{\mathbf{z}}\times\mathbf{B}$$

- TE waves follow the same rules for the propagation modes as TM waves do.

TE Waves in a Rectangular Wave Guide



- The TE wave ($E_z=0$) problem is to solve

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\omega^2}{c^2}-k^2\right)\mathcal{B}_z=0 \quad \text{using} \quad B^{\perp}=0$$

- Do it by separation of variables

$$\mathcal{B}_z = X(x)\,Y(y)$$

$$\Rightarrow \quad Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}+\left(\frac{\omega^2}{c^2}-k^2\right)XY=0$$

$$\Rightarrow \quad \frac{1}{X}\frac{d^2X}{dx^2}=-k_x^2, \quad \frac{1}{Y}\frac{d^2Y}{dy^2}=-k_y^2 \quad \Leftarrow \quad k_x^2+k_y^2=\frac{\omega^2}{c^2}-k^2=\frac{\omega^2}{c^2}-k_z^2$$

$$\Rightarrow \quad X(x)=A\sin k_x x+B\cos k_x x$$

$$\mathcal{B}_{x,\,\text{boundary}}=0 \quad \Rightarrow \quad \frac{dX}{dx}(0)=\frac{dX}{dx}(a)=0 \quad \Rightarrow A=0,\ \ k_x=\frac{m\pi}{a},\ \ m=0,\ 1,\ 2,\cdots$$

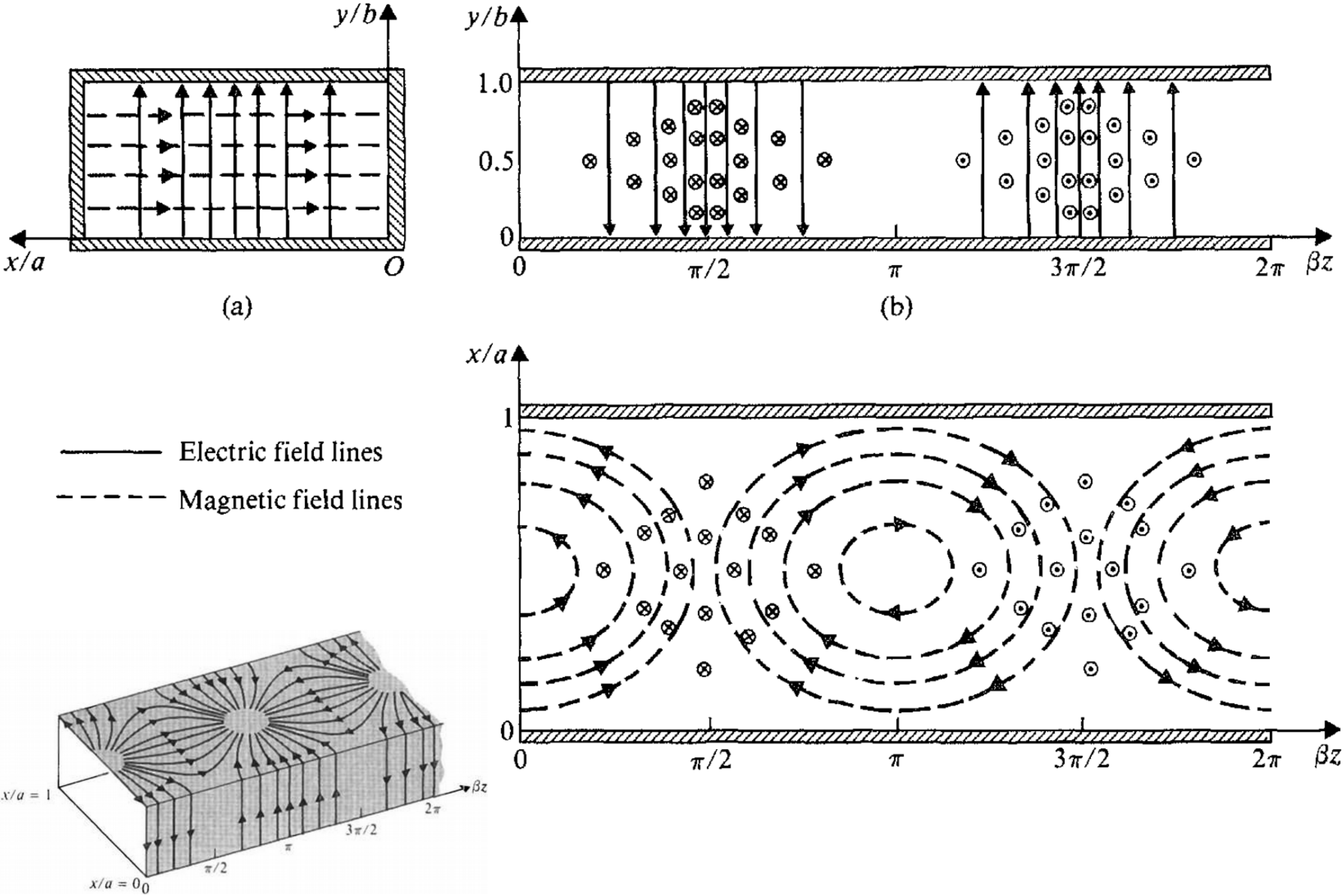
$$\text{similarly} \quad k_y=\frac{n\pi}{b}, \quad n=0,\ 1,\ 2,\cdots \quad \Rightarrow \quad k_z^2=\frac{\omega^2}{c^2}-\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

$$\Rightarrow \quad \mathcal{B}_{z\,m\,n}=B_{0\,m\,n}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \quad \Rightarrow \quad \mathcal{B}_z=\sum_{m,n}B_{0\,m\,n}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

- The other field components are

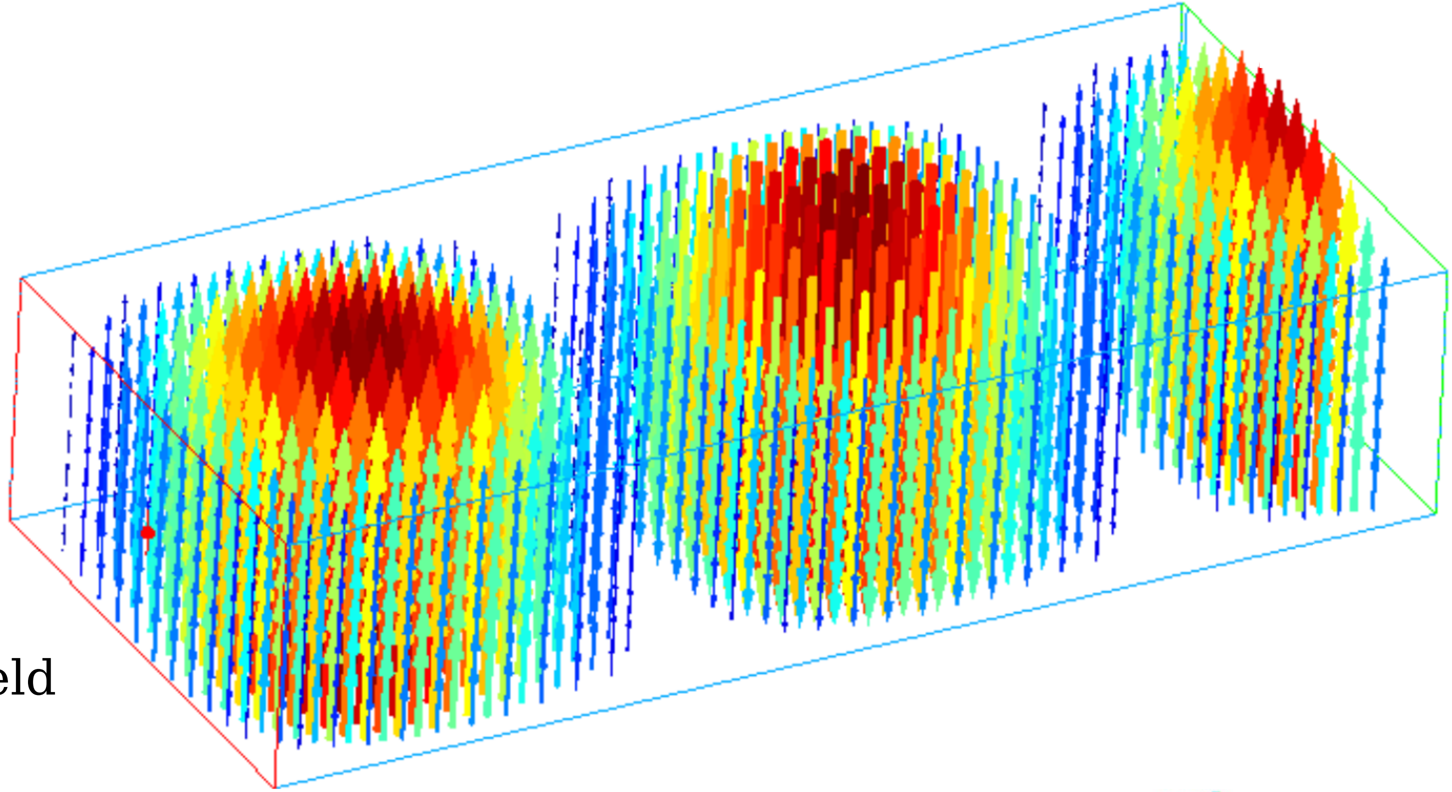
$$\begin{bmatrix} \mathcal{B}_x \\ \mathcal{B}_y \end{bmatrix} = -\sum_{m,n} \frac{i\, c^2 k_z}{\omega^2 - c^2 k_z^2} B_{0\,m\,n} \begin{bmatrix} \frac{m\,\pi}{a} \sin\frac{m\,\pi\,x}{a} \cos\frac{n\,\pi\,y}{b} \\ \frac{n\,\pi}{b} \cos\frac{m\,\pi\,x}{a} \sin\frac{n\,\pi\,y}{b} \end{bmatrix}, \quad \mathbf{E} = -\frac{\omega}{k_z} \hat{\mathbf{z}} \times \mathbf{B}$$

- The solution is called the $\mathrm{TE}_{m\,n}$ mode, assuming $a \geq b$. At least *one* of the indices must be nonzero.
- If $\omega < c\,\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \equiv \omega_{m\,n}$, the wave number k_z is imaginary, and instead of a traveling wave we have exponentially attenuated fields. For this reason, $\omega_{m\,n}$ is called the **cutoff frequency** for the mode in question.
- The **cutoff wavelength**: $\lambda_{m\,n} = \frac{2\,\pi\,c}{\omega_{m\,n}} = \frac{2\,a\,b}{\sqrt{b^2 m^2 + a^2 n^2}}$
- The *lowest* cutoff frequency for a given wave guide occurs for the mode TE_{10}: $\omega_{10} = \frac{c\,\pi}{a}$. Frequencies less than this will not propagate at all.
- The velocity analysis is the same as in the TM modes.

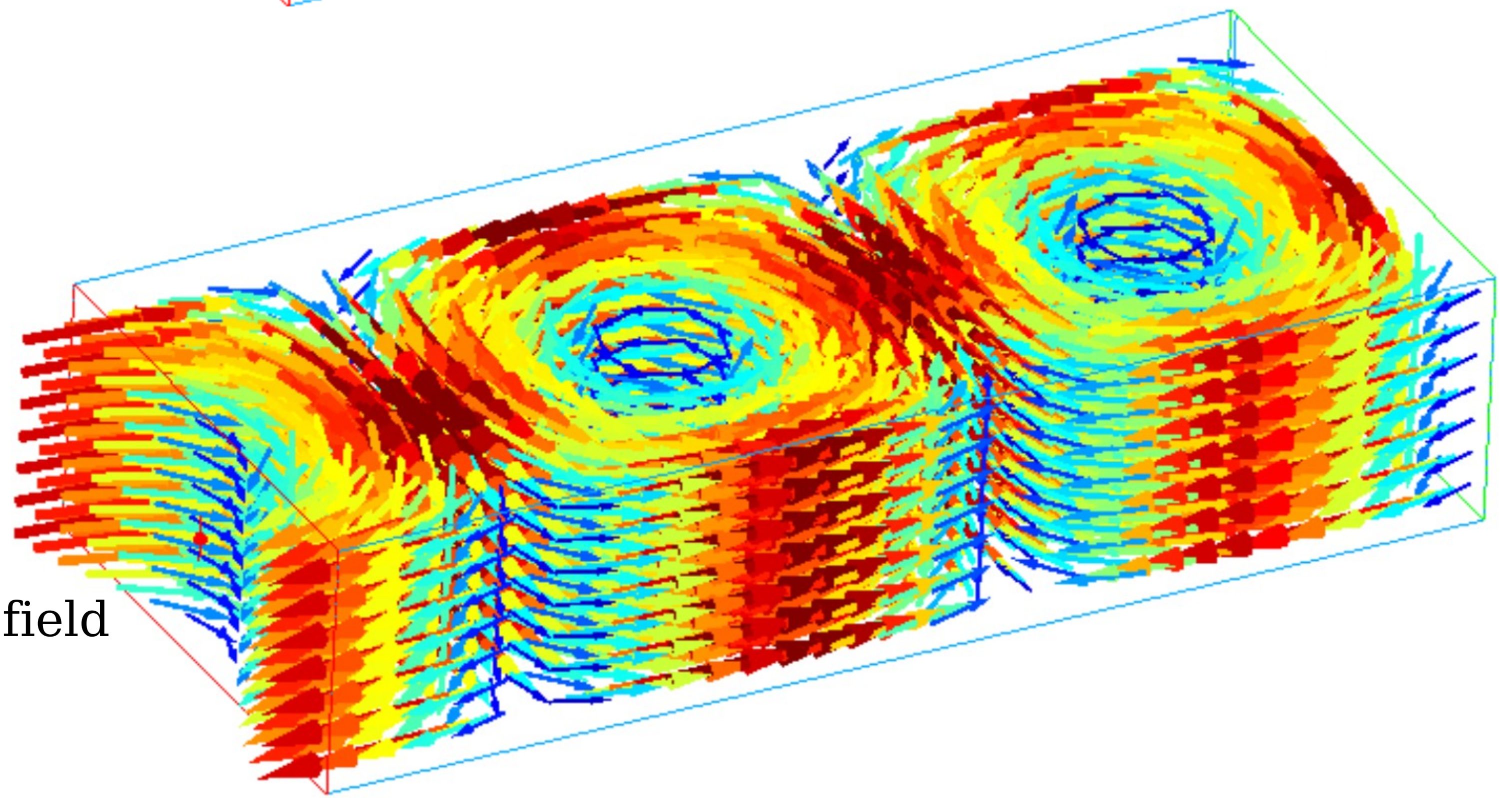


y/b
x/a
O
(a)
y/b
1.0
0.5
0
0
π/2
π
3π/2
2π
βz
(b)
Electric field lines
Magnetic field lines
x/a
1
0
0
π/2
π
3π/2
2π
βz
x/a = 1
x/a = 0
0
π/2
π
3π/2
2π
βz

TE_{10} mode



Electric field



Magnetic field

Example: Standard air-filled waveguides have been designed for the radar bands (300MHz–300 GHz). One type, designated WG-16, is suitable for X-band (8GHz-12.4GHz) applications. Its dimensions are: a=2.29cm and b=1.02cm. If it is desired that a WG-16 waveguide operate only in the dominant TE_{10} mode and that the operating frequency be at least 25% above the cutoff frequency of the TE_{10} mode but no higher than 95% of the next higher cutoff frequency, what is the allowable operating-frequency range?

- $$f_{mn} = \frac{\omega_{mn}}{2\pi} = \frac{c}{2}\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \quad \Rightarrow \quad f_{10} = \frac{c}{2a} = 6.55 \times 10^9 \text{ Hz},\; f_{20} = \frac{c}{a} = 13.1 \times 10^9 \text{ Hz}$$
  $$f_{11} = \frac{c}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = 16.1 \times 10^9 \text{ Hz}$$
- Thus the allowable operating-frequency range under the specified conditions is

$$1.25 f_{10} < f < 0.95 f_{20} \quad \Rightarrow \quad 8.19 \text{ GHz} < f < 12.45 \text{ GHz}$$