Chapter 8 Conservation Laws

Charge and Energy

The Continuity Equation

- **Global** conservation of charge: The total charge in the universe is constant.

- **Local** conservation of charge: If the charge in some region changes, then exactly that amount of charge must have passed in or out through the surface.

- The charge in a volume $\mathcal{V}$ is $Q(t)=\int_{\mathcal{V}}\rho(\mathbf{r},t)\,d\tau$ and the current flowing out through the boundary $\mathcal{S}$ is $\oint_{\mathcal{S}}\boldsymbol{J}\cdot d\boldsymbol{a}$, so the local conservation of charge gives

$$\frac{dQ}{dt}=-\oint_{\mathcal{S}}\boldsymbol{J}\cdot d\boldsymbol{a}\;\Rightarrow\;\int_{\mathcal{V}}\frac{\partial\rho}{\partial t}d\tau=-\int_{\mathcal{V}}\nabla\cdot\boldsymbol{J}\,d\tau$$

$$\Rightarrow\;\boxed{\frac{\partial\rho}{\partial t}=-\nabla\cdot\boldsymbol{J}}\quad\text{continuity equation}$$

- It can be derived from Maxwell's equations—conservation of charge is not an *independent* assumption; it is built into the laws of electrodynamics.

- It serves as a constraint on the sources (ρ and $\boldsymbol{J}$).

Poynting's Theorem

- The work necessary to assemble a static charge distribution and currents going

$$W_e = \frac{\epsilon_0}{2}\int E^2\, d\tau\,, \quad W_m = \frac{1}{2\mu_0}\int B^2\, d\tau$$

- The total energy stored in EM fields, per unit volume, is $u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right)$

- Having some charge and current configuration which, at t, produces $\mathbf{E}$ & $\mathbf{B}$. In dt, the charges move around a bit. How much work, dW, is done by the EM forces acting on these charges, in $d\,t$?

- According to the Lorentz force law, the work done on a charge q is

$$\mathbf{F}\cdot d\boldsymbol{\ell} = q\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot\mathbf{v}\,d\,t = q\,\mathbf{E}\cdot\mathbf{v}\,d\,t \rightarrow \rho\,d\,\tau\,\mathbf{E}\cdot\mathbf{v}\,d\,t \rightarrow \mathbf{E}\cdot\boldsymbol{J}\,d\,t\,d\,\tau \quad \Leftarrow \quad \begin{matrix} q \rightarrow \rho\,d\,\tau \\ \rho\,\mathbf{v} \rightarrow \boldsymbol{J} \end{matrix} \quad \Rightarrow \quad \frac{d\,W}{d\,t} = \int_{\mathcal{V}} \mathbf{E}\cdot\boldsymbol{J}\,d\,\tau$$

- $\mathbf{E}\cdot\boldsymbol{J}$ is the work done/unit time/unit volume, ie, the *power* delivered/unit volume

- We can express this quantity in terms of the fields alone, using the Ampère-Maxwell law to eliminate $\boldsymbol{J}$: $\mathbf{E}\cdot\boldsymbol{J} = \mathbf{E}\cdot\frac{\nabla\times\mathbf{B}}{\mu_0} - \epsilon_0\,\mathbf{E}\cdot\frac{\partial\,\mathbf{E}}{\partial\,t}$

$$\nabla\cdot\left(\mathbf{E}\times\mathbf{B}\right) = \mathbf{B}\cdot\left(\nabla\times\mathbf{E}\right) - \mathbf{E}\cdot\left(\nabla\times\mathbf{B}\right) \quad \Rightarrow \quad \mathbf{E}\cdot\left(\nabla\times\mathbf{B}\right) = \mathbf{B}\cdot\left(-\frac{\partial\,\mathbf{B}}{\partial\,t}\right) - \nabla\cdot\left(\mathbf{E}\times\mathbf{B}\right)$$

$$\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}=\frac{1}{2}\frac{\partial B^2}{\partial t}$$
$$\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}=\frac{1}{2}\frac{\partial E^2}{\partial t}$$
$$\Rightarrow\quad \mathbf{E}\cdot\boldsymbol{J}=-\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2+\frac{B^2}{\mu_0}\right)-\frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B})$$
$$\Rightarrow\quad \frac{\mathrm{d}W}{\mathrm{d}t}=-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}}\frac{1}{2}\left(\epsilon_0 E^2+\frac{B^2}{\mu_0}\right)\mathrm{d}\tau-\frac{1}{\mu_0}\oint_{\mathcal{S}}(\mathbf{E}\times\mathbf{B})\cdot\mathrm{d}\boldsymbol{a}$$

Poynting's theorem: the work-energy theorem of electrodynamics

- The 1st integral is the total energy stored in the fields, $\int u\,\mathrm{d}\tau$. The 2nd term represents the rate at which energy is transported out of $\mathcal{V}$, across its boundary surface, by the EM fields.

- Poynting's theorem says that *the work done on the charges by the EM force is equal to the decrease in energy remaining in the fields, less the energy that flowed out through the surface*.

- The *energy per unit time, per unit area,* transported by the fields is called the **Poynting vector**: $\mathbf{S}\equiv\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}$

- $\mathbf{S}\cdot d\boldsymbol{a}$ is the energy/unit time crossing the infinitesimal surface $d\boldsymbol{a}$—the energy *flux* (so $\mathbf{S}$ is the **energy flux density**) $\Rightarrow \dfrac{dW}{dt} = -\dfrac{d}{dt}\int_{\mathcal{V}} u\, d\tau - \oint_{\mathcal{S}} \mathbf{S}\cdot d\boldsymbol{a}$

- $\dfrac{dW}{dt} = \int_{\mathcal{V}} \boldsymbol{J}\cdot\mathbf{E}\, d\tau, \quad \dfrac{d}{dt}\int_{\mathcal{V}} u\, d\tau = \int_{\mathcal{V}} \dfrac{\partial u}{\partial t} d\tau, \quad \oint_{\mathcal{S}} \mathbf{S}\cdot d\boldsymbol{a} = \int_{\mathcal{V}} \nabla\cdot\mathbf{S}\, d\tau$

$$\Rightarrow \int_{\mathcal{V}} \left(\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} + \boldsymbol{J}\cdot\mathbf{E}\right) d\tau \quad \Rightarrow \quad \boldsymbol{J}\cdot\mathbf{E} = -\frac{\partial u}{\partial t} - \nabla\cdot\mathbf{S}$$

The *differential* form of the *Poynting theorem*

- If no work is done on the charges in $\mathcal{V}$, or no charge in $\mathcal{V}$

$$\frac{dW}{dt} = 0 \quad \Rightarrow \quad \int \frac{\partial u}{\partial t} d\tau = -\oint \mathbf{S}\cdot d\boldsymbol{a} = -\int \nabla\cdot\mathbf{S}\, d\tau \quad \Rightarrow \quad \frac{\partial u}{\partial t} = -\nabla\cdot\mathbf{S}$$

- This is the “continuity equation” for energy—u (energy density) plays the role of ρ (charge density), and $\mathbf{S}$ takes the part of $\boldsymbol{J}$ (current density). It expresses local conservation of EM energy.

- In general, EM energy by itself is not conserved (nor is the energy of the charges). The fields do work on the charges, and the charges create fields—energy is tossed back and forth between them. In the overall energy economy, you must include the contributions of both the matter and the fields.

- $$\boldsymbol{J}\cdot\mathbf{E}=\left(\nabla\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}\right)\cdot\mathbf{E}=\mathbf{E}\cdot\nabla\times\mathbf{H}-\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}$$

$$=-\nabla\cdot(\mathbf{E}\times\mathbf{H})+\mathbf{H}\cdot\nabla\times\mathbf{E}-\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}=-\nabla\cdot(\mathbf{E}\times\mathbf{H})-\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}-\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}$$

$$\Rightarrow\quad\mathbf{S}=\mathbf{E}\times\mathbf{H}\quad\text{Poynting's vector}\ ,\qquad\frac{\partial u}{\partial t}=-\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}-\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}$$

- For linear materials, $\mathbf{B}=\mu\,\mathbf{H}\ ,\quad\mathbf{D}=\epsilon\,\mathbf{E}$

$$\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}=\frac{\mathbf{B}}{\mu}\cdot\frac{\partial\mathbf{B}}{\partial t}=\frac{1}{2\mu}\frac{\partial B^2}{\partial t}=\frac{\partial u_B}{\partial t}\quad\Leftarrow\quad u_B=\frac{B^2}{2\mu}$$

$$\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}=\epsilon\,\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}=\frac{\epsilon}{2}\frac{\partial E^2}{\partial t}=\frac{\partial u_E}{\partial t}\quad\Leftarrow\quad u_E=\frac{\epsilon}{2}E^2$$

$$\Rightarrow\quad u=u_B+u_E=\frac{\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}}{2}$$

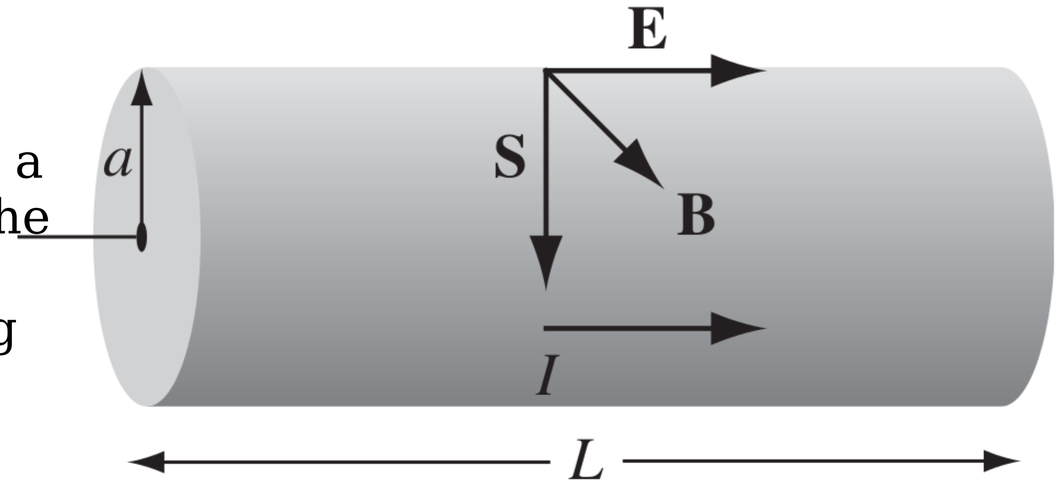
- More generally in nonlinear media, we simply assume

$$\frac{\partial u_E}{\partial t}=\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\quad\Rightarrow\quad u_E=\int_{0,0}^{\mathbf{E},t}\frac{\partial u_E}{\partial t}\,\mathrm{d}t=\int_{0,0}^{\mathbf{E},t}\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\,\mathrm{d}t=\int_0^{\mathbf{E}}\mathbf{E}\cdot\mathrm{d}\mathbf{D}$$

$$\frac{\partial u_B}{\partial t}=\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\quad\Rightarrow\quad u_B=\int_{0,0}^{\mathbf{B},t}\frac{\partial u_B}{\partial t}\,\mathrm{d}t=\int_{0,0}^{\mathbf{B},t}\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\,\mathrm{d}t=\int_0^{\mathbf{B}}\mathbf{H}\cdot\mathrm{d}\mathbf{B}$$

where $\mathbf{D}=\mathbf{D}(\mathbf{E},t)\ ,\quad\mathbf{H}=\mathbf{H}(\mathbf{B},t)$

Example 8.1: When current flows down a wire, work is done as Joule heating of the wire. Calculate the energy/unit time delivered to the wire using the Poynting vector.



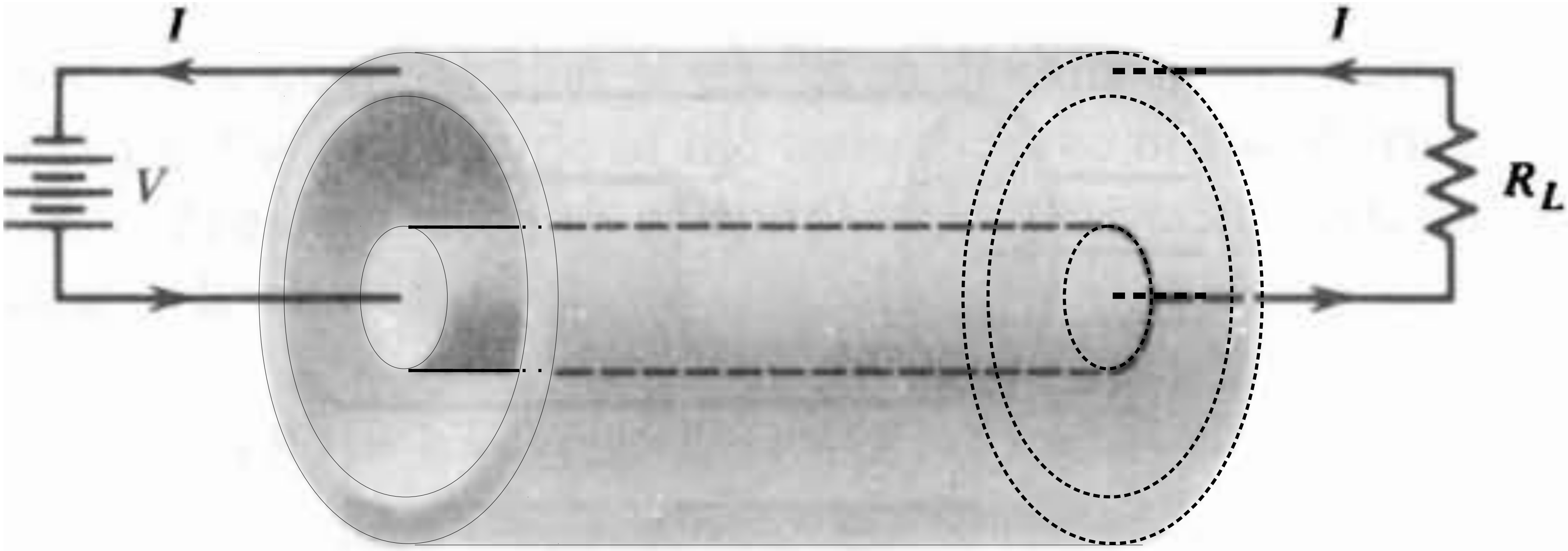
- Assuming the electric field is uniform, $E = \frac{V}{L}$
- The magnetic field is "circumferential" at the surface:

$$B(r=a) = \frac{\mu_0 I}{2\pi a} \quad \Rightarrow \quad S = \frac{1}{\mu_0}\frac{V}{L}\frac{\mu_0 I}{2\pi a} = \frac{V I}{2\pi a L} \quad \text{radially inward}$$

- The energy per unit time passing in through the surface of the wire is

$$\int \mathbf{S}\cdot d\boldsymbol{a} = S\cdot 2\pi a L = V I \quad \text{exactly what we expect}$$

- Since the fields are steady, then $\frac{\partial u}{\partial t} = 0$; hence the conservation of energy expressed by the Poynting's theorem asserts that $\int_{\mathcal{V}} \boldsymbol{J}\cdot\mathbf{E}\, d\tau = -\oint_{\mathcal{S}} \mathbf{S}\cdot d\boldsymbol{a}$.



Example: Consider that a coaxial cable, of radii a (inner) & b (outer), is inserted between a source of constant emf and some load, a steady current I flows down the cable. If the emf provides a constant potential difference V, it will supply power to the cable of magnitude VI. Calculate the rate at which energy passes down the cable.

$$\mathbf{E} = \frac{V}{\ln(b/a)} \frac{\hat{\mathbf{s}}}{s}, \quad \mathbf{B} = \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}}, \quad a \le s \le b \quad \Rightarrow \quad \mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} = \frac{V I}{2\pi \ln(b/a)} \frac{\hat{\mathbf{z}}}{s^2}$$

$$\Rightarrow \int_S \mathbf{S} \cdot \mathrm{d}\boldsymbol{a} = \int_0^{2\pi} \mathrm{d}\phi \int_a^b \frac{V I}{2\pi \ln(b/a)} \frac{s\,\mathrm{d}s}{s^2} = V I$$

- In practice, the conductors of the cable will have a finite resistance, so that energy will also be dissipated as heat in them.

- Assume the charge on the inner cable is $Q = \lambda\,\ell$. Using the Gauss theorem

$$\epsilon_0 \int_0^{\ell} \mathrm{d}\,z \int_0^{2\pi} E\, s\,\mathrm{d}\,\phi = 2\,\pi\,\epsilon_0\,\ell\,s\,E = \int_0^{\ell} \lambda\,\mathrm{d}\,z = \lambda\,\ell \quad\Rightarrow\quad \mathbf{E} = \frac{\lambda}{2\,\pi\,\epsilon_0}\,\frac{\hat{\mathbf{s}}}{s}$$

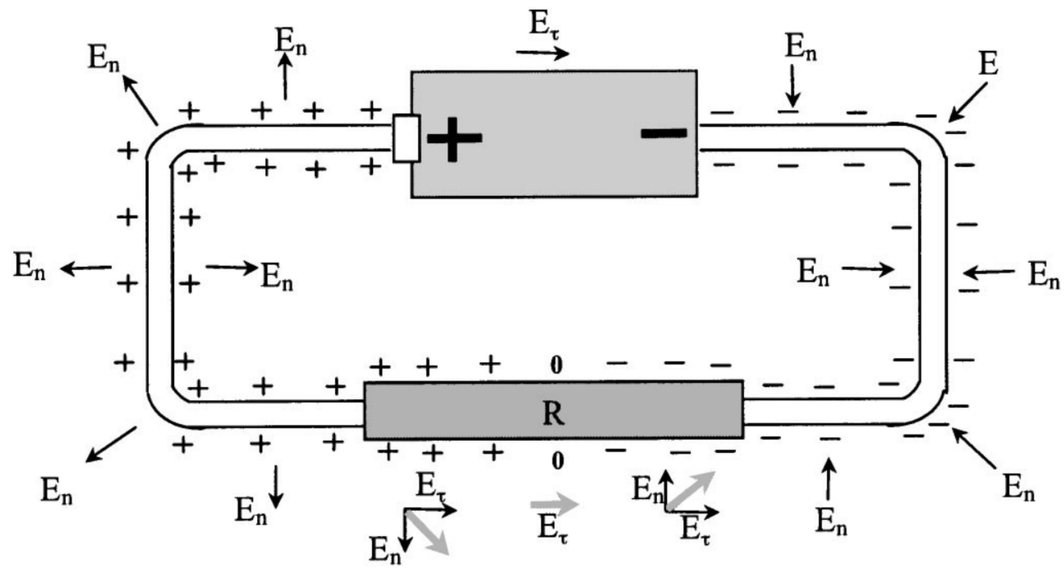
$$\Rightarrow\quad V_b - V_a = 0 - V = -\int_a^b \mathbf{E}\cdot\mathrm{d}\,\mathbf{s} = -\frac{\lambda}{2\,\pi\,\epsilon_0}\int_a^b \frac{\mathrm{d}\,s}{s}$$

$$\Rightarrow\quad \frac{\lambda}{2\,\pi\,\epsilon_0}\ln\frac{b}{a} = V \quad\Rightarrow\quad \lambda = \frac{2\,\pi\,\epsilon_0\,V}{\ln b - \ln a} \quad\Rightarrow\quad \mathbf{E} = \frac{V}{\ln b - \ln a}\,\frac{\hat{\mathbf{s}}}{s}$$

- $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\,\mathbf{B} + (\mathbf{B}\cdot\nabla)\,\mathbf{A}$

  used in Page 13

- The *real* behavior of $\mathbf{E}$ in a wire is the combined result of the 2 examples.
- No way an electric field only exists inside a wire without surface charge, or it violates the Maxwell equations $\nabla\times\mathbf{E}=0$ (in a steady case).

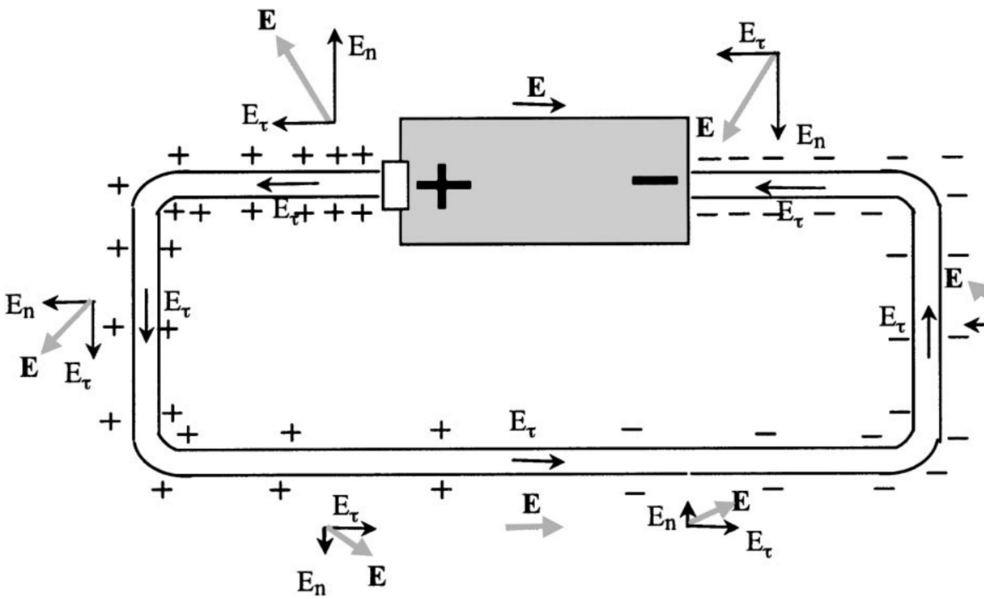


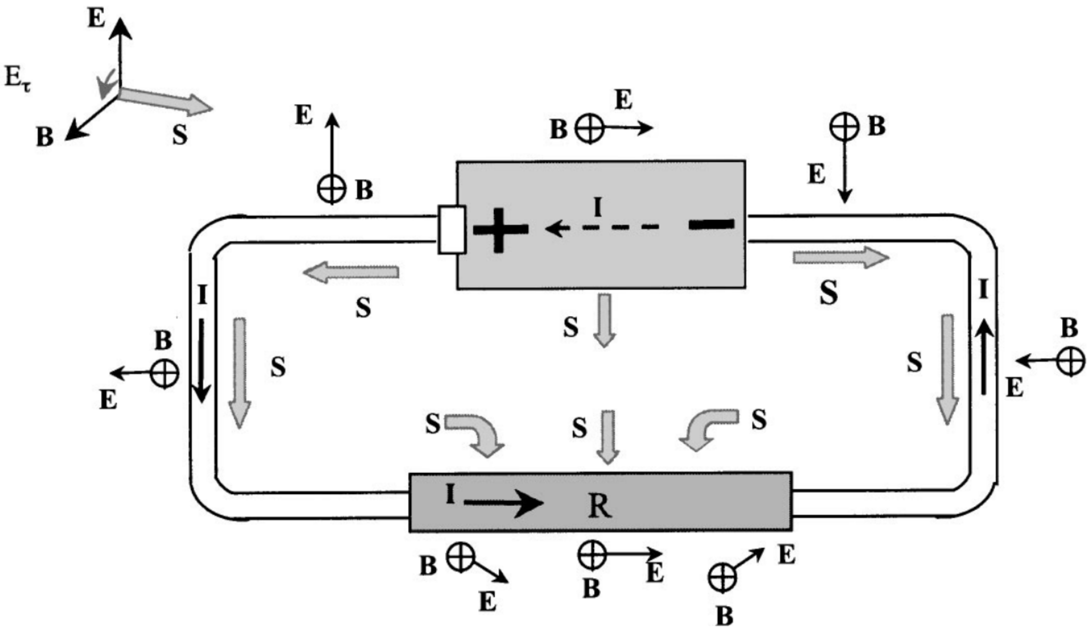
- In general $\mathbf{E}=E_\perp\,\hat{\mathbf{s}}+E_\parallel\,\hat{\mathbf{z}}$

$$\Rightarrow \quad E_\perp\rightarrow E_n\,,\quad E_\parallel\rightarrow E_t$$

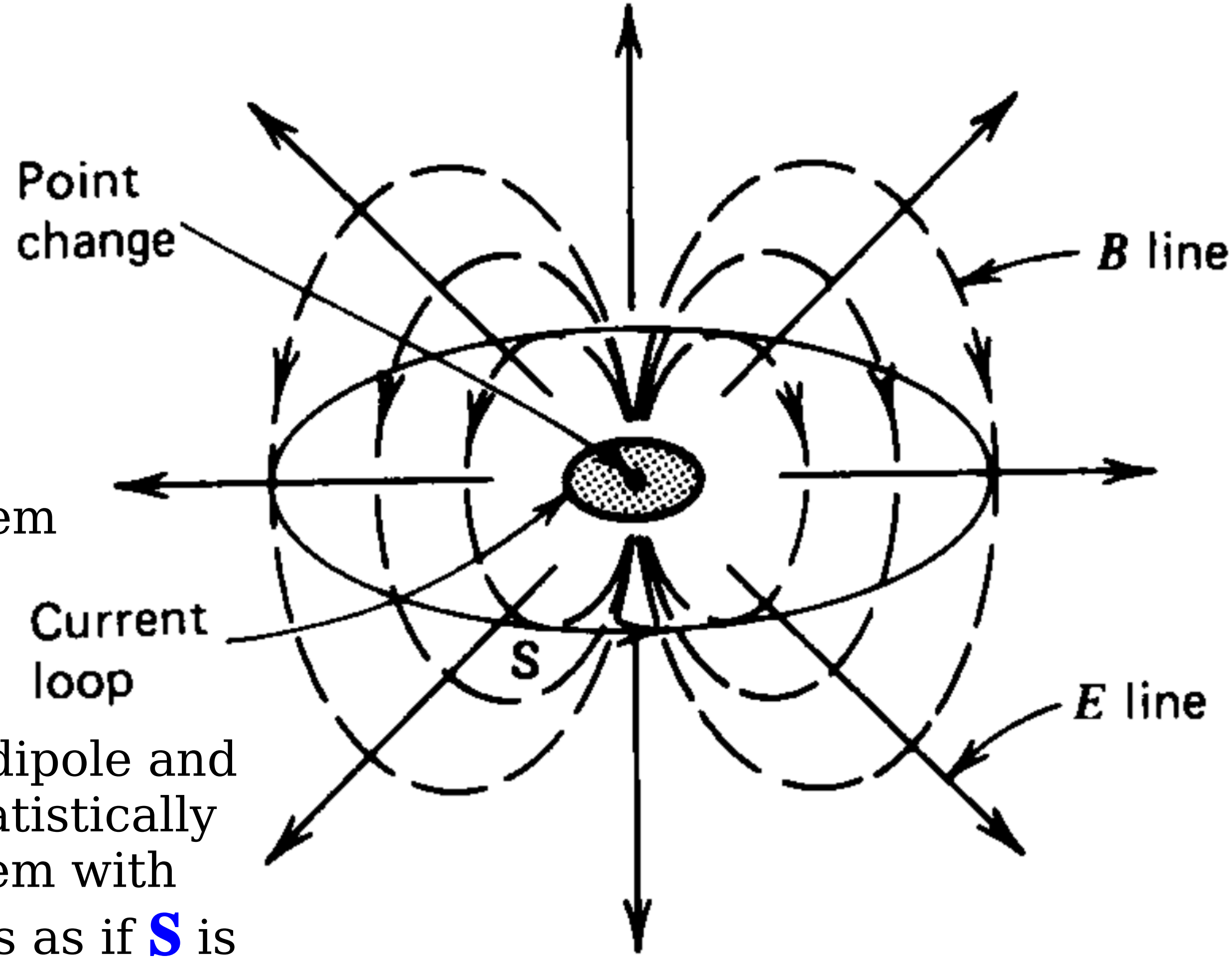
- In a ideal conducting wire

$$\sigma\rightarrow\infty \quad\Rightarrow\quad E_\parallel=0 \quad\Leftarrow\quad J_\parallel=\sigma\,E_\parallel$$





- The interpretation of the Poynting vector as giving the flow of energy density has peculiar effects, especially in static problems, that cannot be resolved.



- The fact that Ponyting's theorem is true does not guarantee that **S** really is an energy flow.

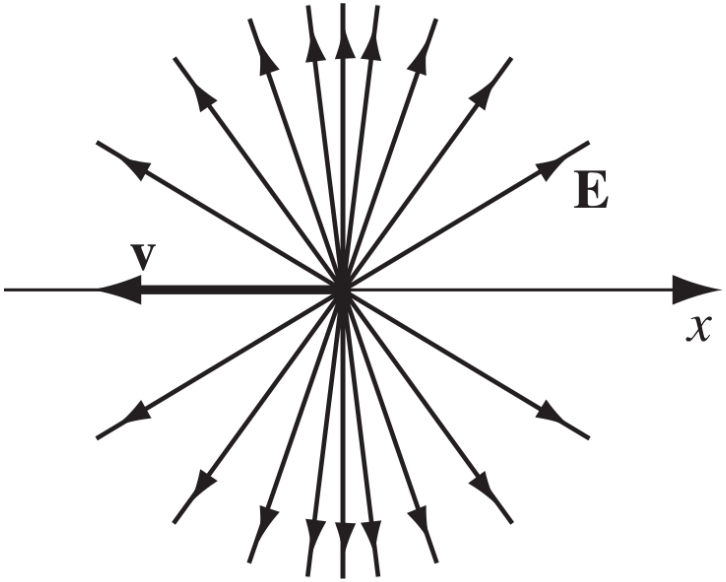
- In the example of a magnetic dipole and a point charge superimposed statistically in space, which is a static problem with constant **E** and **B** fields, it seems as if **S** is flowing around the symmetry axis of the dipole.

- It is hard to believe this is happening, and we cannot verify it experimentally. What is clear is that through any sphere containing the dipole, the integrated energy flow is 0.

- The Poynting's theorem is mainly for nonstatic problems, especially where one wishes to calculate the EM radiation flowing from some energy source.

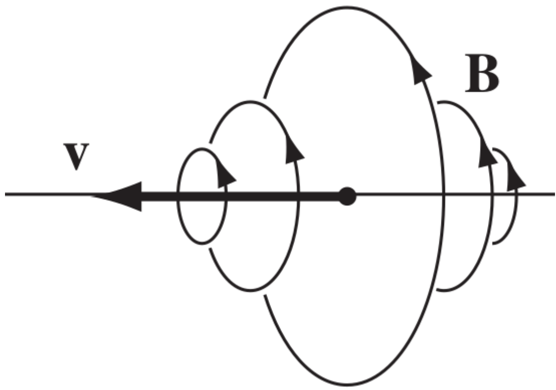
Momentum

Newton's 3rd Law in Electrodynamics

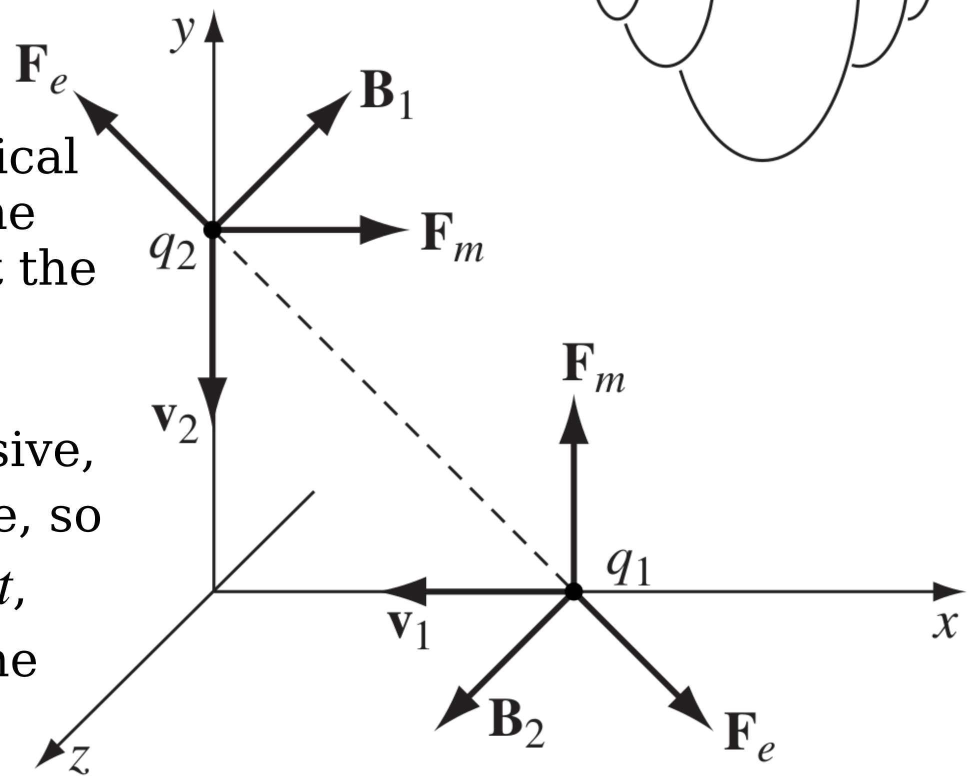


- For a point charge q traveling in along the x axis at a constant speed v, its electric field is not given by Coulomb's law due to its being moving.

- But **E** still points radially outward from the instantaneous position of the charge.



- Since a moving point charge does not constitute a steady current, its magnetic field is not given by the Biot-Savart law. However, **B** still circles around the axis in a manner suggested by the right-hand rule.



- Suppose this charge encounters an identical one, both mounted on tracks to maintain the same direction and speed, proceeding in at the same speed along the y axis.

- The electric force between them is repulsive, the magnetic field of q_1 points into the page, so the magnetic force on q_2 is toward the *right*, whereas the magnetic field of q_2 is *out* of the page, the magnetic force on q_1 is *upward*.

- *The net electromagnetic force of q_1 on q_2 is equal but not opposite to the force of q_2 on q_1, in violation of Newton's 3rd law.* In electro*statics* and magneto*statics* the 3rd law holds, but in electro*dynamics* it does *not*.

- People use the 3rd law all the time. The proof of momentum conservation rests on the cancellation of internal forces, which follows from the 3rd law. When you tamper with the 3rd law, you are placing conservation of momentum in jeopardy.

- Momentum conservation is rescued in electrodynamics by the realization that the *fields themselves carry momentum*, since we have already attributed energy to the fields.

- Whatever momentum is lost to the particles is gained by the fields. Only when the field momentum is added to the mechanical momentum is momentum conservation restored.

$$\text{Identity tensor } \mathbb{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \sum_{i,j=1}^{3} \delta_{ij} \, \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$$

Maxwell's Stress Tensor

- Calculate the total EM force on the charges in volume $\mathcal{V}$:

$$\mathbf{F} = \int_{\mathcal{V}} \rho\,(\mathbf{E} + \mathbf{v}\times\mathbf{B})\,d\tau = \int_{\mathcal{V}} (\rho\,\mathbf{E} + \boldsymbol{J}\times\mathbf{B})\,d\tau = \int_{\mathcal{V}} \boldsymbol{f}\,d\tau \quad \Leftarrow \quad \boldsymbol{f}: \text{force per unit volume}$$

$$\Rightarrow \quad \boldsymbol{f} = \rho\,\mathbf{E} + \boldsymbol{J}\times\mathbf{B} = \epsilon_0 (\nabla\cdot\mathbf{E})\,\mathbf{E} + \left(\frac{1}{\mu_0}\nabla\times\mathbf{B} - \epsilon_0 \frac{\partial\,\mathbf{E}}{\partial\,t}\right)\times\mathbf{B}$$

$$\partial_t (\mathbf{E}\times\mathbf{B}) = \partial_t\,\mathbf{E}\times\mathbf{B} + \mathbf{E}\times\partial_t\,\mathbf{B} \quad + \quad \text{Faraday's law} \quad \partial_t\,\mathbf{B} = -\nabla\times\mathbf{E}$$

$$\Rightarrow \quad \frac{\partial\,\mathbf{E}}{\partial\,t}\times\mathbf{B} = \frac{\partial}{\partial\,t}(\mathbf{E}\times\mathbf{B}) + \mathbf{E}\times(\nabla\times\mathbf{E})$$

$$\Rightarrow \quad \boldsymbol{f} = \epsilon_0 [(\nabla\cdot\mathbf{E})\,\mathbf{E} - \mathbf{E}\times(\nabla\times\mathbf{E})] - \frac{1}{\mu_0}\mathbf{B}\times(\nabla\times\mathbf{B}) - \epsilon_0 \frac{\partial}{\partial\,t}(\mathbf{E}\times\mathbf{B})$$

$$= \epsilon_0 [(\nabla\cdot\mathbf{E})\,\mathbf{E} - \mathbf{E}\times(\nabla\times\mathbf{E})] + \frac{1}{\mu_0}[(\nabla\cdot\mathbf{B})\,\mathbf{B} - \mathbf{B}\times(\nabla\times\mathbf{B})] - \epsilon_0 \frac{\partial}{\partial\,t}(\mathbf{E}\times\mathbf{B})$$

$$\nabla E^2 = \nabla\,\mathbf{E}^2 = 2\,(\mathbf{E}\cdot\nabla)\,\mathbf{E} + 2\,\mathbf{E}\times(\nabla\times\mathbf{E}) \quad \Rightarrow \quad \mathbf{E}\times(\nabla\times\mathbf{E}) = \frac{1}{2}\nabla E^2 - (\mathbf{E}\cdot\nabla)\,\mathbf{E}$$

$$\nabla B^2 = \nabla\,\mathbf{B}^2 = 2\,(\mathbf{B}\cdot\nabla)\,\mathbf{B} + 2\,\mathbf{B}\times(\nabla\times\mathbf{B}) \quad \Rightarrow \quad \mathbf{B}\times(\nabla\times\mathbf{B}) = \frac{1}{2}\nabla B^2 - (\mathbf{B}\cdot\nabla)\,\mathbf{B}$$

$$\Rightarrow \quad \boldsymbol{f} = \epsilon_0 (\nabla\cdot\mathbf{E} + \mathbf{E}\cdot\nabla)\,\mathbf{E} + \frac{(\nabla\cdot\mathbf{B} + \mathbf{B}\cdot\nabla)\,\mathbf{B}}{\mu_0} - \nabla\left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\,\mu_0}\right) - \epsilon_0 \frac{\partial}{\partial\,t}(\mathbf{E}\times\mathbf{B})$$

- Introduce the **Maxwell stress tensor,** $\mathbb{T} = \epsilon_0 \mathbf{E} \otimes \mathbf{E} + \frac{\mathbf{B} \otimes \mathbf{B}}{\mu_0} - \frac{\mathbb{I}}{2} \left(\epsilon_0 E^2 + \frac{B^2}{\mu_0} \right)$

$$\mathbb{T}_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right) \Leftarrow \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \quad \text{Kronecker delta}$$

$$\Rightarrow \quad \mathbb{T}_{xx} = \frac{\epsilon_0}{2} \left(E_x^2 - E_y^2 - E_z^2 \right) + \frac{B_x^2 - B_y^2 - B_z^2}{2 \mu_0} \quad \Leftarrow \quad \mathbb{T}_{ij} = \mathbb{T}_{ji} \quad \text{symmetric}$$

$$\mathbb{T}_{xy} = \epsilon_0 E_x E_y + \frac{B_x B_y}{\mu_0}, \quad \cdots \qquad E^2 = E_x^2 + E_y^2 + E_z^2, \quad B^2 = B_x^2 + B_y^2 + B_z^2$$

- Because it carries 2 indices, where a vector has only one, $\mathbb{T}_{ij}$ is sometimes written with: $\mathbb{T}$. One can form the dot product of $\mathbb{T}$ with a vector $\mathbf{a}$, in 2 ways —on the left, and on the right: $(\mathbf{a} \cdot \mathbb{T})_j = \sum_{i=1}^{3} a_i \mathbb{T}_{ij}$, $(\mathbb{T} \cdot \mathbf{a})_j = \sum_{i=1}^{3} \mathbb{T}_{ji} a_i$

The resulting object, which has one remaining index, is itself a vector.

- The divergence of $\mathbb{T}$ has as its j^{th} component

$$(\nabla \cdot \mathbb{T})_j = \epsilon_0 \left((\nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla) E_j - \frac{1}{2} \nabla_j E^2 \right) + \frac{1}{\mu_0} \left((\nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla) B_j - \frac{1}{2} \nabla_j B^2 \right)$$

$$\Rightarrow \quad \boldsymbol{f} = \nabla \cdot \mathbb{T} - \epsilon_0 \mu_0 \frac{\partial \mathbf{S}}{\partial t} \quad \Rightarrow \quad \text{total EM force} \quad \mathbf{F} = \oint_S \mathbb{T} \cdot \mathrm{d}\boldsymbol{a} - \epsilon_0 \mu_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_V \mathbf{S} \, \mathrm{d}\tau$$

- In the static case the 2nd term drops out, and the EM force on the charge configuration can be expressed entirely with the stress tensor at the boundary:

$$\mathbf{F} = \oint_{\mathcal{S}} \mathbb{T} \cdot \mathrm{d}\,\boldsymbol{a} \quad \text{static} \qquad (\#)$$

- $\mathbb{T}$ is the force per unit area (or **stress**) acting on the surface. $\mathbb{T}_{ij}$ is the force (per unit area) in the i^{th} direction acting on an element of surface oriented in the j^{th} direction—“diagonal” elements ($\mathbb{T}_{xx}$, $\mathbb{T}_{yy}$, $\mathbb{T}_{zz}$) represent *pressures*, and “off-diagonal” elements ($\mathbb{T}_{xy}$, $\mathbb{T}_{xz}$, etc.) are *shears*.

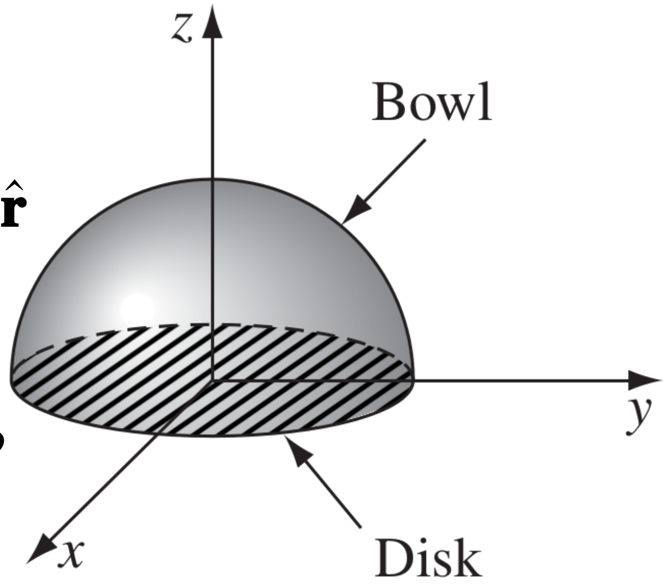
- Let us extend the region to include all space. Since all the sources & are within a finite region, at large distance r, we have, at worst, $E \sim \dfrac{1}{r^2}$ and $B \sim \dfrac{1}{r^3}$, thus $\mathbb{T} \sim \dfrac{1}{r^4}$ at least, while the surface area $\to r^2$ (in the static cases),

$$\int_{\text{all space}} \left(\boldsymbol{f} + \epsilon_0 \mu_0 \frac{\partial \mathbf{S}}{\partial t} \right) \mathrm{d}\,\tau = \oint_{\text{all space}} \mathbb{T} \cdot \mathrm{d}\,\boldsymbol{a} \;\propto \frac{1}{r^2} \to 0$$

- We can define the momentum of the EM field from here in the next section.

Example 8.2: Determine the net force on the "northern" hemisphere of a uniformly charged solid sphere of radius R and charge Q.

- The boundary surface consists of 2 parts—a hemispherical bowl at radius R, and a circular disk at $\theta = \dfrac{\pi}{2}$.



- For the bowl, $\mathrm{d}\,\boldsymbol{a} = R^2 \sin\theta \,\mathrm{d}\phi\,\mathrm{d}\theta\,\hat{\mathbf{r}}$, $\quad \mathbf{E} = \dfrac{Q}{4\pi\epsilon_0 R^2}\hat{\mathbf{r}}$
- $\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$

$$\mathbb{T}_{zx} = \epsilon_0 E_z E_x = \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \sin\theta\cos\theta\cos\phi$$

$$\Rightarrow \quad \mathbb{T}_{zy} = \epsilon_0 E_z E_y = \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \sin\theta\cos\theta\sin\phi$$

$$\mathbb{T}_{zz} = \frac{\epsilon_0}{2}\left(E_z^2 - E_x^2 - E_y^2\right) = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \left(\cos^2\theta - \sin^2\theta\right)$$

$$\Rightarrow \quad \left(\mathbb{T}\cdot\mathrm{d}\,\boldsymbol{a}\right)_z = \mathbb{T}_{zx}\,\mathrm{d}a_x + \mathbb{T}_{zy}\,\mathrm{d}a_y + \mathbb{T}_{zz}\,\mathrm{d}a_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 \sin\theta\cos\theta\,\mathrm{d}\phi\,\mathrm{d}\theta$$

$$\Rightarrow \quad F_{\mathrm{bowl}} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 2\pi \int_0^{\pi/2} \sin\theta\cos\theta\,\mathrm{d}\theta = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{8R^2}$$

$$\mathrm{d}\,\boldsymbol{a} = R^2 \sin\theta\,\mathrm{d}\,\phi\,\mathrm{d}\,\theta\,\hat{\mathbf{r}}\,,\quad \mathbf{E} = \frac{Q}{4\,\pi\,\epsilon_0\,R^2}\,\hat{\mathbf{r}}\,,\quad \mathbb{T} = \epsilon_0\,\mathbf{E}\otimes\mathbf{E} + \frac{\mathbf{B}\otimes\mathbf{B}}{\mu_0} - \frac{\mathbb{I}}{2}\left(\epsilon_0\,E^2 + \frac{B^2}{\mu_0}\right)$$

$$\Rightarrow\quad \mathbb{T}\cdot\mathrm{d}\,\boldsymbol{a} = \epsilon_0\left(\mathbf{E}\cdot\mathrm{d}\,\boldsymbol{a}\right)\mathbf{E} - \frac{\epsilon_0}{2}\,E^2\,\mathbb{I}\cdot\mathrm{d}\,\boldsymbol{a}$$

$$= \epsilon_0\left(\frac{Q\sin\theta}{4\,\pi\,\epsilon_0}\,\mathrm{d}\,\theta\,\mathrm{d}\,\phi\right)\frac{Q\,\hat{\mathbf{r}}}{4\,\pi\,\epsilon_0\,R^2} - \frac{\epsilon_0}{2}\left(\frac{Q}{4\,\pi\,\epsilon_0\,R^2}\right)^2 R^2\sin\theta\,\mathrm{d}\,\phi\,\mathrm{d}\,\theta\,\hat{\mathbf{r}}$$

$$= \frac{\epsilon_0}{2}\left(\frac{Q}{4\,\pi\,\epsilon_0\,R}\right)^2 \sin\theta\,\mathrm{d}\,\phi\,\mathrm{d}\,\theta\,\hat{\mathbf{r}}$$

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$

$$\Rightarrow\quad \left(\mathbb{T}\cdot\mathrm{d}\,\boldsymbol{a}\right)_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\,\pi\,\epsilon_0\,R}\right)^2 \sin\theta\cos\theta\,\mathrm{d}\,\phi\,\mathrm{d}\,\theta \quad \Leftarrow \text{ considering the symmetry}$$

$$\Rightarrow\quad F_{\mathrm{bowl}} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\,\pi\,\epsilon_0\,R}\right)^2 2\,\pi\int_0^{\pi/2}\sin\theta\cos\theta\,\mathrm{d}\,\theta = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{Q^2}{8\,R^2}$$

- For the equatorial disk, *inside* the sphere, $\mathbf{r} = r\,\hat{\mathbf{r}}$

$$\theta = \frac{\pi}{2} \;\Rightarrow\; \mathrm{d}\,\boldsymbol{a} = -r\,\mathrm{d}\phi\,\mathrm{d}r\,\hat{\mathbf{z}} \;\Rightarrow\; \mathbf{E} = \frac{Q}{4\pi\epsilon_0 R^3}\mathbf{r} = \frac{Q\,r}{4\pi\epsilon_0 R^3}(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}})$$

$$\Rightarrow\; \mathbb{T}_{zz} = -\frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 r^2 \;\Rightarrow\; (\mathbb{T}\cdot\mathrm{d}\,\boldsymbol{a})_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 r^3\,\mathrm{d}r\,\mathrm{d}\phi$$

$$\Rightarrow\; F_{\text{disk}} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 2\pi\int_0^R r^3\,\mathrm{d}r = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{16R^2} \;\Rightarrow\; F = \frac{1}{4\pi\epsilon_0}\frac{3Q^2}{16R^2}$$

repulsive

- In applying (#), *any* volume that encloses all of the charge in question (and no *other* charge) will do the job.

- In this case we could use the whole region $z>0$. Then the boundary surface consists of the entire xy plane (+ a hemisphere at $r=\infty \Rightarrow E=0 \Rightarrow F_{\text{bowl}}=0$ there).

- We now have the outer portion of the plane ($r>R$), $\mathrm{d}\,\boldsymbol{a} = -r\,\mathrm{d}\phi\,\mathrm{d}r\,\hat{\mathbf{z}}$

$$\mathbb{T}_{zz} = -\frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2\frac{1}{r^4} \;\Rightarrow\; (\mathbb{T}\cdot\mathrm{d}\,\boldsymbol{a})_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2\frac{1}{r^3}\,\mathrm{d}r\,\mathrm{d}\phi$$

$$\Rightarrow\; F_{r>R} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 2\pi\int_R^\infty\frac{\mathrm{d}r}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{8R^2} \;\Rightarrow\; F = \frac{1}{4\pi\epsilon_0}\frac{3Q^2}{16R^2}$$

the same

<u>Alternative</u> (Problem 2.47): Inside the sphere $\mathbf{E} = \dfrac{Q}{4\pi\epsilon_0 R^3}\mathbf{r}$, $\quad \rho = \dfrac{Q}{4\pi R^3/3}$

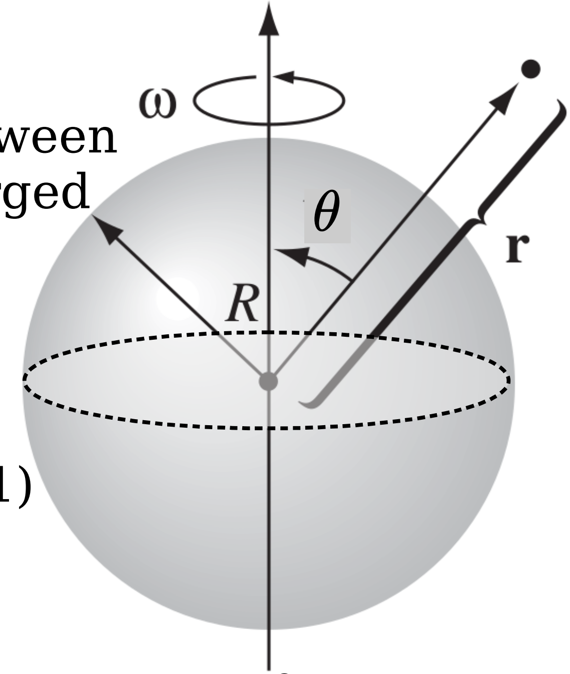
$$\Rightarrow \quad d\mathbf{F} = \mathbf{E}\, dq = \mathbf{E}\,\rho\, d\tau = \frac{Q\,\mathbf{r}}{4\pi\epsilon_0 R^3}\,\frac{3\,Q\,d\tau}{4\pi R^3} = \frac{3}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 \mathbf{r}\, d\tau$$

$$= \frac{3}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 r\left(\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}\right) r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$\Rightarrow \quad \mathbf{F} = \int d\mathbf{F} = \frac{3}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 \hat{\mathbf{z}} \int_0^R r^3\, dr \int_0^{\pi/2} \cos\theta\sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{3}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 \frac{R^4}{4}\,\frac{\sin^2(\pi/2)}{2}\, 2\pi\,\hat{\mathbf{z}} = \frac{3\,Q^2}{64\pi\epsilon_0 R^2}\,\hat{\mathbf{z}} \qquad \text{repulsive}$$

Problem 8.3: Calculate the force of magnetic attraction between the northern and southern hemispheres of a uniformly charged spinning spherical shell, with radius R, angular velocity $\boldsymbol{\omega}$, and surface charge density σ.



- $$\mathbf{B} = \begin{cases} \dfrac{2}{3}\mu_0 \sigma R \omega \hat{\mathbf{z}} & \textit{uniform} \text{ inside} \\ \dfrac{\mu_0 \sigma \omega R^4}{3 r^3}\left(3\cos\theta\,\hat{\mathbf{r}} - \hat{\mathbf{z}}\right) & \textit{dipole} \text{ outside} \end{cases}$$ (from Ex 5.11)

$$\mathrm{d}\,\mathbf{F} = \mathbb{T}_{\text{mag}} \cdot \mathrm{d}\,\boldsymbol{a} = \frac{1}{\mu_0}\left(\mathbf{B}\left(\mathbf{B}\cdot\mathrm{d}\,\boldsymbol{a}\right) - \frac{B^2}{2}\mathbb{I}\cdot\mathrm{d}\,\boldsymbol{a}\right) = \frac{1}{\mu_0}\left(\left(\mathbf{B}\cdot\mathrm{d}\,\boldsymbol{a}\right)\mathbf{B} - \frac{B^2}{2}\mathrm{d}\,\boldsymbol{a}\right)$$

- Consider a surface enclosing the entire upper hemisphere — a hemispherical bowl just *outside* $r{=}R$ + the equatorial disk

- For the bowl, $\mathrm{d}\,\boldsymbol{a} = R^2 \sin\theta\,\mathrm{d}\,\phi\,\mathrm{d}\,\theta\,\hat{\mathbf{r}}$, $\mathbf{B}\cdot\mathrm{d}\,\boldsymbol{a} = \frac{2}{3}\mu_0 \sigma \omega R^3 \cos\theta \sin\theta\,\mathrm{d}\,\theta\,\mathrm{d}\,\phi$

- The force is in the z direction due to symmetry

$$\mathbf{F}_{\text{bowl}} = \int_{\text{bowl}} \mathrm{d}\,\mathbf{F} = \frac{1}{\mu_0}\int_{\text{bowl}}\left[\mathbf{B}\left(\mathbf{B}\cdot\mathrm{d}\,\boldsymbol{a}\right) - \frac{B^2}{2}\mathrm{d}\,\boldsymbol{a}\right] = \frac{\hat{\mathbf{z}}}{\mu_0}\int_{\text{bowl}}\left[B_z\left(\mathbf{B}\cdot\mathrm{d}\,\boldsymbol{a}\right) - \frac{B^2}{2}\mathrm{d}\,a_z\right]$$

$$= \pi\mu_0\left(\frac{\sigma\omega R^2}{3}\right)^2 \hat{\mathbf{z}}\int_0^{\pi/2}\left(9\cos^2\theta - 5\right)\cos\theta\sin\theta\,\mathrm{d}\,\theta = -\frac{\pi\mu_0}{4}\left(\frac{\sigma\omega R^2}{3}\right)^2 \hat{\mathbf{z}}$$

- For the equatorial disk, *inside* the sphere,

$$d\boldsymbol{a} = -r\,d\phi\,d r\,\hat{\mathbf{z}} \quad \Rightarrow \quad \mathbf{B}\cdot d\boldsymbol{a} = -\frac{2}{3}\mu_0\,\sigma\,\omega\,R\,r\,d\phi\,d r\,, \quad B^2 = \frac{4}{9}\left(\mu_0\,\sigma\,\omega\,R\right)^2$$

$$\mathbf{F}_{\text{disk}} = \int_{\text{disk}} d\mathbf{F} = \frac{1}{\mu_0}\int_{\text{disk}}\left[\mathbf{B}\left(\mathbf{B}\cdot d\boldsymbol{a}\right) - \frac{B^2}{2}\,d\boldsymbol{a}\right] = \frac{\hat{\mathbf{z}}}{\mu_0}\int_{\text{disk}}\left[B_z\left(\mathbf{B}\cdot d\boldsymbol{a}\right) - \frac{B^2}{2}\,d a_z\right]$$

$$= -\frac{4}{9}\pi\,\mu_0\,\sigma^2\,\omega^2\,R^4\,\hat{\mathbf{z}}\int_0^R r\,d r = -2\,\pi\,\mu_0\left(\frac{\sigma\,\omega\,R^2}{3}\right)^2\hat{\mathbf{z}}$$

$$\Rightarrow \quad \mathbf{F} = \mathbf{F}_{\text{bowl}} + \mathbf{F}_{\text{disk}} = -\pi\,\mu_0\left(\frac{\sigma\,\omega\,R^2}{2}\right)^2\hat{\mathbf{z}} \quad \text{attractive}$$

<u>Alternative</u> (Problem 5.44):

$$r = R \;\Rightarrow\; \mathbf{B}_{\text{ave}} = \frac{\mathbf{B}_{\text{inside}} + \mathbf{B}_{\text{outside}}}{2} = \frac{\mu_0\,\sigma\,\omega\,R}{6}\left(3\cos\theta\,\hat{\mathbf{r}} + \hat{\mathbf{z}}\right)$$

$$\mathbf{F} = \int_{\text{bowl}} d\mathbf{F} = \int_{\text{bowl}} \sigma\,\mathbf{v}\times\mathbf{B}_{\text{ave}}\,d a = \int_{\text{bowl}} \sigma\,\omega\,R\sin\theta\,\hat{\boldsymbol{\phi}}\times\mathbf{B}_{\text{ave}}\,R^2\sin\theta\,d\theta\,d\phi$$

$$= -\mu_0\,\pi\,\sigma^2\,\omega^2\,R^4\,\hat{\mathbf{z}}\int_0^{\pi/2}\cos\theta\,\sin^3\theta\,d\theta \quad \Leftarrow \quad \text{due to axial-symmetry}$$

$$= -\pi\,\mu_0\left(\frac{\sigma\,\omega\,R^2}{2}\right)^2\hat{\mathbf{z}}$$

Conservation of Momentum

- According to Newton's 2nd law, the force on an object is equal to the rate of change of its momentum: $\mathbf{F} = \dfrac{d\,\mathbf{p}_{\text{mech}}}{d\,t} = -\epsilon_0\,\mu_0 \dfrac{d}{d\,t}\displaystyle\int_{\mathcal{V}} \mathbf{S}\,d\tau + \oint_{\mathcal{S}} \mathbb{T}\cdot d\boldsymbol{a} \quad (\$)$
- This expression is similar in structure to Poynting's theorem, and it invites an analogous interpretation: The 1st integral represents *momentum stored in the fields*: $\mathbf{p}_{\text{EM}} = \epsilon_0\,\mu_0 \displaystyle\int_{\mathcal{V}} \mathbf{S}\,d\tau$, while the 2nd integral is the *momentum per unit time flowing in through the surface* $\Rightarrow \quad \dfrac{d}{d\,t}\left(\mathbf{p}_{\text{mech}} + \mathbf{p}_{\text{EM}}\right) = \displaystyle\oint_{\mathcal{S}} \mathbb{T}\cdot d\boldsymbol{a}$
- ($) is the statement of *conservation of momentum* in electrodynamics: If the mechanical momentum increases, either the field momentum decreases, or else the fields are carrying momentum into the volume through the surface, or both.
- The momentum *density* in the fields: $\boldsymbol{g} = \epsilon_0\,\mu_0\,\mathbf{S} = \epsilon_0\,\mathbf{E}\times\mathbf{B} = \dfrac{\mathbf{S}}{c^2} \quad \Leftarrow \quad \epsilon_0\,\mu_0 = \dfrac{1}{c^2}$
- The momentum flux transported by the fields is $-\mathbb{T}$, and $-\mathbb{T}\cdot d\boldsymbol{a}$ is the EM momentum per unit time passing through the area $d\boldsymbol{a}$.
- If the mechanical momentum in $\mathcal{V}$ is not changing, then

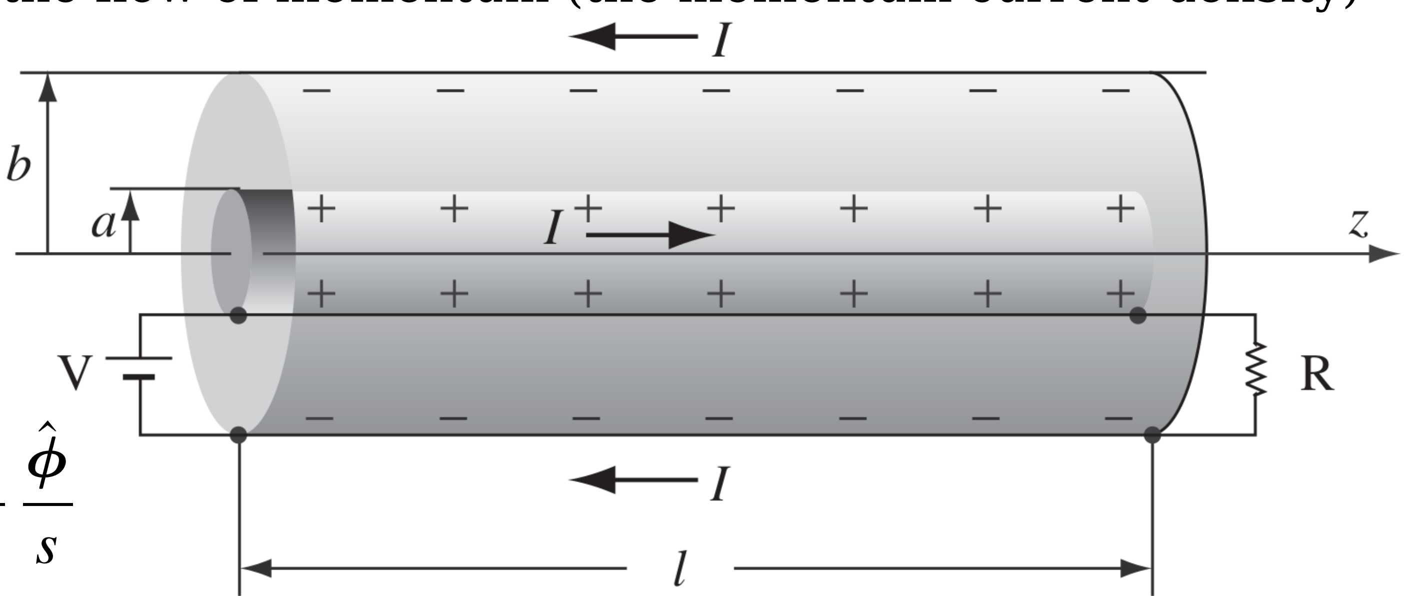
$$\int \frac{\partial \boldsymbol{g}}{\partial t}\,d\tau = \oint \mathbb{T}\cdot d\boldsymbol{a} = \int \nabla\cdot\mathbb{T}\,d\tau \quad \Rightarrow \quad \frac{\partial \boldsymbol{g}}{\partial t} = \nabla\cdot\mathbb{T}$$

- This is the "continuity eqn" for electromagnetic momentum, with $\boldsymbol{g}$ (momentum density) in the role of ρ (charge density) & $-\mathbb{T}$ playing the part of $\boldsymbol{J}$; it expresses the local conservation of field momentum.

- In general the field momentum by itself, and the mechanical momentum by itself, are *not* conserved—charges and fields exchange momentum, and only the *total* is conserved.

- The Poynting vector has appeared in 2 quite different roles: $\mathbf{S}$ is the energy per unit area, per unit time, transported by the EM fields, while $\mu_0 \epsilon_0 \mathbf{S}$ is the momentum per unit volume stored in those fields.

- $\mathbb{T}$ also plays a dual role: $\mathbb{T}$ itself is the EM stress (force per unit area) acting on a surface, $-\mathbb{T}$ describes the flow of momentum (the momentum current density) carried by the fields.



Example 8.3: What is the EM momentum stored in the fields?

- The fields are

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{\hat{\mathbf{s}}}{s}, \quad \mathbf{B} = \frac{\mu_0 I}{2\pi}\frac{\hat{\phi}}{s}$$

- The Poynting vector is $\mathbf{S} = \frac{\lambda I}{4\pi^2 \epsilon_0 s^2} \hat{\mathbf{z}}$
- Energy is flowing down the line, from the battery to the resistor. The power transported is $P = \int \mathbf{S} \cdot \mathrm{d}\boldsymbol{a} = \frac{\lambda I}{4\pi^2 \epsilon_0} \int_a^b \frac{1}{s^2} 2\pi s \, \mathrm{d}s = \frac{\lambda I}{2\pi \epsilon_0} \ln \frac{b}{a} = I V$
- The *momentum* in the fields is

$$\mathbf{p}_{\mathrm{EM}} = \mu_0 \epsilon_0 \int \mathbf{S} \, \mathrm{d}\tau = \frac{\mu_0 \lambda I}{4\pi^2} \hat{\mathbf{z}} \int_a^b \frac{\ell}{s^2} 2\pi s \, \mathrm{d}s = \frac{\mu_0 \lambda I \ell}{2\pi} \ln \frac{b}{a} \hat{\mathbf{z}} = \frac{I V \ell}{c^2} \hat{\mathbf{z}}$$

- The cable is not moving, $\mathbf{E}$ & $\mathbf{B}$ are static, yet there is momentum in the fields.

- If now we turn up the resistance, so the current decreases. The changing magnetic field will induce an electric field (Ex. 7.9) $\mathbf{E} = \left(\frac{\mu_0}{2\pi} \frac{\mathrm{d}I}{\mathrm{d}t} \ln s + K \right) \hat{\mathbf{z}}$
- This field exerts a force on $\pm\lambda$:

$$\mathbf{F} = \lambda \ell \left(\frac{\mu_0}{2\pi} \frac{\mathrm{d}I}{\mathrm{d}t} \ln a + K \right) \hat{\mathbf{z}} - \lambda \ell \left(\frac{\mu_0}{2\pi} \frac{\mathrm{d}I}{\mathrm{d}t} \ln b + K \right) \hat{\mathbf{z}} = -\frac{\mu_0 \lambda \ell}{2\pi} \frac{\mathrm{d}I}{\mathrm{d}t} \ln \frac{b}{a} \hat{\mathbf{z}}$$

- The total momentum imparted to the cable, as the current drops from I to 0

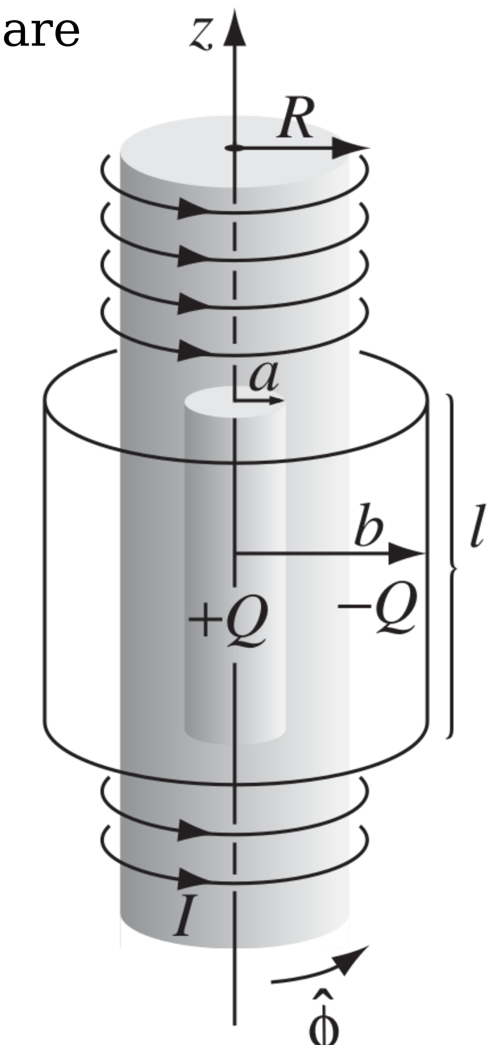
$$\mathbf{p}_{\mathrm{mech}} = \int \mathbf{F} \, \mathrm{d}t = \frac{\mu_0 \lambda I \ell}{2\pi} \ln \frac{b}{a} \hat{\mathbf{z}}$$ the momentum originally stored in the fields.

Angular Momentum

- The EM fields carry *energy* $u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$ and *momentum* $\boldsymbol{g} = \epsilon_0 \mathbf{E} \times \mathbf{B}$

- The *angular* momentum density: $\boldsymbol{\vartheta} \equiv \mathbf{r} \times \boldsymbol{g} = \epsilon_0 \mathbf{r} \times (\mathbf{E} \times \mathbf{B})$
$$= \epsilon_0 \left[(\mathbf{r}\cdot\mathbf{B})\mathbf{E} - (\mathbf{r}\cdot\mathbf{E})\mathbf{B}\right]$$

- Even perfectly *static* fields can harbor momentum and angular momentum, as long as $\mathbf{E}\times\mathbf{B} \neq 0$, and it is only when these field contributions are included that the conservation laws are sustained.



Example 8.4: When the current in the solenoid is gradually reduced, the cylinders begin to rotate, as in Ex. 7.8.
Question: Where does the angular momentum come from?

- The angular momentum was initially stored in the fields. Before the current was switched off,

$$\mathbf{E} = \frac{Q}{2\pi\epsilon_0 \ell}\frac{\hat{\mathbf{s}}}{s}, \quad a < s < b \quad \Rightarrow \quad \boldsymbol{g} = -\frac{\mu_0 n I Q}{2\pi \ell s}\hat{\boldsymbol{\phi}}, \quad a < s < R$$

$$\mathbf{B} = \mu_0 n I \hat{\mathbf{z}}, \quad s < R$$

momentum density

- The *angular* momentum density with $\mathbf{r} = s\,\hat{\mathbf{s}} + z\,\hat{\mathbf{z}}$

$$\boldsymbol{\vartheta} = \mathbf{r} \times \boldsymbol{g} = \frac{\mu_0 n I Q}{2\pi\ell}\left(\frac{z}{s}\hat{\mathbf{s}} - \hat{\mathbf{z}}\right) \Leftarrow$$

z -component constant independent of s

- Get the total angular momentum in the fields by integrating with the volume,

$$\mathbf{L} = \int \boldsymbol{\vartheta}\, \mathrm{d}\,\tau = \frac{\mu_0\, n\, I\, Q}{2\,\pi\,\ell} \int \left(\frac{z}{s}\,\hat{\mathbf{s}} - \hat{\mathbf{z}} \right) \mathrm{d}\,\tau = -\,\mu_0\, n\, I\, Q\, \frac{R^2 - a^2}{2}\,\hat{\mathbf{z}}$$

- When the current is turned off, the changing magnetic field induces a circumferential electric field, by Faraday's law: $\mathbf{E} = -\,\frac{\mu_0\, n}{2}\,\frac{\mathrm{d}\,I}{\mathrm{d}\,t}\,\frac{s_<^2}{s}\,\hat{\boldsymbol{\phi}}\,, \quad s_< = \min\left(s\,,\, R\right)$

$$\Rightarrow \quad \mathbf{N}_b = \int_{\mathcal{S}} \mathbf{r} \times \left(-\,\mathrm{d}\,q\,\mathbf{E}\right) = \frac{\mu_0\, n\, Q\, R^2}{2}\,\frac{\mathrm{d}\,I}{\mathrm{d}\,t}\,\hat{\mathbf{z}} \quad \text{torque on the outer cylinder}$$

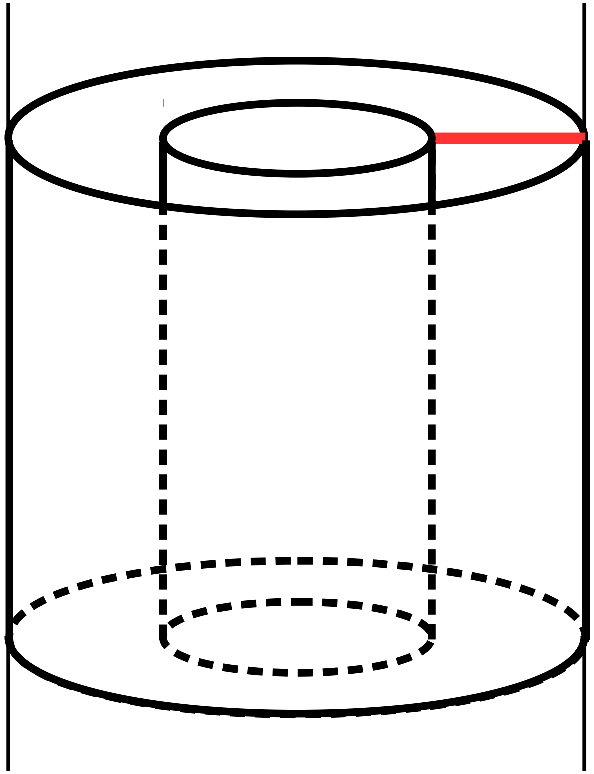
$$\Rightarrow \quad \mathbf{L}_b = \int \mathbf{N}_b\, \mathrm{d}\,t = \frac{\mu_0\, n\, Q\, R^2}{2}\,\hat{\mathbf{z}} \int_I^0 \frac{\mathrm{d}\,I}{\mathrm{d}\,t}\, \mathrm{d}\,t = -\,\frac{1}{2}\,\mu_0\, n\, I\, Q\, R^2\,\hat{\mathbf{z}}$$

- Similarly, the torque on the inner cylinder is $\mathbf{N}_a = -\,\frac{\mu_0\, n\, Q\, a^2}{2}\,\frac{\mathrm{d}\,I}{\mathrm{d}\,t}\,\hat{\mathbf{z}}$

$$\Rightarrow \quad \mathbf{L}_a = \int \mathbf{N}_a\, \mathrm{d}\,t = \frac{1}{2}\,\mu_0\, n\, I\, Q\, a^2\,\hat{\mathbf{z}} \quad \Rightarrow \quad \mathbf{L}_{\text{em}} = \mathbf{L}_a + \mathbf{L}_b$$

- The angular momentum *lost* by the fields is precisely equal to the angular momentum *gained* by the cylinders, the *total* angular momentum (fields + matter) is conserved.

Selected problems: 2, 6, 7, 14, 22



Problem 8.8: As in Ex 8.4, now we turn off the electric field instead, by connecting a weakly conducting radial spoke between the cylinders. So they are now rigidly connected to rotate together. From the magnetic force on the current in the spoke, determine the total angular momentum delivered to the cylinders, as they discharge.

- The *Lorentz* force on a segment ds of spoke is

$$\mathrm{d}\mathbf{F} = \frac{\mathrm{d}Q}{\mathrm{d}t}\,\mathrm{d}\boldsymbol{\ell}\times\mathbf{B} = -\mu_0 n I \frac{\mathrm{d}Q}{\mathrm{d}t}\,\mathrm{d}s\,\hat{\boldsymbol{\phi}} \quad \Leftarrow \quad \mathbf{B} = \mu_0 n I\,\hat{\mathbf{z}},\quad \mathrm{d}\boldsymbol{\ell} = \mathrm{d}s\,\hat{\mathbf{s}}$$

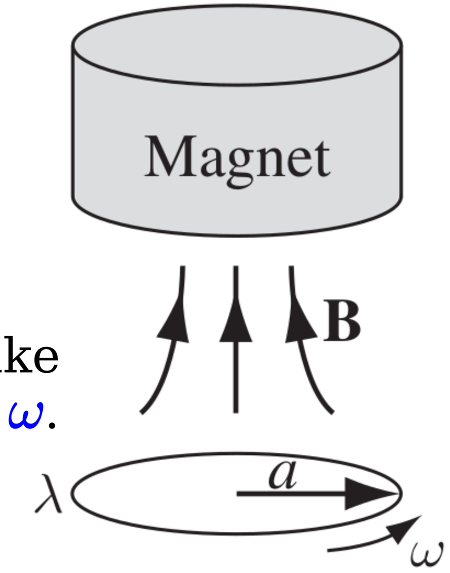
$$\Rightarrow\quad \mathbf{N} = \int \mathbf{r}\times\mathrm{d}\mathbf{F} = -\mu_0 n I \frac{\mathrm{d}Q}{\mathrm{d}t}\int_a^R \mathrm{d}s\,(s\,\hat{\mathbf{s}} + z\,\hat{\mathbf{z}})\times\hat{\boldsymbol{\phi}}$$

$$\Rightarrow\quad N_z = -\mu_0 n I \frac{\mathrm{d}Q}{\mathrm{d}t}\int_a^R s\,\mathrm{d}s = -\mu_0 n I \frac{R^2 - a^2}{2}\frac{\mathrm{d}Q}{\mathrm{d}t} \quad \Leftarrow \text{only rotate along } z\text{-axis}$$

$$\Rightarrow\quad \mathbf{L} = \hat{\mathbf{z}}\int N_z\,\mathrm{d}t = -\mu_0 n I Q \frac{R^2 - a^2}{2}\,\hat{\mathbf{z}} \quad \text{as expected}$$

- The mechanism by which angular momentum is transferred from the EM fields to the cylinders is entirely different in the 2 cases: in Ex. 8.4 it was Faraday's law, but here it is the Lorentz force law.

Magnetic Forces Do No Work



- If magnetic forces do no work, what about the magnetic crane lifting the carcass of a junked car?

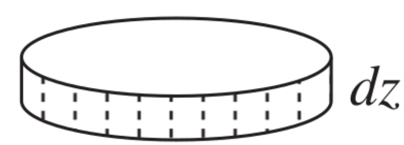
- Let’s model the car as a circular current loop—in fact, let’s make it an insulating ring of line charge λ rotating at angular velocity ω.

- The upward magnetic force on the loop is

$$F = 2\pi I a B_s \;\Leftarrow\; I = \lambda \omega a \;\Rightarrow\; \mathrm{d}W = 2\pi a^2 \lambda \omega B_s \,\mathrm{d}z \;\Rightarrow\; \text{Ring's potential energy increases}$$

- As the ring rises, the magnetic force is ⊥ the net velocity of the charges in the ring, so it does no work on them.

- At the same time a motional emf is induced in the ring, which opposes the flow of charge, and hence reduces its angular velocity:



$$\mathscr{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \;\Leftarrow\; \mathrm{d}\Phi = B_s 2\pi a \,\mathrm{d}z \;\text{ in }\; \mathrm{d}t$$

- $\mathscr{E} = \oint \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{\ell} = f \cdot 2\pi a \;\Leftarrow\; \boldsymbol{f} : \frac{\text{force}}{\text{charge}} \;\Rightarrow\; f = -B_s \frac{\mathrm{d}z}{\mathrm{d}t}$

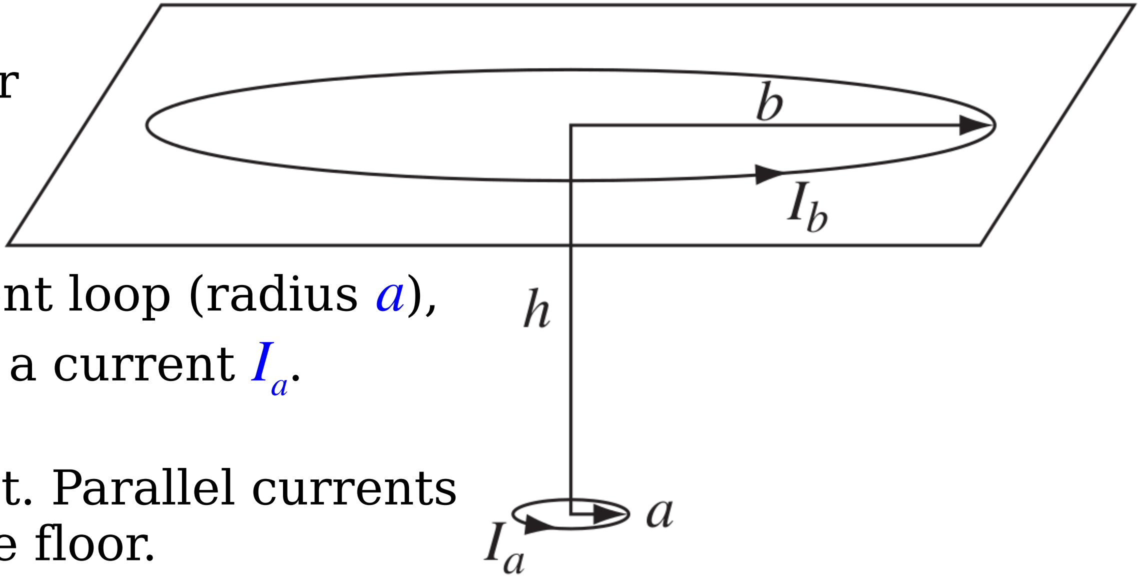
$$\Rightarrow \;\text{force on }\; \mathrm{d}\ell = f \lambda \,\mathrm{d}\ell \;\Rightarrow\; \begin{matrix}\text{torque on}\\ \text{the ring}\end{matrix} \; N = a\left(-B_z \frac{\mathrm{d}z}{\mathrm{d}t}\right)\lambda \cdot 2\pi a$$

$$\Rightarrow \; \mathrm{d}W = N \,\mathrm{d}\phi = N \omega \,\mathrm{d}t = -2\pi a^2 \lambda \omega B_s \,\mathrm{d}z$$

- The ring slows down, and the rotational energy it loses is precisely equal to the potential energy it gains.

- All the magnetic field did was convert energy from one form to another. Or the work done by the vertical component of the magnetic force is equal and opposite to the work done by its horizontal component.

- Model the magnet as a big circular loop (radius b), resting on a table and carrying a current I_b; the "junk car" is a relatively small current loop (radius a), on the floor directly below, carrying a current I_a.



- Assume both currents are constant. Parallel currents attract, and lift the small loop off the floor.

- Start by adjusting the currents to let the small ring just "float," a distance h below the table, with the magnetic force exactly balancing the weight ($m_a g$),

$$\Rightarrow \quad F_{\text{mag}} = \frac{3\pi}{2}\mu_0 I_a I_b \frac{a^2 b^2 h}{\left(b^2+h^2\right)^{5/2}} = m_a g \quad \Leftarrow \quad \text{Problem 8.11}$$

$$\Rightarrow \quad \mathrm{d}W_g = m_a g\,\mathrm{d}z = \frac{3\pi}{2}\mu_0 I_a I_b \frac{a^2 b^2 h}{\left(b^2+h^2\right)^{5/2}}\mathrm{d}z$$

Problem 8.11:

- For current I_b in the big loop, by Ex. 5.6, its $\mathbf{B}$ along the z axis is $$\mathbf{B}=\frac{\mu_0\, b^2\, I_b}{2\left(z^2+b^2\right)^{3/2}}\,\hat{\mathbf{z}}$$

- The little loop is small to be treated as a magnetic dipole $\mathbf{m}=\pi\, a^2\, I_a\,\hat{\mathbf{z}}$

- The magnetic force on $\mathbf{m}$ is $$\mathbf{F}_{\text{mag}}=\nabla\left(\mathbf{m}\cdot\mathbf{B}\right)=\nabla\,\frac{\mu_0\,\pi\, a^2\, b^2\, I_a\, I_b}{2\left(z^2+b^2\right)^{3/2}}=-\frac{3\,\pi\,\mu_0\, a^2\, b^2\, I_a\, I_b\, z}{2\left(b^2+z^2\right)^{5/2}}\,\hat{\mathbf{z}}$$

Problem 8.12:

$$\frac{\mathrm{d}\,W}{\mathrm{d}\,t}=-\mathscr{E}_a\, I_a-\mathscr{E}_b\, I_b \quad ⇦\ \mathscr{E}_a=-L_a\,\frac{\mathrm{d}\,I_a}{\mathrm{d}\,t}-M\,\frac{\mathrm{d}\,I_b}{\mathrm{d}\,t},\quad \mathscr{E}_b=-L_b\,\frac{\mathrm{d}\,I_b}{\mathrm{d}\,t}-M\,\frac{\mathrm{d}\,I_a}{\mathrm{d}\,t}$$

$$=\left(L_a\,\frac{\mathrm{d}\,I_a}{\mathrm{d}\,t}+M\,\frac{\mathrm{d}\,I_b}{\mathrm{d}\,t}\right) I_a+\left(L_b\,\frac{\mathrm{d}\,I_b}{\mathrm{d}\,t}+M\,\frac{\mathrm{d}\,I_a}{\mathrm{d}\,t}\right) I_b$$

$$=\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{1}{2}\,L_a\, I_a^2+\frac{1}{2}\,L_b\, I_b^2+M\, I_a\, I_b\right) \quad\Rightarrow\quad W=\frac{1}{2}\,L_a\, I_a^2+\frac{1}{2}\,L_b\, I_b^2+M\, I_a\, I_b$$

- The work was done by the power supply that sustains the current in loop *a*, but not by the magnetic field.

- As the loop rises, a motional emf is induced in it. The flux through the loop

$$\Phi_a = M\, I_b \quad \Leftarrow \quad M = \frac{\pi\,\mu_0}{2}\,\frac{a^2\, b^2}{\left(b^2 + h^2\right)^{3/2}} \quad \text{mutual inductance} \quad \Leftarrow \quad \text{Problem 7.22}$$

$$\Rightarrow \quad \mathscr{E}_a = -\frac{\mathrm{d}\,\Phi_a}{\mathrm{d}\,t} = -I_b\,\frac{\mathrm{d}\,M}{\mathrm{d}\,t} = -I_b\,\frac{\mathrm{d}\,M}{\mathrm{d}\,h}\,\frac{\mathrm{d}\,h}{\mathrm{d}\,t} = -I_b\,(-3)\,\frac{\pi\,\mu_0}{2}\,\frac{a^2\, b^2\, h}{\left(b^2 + h^2\right)^{5/2}}\left(-\frac{\mathrm{d}\,z}{\mathrm{d}\,t}\right)$$

$$\Rightarrow \quad \text{work by the power supply} \quad \mathrm{d}\,W_a = -\mathscr{E}_a\, I_a\, \mathrm{d}\,t = \frac{3\,\pi}{2}\,\mu_0\, I_a\, I_b\,\frac{a^2\, b^2\, h}{\left(b^2 + h^2\right)^{5/2}}\,\mathrm{d}\,z \quad \text{same as the work for lifting loop } a$$

- A Faraday emf is also induced in the upper loop, due to the changing flux from the lower loop: $\Phi_b = M\, I_a \quad \Rightarrow \quad \mathscr{E}_a = -I_a\,\dfrac{\mathrm{d}\,M}{\mathrm{d}\,t}$

$$\Rightarrow \quad \mathrm{d}\,W_b = -\mathscr{E}_b\, I_b\, \mathrm{d}\,t = \frac{3\,\pi}{2}\,\mu_0\, I_a\, I_b\,\frac{a^2\, b^2\, h}{\left(b^2 + h^2\right)^{5/2}}\,\mathrm{d}\,z = \mathrm{d}\,W_a$$

- So the power supplies have done *twice* as much work as was necessary to lift the junk car! It increased the energy stored in the fields.

- The energy in a system of 2 current-carrying loops is (see Problem 8.12)

$$U = \frac{1}{2} L_a I_a^2 + \frac{1}{2} L_b I_b^2 + M I_a I_b \quad \Rightarrow \quad \mathrm{d}U = I_a I_b \frac{\mathrm{d}M}{\mathrm{d}t} \mathrm{d}t = \mathrm{d}W_b$$

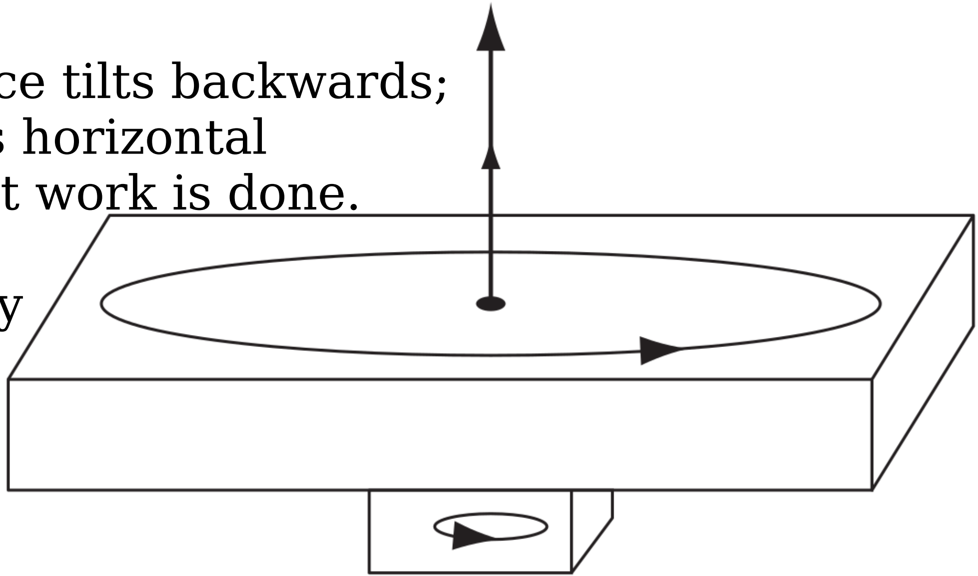
- All 4 energy increments are the same. The power supply in loop *a* contributes the energy necessary to lift the lower ring, while the power supply in loop *b* provides the extra energy for the fields.

- If all we're interested in is the work done to raise the ring, we can ignore the upper loop (and the energy in the fields) altogether.

- In both these models, the magnet itself was stationary. As a model, we stick the upper loop in a big box, the lower loop in a little box, and crank up the currents so the force of attraction is much greater than $m_a g$; the 2 boxes snap together, and we attach a string to the upper box and pull up on it.

- As the lower loop rises, the magnetic force tilts backwards; its vertical component lifts the loop, but its horizontal component opposes the current, and no net work is done.

- The motional emf is perfectly balanced by the Faraday emf fighting to keep the current going—the flux through the lower loop is not changing.



- Another way of thinking: the flux is *increasing* because the loop is moving upward, into a region of higher magnetic field, but it is *decreasing* because the magnetic field of the upper loop—at any give point in space—is decreasing as that loop moves up.

- No power supply is needed to sustain the current, since the energy in the fields is not changing. So the person pulling up on the rope did the work to lift the car.

- The role of the magnetic field was to transmit this energy to the car, via the vertical component of the magnetic force. But the magnetic field did no work.

- The fact that magnetic fields do no work follows directly from the Lorentz force law. If magnetic monopoles exist, the force on a particle with electric charge q_e and magnetic charge q_m becomes $\mathbf{F} = q_e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) + q_m\left(\mathbf{B} - \epsilon_0 \mu_0 \mathbf{v} \times \mathbf{E}\right)$. In that case, magnetic fields *can* do work *only on magnetic charges*.

- Another possibility is that in addition to electric charges there exist permanent point magnetic dipoles, whose dipole moment $\mathbf{m}$ is not associated with any electric current, but *intrinsic*.

- The Lorentz force law acquires an extra term $\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) + \nabla\left(\mathbf{m} \cdot \mathbf{B}\right)$

- The magnetic field can do work on these “intrinsic” dipoles, but it is beyond classical electrodynamics.