# Chapter 7 Electrodynamics

## Electromotive Force

### Ohm's Law

- For most substances, the current density $\boldsymbol{J}$ is proportional to the *force per unit charge*, $\boldsymbol{f}$: $\boldsymbol{J} = \sigma \boldsymbol{f}$ $\Leftarrow$ $\sigma$ : conductivity not surface charge , $\boldsymbol{f} = \dfrac{\mathbf{F}}{q}$

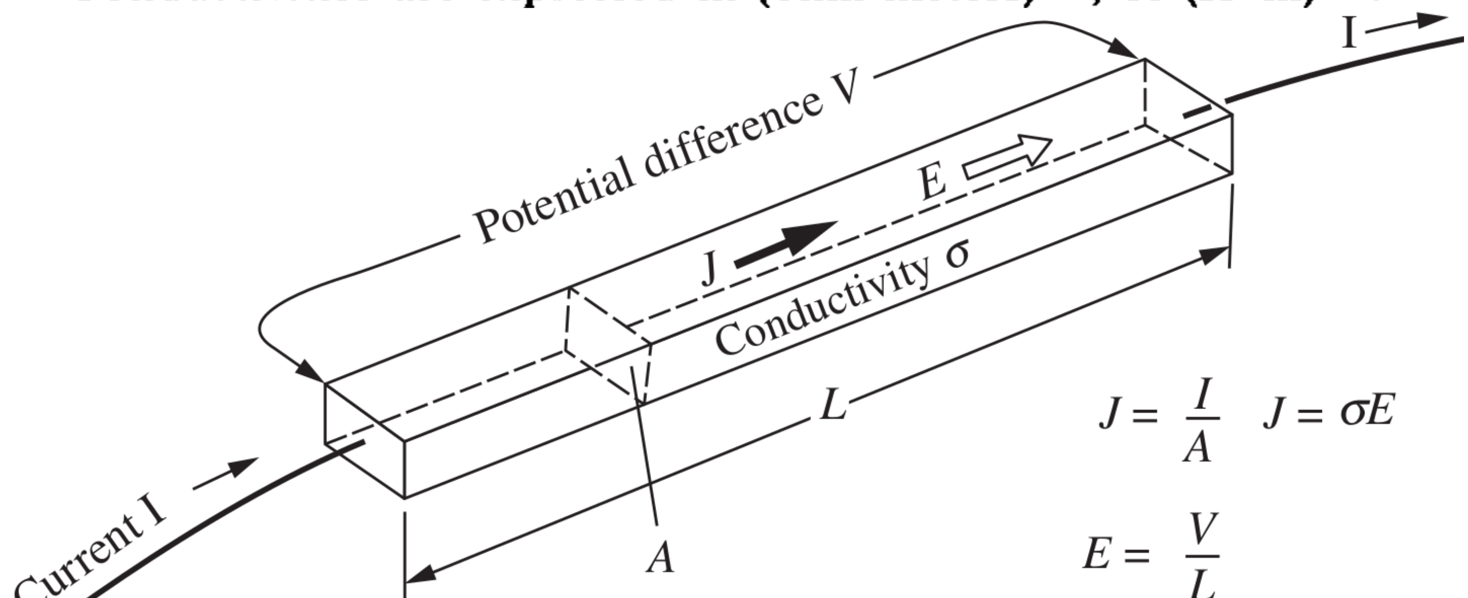
- The *reciprocal* of $\sigma$ is called the **resistivity**: $\rho = \dfrac{1}{\sigma}$ not charge density

- Even *insulators* conduct slightly, though the conductivity of a metal is much greater; in fact, for most purposes metals can be regarded as **perfect conductors**, with $\sigma = \infty$, while for insulators we can pretend $\sigma = 0$.

| Material | Resistivity | Material | Resistivity |
|---|---|---|---|
| *Conductors:* | | *Semiconductors:* | |
| Silver | $1.59 \times 10^{-8}$ | Sea water | 0.2 |
| Copper | $1.68 \times 10^{-8}$ | Germanium | 0.46 |
| Gold | $2.21 \times 10^{-8}$ | Diamond | 2.7 |
| Aluminum | $2.65 \times 10^{-8}$ | Silicon | 2500 |
| Iron | $9.61 \times 10^{-8}$ | *Insulators:* | |
| Mercury | $9.61 \times 10^{-7}$ | Water (pure) | $8.3 \times 10^{3}$ |
| Nichrome | $1.08 \times 10^{-6}$ | Glass | $10^{9} - 10^{14}$ |
| Manganese | $1.44 \times 10^{-6}$ | Rubber | $10^{13} - 10^{15}$ |
| Graphite | $1.6 \times 10^{-5}$ | Teflon | $10^{22} - 10^{24}$ |

| Material | Conductivity[a] | Material | Conductivity[a] |
|---|---|---|---|
| Silver | $6.2 \times 10^{7}$ | $H_2O$ | $2 \times 10^{-4}$ |
| Copper | $5.8 \times 10^{7}$ | Marble | $10^{-5}$ |
| Pure iron | $1.0 \times 10^{7}$ | Wood | $10^{-9}$ |
| Steel | $0.2 \times 10^{7}$ | Glass | $10^{-11}$ |
| Mercury | $10^{6}$ | Oil | $10^{-14}$ |
| Carbon | $10^{4}$ | Polyethylene | $10^{-15}$ |
| Silicon | $10^{-2}$ | Fused quartz | $10^{-17}$ |
| Alcohol | $3 \times 10^{-4}$ | True vacuum | ? |

[a] Conductivities are expressed in (ohm-meters)$^{-1}$, or $(\Omega \cdot m)^{-1}$.




$$R = \frac{V}{I} = \frac{E\,L}{J\,A} = \frac{L}{\sigma\,A} = \rho\,\frac{L}{A}$$

- It’s usually an EM force that drives the charges to produce the current, so

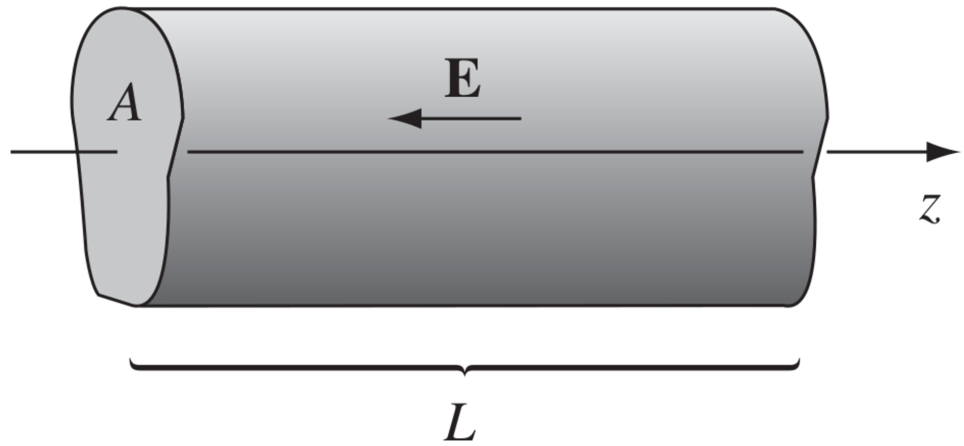
$$\boldsymbol{J} = \sigma\, \boldsymbol{f} = \sigma \frac{\mathbf{F}_L}{q} = \sigma\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \;\Rightarrow\; \boxed{\boldsymbol{J} = \sigma\, \mathbf{E}} \text{ Ohm's law } \Leftarrow \; v \ll 1$$

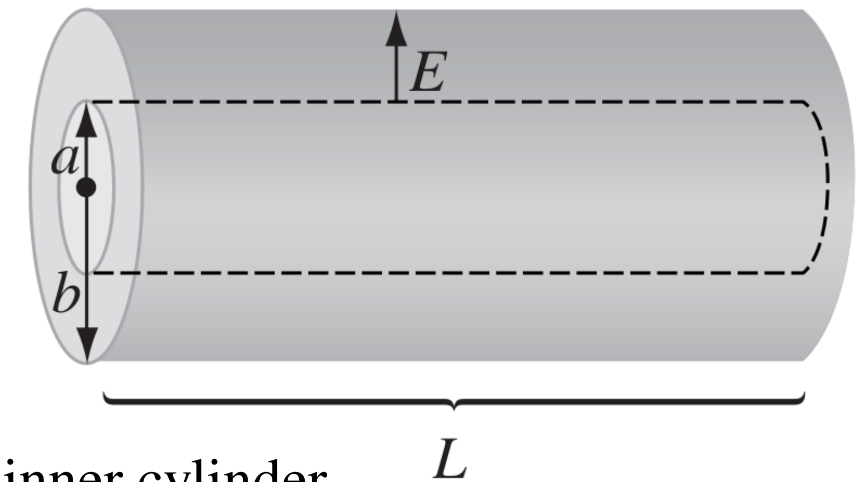
- $\mathbf{E}=0$ in a conductor for *stationary* charges ($\boldsymbol{J}=0$) but not true for current≠0.
- For *perfect* conductors, $\mathbf{E} = \frac{\boldsymbol{J}}{\sigma} \sim 0$ even if current exists. Metals are usually

good conductors, the electric field required to drive current in them is negligible.

- We routinely treat the connecting wires in electric circuits as equipotentials.
- **Resistors** are made from poorly conducting materials.

Example 7.1




Example 7.2: 2 long coaxial metal cylinders (radii $a$ and $b$) are separated by material of conductivity $\sigma$. If they are maintained at a potential difference $V$, what current flows from one to the other, in a length $L$?

- $\mathbf{E} = \dfrac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}} \quad \Leftarrow \quad \lambda:$ charge/length on the inner cylinder

$$\Rightarrow \quad I = \int \boldsymbol{J}\cdot d\,\boldsymbol{a} = \sigma \int \mathbf{E}\cdot d\,\boldsymbol{a} = \sigma \iint \frac{\lambda}{2\pi\epsilon_0 s}\, s\, d\phi\, d z = \frac{\sigma}{\epsilon_0}\lambda L \quad \Rightarrow \quad \lambda = \epsilon_0 \frac{I}{\sigma L}$$

$$\Rightarrow \quad V = \int_a^b \mathbf{E}\cdot d\,\boldsymbol{\ell} = \frac{\lambda}{2\pi\epsilon_0}\int_b^a \frac{d\,s}{s} = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{b}{a} \quad \Rightarrow \quad I = \frac{2\pi\sigma L}{\ln b - \ln a} V$$

- The total current flowing from one **electrode** to the other is proportional to the potential difference between them: $V = I\ R$

- The constant $R$ is called the **resistance**; it's a function of the geometry of the arrangement and the conductivity of the medium between the electrodes.

- In Ex. 7.1, $R = \dfrac{L}{\sigma\ A}$; in Ex. 7.2, $R = \dfrac{1}{2\ \pi\ \sigma\ L} \ln \dfrac{b}{a}$ .

- Resistance is measured in **ohms** ($\Omega$): an ohm is a volt per ampere.

- The proportionality between $V$ & $I$ is a direct consequence of $\boldsymbol{J} = \sigma\ \mathbf{E}$: if you want to double $V$, you double the charge on the electrodes—that doubles $\mathbf{E}$, which doubles $\boldsymbol{J}$, which doubles $I$.

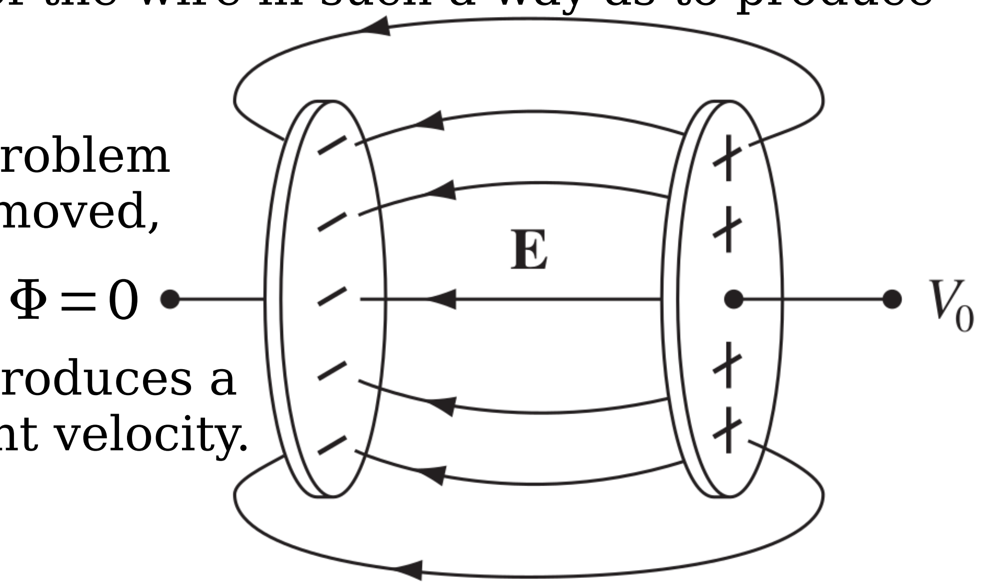
- For *steady* currents and *uniform* conductivity, $\nabla \cdot \mathbf{E} = \dfrac{1}{\sigma} \nabla \cdot \boldsymbol{J} = 0$. Therefore the charge density inside is 0; any unbalanced charge resides on the surface.

- We proved this using the fact $\mathbf{E}=0$ for the case of *stationary* charges; evidently, it is still true when the charges are allowed to move.

- It follows that **Laplace's equation** holds within a homogeneous ohmic material carrying a steady current, so all the tools/tricks of Chapter 3 are available for calculating the potential.

Example 7.3: Prove that the field in Ex. 7.1 is *uniform*.

- At the left end let the potential $\Phi=0$ and at the right end the potential $\Phi=V_0$.
- On the cylindrical surface, $\boldsymbol{J}\cdot\hat{\mathbf{n}}=0$ no leaking $\Rightarrow$ $\mathbf{E}\cdot\hat{\mathbf{n}}=-\dfrac{\partial\Phi}{\partial n}=0$
- With $\Phi$ or its normal derivative specified on all surfaces, the potential is *uniquely* determined (Prob. 3.5).
- One potential under Laplace's eqn and the boundary conditions: $\Phi(z)=\dfrac{V_0 z}{L}$
- The uniqueness theorem guarantees that it's the solution $\Rightarrow \mathbf{E}=-\nabla\Phi=-\dfrac{V_0}{L}\hat{\mathbf{z}}$
- Charge arranges itself over the surface of the wire in such a way as to produce a nice uniform field within.
- Contrast the enormously more difficult problem that arises if the conducting material is removed, leaving only metal plates.
- Ohm's law implies that a constant field produces a constant current, which suggests a constant velocity. Isn't that a contradiction to Newton's law?



- No, because of the frequent collisions of electrons as they pass down the wire.

- If the length of a block is $\lambda$ and your acceleration is $a$, the time it takes to go a block is

$$t = \sqrt{\frac{2\lambda}{a}} \;\Rightarrow\; v_{\text{ave}} = \frac{1}{2} a t = \sqrt{\frac{\lambda a}{2}} \propto \sqrt{\frac{\lambda E}{2}} \;\Leftarrow\; \text{NG}$$

- In practice, the charges are already moving very fast because of their thermal energy. But the thermal velocities have random directions, and average to 0. The **drift velocity** is a tiny extra bit:

$$t = \frac{\lambda}{v_{\text{thermal}}} \;\Rightarrow\; v_{\text{ave}} = \frac{a\lambda}{2 v_{\text{thermal}}} \;\Rightarrow\; \boldsymbol{J} = n f q \mathbf{v}_{\text{ave}} = \frac{n f q \lambda}{2 v_{\text{thermal}}} \frac{\mathbf{F}}{m} = \frac{n f \lambda q^2}{2 m v_{\text{thermal}}} \mathbf{E}$$

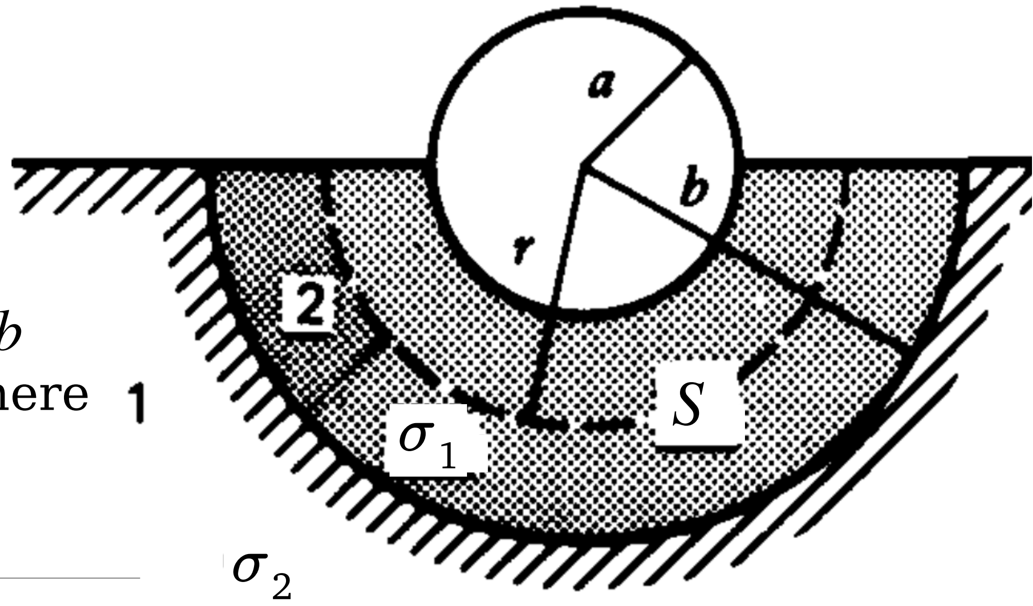
- The eqn correctly predicts that conductivity is proportional to the density of the moving charges and decreases with increasing temperature.

- Due to all the collisions, the work done by the electrical force is converted into heat in the resistor.

- Since the work done per unit charge is $V$ and the charge flowing per unit time is $I$, the power delivered is $P = V I = I R^2$ Joule heating law

- With $I$ in amperes and $R$ in ohms, $P$ comes out in watts (joules/second).

Example: A system is grounded by using a perfectly conducting sphere of radius $a$ with half of the sphere in contact with the ground. The layer of earth of radius $b$ that is in immediate contact with the sphere has a conductivity $\sigma_1$, and the rest of the ground has a conductivity $\sigma_2$.




Assuming that there is a current $I$ flowing from the sphere to the ground

$$\int_S \boldsymbol{J}\cdot d\boldsymbol{a} = I \quad \Rightarrow \quad \boldsymbol{J} = \frac{I}{2\pi r^2}\hat{\mathbf{r}} \quad \Rightarrow \quad \mathbf{E}_{1,2} = \frac{\boldsymbol{J}}{\sigma_{1,2}} = \frac{I}{2\pi\sigma_{1,2}}\frac{\hat{\mathbf{r}}}{r^2}$$

$$\Rightarrow \quad V = -\int_\infty^a \mathbf{E}\cdot d\boldsymbol{\ell} = \int_a^\infty \mathbf{E}\cdot d\mathbf{r} = \int_a^b \frac{I\,dr}{2\pi\sigma_1 r^2} + \int_b^\infty \frac{I\,dr}{2\pi\sigma_2 r^2}$$

$$= \frac{I}{2\pi}\left(\frac{1}{\sigma_1}\int_a^b \frac{dr}{r^2} + \frac{1}{\sigma_2}\int_b^\infty \frac{dr}{r^2}\right) = \frac{I}{2\pi\sigma_1}\left(\frac{1}{a}-\frac{1}{b}\right) + \frac{I}{2\pi\sigma_2}\frac{1}{b}$$

$$\Rightarrow \quad R = \frac{V}{I} = \frac{1}{2\pi\sigma_2 b} + \frac{1}{2\pi\sigma_1}\left(\frac{1}{a}-\frac{1}{b}\right)$$

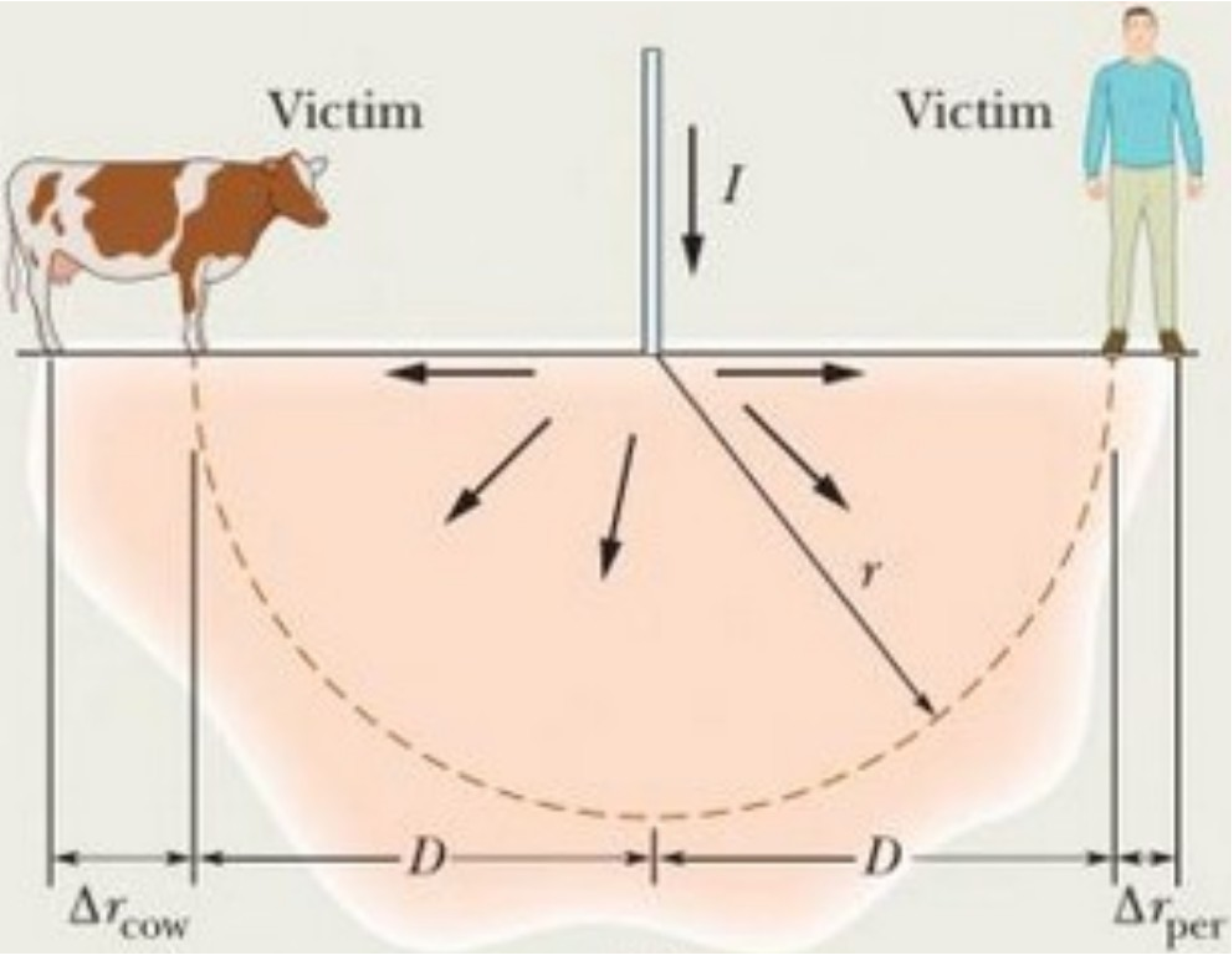
$$\Delta V = -\int_{D+\Delta r}^{D} \mathbf{E}\cdot \mathrm{d}\,\mathbf{r} = -\int_{D+\Delta r}^{D} \frac{I\,\mathrm{d}\,r}{2\,\pi\,\sigma\,r^2} = \frac{I}{2\,\pi\,\sigma}\,\frac{\Delta\,r}{D\,(D+\Delta\,r)}\,, \quad R_{\text{man}} \simeq R_{\text{cow}} = R'$$

$$\Rightarrow \quad i_{\text{man/cow}} = \frac{\Delta\,V_{\text{man/cow}}}{R_{\text{man/cow}}} \approx \frac{I}{2\,\pi\,\sigma\,R'}\,\frac{\Delta\,r_{\text{man/cow}}}{D^2} \;\text{ for }\; D \gg \Delta\,r \quad \Rightarrow \quad \frac{i_{\text{cow}}}{i_{\text{man}}} \approx \frac{\Delta\,r_{\text{cow}}}{\Delta\,r_{\text{man}}}$$

$$\Delta\,r_{\text{cow}} \sim 3\,\Delta\,r_{\text{man}}$$

$$\Rightarrow \quad i_{\text{cow}} \sim 3\,i_{\text{man}}$$

So the ground current in a lightning strike is dangerous, especially to cows.

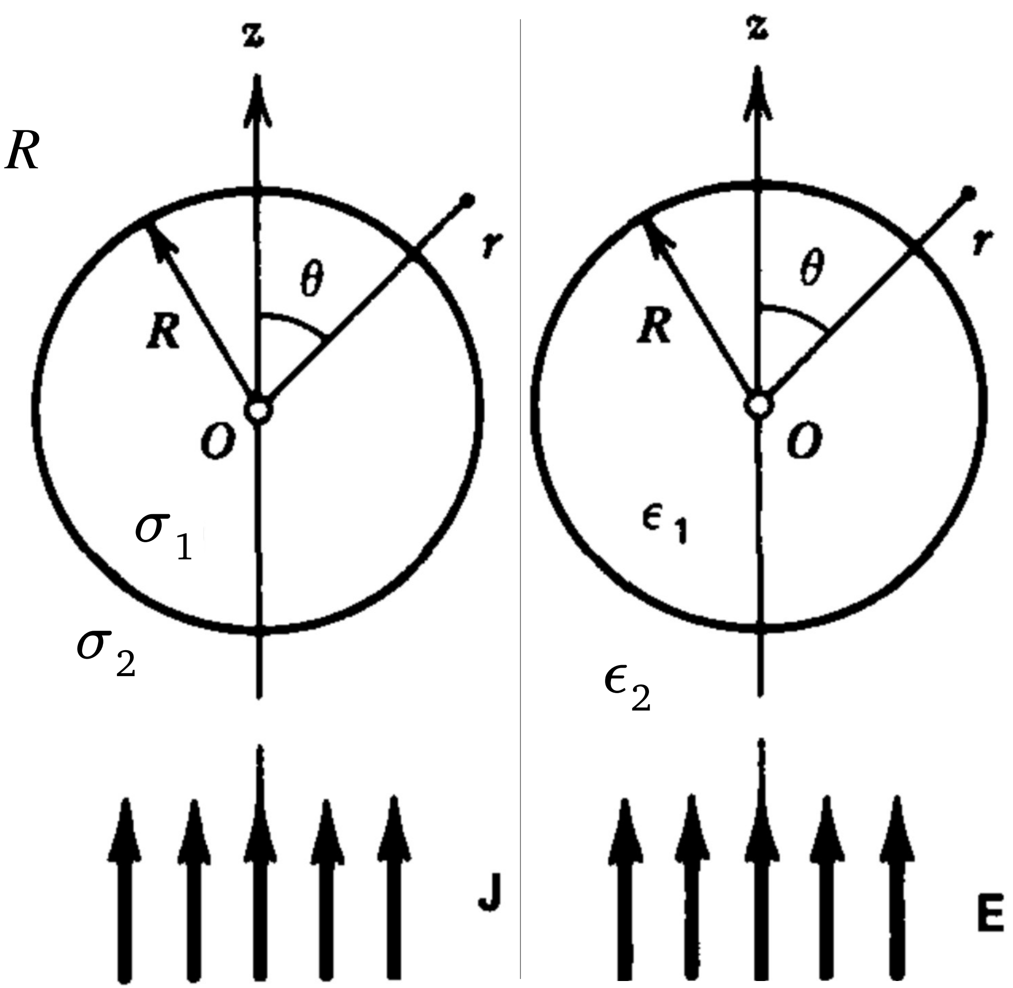


Example: Consider a sphere of radius $R$ and conductivity $\sigma_1$, placed in an initially uniform current with density $\boldsymbol{J}_0 = J_0\,\hat{\mathbf{z}}$. The medium surrounding the sphere is of conductivity $\sigma_2$.

This problem is analogous to the dielectric problem (Ex. 4.7) where a dielectric sphere is placed in an external electric field, $\mathbf{E}_0 = E_0\,\hat{\mathbf{z}}$. The potentials in the dielectric case are as follows:

$$\Phi_{\text{out}} = -E_0\,z + \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\,\epsilon_2}\,E_0\,\frac{R^3}{r^2}\cos\theta\,, \quad r > R$$

$$\Phi_{\text{in}} = -\frac{3\,\epsilon_2}{\epsilon_1 + 2\,\epsilon_2}\,E_0\,z\,, \qquad r < R$$




We follow the similar track:

$$\nabla\cdot\boldsymbol{J} = \sigma\,\nabla\cdot\mathbf{E} = 0 \quad \Leftarrow \quad \boldsymbol{J} = \sigma\,\mathbf{E}$$

$$\Rightarrow \quad \boldsymbol{J} = -\sigma\,\nabla\,\Phi \quad \Rightarrow \quad \nabla^2\Phi = 0$$

$$\Phi_{\text{in}} = \sum A_\ell\, r^\ell\, P_\ell(\cos\theta)$$

$$\Rightarrow \quad \Phi_{\text{out}} = -\frac{J_0}{\sigma_2}\,z + \sum \frac{B_\ell}{r^{\ell+1}}\; P_\ell(\cos\theta)$$

$$(1)\;\; \Phi_{\text{in}}(r=R) = \Phi_{\text{out}}(r=R) \quad \text{boundary}$$

$$(2)\; J_{\text{in},\,r}(R) = -\sigma_1\,\partial_r\,\Phi_{\text{in}}(R) \quad \text{conditions}$$

$$= -\sigma_2\,\partial_r\,\Phi_{\text{out}}(R) = J_{\text{out},\,r}$$

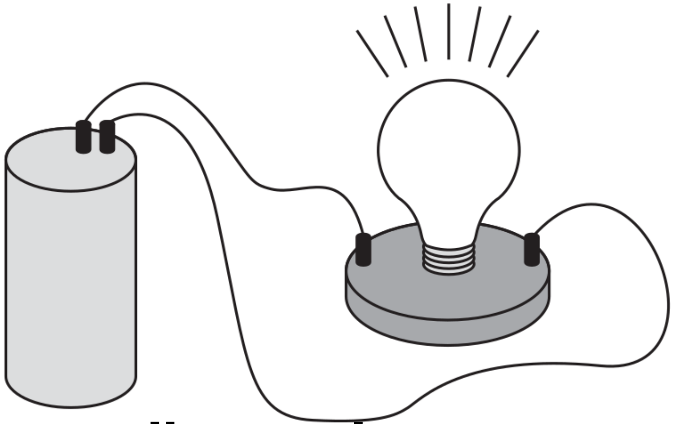
$$\Rightarrow \begin{cases} (1) \quad A_1 = \dfrac{B_1}{R^3} - \dfrac{J_0}{\sigma_2}, \qquad A_\ell = \dfrac{B_\ell}{R^{2\ell+1}} \text{ for } \ell \neq 1 \\ (2) \quad \dfrac{\sigma_1}{\sigma_2} A_1 = -2\dfrac{B_1}{R^3} - \dfrac{J_0}{\sigma_2}, \quad \dfrac{\sigma_1}{\sigma_2} \ell A_\ell = -(\ell+1)\dfrac{B_\ell}{R^{2\ell+1}} \text{ for } \ell \neq 1 \end{cases}$$

$$\Rightarrow \quad A_1 = -\frac{3 J_0}{\sigma_1 + 2\sigma_2}, \quad B_1 = \frac{\sigma_1 - \sigma_2}{\sigma_1 + 2\sigma_2} \frac{J_0}{\sigma_2} R^3, \quad A_\ell = B_\ell = 0 \text{ for } \ell \neq 1$$

$$\Rightarrow \quad \Phi_{\text{in}} = -\frac{3 J_0}{\sigma_1 + 2\sigma_2} z, \quad \Phi_{\text{out}} = -\frac{J_0}{\sigma_2} z + \frac{\sigma_1 - \sigma_2}{\sigma_1 + 2\sigma_2} \frac{J_0}{\sigma_2} \frac{R^3}{r^2} \cos\theta$$

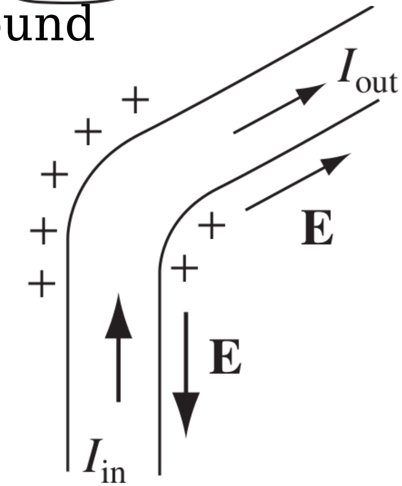
$$\Rightarrow \begin{cases} \boldsymbol{J}_{\text{in}} = \dfrac{3\sigma_1}{\sigma_1 + 2\sigma_2} \boldsymbol{J}_0, & \text{inside the sphere } r < R \\ \boldsymbol{J}_{\text{out}} = \boldsymbol{J}_0 + \dfrac{\sigma_1 - \sigma_2}{\sigma_1 + 2\sigma_2} \dfrac{R^3}{r^3} \left[ 3 (\hat{\mathbf{r}} \cdot \boldsymbol{J}_0) \hat{\mathbf{r}} - \boldsymbol{J}_0 \right], & \text{outside the sphere } r > R \end{cases}$$

## Electromotive Force



- The *current is the same all the way around the loop*; otherwise charge would be piling up somewhere, and the electric field of this accumulating charge is in such a direction as to even out the flow.

- In practice, we can safely assume the current is the same all around the circuit, even in systems that oscillate at radio frequencies.




- 2 forces are involved in driving current around a circuit: the *source*, $\boldsymbol{f}_s$, confined to one portion of the loop (battery), and an *electrostatic* force, which smooths out the flow and communicate the influence of the source to distant parts of the circuit: $\boldsymbol{f}=\boldsymbol{f}_s+\mathbf{E}$.

- The physical agency responsible for $\boldsymbol{f}_s$ can be many different things. Whatever the *mechanism*, its net effect is determined by the line integral of $\boldsymbol{f}$ around the circuit:

$$\mathscr{E} = \oint \boldsymbol{f} \cdot d\boldsymbol{\ell} = \oint \boldsymbol{f}_s \cdot d\boldsymbol{\ell} \Leftarrow \oint \mathbf{E} \cdot d\boldsymbol{\ell} = 0 \text{ for electrostatic fields}$$

- $\mathscr{E}$ is called the **electromotive force** (or **electromotance**), or **emf**, of the circuit. It’s the *integral of a force per unit charge*.

- If work is done, $\boldsymbol{f}_s$ must be *nonconservative* in the region containing the loop.

- Since, usually, $\boldsymbol{f}_s \neq 0$ only "inside" the source of emf, ie, localized, one writes

$$\mathscr{E} = \oint \boldsymbol{f}_s \cdot d\boldsymbol{\ell} = \int_a^b \boldsymbol{f}_s \cdot d\boldsymbol{\ell} \Leftarrow$$ where $a$ and $b$ are points at the terminals of the source

$$\boldsymbol{f} = \boldsymbol{f}_s + \mathbf{E}, \quad \sigma \to \infty$$
$$\boldsymbol{J} = \sigma \boldsymbol{f} \quad \Rightarrow \quad \boldsymbol{f} = 0$$

$$\Rightarrow \int_a^b \boldsymbol{f}_s \cdot d\boldsymbol{\ell} + \int_a^b \mathbf{E} \cdot d\boldsymbol{\ell} = 0 \Rightarrow$$ the work done by both $\boldsymbol{f}_s$ and $\mathbf{E}$ in transporting charge from $a$ to $b$ is 0

$$\Rightarrow V = -\int_a^b \mathbf{E} \cdot d\boldsymbol{\ell} = \int_a^b \boldsymbol{f}_s \cdot d\boldsymbol{\ell} = \oint \boldsymbol{f}_s \cdot d\boldsymbol{\ell} = \mathscr{E} \quad \Leftarrow \quad \mathbf{E} = -\boldsymbol{f}_s$$

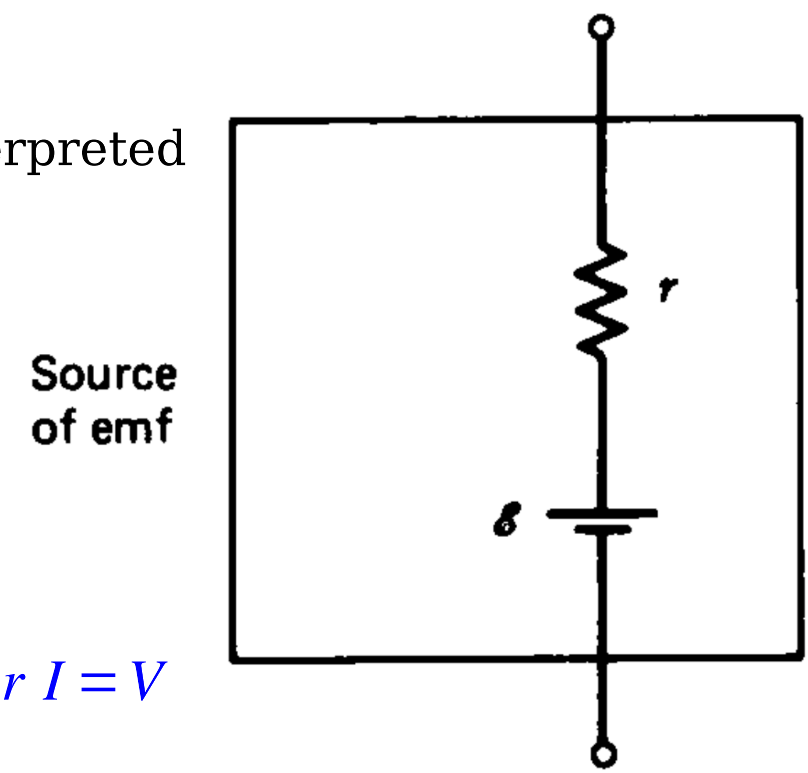
- The function of a battery is to establish and maintain a voltage difference equal to the electromotive force. The resulting electrostatic field drives current around the rest of the circuit.

- Because it's the line integral of $\boldsymbol{f}_s$, $\mathscr{E}$ can be interpreted as the *work done per unit charge*, by the source.

- If there is resistance to the current flow inside the source, then current can flow only if

$$\int_a^b (\boldsymbol{f}_s + \mathbf{E}) \cdot d\boldsymbol{\ell} = \int_a^b \boldsymbol{f} \cdot d\boldsymbol{\ell} > 0$$

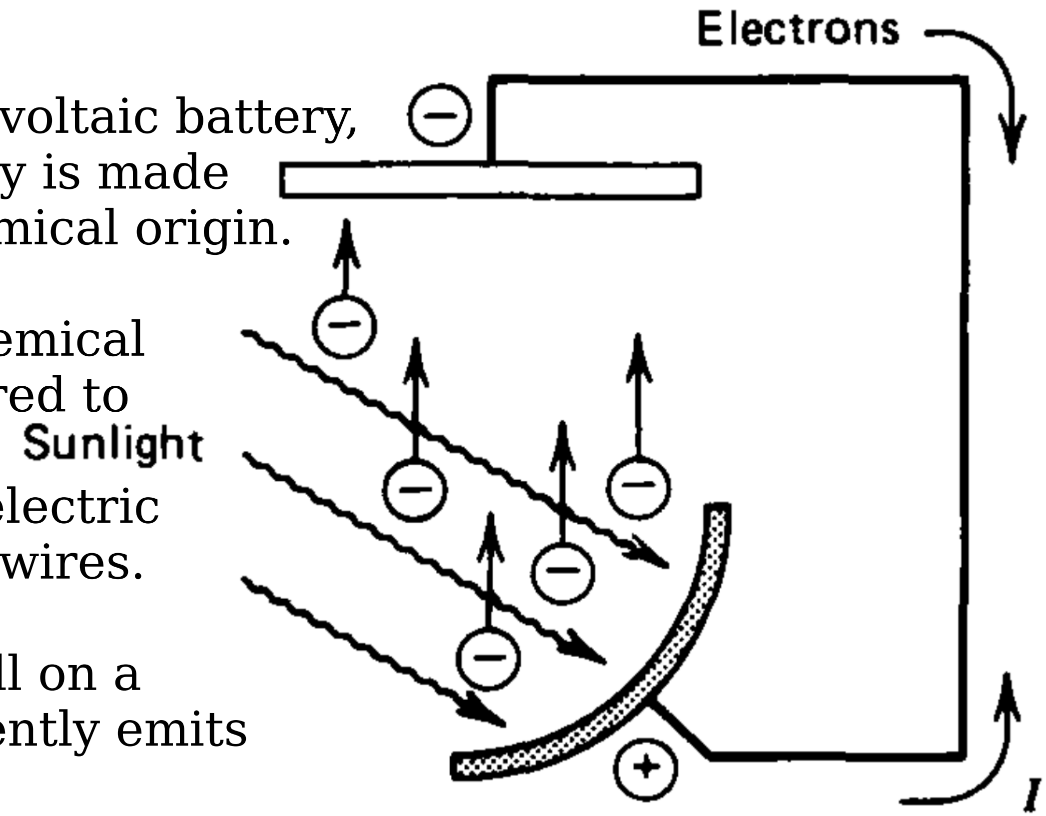
$$\Rightarrow V_{ab} = \int_a^b \boldsymbol{f} \cdot d\boldsymbol{\ell} = r I \Leftarrow r:$$ internal resistance $$\Rightarrow \mathscr{E} - r I = V$$



## Batteries

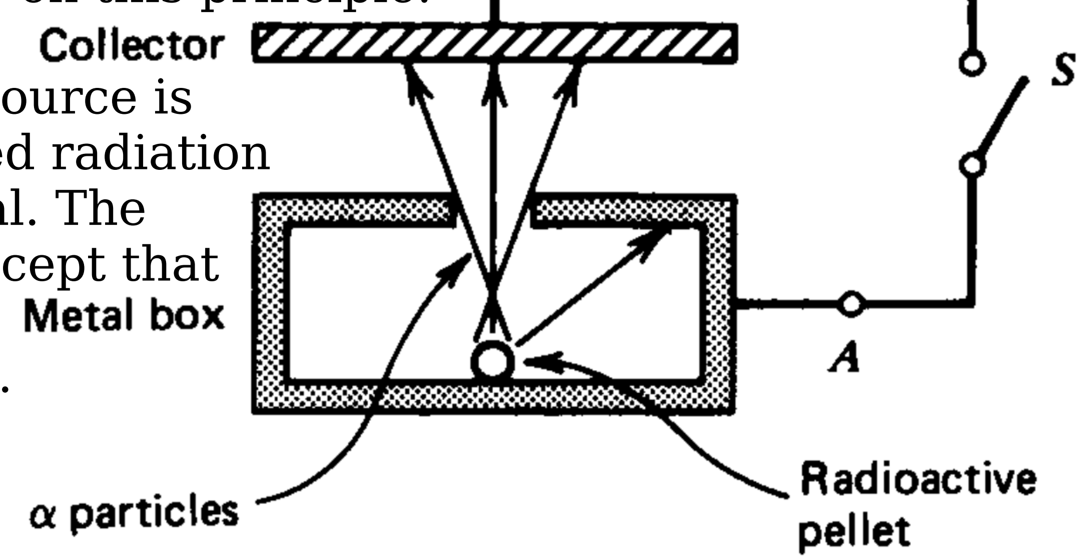
- A common emf source is the ordinary voltaic battery, in which the mechanism whereby energy is made available to produce currents has a chemical origin.

- Chemical reactions occur in which chemical energy is released to do the work required to produce a charge separation. The charge separation in turn produces an electric potential difference to move charges in wires.




- For solar batteries, rays of sunlight fall on a sensitive metal surface, which consequently emits electrons via the photoelectric effect.

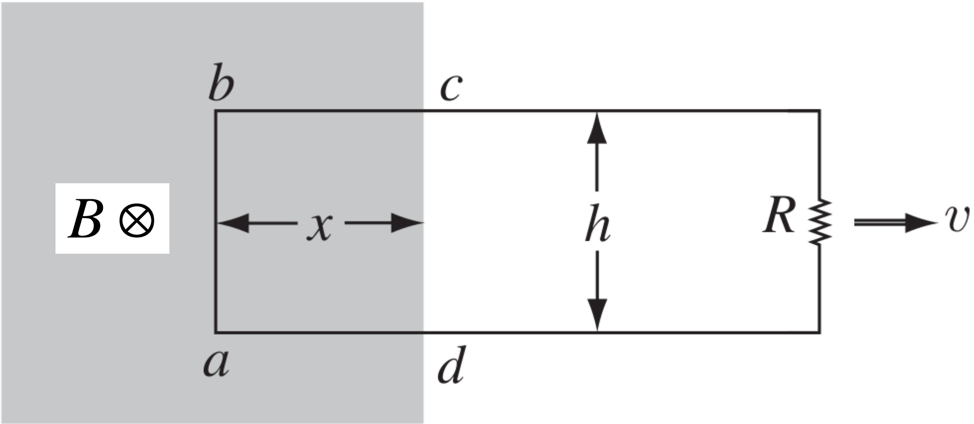
- The light meters on cameras operate on this principle.



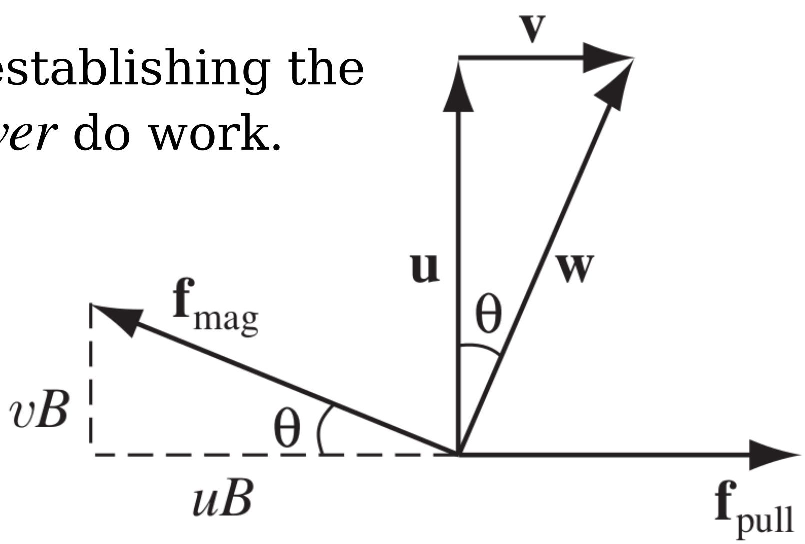

- For a nuclear battery, a radioactive source is placed at one terminal, and the charged radiation emitted is collected at another terminal. The action is similar to the solar battery except that the source of energy here is nuclear rather than electromagnetic (sunlight).

- The common characteristic of all sources of emf is their ability to effect a charge separation.

# Motional emf




- **Generators** exploit **motional emfs**, which arise when you *move a wire through a magnetic field*.
- $\mathcal{E} = \oint \boldsymbol{f}_{\text{mag}} \cdot d\boldsymbol{\ell} = v B h = \left| \frac{d\Phi_B}{dt} \right|$
- The integrals performed to calculate $\mathcal{E}$ is carried out at *one instant of time*.
- Although the magnetic force is responsible for establishing the emf, it is *not* doing any work—magnetic forces *never* do work.



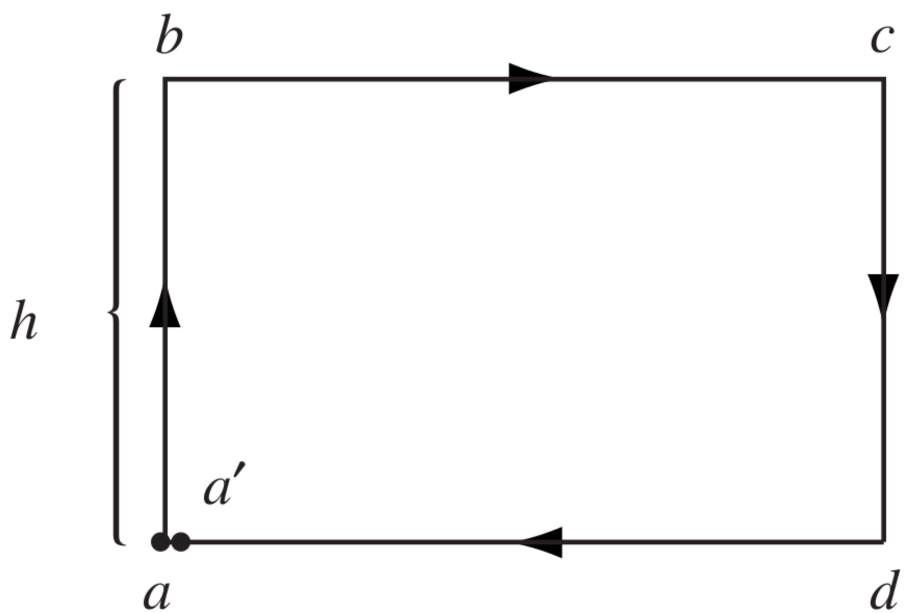

- The person pulling on the loop *is* supplying the energy that heats the resistor.
- With the current flowing, the free charges in segment *ab* have a vertical velocity (call it **u**) in addition to the horizontal velocity **v** from the motion of the loop. So the magnetic force has a component $q u B$ to the left.
- To counteract this, the person pulling on the wire must exert a force per unit charge $f_{\text{pull}} = u B$ to the *right*. This force is transmitted to the charge by the structure of the wire.

- Meanwhile, the particle is actually *moving* in the direction of the resultant velocity **w**, and the distance it goes is $h \sec\theta$. The work done per unit charge is

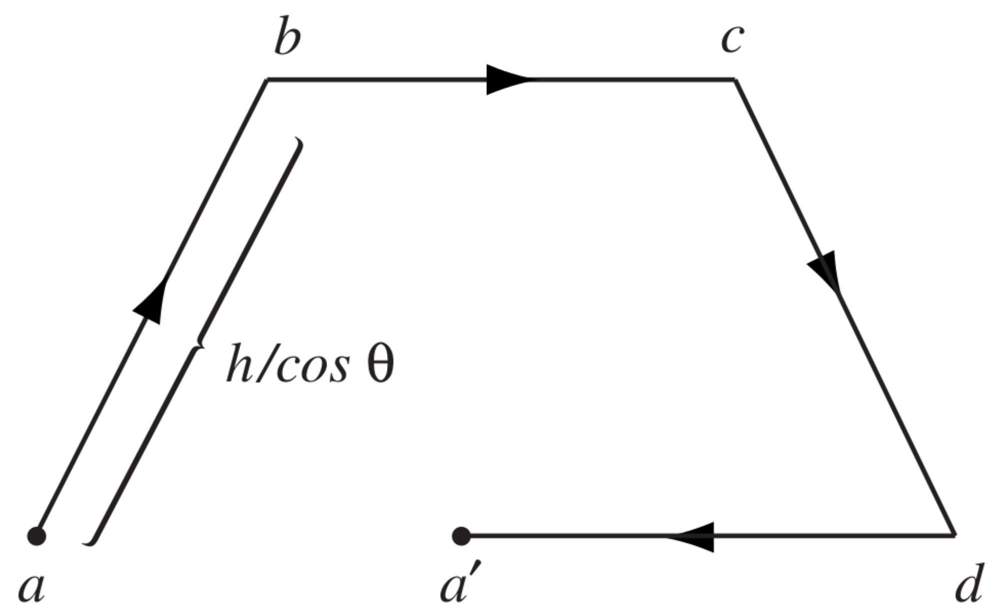
$$\mathbb{W} = \int \boldsymbol{f}_{\text{pull}} \cdot d\boldsymbol{\ell} = u\,B\,h \sec\theta \sin\theta = v\,B\,h = \mathcal{E}$$

- So the *work done per unit charge is exactly equal to the emf*, though the integrals are taken along different paths, and completely different forces are involved.

- To calculate the emf, you integrate around the loop at *one instant*, but to calculate the work done you follow a charge in its journey around the loop.




(a) Integration path for computing $\mathcal{E}$ (follow the wire at one instant of time).




(b) Integration path for calculating work done (follow the charge around the loop)

- $\boldsymbol{f}_{\text{pull}}$ contributes nothing to the emf $\mathscr{E}$, because it $\perp$ the wire, whereas $\boldsymbol{f}_{\text{mag}}$ contributes nothing to work $\mathbb{W}$ because it $\perp$ the motion of the charge.

- Let $\Phi_B$ be the flux of $\mathbf{B}$ through the loop: $\Phi_B \equiv \int \mathbf{B} \cdot \mathrm{d}\boldsymbol{a} \Rightarrow \Phi_B = B\,h\,x$

$$\Rightarrow \frac{\mathrm{d}\,\Phi_B}{\mathrm{d}\,t} = B\,h\,\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = -B\,h\,v \Rightarrow \mathscr{E} = -\frac{\mathrm{d}\,\Phi_B}{\mathrm{d}\,t}$$

- The **flux rule** for motional emf: the emf generated in the loop is minus the rate of change of flux through the loop.

- The flux rule has the virtue of applying to *non*rectangular loops moving in *arbitrary* directions through *non*uniform magnetic fields; the loop need not even maintain a fixed shape.

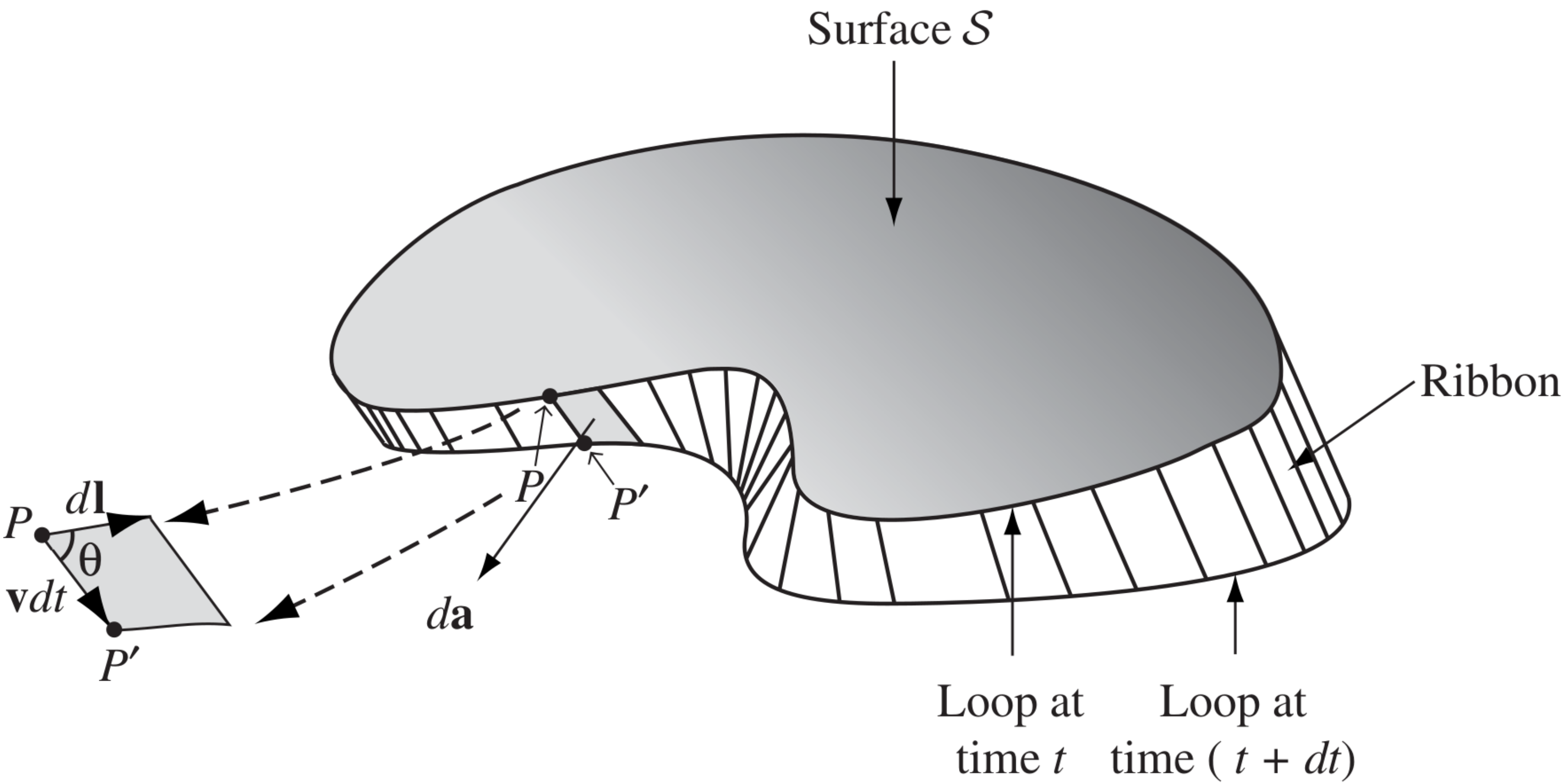
Proof: the figure shows a loop of wire at time $t$, and also a short time $\mathrm{d}\,t$ later.

- The change in flux: $\mathrm{d}\,\Phi_B = \Phi_B(t + \mathrm{d}\,t) - \Phi_B(t) = \Phi_{\text{ribbon}} = \int_{\text{ribbon}} \mathbf{B} \cdot \mathrm{d}\boldsymbol{a}$

- Point $P$ moves to $P'$ in time $\mathrm{d}\,t$. $\mathbf{v}$ is the velocity of the wire, and $\mathbf{u}$ is the velocity of a charge down the wire; $\mathbf{w} = \mathbf{v} + \mathbf{u}$ is the resultant velocity of a charge at $P$.

- The infinitesimal element of area on the ribbon: $\mathrm{d}\boldsymbol{a} = (\mathbf{v} \times \mathrm{d}\boldsymbol{\ell})\,\mathrm{d}\,t$ and $\mathbf{u} \parallel \mathrm{d}\boldsymbol{\ell}$

$$\Rightarrow \frac{\mathrm{d}\,\Phi_B}{\mathrm{d}\,t} = \oint \mathbf{B} \cdot \mathbf{v} \times \mathrm{d}\boldsymbol{\ell} = \oint \mathbf{B} \cdot \mathbf{w} \times \mathrm{d}\boldsymbol{\ell} = -\oint \mathbf{w} \times \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = -\oint \boldsymbol{f}_{\text{mag}} \cdot \mathrm{d}\boldsymbol{\ell} = -\mathscr{E}$$

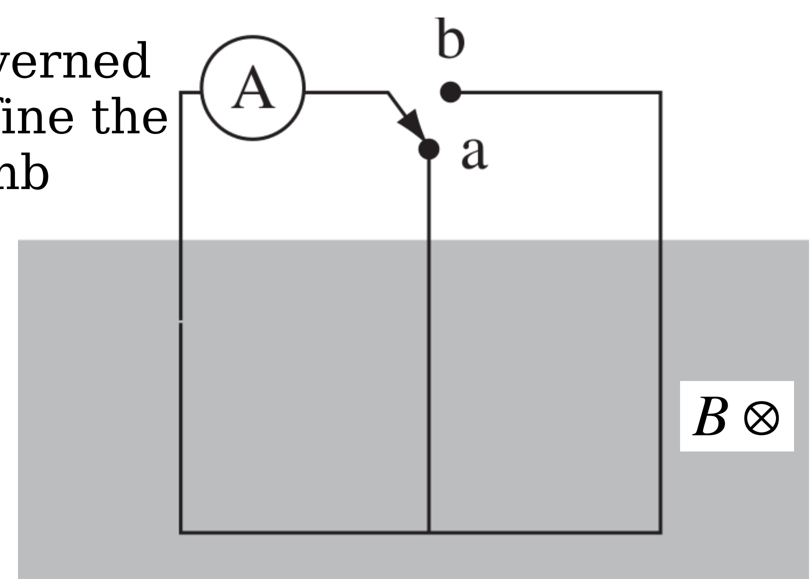



Enlargement of $d\mathbf{a}$

- In applying the flux rule, sign consistency is governed (as always) by your right hand: If your fingers define the positive direction around the loop, then your thumb indicates the direction of $d\boldsymbol{a}$.

- If the emf comes out negative, the current will flow in the negative direction around the circuit.

- A "flux rule paradox" involves the circuit.



- When the switch is thrown (from $a$ to $b$) the flux through the circuit doubles, but there's no motional emf (no conductor moving through a magnetic field), and the ammeter ($A$) records no current.
- $\mathscr{E} = -\frac{d\Phi_B}{dt}$ may represent

**1**. The change in magnetic flux through a loop fixed in space (in our reference system) due to variation of $\mathbf{B}(t)$ in time.

**2**. The change in magnetic flux through a well-defined conducting loop which moves (relative to our reference system) through $\mathbf{B}$ constant in time.

**3**. The flux "swept out" by a conducting loop as it changes its dimensions in the presence of $\mathbf{B}$ constant in time.

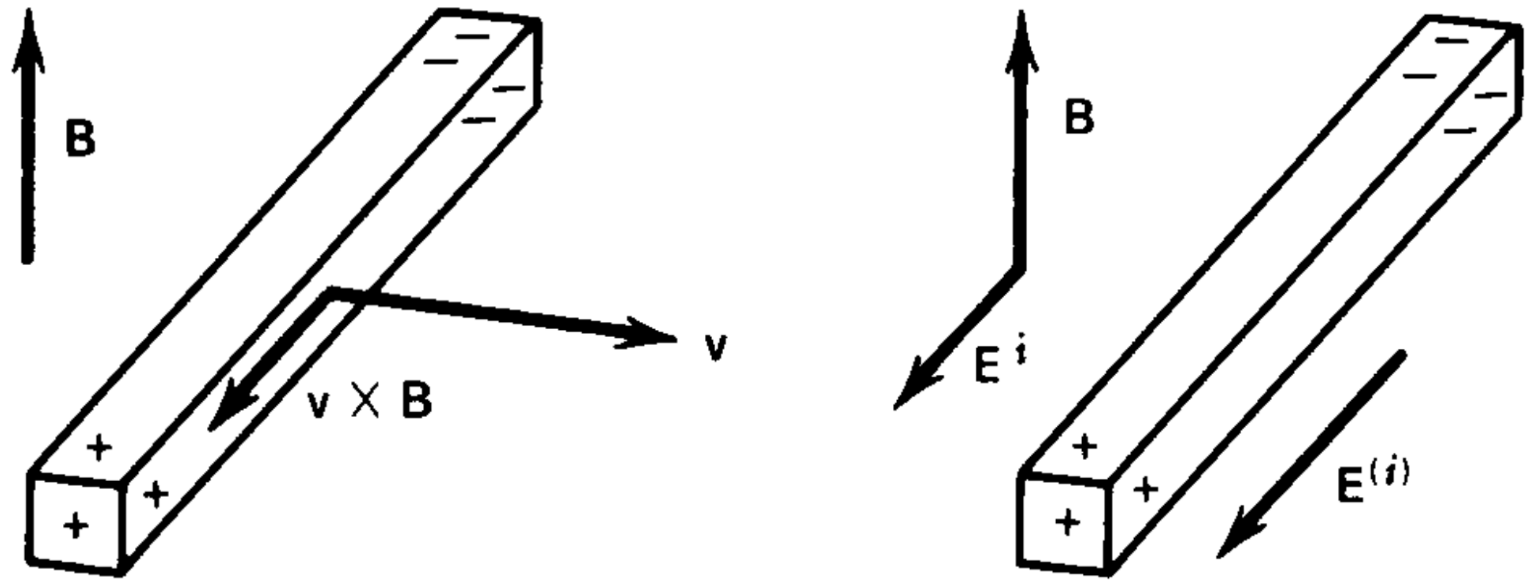
**4**. A linear combination of items 1 & 2 or 1 & 3 above.

- $$\mathscr{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int \mathbf{B}\cdot d\boldsymbol{a} = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\boldsymbol{a} - \int \mathbf{B}\cdot\frac{\partial}{\partial t} d\boldsymbol{a}$$

$$= \int \nabla\times\mathbf{E}_F\cdot d\boldsymbol{a} - \oint \mathbf{B}\cdot\mathbf{v}\times d\boldsymbol{\ell} \quad \Leftarrow \quad \text{Faraday's Law } \frac{\partial \mathbf{B}}{\partial t} + \nabla\times\mathbf{E} = 0$$ mention later

$$= \oint(\mathbf{E}_F + \mathbf{v}\times\mathbf{B})\cdot d\boldsymbol{\ell} = \oint(\mathbf{E}_F + \mathbf{E}_L)\cdot d\boldsymbol{\ell} = \oint \mathbf{E}_i\cdot d\boldsymbol{\ell}$$

Example: Conducting Bar Moving Through a Constant Magnetic Field




- From the rest frame, charges under the influence of the Lorentz force and the forces constraining the charges to remain in the bar will move until equilibrium is established,

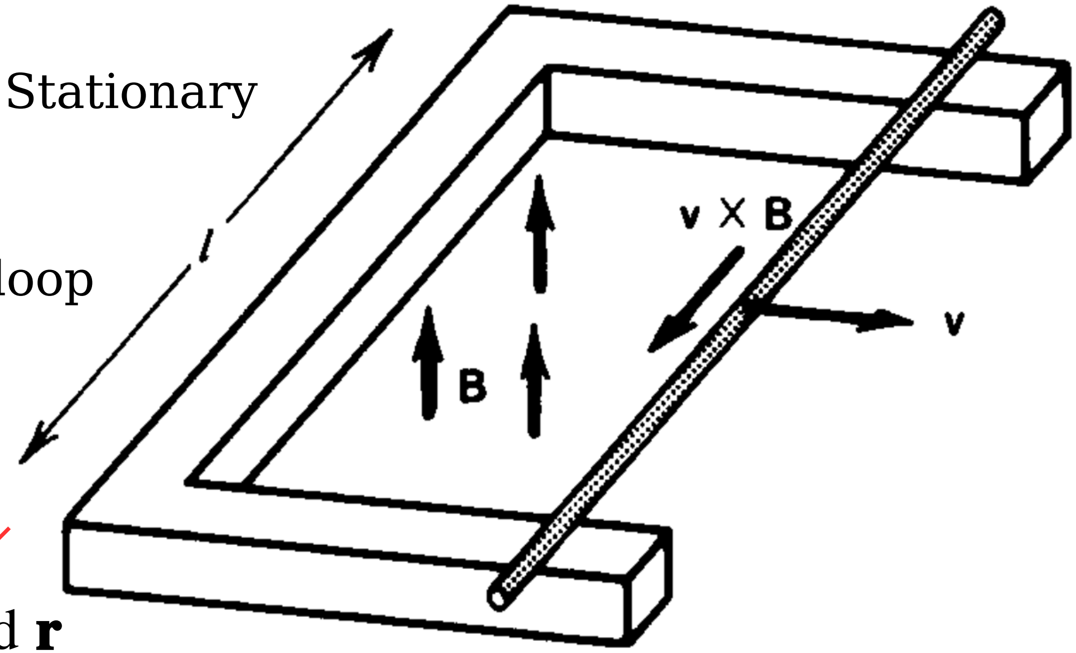
$$\mathbf{F} = q\,\mathbf{v}\times\mathbf{B} + q\,\mathbf{E} = 0 \quad \Rightarrow \quad \mathbf{E} = -\mathbf{v}\times\mathbf{B}$$

- From the observer moving with the bar, an electric field $\mathbf{E}_i$ is observed everywhere in space having the constant value

$$\mathbf{F} = q\,\mathbf{E}_i + q\,\mathbf{E} = 0 \quad \Rightarrow \quad \mathbf{E} = -\mathbf{E}_i \quad \Rightarrow \quad \mathbf{E}_i = \mathbf{v}\times\mathbf{B}$$

- This argument stands in the nonrelativistic cases, ie, $v \ll c$.

Example: Conducting Bar Moving on Stationary Tracks Through $\mathbf{B}$ Field




- If we calculate the emf around the loop at any instant, $\mathbf{u}$: drift velocity,

$$\mathscr{E} = \int_{\text{bar}} (\mathbf{v}\times\mathbf{B})\cdot \mathrm{d}\,\mathbf{r} - \int_{\text{bar}} \mathbf{E}\cdot \mathrm{d}\,\mathbf{r} - \int_{\text{track}} \mathbf{E}\cdot \mathrm{d}\,\mathbf{r} + \int_{\text{bar + track}} (\mathbf{u}\times\mathbf{B})\cdot \mathrm{d}\,\mathbf{r}$$

$$= \int_{\text{bar}} (\mathbf{v}\times\mathbf{B})\cdot \mathrm{d}\,\mathbf{r} - \oint \mathbf{E}_q\cdot \mathrm{d}\,\mathbf{r} = -v\,B\,\ell = -\frac{\mathrm{d}\,\Phi_B}{\mathrm{d}\,t} \;\Rightarrow\; I = \frac{\mathscr{E}}{R} = -\frac{v\,B\,\ell}{R}$$

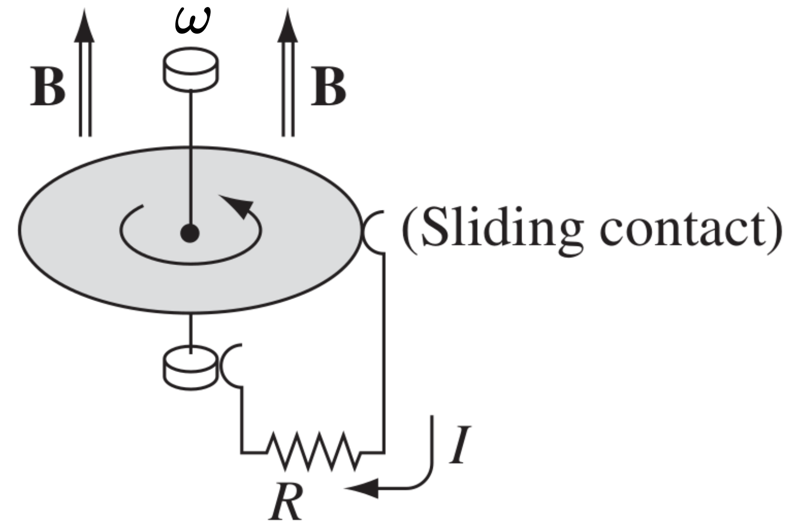
- From the co-moving frame with the bar, the observer sees the electric field as

$$\mathbf{E}' = \mathbf{E} + \mathbf{E}_i = \mathbf{E} + \mathbf{v}\times\mathbf{B} \;\Rightarrow\; \int_{\text{bar}} \mathbf{E}'\cdot \mathrm{d}\,\mathbf{r} = \int_{\text{bar}} \mathbf{E}\cdot \mathrm{d}\,\mathbf{r} + \int_{\text{bar}} (\mathbf{v}\times\mathbf{B})\cdot \mathrm{d}\,\mathbf{r} = 0$$

- However, the moving observer sees the U-shaped section of the loop moving with velocity $\mathbf{v}' = -\mathbf{v}$. So the moving observer will find an emf $|\,v\,B\,\ell\,|$ due to the moving loop.

- The sense of current flow will be the same as found above. Again, consistent results are obtained for the 2 observers.

Example 7.4: Find the current in the resistor.




- The velocity at a distance $s$ from the axis is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{s} \quad \Rightarrow \quad \boldsymbol{f}_{\text{mag}} = \mathbf{v} \times \mathbf{B} = \omega\, B\, \mathbf{s}$$

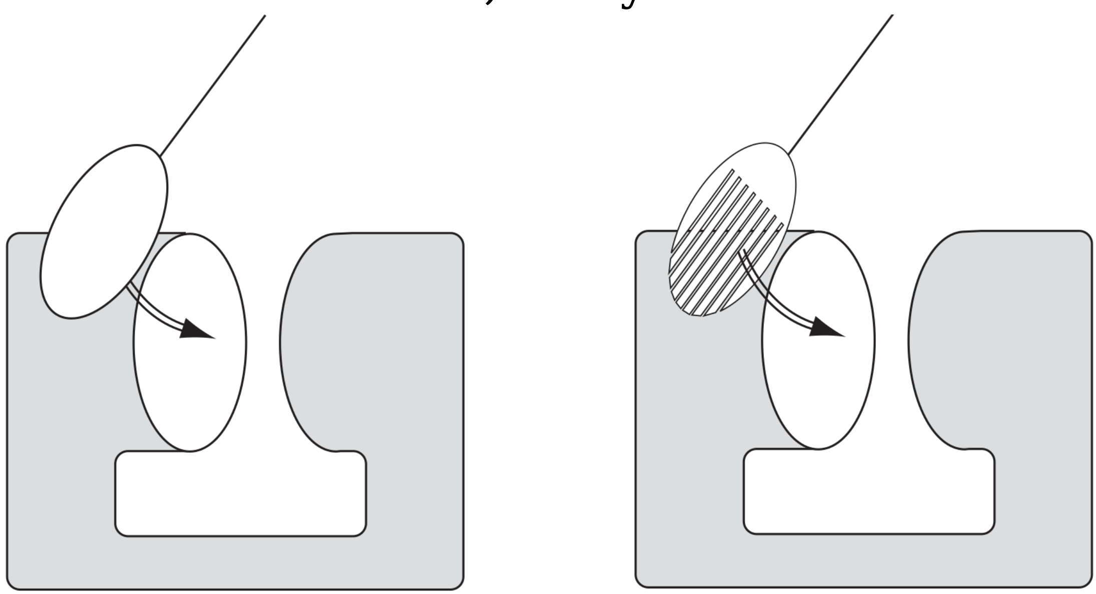
$$\Rightarrow \quad \mathcal{E} = \int \boldsymbol{f}_{\text{mag}} \cdot \mathrm{d}\,\mathbf{s} = \omega\, B \int_0^a s\, \mathrm{d}\, s = \frac{\omega\, B\, a^2}{2}$$

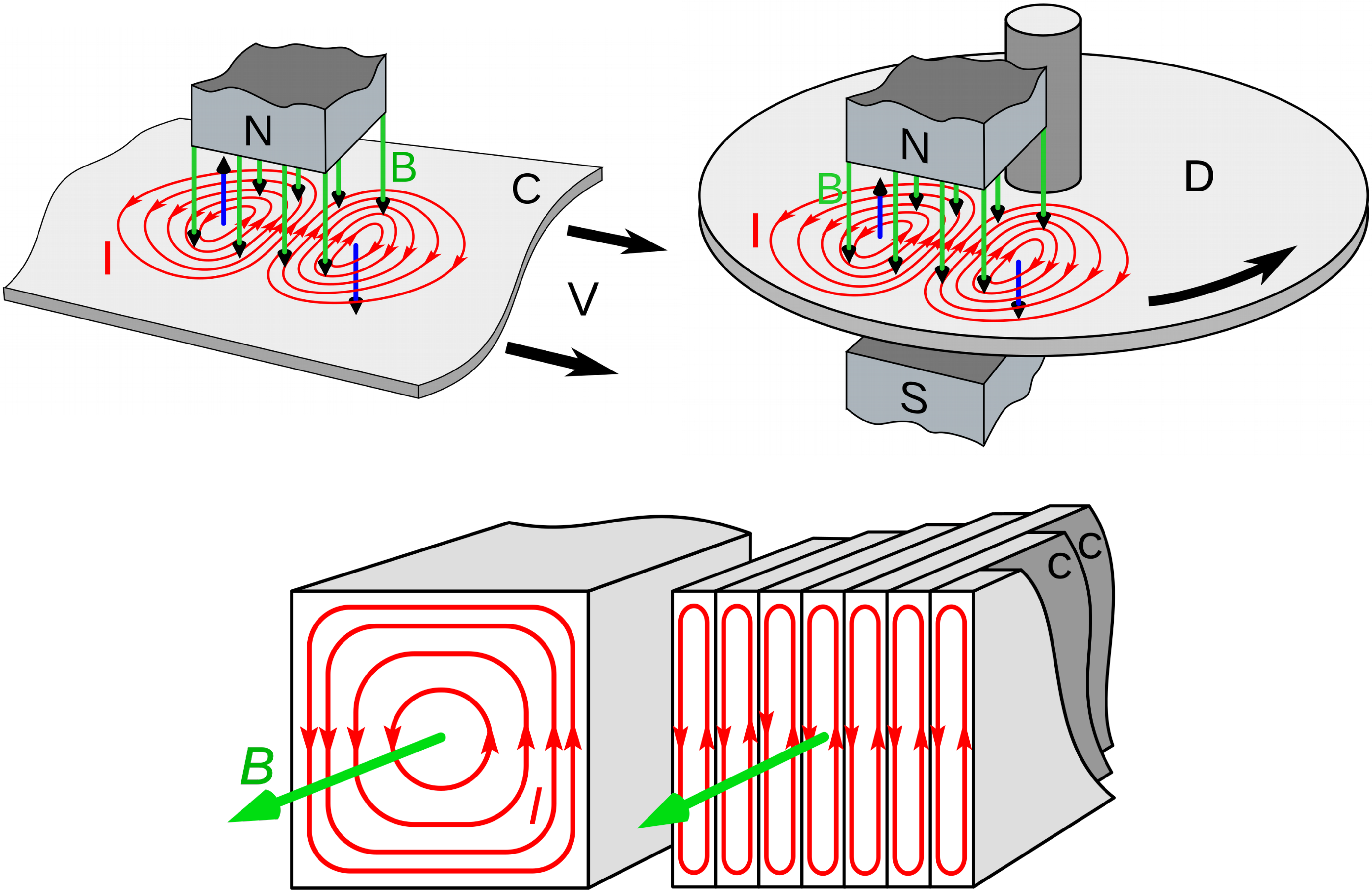
$$\Rightarrow \quad I = \frac{\mathcal{E}}{R} = \frac{\omega\, B\, a^2}{2\, R}$$

- This example (the **Faraday disk**, or **Faraday dynamo**) involves a motional emf that you can't calculate from the flux rule.

- For **eddy currents**, take a chunk of metal and shake it around in a nonuniform magnetic field. Currents will be generated in the material, and you will feel a kind of "viscous drag."

- To confirm that eddy currents are responsible, one repeats the demonstration using a disk that has many slots cut in it, to prevent the flow of large-scale currents. This time the disk swings freely, unimpeded by the field.

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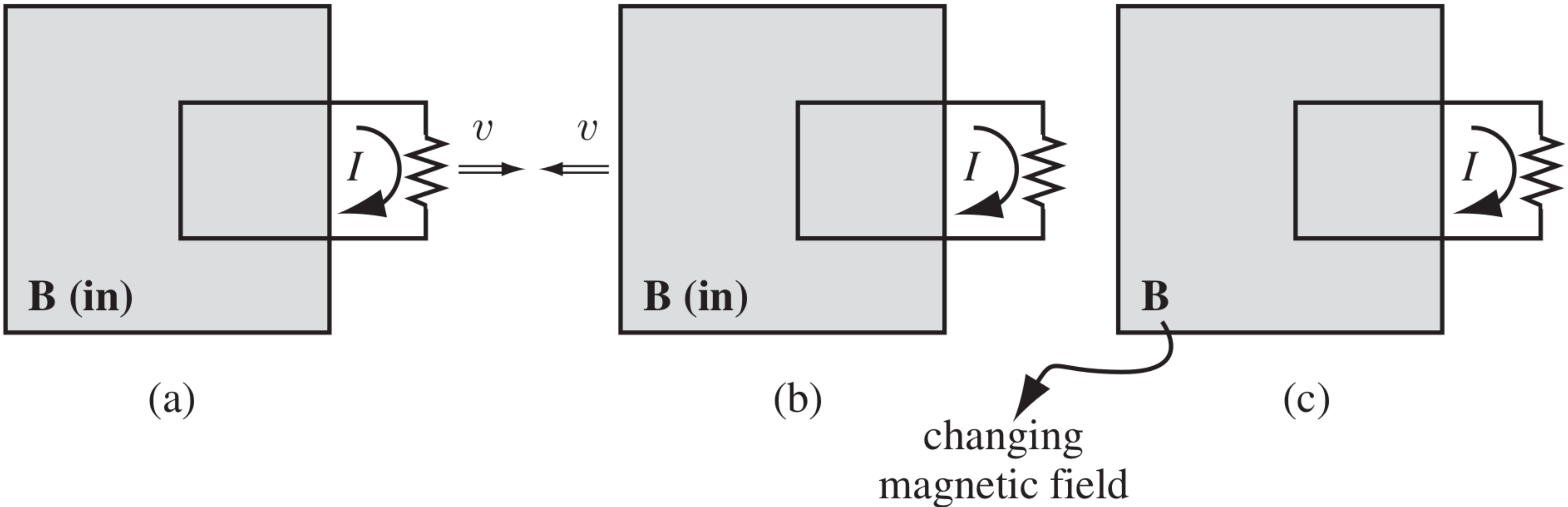
## Electromagnetic Induction

### Faraday's Law

- Faraday's 3 experiments:

**Experiment 1**: Pull a loop of wire to the right through a magnetic field.
A current flows in the loop.
**Experiment 2**: Move the magnet to the left, holding the loop still.
A current flows in the loop.
**Experiment 3**: With both the loop and the magnet at rest, change the strength of the field. Once again, current flows in the loop.

- The 1st experiment is a straightforward case of motional emf; the flux rule gives $\mathscr{E} = -\frac{d\Phi_B}{dt}$



- The first 2 cases show that all the idea matters is the *relative* motion of the magnet and the loop. Indeed, in the light of special relativity it has to be so.

- If the *loop* moves, it's a *magnetic* force that sets up the emf, but if the loop is *stationary*, the force *cannot* be magnetic—stationary charges experience no magnetic forces.

- Faraday came up an ingenious inspiration:

A changing magnetic field induces an electric field.

- This induced electric field accounts for the emf in Experiment 2. And the emf is again equal to the rate of change of the flux,

$$\mathscr{E} = \oint \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt} \Rightarrow \oint \mathbf{E} \cdot d\boldsymbol{\ell} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\boldsymbol{a} \quad \text{Faraday's law}$$

$$\Rightarrow \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Leftarrow \text{Stokes' theorem}$$

- Faraday's law reduces to the old rule $\oint \mathbf{E} \cdot d\boldsymbol{\ell} = 0$ (or, $\nabla \times \mathbf{E} = 0$) in the static case (constant $\mathbf{B}$).

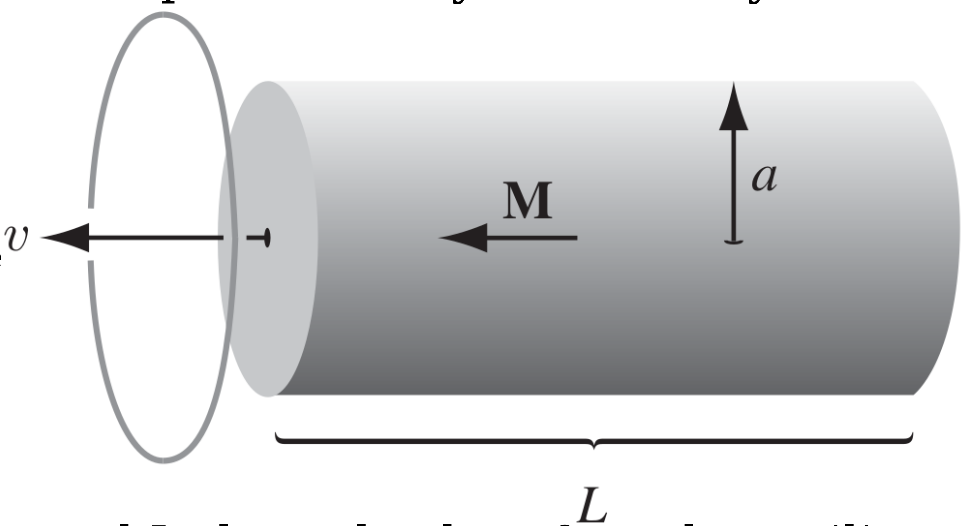
- In Experiment 3, the magnetic field changes for totally different reasons, but according to Faraday's law an electric field is induced, giving rise to emf $-\frac{d\Phi_B}{dt}$

- All 3 cases can be subsumed into a kind of **universal flux rule**:

**Whenever (and for whatever reason) the magnetic flux through a loop changes, an emf $\mathcal{E} = -\frac{d\,\Phi_B}{d\,t}$ will appear in the loop.**

- In Experiment 1 it's the Lorentz force law at work; the emf is *magnetic*. But in the other two it's an *electric* field (induced by the changing magnetic field) that does the job.

- It is astonishing that all 3 processes yield the same formula for the emf. In fact, it was this "coincidence" that led Einstein to the special theory of relativity.



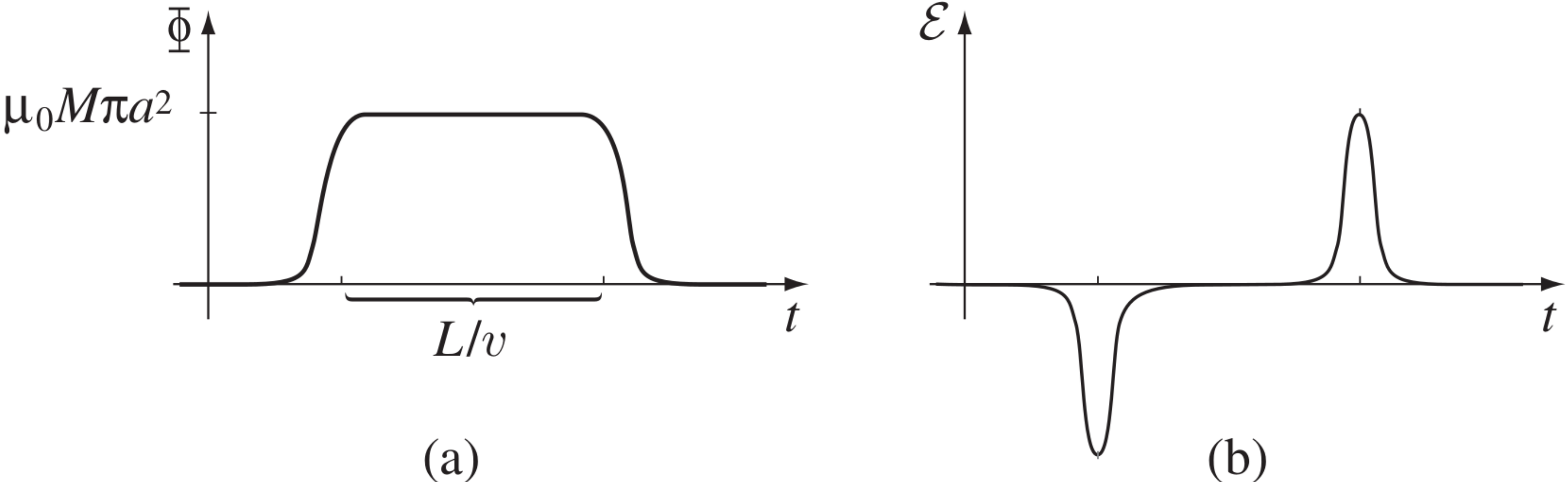

Example 7.5: Graph the emf induced in the ring, as a function of time.

- Surface current and magnetic field inside

$$\boldsymbol{K}_b = M\,\hat{\boldsymbol{\phi}} \qquad \mathbf{B} = \mu_0\,\mathbf{M}$$

- The flux through the ring is 0 when the magnet is far; it builds up to a maximum of $\mu_0 M \pi a^2$ as the leading end passes through; and It drops back to 0 as the trailing end emerges.

- The emf is (minus) the time derivative of $\Phi_B$, so it consists of 2 spikes.




- The right-hand rule does the job to know which *way* around the ring the induced current flows.

- $\Phi_B$ is positive to the left in the figure, the positive direction for current in the ring is counterclockwise; since the first spike is *negative*, the 1st current pulse flows *clockwise*, and the 2nd counterclockwise.

- **Lenz's law** helps to get the directions right: **Nature abhors a change in flux.**

- The induced current will flow in such a direction that the flux it produces tends to cancel the change.

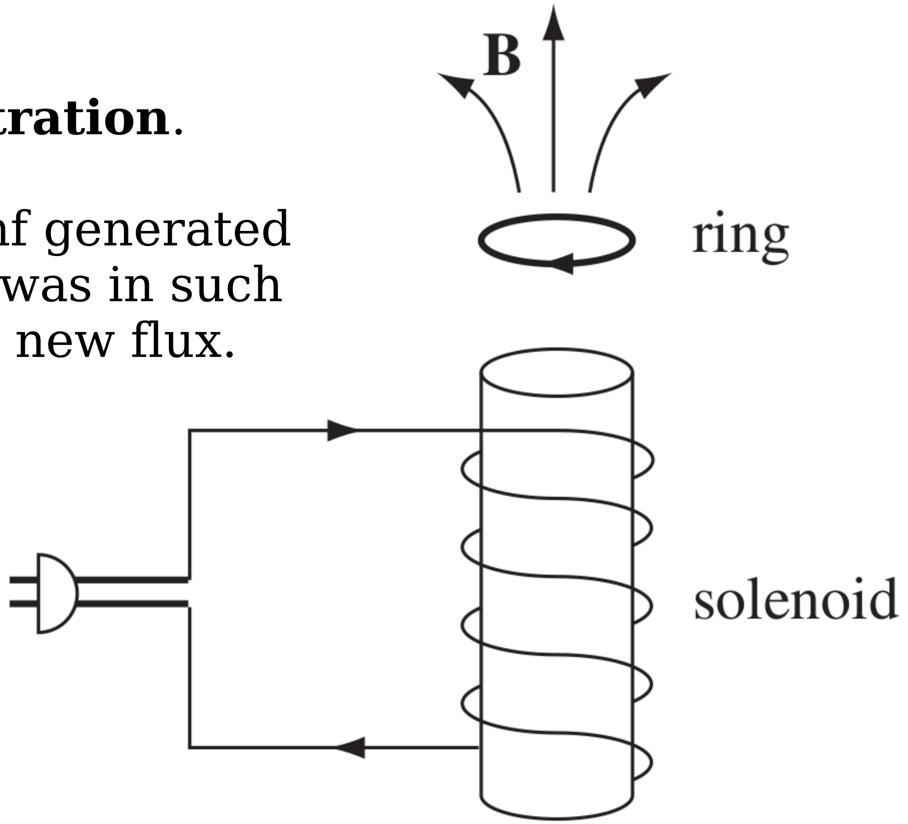
- Notice that it is the *change* in flux, not the flux itself, that nature abhors.

- Faraday induction is a kind of "inertial" phenomenon: A conducting loop "likes" to maintain a constant flux through it; if you change the flux, the loop responds by sending a current around in such a direction as to frustrate your efforts.

Example 7.6: **The “jumping ring” demonstration**.

- When a flux appeared upwards, and the emf generated in the ring led to a current in the ring which was in such a direction that its field tended to cancel this new flux.

- Thus the current in the loop is opposite to the current in the solenoid. And opposite currents repel, so the ring flies off.



## The Induced Electric Field

- Faraday's law generalizes the electrostatic rule $\nabla\times\mathbf{E}=0$ to the time-dependent régime. The *divergence* of $\mathbf{E}$ is still given by Gauss's law ( $\nabla\cdot\mathbf{E}=\frac{\rho}{\epsilon_0}$ ).

- If $\mathbf{E}$ is a *pure* Faraday field (due exclusively to a changing $\mathbf{B}$, with $\rho=0$),

$$\nabla\cdot\mathbf{E}=0\,,\quad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\quad [\text{vs}\quad \nabla\cdot\mathbf{B}=0\,,\quad \nabla\times\mathbf{B}=\mu_0\boldsymbol{J}\quad (\text{magnetostatics})]$$

- Faraday-induced electric fields are determined by $-\frac{\partial\mathbf{B}}{\partial t}$ in exactly the same way as magnetostatic fields are determined by $\mu_0\boldsymbol{J}$.

- The analog to Biot-Savart is $\mathbf{E}=-\frac{1}{4\pi}\int\frac{\partial_t\mathbf{B}\times\hat{\mathbb{r}}}{\mathbb{r}^2}\,\mathrm{d}\tau=-\frac{1}{4\pi}\frac{\partial}{\partial t}\int\frac{\mathbf{B}\times\hat{\mathbb{r}}}{\mathbb{r}^2}\,\mathrm{d}\tau$

- If symmetry permits, we can use all the tricks associated with Ampère's law in integral form ($\oint\mathbf{B}\cdot\mathrm{d}\boldsymbol{\ell}=\mu_0\boldsymbol{I}_{\mathrm{enc}}$), only now it's *Faraday's* law in integral form:

$$\oint\mathbf{E}\cdot\mathrm{d}\boldsymbol{\ell}=-\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$$

- The rate of change of (magnetic) flux through the Amperian loop plays the role formerly assigned to $\mu_0\boldsymbol{I}_{\mathrm{enc}}$ .

- $\nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}=-\dfrac{\partial}{\partial t}\nabla\times\mathbf{A}=\nabla\times\left(-\dfrac{\partial\mathbf{A}}{\partial t}\right)\quad\Leftarrow\quad\mathbf{B}=\nabla\times\mathbf{A}$

$$\Rightarrow\quad\nabla\times\left(\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}\right)=0\quad\Rightarrow\quad\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}=-\nabla\Phi\quad\Rightarrow\quad\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\Rightarrow\quad\mathbf{E}=\mathbf{E}_q+\mathbf{E}_i\quad\Leftarrow$$

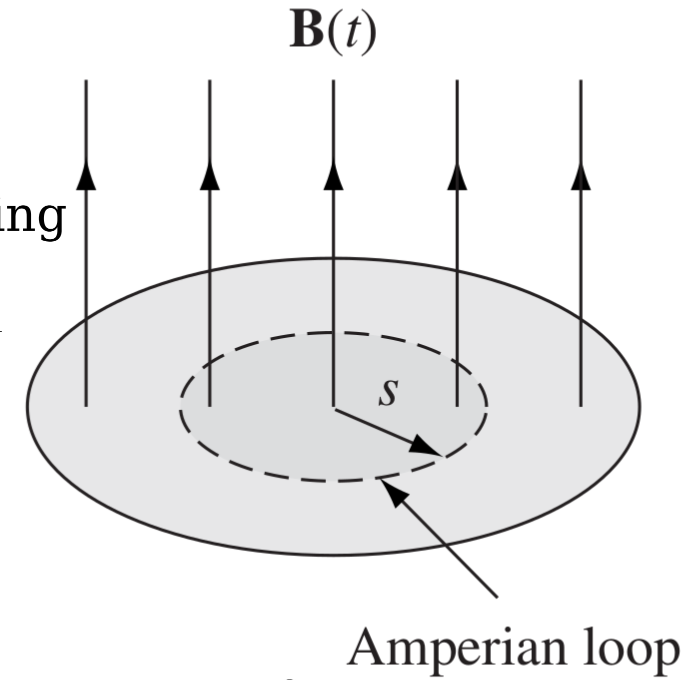
$\mathbf{E}_q\equiv-\nabla\Phi$ : from charge, conservative (nonsolenoidal)

$\mathbf{E}_i\equiv-\dfrac{\partial\mathbf{A}}{\partial t}$ : induced field, nonconservative (solenoidal)

$$\Rightarrow\quad\nabla\times\mathbf{E}_q=0\,,\quad\nabla\times\mathbf{E}_i=-\frac{\partial\mathbf{B}}{\partial t}$$

$$\Rightarrow\quad\oint_c\mathbf{E}\cdot d\boldsymbol{\ell}=\oint_c\mathbf{E}_i\cdot d\boldsymbol{\ell}=-\frac{d\Phi_B}{dt}=-\frac{d}{dt}\int_S\mathbf{B}\cdot d\boldsymbol{a}=-\frac{d}{dt}\oint_c\mathbf{A}\cdot d\boldsymbol{\ell}$$

<u>Example 7.7</u>: A uniform magnetic field $\mathbf{B}(t)$, pointing upwards, fills the shaded circular region. If $\mathbf{B}$ is changing with time, what is the induced electric field?

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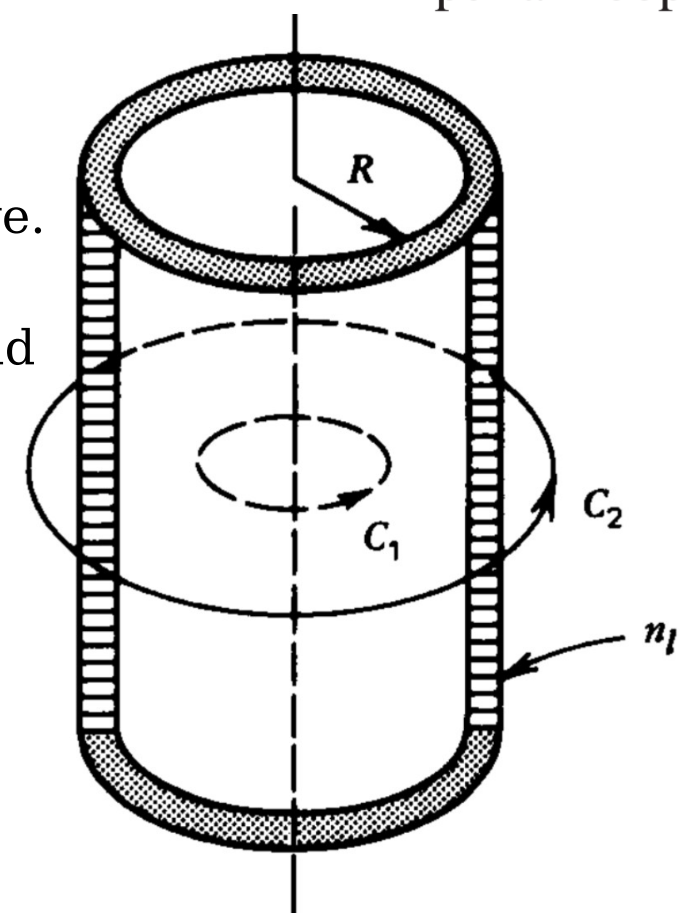
- $\mathbf{E}$ points in the circumferential direction. Draw an Amperian loop of radius $s$, and apply Faraday's law:

$$\mathscr{E} = \oint \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = E\cdot 2\pi s = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\left[\pi s^2 B(t)\right]$$

$$= -\pi s^2 \frac{\mathrm{d}B}{\mathrm{d}t} \quad \Rightarrow \quad \mathbf{E} = -\frac{s}{2}\frac{\mathrm{d}B}{\mathrm{d}t}\hat{\phi}$$

- If $\mathbf{B}$ is *increasing*, $\mathbf{E}$ runs *clockwise* viewed from above.

- If $\mathbf{B}$ is produced inside a long solenoid of radius $R$ and of turn density $n$, then

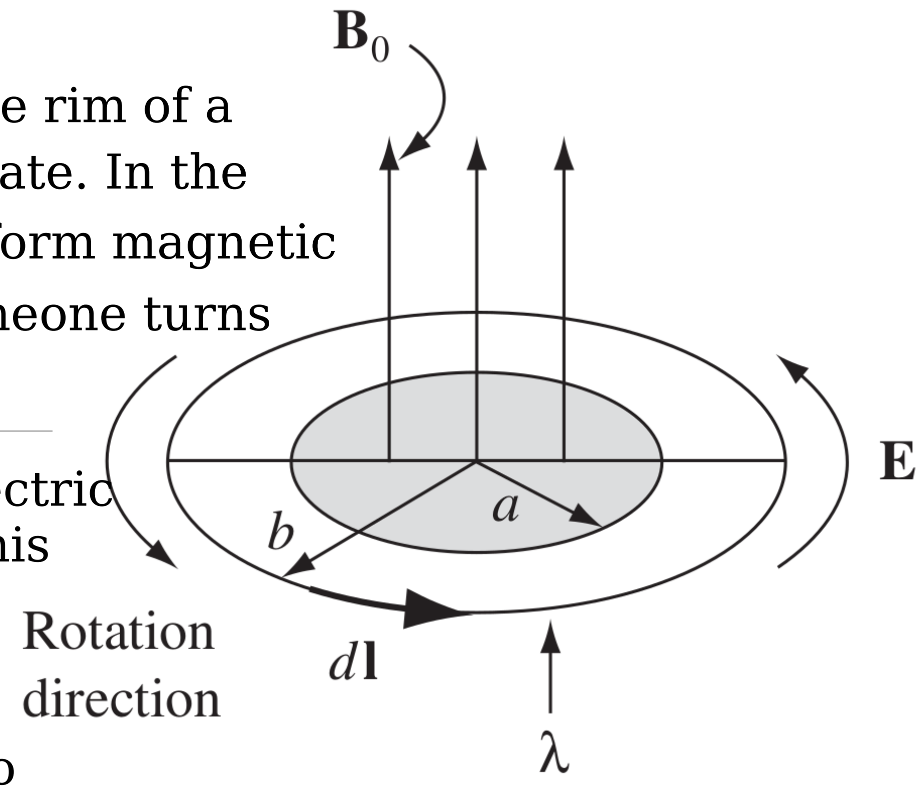



$$\frac{\mathrm{d}B}{\mathrm{d}t} = \mu_0 n \frac{\mathrm{d}I}{\mathrm{d}t} \quad \Rightarrow \quad \mathscr{E} = -\mu_0 n \pi s^2 \frac{\mathrm{d}I}{\mathrm{d}t}$$

$$\Rightarrow \quad \mathbf{E} = -\frac{\mu_0 n s}{2}\frac{\mathrm{d}I}{\mathrm{d}t}\hat{\phi} \quad \text{for} \quad s < R$$

- $\mathscr{E}$ and $\mathbf{E}$ for $s > R$ can be calculated in a similar way.

Example 7.8: A line charge $\lambda$ is glued onto the rim of a wheel of radius $b$ horizontally, and free to rotate. In the central region, out to radius $a$, there is a uniform magnetic field $\mathbf{B}_0$ pointing up. What happens when someone turns the field off?

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- The changing magnetic field induces an electric field, curling around the axis of the wheel. This electric field exerts a force on the charges at the rim, and the wheel starts to turn.

- By Lenz's law, its rotation direction tends to restore the upward flux; it's counterclockwise.

- $$\mathcal{E} = \oint \mathbf{E} \cdot d\boldsymbol{\ell} = E \cdot 2\pi b = -\frac{d\Phi_B}{dt} = -\pi a^2 \frac{dB}{dt} \quad \Rightarrow \quad \mathbf{E} = -\frac{a^2}{2b}\frac{dB}{dt}\hat{\boldsymbol{\phi}}$$

- The torque on a segment of length $d\boldsymbol{\ell}$ is $\mathbf{r}\times\mathbf{F}$, or $b\lambda E\, d\ell$. The total torque on the wheel is $N = b\lambda\left(-\frac{a^2}{2b}\frac{dB}{dt}\right)\oint d\boldsymbol{\ell} = -b\lambda\pi a^2 \frac{dB}{dt}$

$$\Rightarrow \quad L = \int N\, dt = -\lambda\pi a^2 b \int_{B_0}^{0} dB = \lambda\pi a^2 b B_0 \quad \Leftarrow$$ angular momentum

- No matter how quickly or slowly you turn off the field, the resulting angular velocity of the wheel is the same regardless.

- It's the *electric* field that did the rotating because the magnetic field is 0 at the location of the charge.

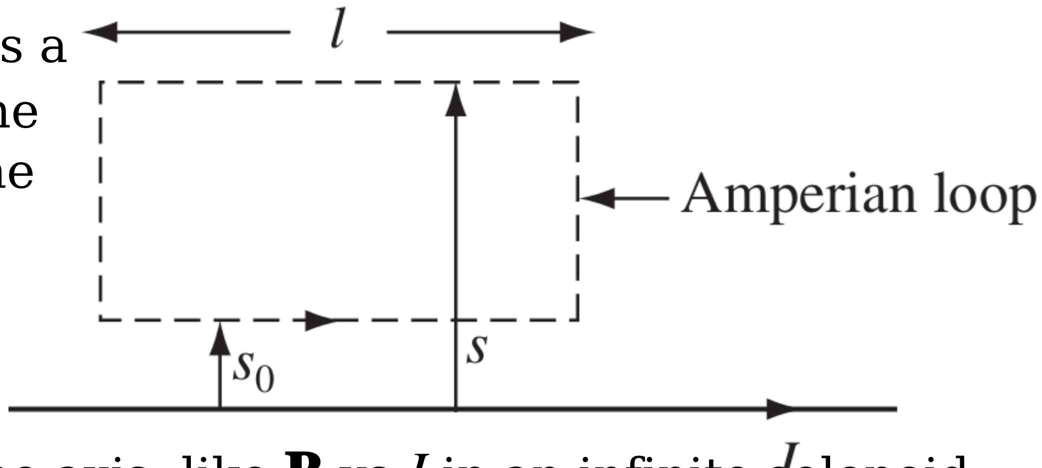
- Electromagnetic induction occurs only when the magnetic fields are changing, but we use the apparatus of magnetostatics (Ampère's law, the Biot-Savart law, etc) to calculate those magnetic fields.

- Technically, any result derived in this way is only approximately correct. But in practice the error is usually negligible, unless the field fluctuates extremely rapidly, or you are interested in points very far from the source.

- This régime, in which magnetostatic rules can be used to calculate the magnetic field on the right hand side of Faraday's law, is called **quasistatic**.

- It is only when we come to EM waves and radiation that we must worry seriously about the breakdown of magnetostatics.

Example 7.9: A long straight wire carries a slowly varying current $I(t)$. Determine the induced electric field, as a function of the distance $s$ from the wire.




- The magnetic field is $\frac{\mu_0 I}{2\pi s}$, circling around the wire. So $\mathbf{E}$ runs parallel to the axis, like $\mathbf{B}$ vs $I$ in an infinite solenoid.

- For the rectangular "Amperian loop," Faraday's law gives:

$$\oint \mathbf{E}\cdot d\boldsymbol{\ell} = E(s_0)\,\ell - E(s)\,\ell = -\frac{d}{dt}\int \mathbf{B}\cdot d\boldsymbol{a} = -\frac{\mu_0 \ell}{2\pi}\frac{dI}{dt}\int_{s_0}^{s}\frac{ds'}{s'} = \frac{\mu_0 \ell}{2\pi}\ln\frac{s_0}{s}\frac{dI}{dt}$$

$$\Rightarrow \quad \mathbf{E}(s) = \left(\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln\frac{s}{s_0} + E(s_0)\right)\hat{\mathbf{z}} = \left(\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln s + K\right)\hat{\mathbf{z}}$$

- The actual *value* of $K$ depends on the whole history of the function $I(t)$.

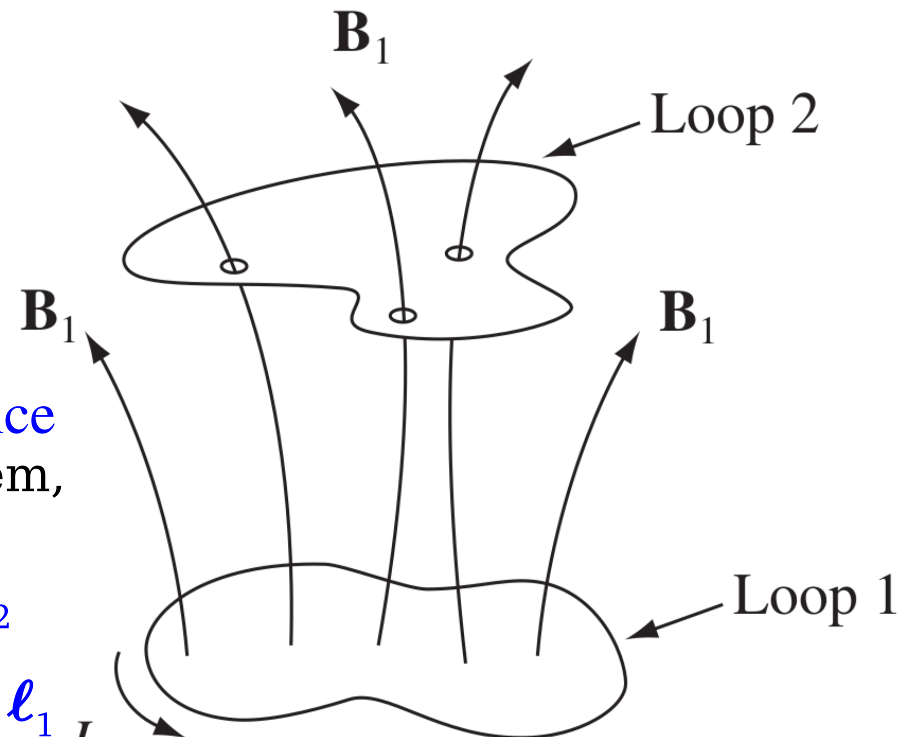
- The equation has the peculiar implication that $E$ blows up as $s$ goes $\infty$. That is because we have overstepped the limits of the quasi-static approximation. EM "news" travels at the speed of light, and at large distances $\mathbf{B}$ depends not on the current now, but on the current as it was at earlier time.

- If $\tau$ is the time it takes $I$ to change substantially, the quasi-static approximation should hold only for $s \ll c\tau$, hence the eqn does not apply, at extremely large $s$.

# Inductance

- $\mathbf{B}_1 = \dfrac{\mu_0}{4\pi} I_1 \oint \dfrac{d\boldsymbol{\ell}_1 \times \hat{\mathscr{r}}}{\mathscr{r}^2}$ $\propto I_1$

- $\Phi_2 = \int \mathbf{B}_1 \cdot d\boldsymbol{a}_2 = M_{21} I_1 \;\Leftarrow\; M_{21}$ : mutual inductance

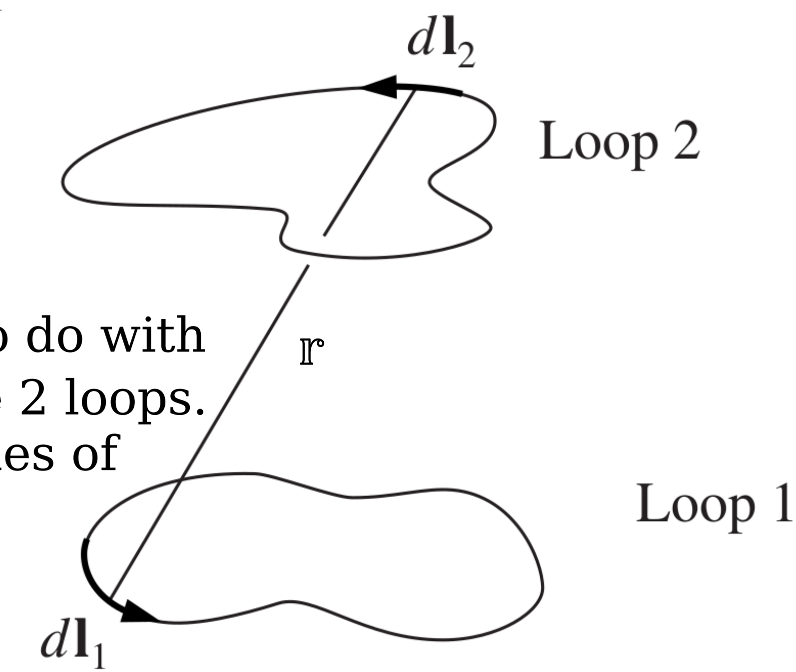



- With the vector potential and Stokes' theorem,

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\boldsymbol{a}_2 = \int \nabla \times \mathbf{A}_1 \cdot d\boldsymbol{a}_2 = \oint \mathbf{A}_1 \cdot d\boldsymbol{\ell}_2$$

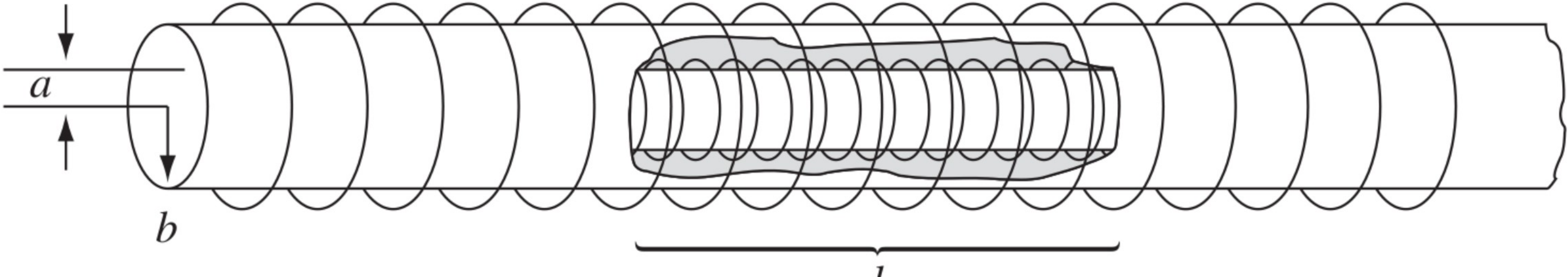
$$= \frac{\mu_0 I_1}{4\pi} \oint \oint \frac{d\boldsymbol{\ell}_1}{\mathscr{r}} \cdot d\boldsymbol{\ell}_2 \;\Leftarrow\; \mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\boldsymbol{\ell}_1}{\mathscr{r}} I_1$$

$$\Rightarrow \quad M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\boldsymbol{\ell}_1 \cdot d\boldsymbol{\ell}_2}{\mathscr{r}}$$ Neumann formula




- 2 important things about mutual inductance:
  1: $M_{21}$ is a purely geometrical quantity, having to do with the sizes, shapes, and relative positions of the 2 loops.
  2: The integral is unchanged if we switch the roles of loops 1 & 2, ie, $M_{21}=M_{12}$, symmetric.

- *Whatever the shapes and positions of the loops, the flux through 2 when we run a current I around 1 is identical to the flux through 1 when we send the same current I around 2.*




Example 7.10: A short solenoid ($\ell$, $a$, $n_1$) lies on the axis of a long solenoid ($b$, $n_2$). Current $I$ flows in the short solenoid. What is the flux through the long solenoid?

- Since the inner solenoid is short, it has a complicated field; it puts a different flux through each turn of the outer solenoid. It is hard to compute the total flux.

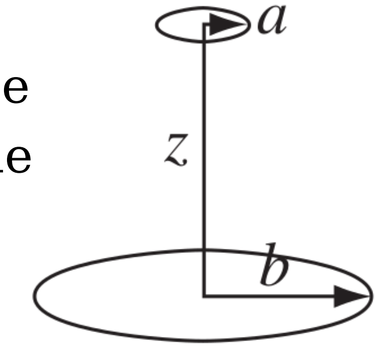
- If we exploit the equality of the mutual inductances, the problem becomes easy. Just look at the reverse situation: run the current $I$ through the outer solenoid, and calculate the flux through the inner one.

- The field inside the long solenoid is constant:

$$B = \mu_0 n_2 I \quad \Rightarrow \quad \Phi_{\text{single}} = B \cdot \pi a^2 = \mu_0 n_2 I \pi a^2 \quad \Rightarrow \quad \Phi_{\text{total}} = \mu_0 \pi a^2 n_1 n_2 \ell I$$

- It's also the flux a current $I$ in the short solenoid putting through the long one.

- $M = \mu_0 \pi a^2 n_1 n_2 \ell$




<u>Problem 7.22</u>: A small loop of radius $a$ is held a distance $z$ above the center of a large radius $b$. The planes of the 2 loops are ∥, and ⊥ the common axis.

- Let current $I_b$ flows in the big loop, by Ex. 5.6, its $\mathbf{B}_b$ along the $z$ axis is $$\mathbf{B}_b = \frac{\mu_0\, b^2\, I_b}{2\left(z^2+b^2\right)^{3/2}}\,\hat{\mathbf{z}}$$

- The little loop is small so the field of the big loop to be essentially constant.

Then the flux through it is $$\Phi_a = \int_{\mathcal{S}_a} \mathbf{B}_b \cdot \mathrm{d}\,\boldsymbol{a} = \frac{\mu_0\,\pi\, a^2\, b^2\, I_b}{2\left(z^2+b^2\right)^{3/2}}$$

- Current $I_a$ flows in the little loop, considering it as a dipole, its $\mathbf{B}_a$ is $$\mathbf{B}_a = \frac{\mu_0\, m}{4\,\pi\, r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right) \Leftarrow m = \pi\, a^2\, I_a$$

$$\Rightarrow\quad \Phi_b = \int_{\mathcal{S}_b} \mathbf{B}_a \cdot \mathrm{d}\,\boldsymbol{a} = \frac{\mu_0\, m}{4\,\pi}\int \frac{2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}}{\left(s^2+z^2\right)^{3/2}}\cdot s\,\mathrm{d}\,s\,\mathrm{d}\,\phi\,\hat{\mathbf{z}} \quad\Leftarrow\quad \hat{\mathbf{z}} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$$

$$= \frac{\mu_0\, m}{2}\int_0^b \frac{3\cos^2\theta - 1}{\left(s^2+z^2\right)^{3/2}}\, s\,\mathrm{d}\,s = \frac{\mu_0\, m}{4}\int_0^b \left(\frac{3\,z^2}{\left(s^2+z^2\right)^{5/2}} - \frac{1}{\left(s^2+z^2\right)^{3/2}}\right)\mathrm{d}\,s^2$$

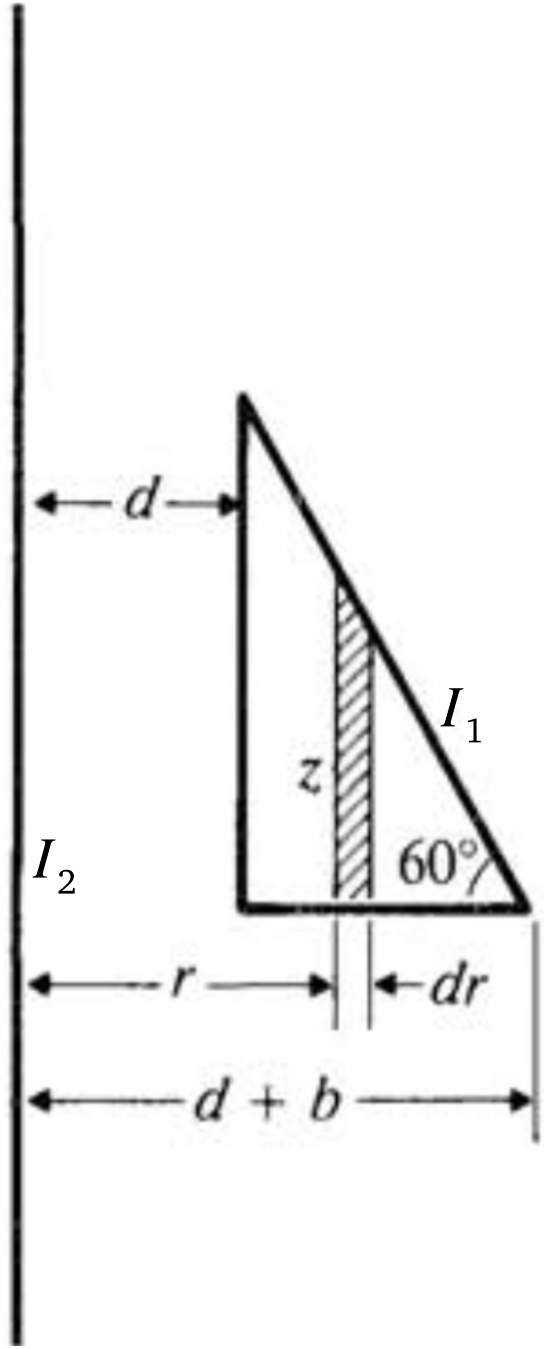
$$= \frac{\mu_0\, m\, b^2}{2\left(b^2+z^2\right)^{3/2}} = \frac{\mu_0\,\pi\, a^2\, b^2\, I_a}{2\left(z^2+b^2\right)^{3/2}}$$

$$\Phi_a = M_{ab}\, I_b\,,\quad \Phi_b = M_{ba}\, I_a \quad\Rightarrow\quad M_{ab} = \frac{\Phi_a}{I_b} = \frac{\mu_0\,\pi\, a^2\, b^2}{2\left(z^2+b^2\right)^{3/2}} = \frac{\Phi_b}{I_a} = M_{ba}$$

Example: Determine the mutual inductance between a conducting triangular loop and a very long straight wire.

Apply Ampere's law and write the expression for $\mathbf{B}_2$, caused by a current $I_2$ in the long straight wire:

$$\oint \mathbf{B}_2 \cdot d\boldsymbol{\ell} = \mu_0 I_2 \quad \Rightarrow \quad \mathbf{B}_2 = \frac{\mu_0 I_2}{2\pi r}\hat{\boldsymbol{\phi}} \quad \Rightarrow \quad \Phi_1 = \int \mathbf{B}_2 \cdot d\boldsymbol{a}_1$$




The equation of the sloped line of the triangle is

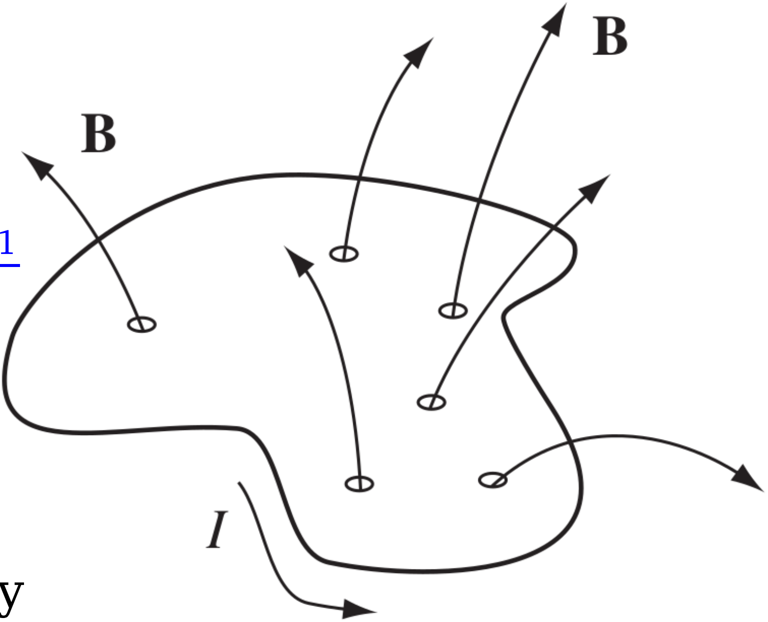
$$z = [(d+b) - r]\tan\frac{\pi}{3} = \sqrt{3}\,(d+b-r) \quad \Rightarrow \quad d\boldsymbol{a}_1 = z\,dr\,\hat{\boldsymbol{\phi}}$$

$$\Rightarrow \quad \Phi_1 = \int \frac{\mu_0 I_2}{2\pi r} z\,dr = \sqrt{3}\frac{\mu_0 I_2}{2\pi}\int_d^{d+b}\frac{d+b-r}{r}\,dr$$

$$= \frac{\sqrt{3}\,\mu_0 I_2}{2\pi}\left((d+b)\ln\frac{d+b}{d} - b\right) \Rightarrow M_{12} I_2$$

$$\Rightarrow \quad M = M_{12} = \frac{\sqrt{3}\,\mu_0}{2\pi}\left((d+b)\ln\frac{d+b}{d} - b\right)$$

- If you *vary* the current in loop 1, the flux through loop 2 will vary accordingly, $\mathscr{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}$




- Every time you change the current in loop 1, an induced current flows in loop 2—even though there are no wires connecting them!

- A changing current not only induces an emf in any nearby loops, it also induces an emf in the loop *itself*.

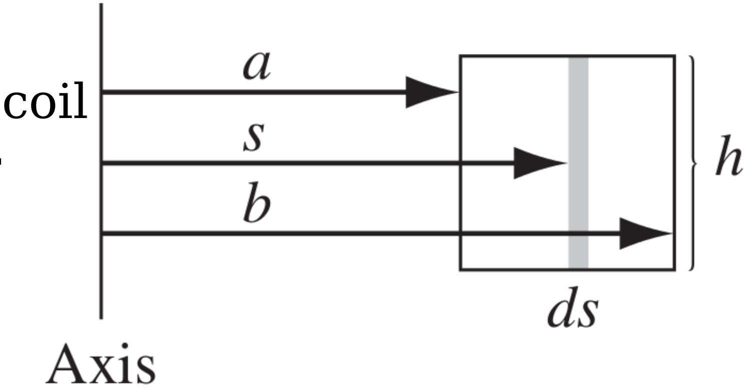
- The field (thus the flux) ∝ the current:

$$\Phi = L\,I \quad \Leftarrow \quad L: \text{ self inductance} \quad \Rightarrow \quad \mathscr{E} = -L\frac{dI}{dt}$$

- Inductance is measured in **henries** (H); a henry is a volt-second per ampere.

- The emf from the self-induction and the mutual induction combined can be expressed as $\mathscr{E} = -\frac{d}{dt}\Phi_{1,\,\text{total}} = -L\frac{dI_1}{dt} - \sum_{i=2} M_{1i}\frac{dI_i}{dt}$

<u>Example 7.11</u>: Find the self-inductance of a toroidal coil with rectangular cross section (inner radius $a$, outer radius $b$, height $h$), that carries a total of $N$ turns.




- The magnetic field inside the toroid is $B = \dfrac{\mu_0 N I}{2 \pi s}$

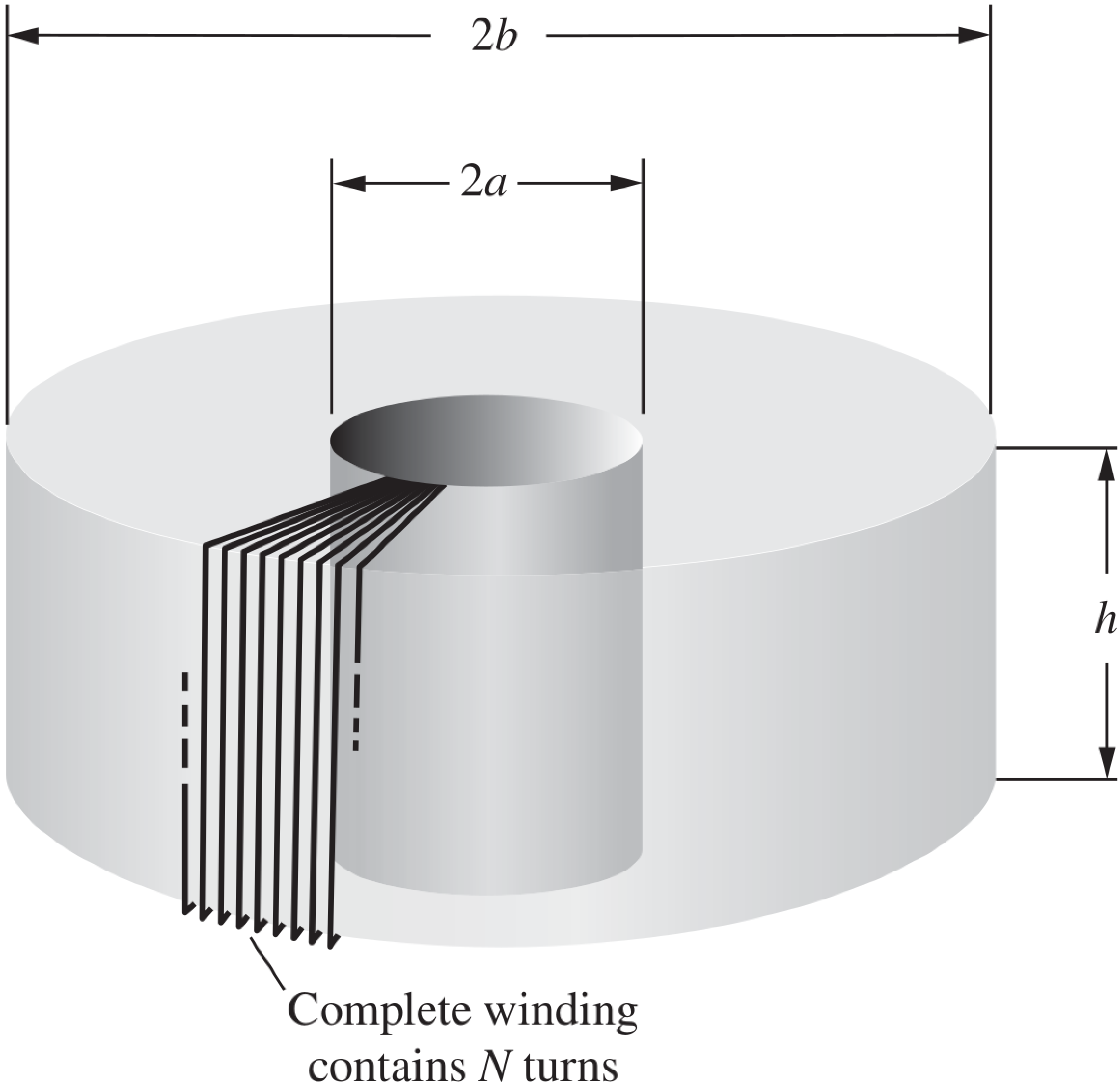
$$\Rightarrow \Phi_{\text{single}} = \int \mathbf{B} \cdot d\boldsymbol{a} = \frac{\mu_0 N I}{2\pi} h \int_a^b \frac{ds}{s} = \frac{\mu_0 N I h}{2\pi} \ln \frac{b}{a}$$

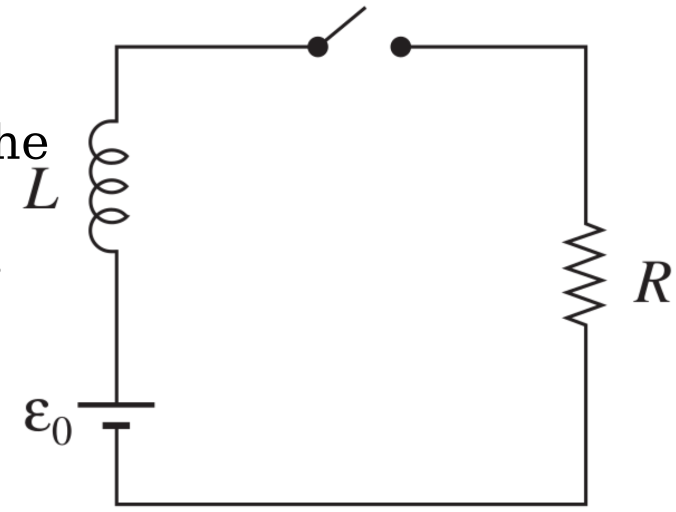
- The *total* flux is $N$ times this, so the self-inductance $L = \dfrac{\mu_0 N^2 h}{2\pi} \ln \dfrac{b}{a}$

- Lenz's law dictates that the emf is in such a direction as to *oppose* any *change in current*. For this reason, it is called a **back emf**.

- Whenever you alter the current in a wire, you must fight against this back emf.

- Inductance plays the same role in electric circuits that *mass* does in mechanical systems: The greater $L$ is, the harder it is to change the current, as the larger the mass, the harder it is to change an object's velocity.


2b
2a
h
Complete winding
contains N turns

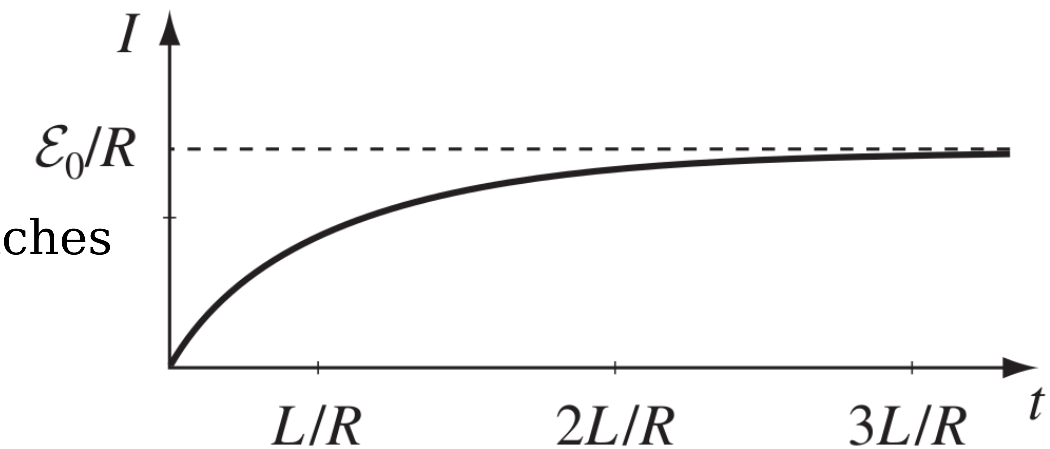



Example 7.12: The total emf in this circuit is $\mathcal{E}_0$ from the battery plus $-L\frac{dI}{dt}$ from the inductance. Ohm's law is

$$\mathcal{E}_0 - L\frac{dI}{dt} = IR \quad \Rightarrow \quad I(t) = \frac{\mathcal{E}_0}{R} + k\,e^{-\frac{Rt}{L}}$$

$$I(0) = 0 \quad \Rightarrow \quad k = -\frac{\mathcal{E}_0}{R} \quad \Rightarrow \quad I(t) = \frac{\mathcal{E}_0}{R}\left(1 - e^{-\frac{Rt}{L}}\right) = \frac{\mathcal{E}_0}{R}\left(1 - e^{-\frac{t}{\tau}}\right) \quad \Leftarrow \quad \tau \equiv \frac{L}{R}$$

- Had there been no inductance in the circuit, the current would have jumped immediately to $\frac{\mathcal{E}_0}{R}$.
- In practice, every circuit has some self-inductance, and the current approaches $\frac{\mathcal{E}_0}{R}$ asymptotically.




- The quantity $\tau = \frac{L}{R}$ is the **time constant**; it tells you how long the current takes to reach a substantial fraction (roughly $\frac{2}{3}$) of its final value.

## Energy in Magnetic Fields

- Energy delivered to the resistors and converted into heat is irretrievably lost.

- The work done *against the back emf* to get the current going is a fixed amount, and it is *recoverable*: you get it back when the current is turned off.

- It represents energy latent in the circuit; it can be regarded as energy stored in the magnetic field.

- The work done on a unit charge, against the back emf, in one trip around the circuit is $-\mathcal{E}$. The amount of charge per unit time passing the wire is $I$. So the total work done per unit time is $\frac{dW}{dt} = -\mathcal{E}\, I = L\, I\, \frac{dI}{dt} \Rightarrow W = \frac{1}{2} L\, I^2$

- It does not depend on how *long* we take to crank up the current, only on the geometry of the loop, ie, $L$, and the final current $I$.

- For a nicer way to write $W$, $L\, I = \Phi = \int \mathbf{B} \cdot d\boldsymbol{a} = \int \nabla \times \mathbf{A} \cdot d\boldsymbol{a} = \oint \mathbf{A} \cdot d\boldsymbol{\ell}$

$$\Rightarrow \quad W = \frac{1}{2} I \oint \mathbf{A} \cdot d\boldsymbol{\ell} = \frac{1}{2} \oint \mathbf{A} \cdot \boldsymbol{I}\, d\ell$$

$$= \frac{1}{2} \int_{\mathcal{V}} \mathbf{A} \cdot \boldsymbol{J}\, d\tau = \frac{1}{2\mu_0} \int_{\mathcal{V}} \mathbf{A} \cdot \nabla \times \mathbf{B}\, d\tau \quad \Leftarrow \quad \nabla \times \mathbf{B} = \mu_0 \boldsymbol{J}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B}) \quad\Rightarrow\quad \mathbf{A}\cdot(\nabla\times\mathbf{B})=\mathbf{B}^2-\nabla\cdot(\mathbf{A}\times\mathbf{B})$$

$$\Rightarrow\quad W=\frac{1}{2\mu_0}\left[\int B^2\,\mathrm{d}\tau-\int\nabla\cdot(\mathbf{A}\times\mathbf{B})\,\mathrm{d}\tau\right]=\frac{1}{2\mu_0}\left[\int_{\mathcal{V}} B^2\,\mathrm{d}\tau-\oint_{\mathcal{S}}(\mathbf{A}\times\mathbf{B})\cdot\mathrm{d}\boldsymbol{a}\right]$$

- The integration with current density is taken over the *entire volume occupied by the current*. But any region *larger* than this will do, for $\boldsymbol{J}$ is 0 out there anyway.

- The larger the region we pick the greater is the contribution from the volume integral, and therefore the smaller is that of the surface integral.

- If we integrate over *all* space, the surface integral=0, $W=\dfrac{1}{2\mu_0}\displaystyle\int_{\text{all space}} B^2\,\mathrm{d}\tau$

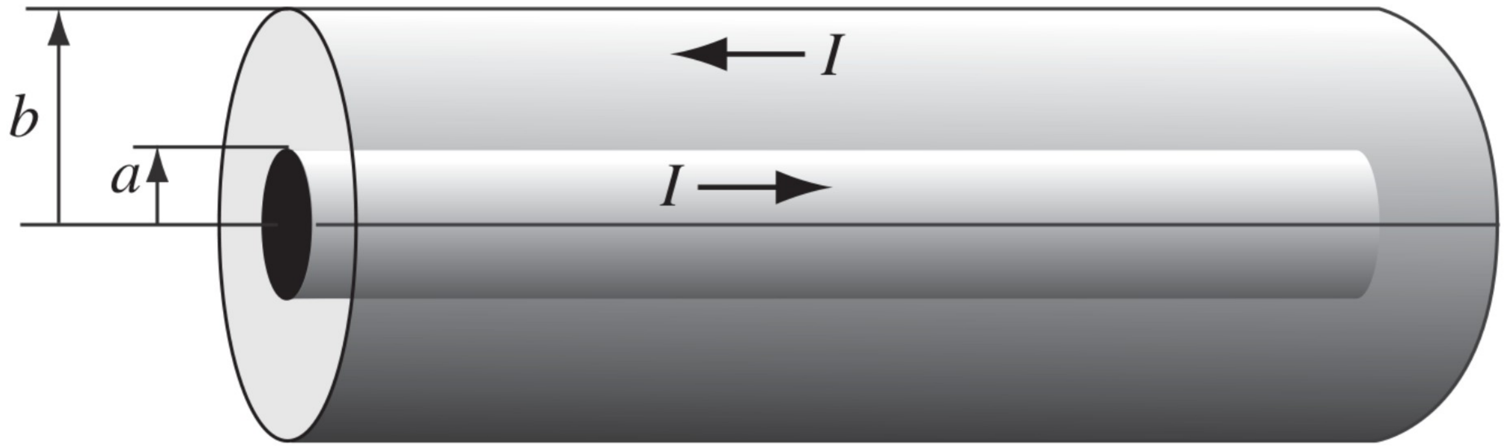
- This result indicates that the energy is "stored in the magnetic field," in the amount $\dfrac{B^2}{2\mu_0}$ per unit volume. This is a nice way to think of it.

- Someone prefers to say that the energy is stored in the current distribution, in the amount $\dfrac{\mathbf{A}\cdot\boldsymbol{J}}{2}$ per unit volume. Both are fine.

- Magnetic fields do no work, but producing a magnetic field requires *changing* the field, and a changing $\mathbf{B}$-field induces an *electric* field. The latter *does* work.

- In the beginning and at the end, there is no $\mathbf{E}$; but in between, while $\mathbf{B}$ is building up, there is an $\mathbf{E}$, and it is against *this* that the work is done.

- $W_{\text{mag}} = \frac{1}{2}\int \mathbf{A}\cdot\boldsymbol{J}\, d\tau = \frac{1}{2\mu_0}\int B^2\, d\tau \quad \Leftrightarrow \quad W_{\text{elec}} = \frac{1}{2}\int \rho\,\Phi\, d\tau = \frac{\epsilon_0}{2}\int E^2\, d\tau$




<u>Example 7.13</u>: Find the magnetic energy stored in a section of length $\ell$.

- According to Ampère's law, only the field between the cylinders is nonzero,

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi} \quad \Rightarrow \quad \text{energy density } \; w = \frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi s}\right)^2 = \frac{\mu_0 I^2}{8\pi^2 s^2}$$

$$\Rightarrow \quad W = \int w\, d\tau = \int \frac{\mu_0 I^2}{8\pi^2 s^2} 2\pi\ell s\, d s = \frac{\mu_0 I^2 \ell}{4\pi}\int_a^b \frac{d s}{s} = \frac{\mu_0 I^2 \ell}{4\pi}\ln\frac{b}{a}$$

$W = \frac{1}{2} L I^2 \quad \Rightarrow \quad L = \frac{\mu_0 \ell}{2\pi}\ln\frac{b}{a}$ **external inductance** of a coaxial line

- This method of calculating self-inductance is especially useful when the current is not confined to a single path, but spreads over some surface or volume, so that different parts of the current enclose different amounts of flux.

Example 7.13′: Same as Ex. 7.13, but consider the current $I$ is uniformly distributed in a solid inner conductor of radius $a$.

- According to Ampère's law, $\oint \mathbf{B}\cdot d\boldsymbol{\ell}=\mu_0 I \;\Rightarrow\; \mathbf{B}=\frac{\mu_0 I}{2\pi}\frac{s}{s_>^2}\hat{\boldsymbol{\phi}} \;\Leftarrow\; s_>=\max(s,a)$

$$\text{energy density } w=\frac{1}{2\mu_0}\left(\frac{\mu_0 I s}{2\pi s_>^2}\right)^2=\frac{\mu_0 I^2 s^2}{8\pi^2 s_>^4}$$

$$\Rightarrow\quad W=\int w\,d\tau=\int_0^b \frac{\mu_0 I^2 s^2}{8\pi^2 s_>^4}2\pi\ell s\,ds=\frac{\mu_0 I^2\ell}{4\pi}\left(\int_0^a \frac{s^3}{a^4}ds+\int_a^b \frac{ds}{s}\right)$$

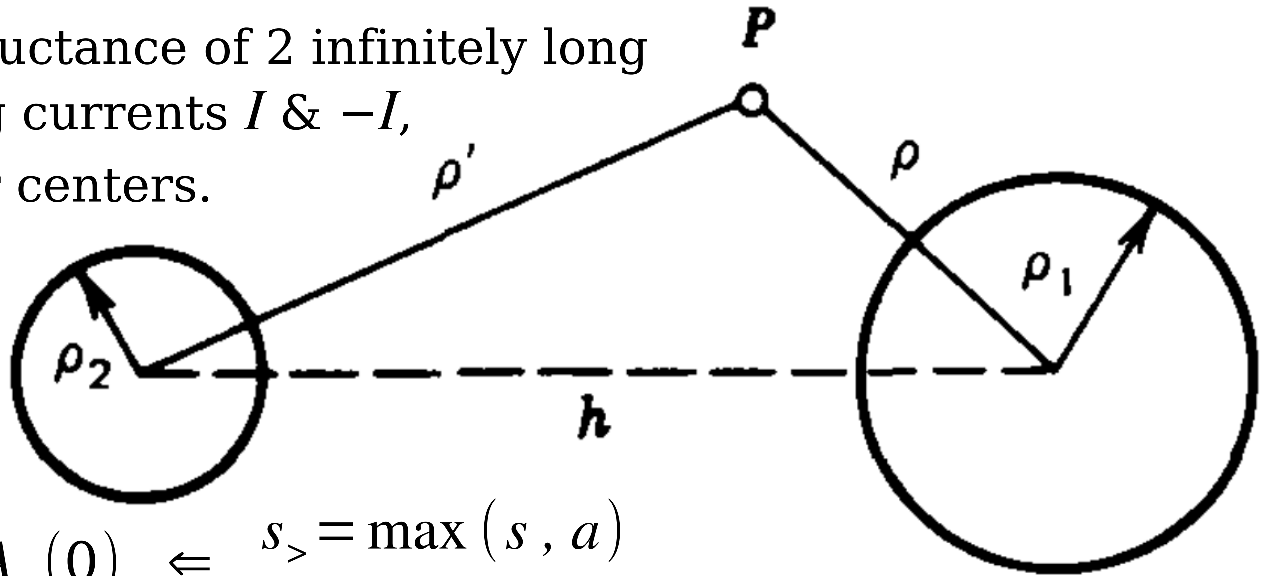
$$=\frac{\mu_0 I^2\ell}{4\pi}\left(\frac{1}{4}+\ln\frac{b}{a}\right)\quad\Rightarrow\quad L=\frac{\mu_0\ell}{2\pi}\left(\frac{1}{4}+\ln\frac{b}{a}\right)\quad\Leftarrow\quad W=\frac{1}{2}LI^2$$

- The new term $\frac{\mu_0\ell}{8\pi}$ from the flux linkage internal to the solid inner conductor is known as the **internal inductance** of the inner conductor.

- In high-frequency cases the current in a good conductor tends to shift to the surface of the conductor (due to **skin effect**, mentioned later), resulting in an uneven current distribution in the inner conductor an thus changing the value of the internal inductance.

- In the extreme case the current essentially concentrate in the "skin" of the inner conductor as a surface current, the internal self-inductance is reduced to 0.

Example: calculate the self-inductance of 2 infinitely long wires, of radii $\rho_1$ & $\rho_2$, carrying currents $I$ & $-I$, and a distance $h$ between their centers.




- The vector potential of a cylindrical wire was showed in p.43 of Chapter 5 note as

$$A_z=-\frac{\mu_0 I}{4\pi}\left(\frac{s^2}{s_>^2}+2\ln\frac{s_>}{a}\right)+A_z(0) \Leftarrow \begin{array}{l} s_>=\max(s,a) \\ \text{let } A_z(0)=0 \text{ for convenience}\end{array}$$

$$\Rightarrow \quad \mathbf{A}_1=-\frac{\mu_0 I}{4\pi}\begin{bmatrix}\dfrac{\rho^2}{\rho_1^2} \\ \ln\dfrac{\rho^2}{\rho_1^2}+1\end{bmatrix}\hat{\mathbf{z}} \quad \text{for} \begin{bmatrix}\rho\le\rho_1 \\ \rho>\rho_1\end{bmatrix}, \quad \mathbf{A}_2=\frac{\mu_0 I}{4\pi}\begin{bmatrix}\dfrac{\rho'^2}{\rho_2^2} \\ \ln\dfrac{\rho'^2}{\rho_2^2}+1\end{bmatrix}\hat{\mathbf{z}} \quad \text{for} \begin{bmatrix}\rho'\le\rho_2 \\ \rho'>\rho_2\end{bmatrix}$$

The magnetic energy of the system can be calculated as

$$U=\frac{1}{2}\int \boldsymbol{J}\cdot\mathbf{A}\,\mathrm{d}\tau=\frac{1}{2}\left(\int_{S_1}\boldsymbol{J}_1\cdot(\mathbf{A}_1+\mathbf{A}_2)\,\mathrm{d}a_1\,\mathrm{d}z+\int_{S_2}\boldsymbol{J}_2\cdot(\mathbf{A}_1+\mathbf{A}_2)\,\mathrm{d}a_2\,\mathrm{d}z\right)$$

$$=\frac{I}{2\pi\rho_1^2}\int_{S_1}(A_1+A_2)\,\mathrm{d}a_1\,\mathrm{d}z-\frac{I}{2\pi\rho_2^2}\int_{S_2}(A_1+A_2)\,\mathrm{d}a_2\,\mathrm{d}z \quad \Leftarrow \boldsymbol{J}_{1,2}=\pm\frac{I\,\hat{\mathbf{z}}}{\pi\rho_{1,2}^2}$$

- Since the potentials are independent of $z$ and are in the direction of $z$, then the energy per unit length $u \equiv \frac{U}{\ell} = \frac{I}{2\pi}\left[\frac{1}{\rho_1^2}\int_{S_1}(A_1+A_2)\,\mathrm{d}\,a_1 - \frac{1}{\rho_2^2}\int_{S_2}(A_1+A_2)\,\mathrm{d}\,a_2\right]$

$$\rho'^2 = \rho^2 + h^2 + 2\rho h\cos\phi \quad \text{in} \quad S_1$$

$$\Rightarrow \int_{S_1}(A_1+A_2)\,\mathrm{d}\,a_1 = \frac{\mu_0 I}{4\pi}\int_{S_1}\left(1+\ln\frac{\rho^2+h^2+2\rho h\cos\phi}{\rho_2^2} - \frac{\rho^2}{\rho_1^2}\right)\rho\,\mathrm{d}\rho\,\mathrm{d}\phi$$

$$= \frac{\mu_0 I\rho_1^2}{8} + \frac{\mu_0 I}{4\pi}\int_0^{\rho_1}\rho\,\mathrm{d}\rho\int_0^{2\pi}\left[\ln\frac{\rho^2+h^2}{\rho_2^2} + \ln\left(1+\frac{2h\rho}{\rho^2+h^2}\cos\phi\right)\right]\mathrm{d}\phi$$

$$= \frac{\mu_0 I\rho_1^2}{8} + \frac{\mu_0 I}{2}\int_0^{\rho_1}\left(\ln\frac{\rho^2+h^2}{\rho_2^2} + \ln\frac{h^2}{\rho^2+h^2}\right)\rho\,\mathrm{d}\rho = \frac{\mu_0 I\rho_1^2}{4}\left(\frac{1}{2}+\ln\frac{h^2}{\rho_2^2}\right)$$

$$\Rightarrow \text{Similarly} \int_{S_2}(A_1+A_2)\,\mathrm{d}\,a_2 = -\frac{\mu_0 I\rho_2^2}{4}\left(\frac{1}{2}+\ln\frac{h^2}{\rho_1^2}\right)$$

$$\Rightarrow u = \frac{\mu_0 I^2}{8\pi}\left(1+2\ln\frac{h^2}{\rho_1\rho_2}\right) \Rightarrow \frac{1}{2}\frac{L}{\ell}I^2 \quad \Rightarrow \quad \frac{L}{\ell} = \frac{\mu_0}{4\pi}\left(1+2\ln\frac{h^2}{\rho_1\rho_2}\right)$$

$$S=\int_0^{2\pi}\ln(1+\beta\cos\phi)\,d\phi=2\int_0^{\pi}\ln(1+\beta\cos\phi)\,d\phi$$

$$=2\left(\int_0^{\pi/2}\ln(1+\beta\cos\phi)\,d\phi+\int_0^{\pi/2}\ln(1-\beta\cos\phi)\,d\phi\right)$$

$$=2\int_0^{\pi/2}\ln(1-\beta^2\cos^2\phi)\,d\phi=2\int_0^{\pi/2}\int_0^{\beta^2}\frac{-\cos^2\phi}{1-x\cos^2\phi}\,dx\,d\phi$$

$$=2\int_0^{\beta^2}\int_0^{\pi/2}\left(1-\frac{1}{1-x\cos^2\phi}\right)d\phi\,\frac{dx}{x}=\int_0^{\beta^2}\left(\pi-2\int_0^{\pi/2}\frac{\sec^2\phi}{\sec^2\phi-x}\,d\phi\right)\frac{dx}{x}$$

$$\int_0^{\pi/2}\frac{\sec^2\phi\,d\phi}{\sec^2\phi-x}=\int_0^{\pi/2}\frac{d\tan\phi}{1-x+\tan^2\phi}=\frac{1}{\sqrt{1-x}}\int_0^{\infty}\frac{dy}{1+y^2}\quad\Leftarrow\quad y=\frac{\tan\phi}{\sqrt{1-x}}$$

$$=\left.\frac{\tan^{-1}y}{\sqrt{1-x}}\right|_0^{\infty}=\frac{\pi/2}{\sqrt{1-x}}$$

$$\Rightarrow\quad S=\pi\int_0^{\beta^2}\left(\frac{1}{x}-\frac{1}{x\sqrt{1-x}}\right)dx=2\pi\int_1^{\sqrt{1-\beta^2}}\frac{dt}{t+1}\quad\Leftarrow\quad x=1-t^2$$

$$=2\pi\ln\frac{1+\sqrt{1-\beta^2}}{2},\quad -1<\beta<1$$

## Magnetic Energy in Matter

- $W = \frac{1}{2}\int_{\mathcal{V}} \mathbf{A}\cdot\boldsymbol{J}\,\mathrm{d}\tau = \frac{1}{2}\int_{\mathcal{V}} \mathbf{A}\cdot\nabla\times\mathbf{H}\,\mathrm{d}\tau \quad \Leftarrow \quad \nabla\times\mathbf{H} = \boldsymbol{J}$

$$= \frac{1}{2}\int_{\mathcal{V}} \mathbf{H}\cdot\nabla\times\mathbf{A}\,\mathrm{d}\tau - \frac{1}{2}\int_{\mathcal{V}} \nabla\cdot\left(\mathbf{A}\times\mathbf{H}\right)\mathrm{d}\tau \quad \Leftarrow \quad \nabla\cdot\left(\mathbf{A}\times\mathbf{H}\right) = \mathbf{H}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{H}$$

$$= \frac{1}{2}\int_{\mathcal{V}} \mathbf{H}\cdot\mathbf{B}\,\mathrm{d}\tau - \frac{1}{2}\oint_{\mathcal{S}} \mathbf{A}\times\mathbf{H}\cdot\mathrm{d}\boldsymbol{a} = \frac{1}{2}\int_{\text{all space}} \mathbf{H}\cdot\mathbf{B}\,\mathrm{d}\tau = \int_{\text{all space}} u\,\mathrm{d}\tau$$

- In terms of $\mathbf{H}$ and the magnetization $\mathbf{M}$

$$\mathbf{B} = \mu_0\left(\mathbf{H}+\mathbf{M}\right) \quad \Rightarrow \quad \text{energy density} \;\; u = \frac{1}{2}\mathbf{H}\cdot\mathbf{B} = \frac{\mu_0}{2}H^2 + \frac{\mu_0}{2}\mathbf{H}\cdot\mathbf{M}$$

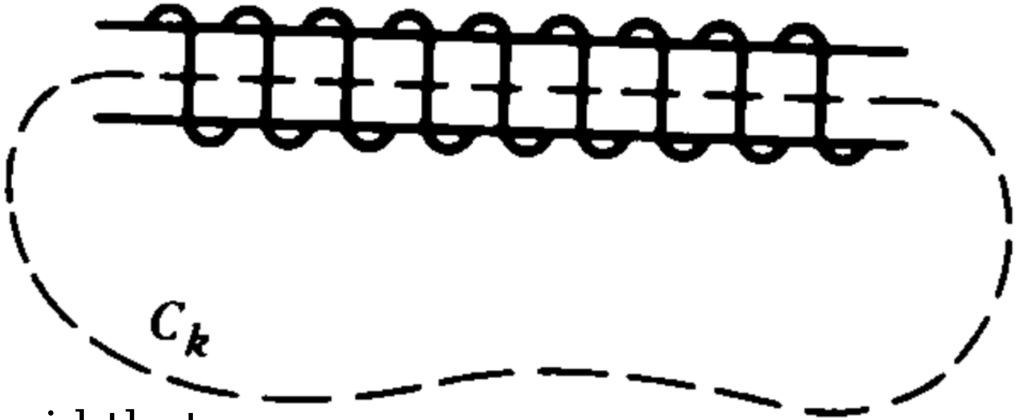
- The 1st term is the energy density of the magnetic field in vacuum. The 2nd term is the energy density stored in the material itself.

- For *linear isotropic* materials, $\mathbf{B} = \mu\,\mathbf{H} \quad \Rightarrow \quad \text{energy density} \;\; u = \frac{\mu}{2}H^2 = \frac{B^2}{2\,\mu}$

- In linear materials the work for building a magnetic field depends only on the final value of the magnetic field. This implies that these systems are *reversible*, ie, that the energy in establishing the magnetic system can be recovered as the field is switched off.

- In *nonlinear* materials it is not true because *hysteresis* plays an important role.

- The *irreversible* changes in the domain configurations that are responsible for the hysteresis cause energy losses in the form of heat.




- Consider a circuit in the form of a solenoid that has $N$ current turns and negligible resistance, and completely filled with a ferro-magnetic material, the work done by an *external source* $\Delta W$ in a time interval $\Delta t$

$$\Delta W_a = -\mathscr{E}\, I\, \Delta t = I\, \Delta \Phi_B \quad \Leftarrow \quad \mathscr{E} = -\frac{\mathrm{d}\,\Phi_B}{\mathrm{d}\,t}$$

$$= \left( \frac{1}{N} \oint_c \mathbf{H} \cdot \mathrm{d}\,\boldsymbol{\ell} \right) \left( N\, A\, \Delta\, B \right) \quad \Leftarrow \quad \oint \mathbf{H} \cdot \mathrm{d}\,\boldsymbol{\ell} = N\, I\,, \quad \Delta\,\Phi_B = N\, A\, \Delta\, B$$

$$= \oint_c \mathbf{H} \cdot (\Delta\, B\, \mathrm{d}\,\boldsymbol{\ell})\, A = \oint_c \mathbf{H} \cdot (\Delta\, \mathbf{B}\, \mathrm{d}\,\ell)\, A \quad \Leftarrow \quad \begin{array}{l} \text{choose } \mathrm{d}\,\boldsymbol{\ell} \text{ along } \Delta\,\mathbf{B} \\ \Rightarrow \quad \Delta\, B\, \mathrm{d}\,\boldsymbol{\ell} = \Delta\, \mathbf{B}\, \mathrm{d}\,\ell \end{array}$$

$$= \oint_c \mathbf{H} \cdot \Delta\, \mathbf{B}\, A\, \mathrm{d}\,\ell = \int_v \mathbf{H} \cdot \Delta\, \mathbf{B}\, \mathrm{d}\,\tau \quad \Leftarrow \quad \text{Replace } \oint_c A\, \mathrm{d}\,\ell \text{ to } \int_v \mathrm{d}\,\tau$$

$$\Rightarrow \quad \mathrm{d}\,w = \mathbf{H} \cdot \mathrm{d}\,\mathbf{B} = \mu_0\, H\, \mathrm{d}\,H + \mu_0\, \mathbf{H} \cdot \mathrm{d}\,\mathbf{M} \quad \Leftarrow \quad \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})\,, \quad w \equiv \frac{\mathrm{d}\,W_a}{\mathrm{d}\,\tau}$$

- $\mathrm{d}\,w$ represents the work necessary to establish the magnetic field from $\mathbf{H}$ to $\mathbf{H}+\mathrm{d}\,\mathbf{H}$ and to magnetize the material from $\mathbf{M}$ to $\mathbf{M}+\mathrm{d}\,\mathbf{M}$.

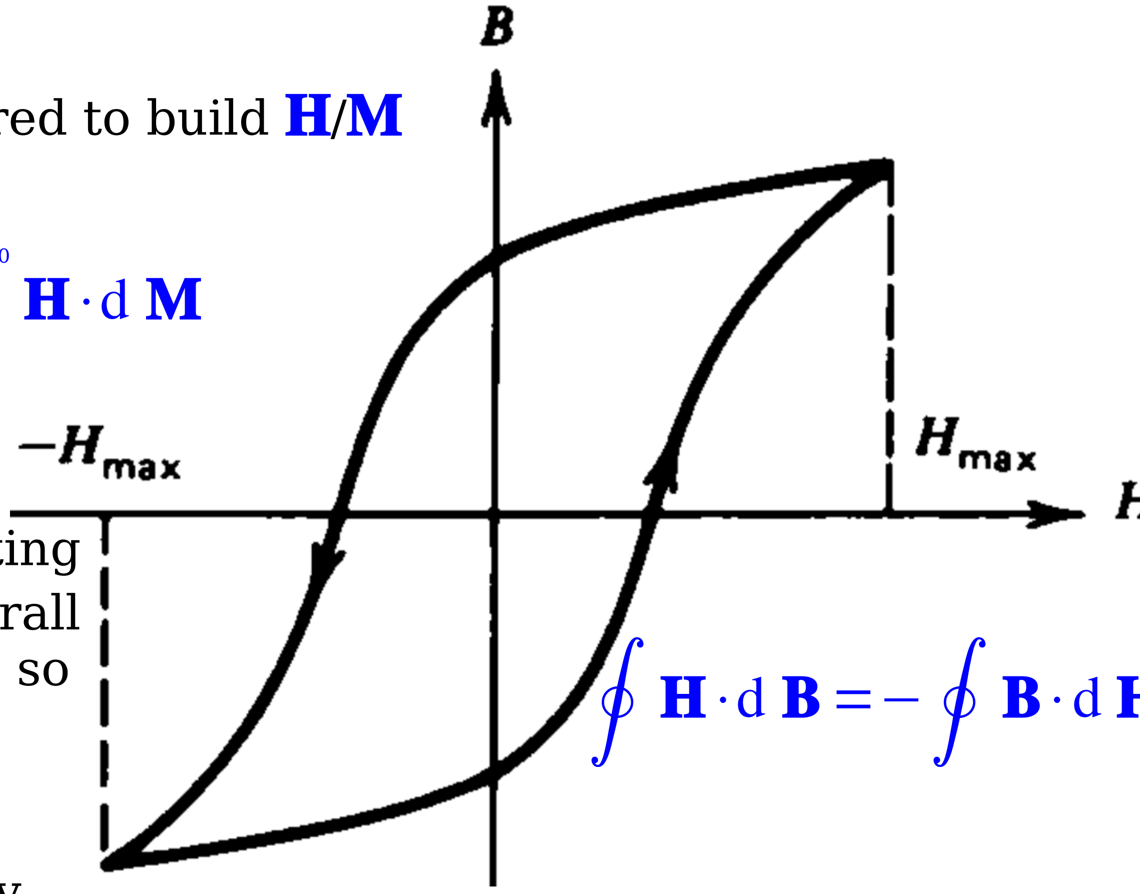
- The total energy density $w$ required to build $\mathbf{H}/\mathbf{M}$ from $0$ to $H_0/\mathbf{M}(H_0)$ is

$$w = \int_0^{B_0} \mathbf{H} \cdot \mathrm{d}\,\mathbf{B} = \frac{\mu_0}{2} H_0^2 + \mu_0 \int_0^{H_0} \mathbf{H} \cdot \mathrm{d}\,\mathbf{M}$$

- For a ferromagnetic material, the system returns to its initial magnetic state ($\mathbf{H}$,$\mathbf{B}$) after completing its whose hysteresis curve, the overall change in its magnetic energy is 0, so

$$w = \oint_c \mathbf{H} \cdot \mathrm{d}\,\mathbf{B} = \mu_0 \oint_c \mathbf{H} \cdot \mathrm{d}\,\mathbf{M}$$

The integral is the area enclosed by the hysteresis loop.




$$\oint \mathbf{H} \cdot \mathrm{d}\,\mathbf{B} = -\oint \mathbf{B} \cdot \mathrm{d}\,\mathbf{H}$$

- It takes more energy to produce the magnetization than is returned when the magnetization is reduced. The energy lost is the *hysteresis loss*; it goes into heat.

- By varying the magnetic field applied to material, it is possible to change the temperature of the sample (heat it or cool it). This principle has been used to provide a means for attaining very low temperatures, ie, *magnetic cooling*.

## Forces & Torques Using the Magnetostatic Energy

- Use the magnetic-energy formalism to determine applied forces and torques.

- Consider a rigid circuit. Let the circuit make a virtual displacement $\mathrm{d}\mathbf{r}$ while the current is kept *constant*. Only consider static situations where the circuits are stationary. For this displacement, a mechanical work $\mathrm{d}W_{\mathrm{mech}}$ is done by the magnetic force $\mathbf{F}_{\mathrm{mag}}$, and an electrical work $\mathrm{d}W_b$ is done by the battery against the induced electromotive force to maintain the current in the circuit.

- The sum of these 2 effects is equal to the change in the magnetic energy of the system: $-\mathrm{d}W_{\mathrm{mech}} + \mathrm{d}W_b = \mathrm{d}U$

- $\mathrm{d}W_b = -\mathscr{E}\, I\, \mathrm{d}t = I\, \mathrm{d}\Phi_B, \quad \mathrm{d}U = \mathrm{d}\left(\frac{1}{2} L I^2\right) = \mathrm{d}\left(\frac{1}{2} I \Phi_B\right) = \frac{I}{2}\mathrm{d}\Phi_B$

$$\Rightarrow \quad \mathrm{d}W_b = 2\,\mathrm{d}U \quad \Rightarrow \quad \mathrm{d}W_{\mathrm{mech}} = \mathrm{d}U \quad (\text{constant current}) \quad + \quad \mathrm{d}W_{\mathrm{mech}} = \mathbf{F}_{\mathrm{mag}} \cdot \mathrm{d}\mathbf{r}$$

$$\Rightarrow \quad \mathrm{d}U = \mathbf{F}_{\mathrm{mag}} \cdot \mathrm{d}\mathbf{r} \quad \Leftarrow \quad \Rightarrow \quad \mathbf{F}_{\mathrm{mag}} = \nabla U \quad (\text{constant current}) \quad \Rightarrow \quad F_{\mathrm{mag},\,i} = + \left.\frac{\partial U}{\partial x^i}\right|_I$$

- Another physical situation arises when the circuit is isolated from the external sources (batteries). So a virtual rigid movement of the circuit results in a change in the current. According to Lenz's law, the amount of change in the induced current due to the induced emf is such that the magnetic flux $\Phi_B$ passing through the circuits stays the same.

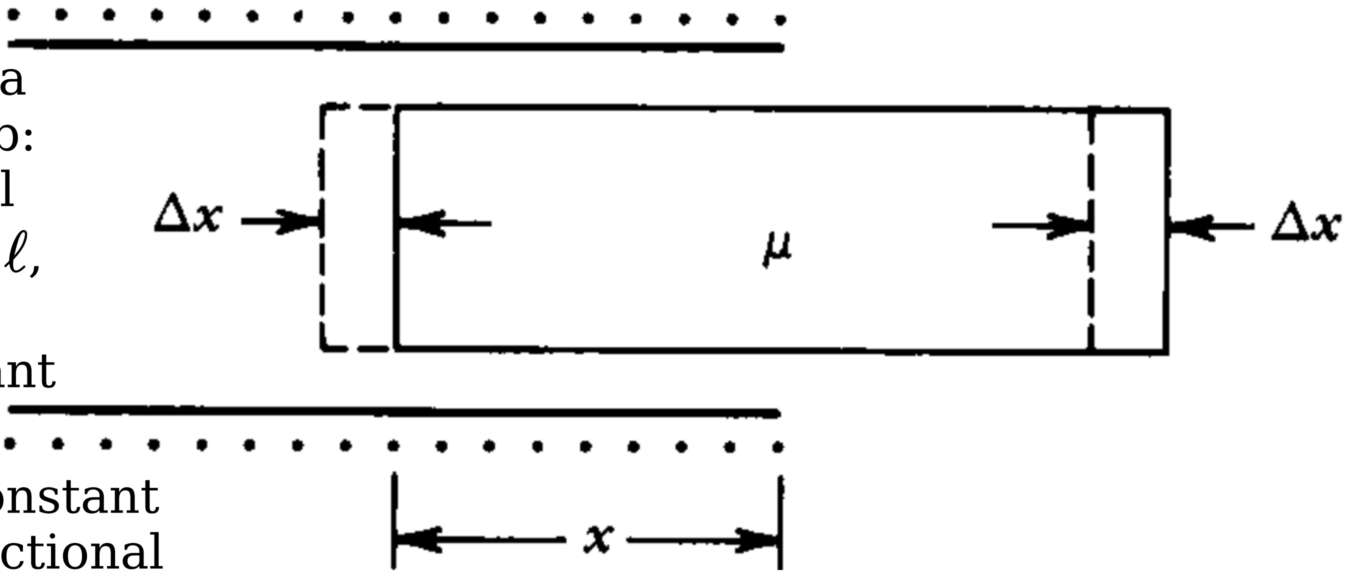
- $\mathrm{d}W_b = 0 \;\Rightarrow\; \mathrm{d}U = -\mathrm{d}W_{\text{mech}} \;\left(\text{constant flux}\right) \;+\; \mathrm{d}W_{\text{mech}} = \mathbf{F}_{\text{mag}} \cdot \mathrm{d}\mathbf{r}$

$$\Rightarrow \mathbf{F}_{\text{mag}} = -\nabla U \;\left(\text{constant flux}\right) \;\Rightarrow\; F_{\text{mag},\,i} = -\frac{\partial U}{\partial x^i}\Big|_{\Phi_B}$$

- The same procedure followed above can be used to find the magnetic torques acting on the circuits. If the circuit is allowed to have a rigid virtual rotation $\mathrm{d}\theta$,

then $\tau_\theta = +\dfrac{\partial U}{\partial \theta}\Big|_I$ gives the torque in the increasing $\theta$ direction in the case of constant current,

and $\tau_\theta = -\dfrac{\partial U}{\partial \theta}\Big|_{\Phi_B}$ gives the torque in the increasing $\theta$ direction in the case of constant flux.




Example: Force Exerted by a Solenoid on a Magnetic Slab: A solenoid of cross-sectional area $A$, $N$ turns, and length $\ell$, is connected to an external source that sets up a constant current $I$ through it. A rod of a magnetic material of constant permeability $\mu$ and cross-sectional area $A$ is partially inserted in the solenoid.

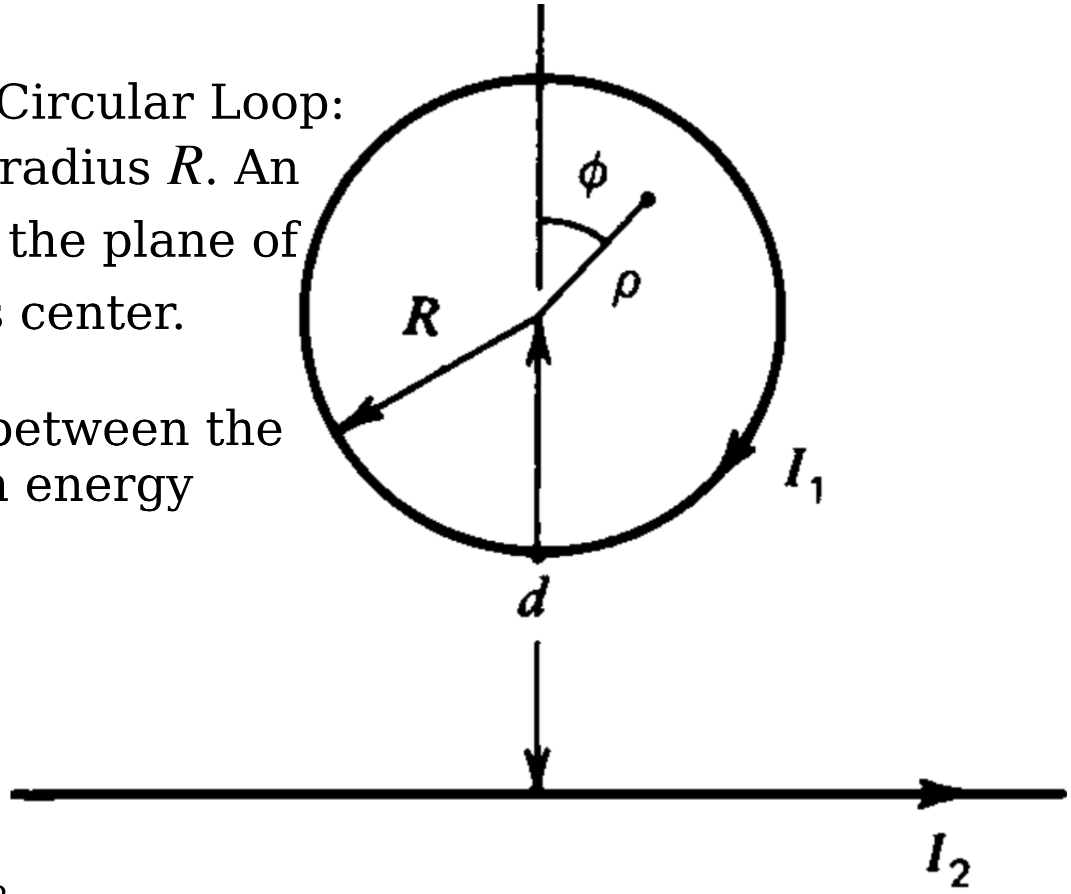
- To calculate the force, we need to calculate the magnetic energy of the system as a function of $x$. Assume that when the slab is slightly moved by $\Delta x$ from its position, the structure of the field remains the same, the only difference being that a $\Delta x$ of the slab is effectively transferred from the very outer region to the region well inside the solenoid.

$$\Rightarrow \quad \Delta U = U(x+\Delta x) - U(x) \approx \frac{\mu-\mu_0}{2}\int_{\mathcal{V}} H^2 \, d\tau$$

$$\approx \frac{\mu-\mu_0}{2}\left(\frac{NI}{\ell}\right)^2 A\,\Delta x \quad \Leftarrow \quad \mathcal{V} = A\,\Delta x\,, \quad H = \frac{NI}{\ell}$$

$$\Rightarrow \quad \text{The force on the slab } \; F \approx \frac{\Delta U}{\Delta x}\Big|_I = \frac{\mu-\mu_0}{2}\left(\frac{NI}{\ell}\right)^2 A$$

<u>Example</u>: Force Between a Wire and a Circular Loop: A current $I_1$, flows in a circular loop of radius $R$. An infinite wire carrying a current $I_2$, is in the plane of the loop and at a distance $d>R$ from its center.




- Since we are interested in the force between the wires, we calculate only the interaction energy

$$U = I_1 \Phi_1 \quad \Leftarrow \quad \Phi_1 = \int_{\text{loop}} \mathbf{B}_2 \cdot \mathrm{d}\boldsymbol{a}$$

$$\Rightarrow \quad \Phi_1 = -\frac{\mu_0 I_2}{2\pi} \int_0^R \int_0^{2\pi} \frac{\rho \, \mathrm{d}\rho \, \mathrm{d}\phi}{d + \rho \cos\phi}$$

$$= -\mu_0 I_2 \int_0^R \frac{\rho \, \mathrm{d}\rho}{\sqrt{d^2 - \rho^2}} \quad \Leftarrow \quad \int_0^{2\pi} \frac{\mathrm{d}\phi}{1 + \beta\cos\phi} = \frac{2\pi}{\sqrt{1-\beta^2}}, \quad -1 < \beta < 1$$

$$= \mu_0 I_2 \left(\sqrt{d^2 - R^2} - d\right) \quad \Rightarrow \quad U = \mu_0 I_1 I_2 \left(\sqrt{d^2 - R^2} - d\right)$$

$$\Rightarrow \quad \text{the force between the wires} \quad F = \frac{\partial U}{\partial d}\Big|_{I_1, I_2} = \mu_0 I_1 I_2 \left( \frac{d}{\sqrt{d^2 - R^2}} - 1 \right) > 0$$

$$\int_0^{\pi} \frac{d\phi}{1+\beta\cos\phi} = \int_0^{\pi/2} \frac{d\phi}{1+\beta\cos\phi} + \int_{\pi/2}^{\pi} \frac{d\phi}{1+\beta\cos\phi} \quad \Leftarrow \quad |\beta|<1$$

$$= \int_0^{\pi/2} \frac{d\phi}{1+\beta\cos\phi} + \int_0^{\pi/2} \frac{d\phi}{1-\beta\cos\phi} = 2\int_0^{\pi/2} \frac{d\phi}{1-\beta^2\cos^2\phi}$$

$$= 2\int_0^{\pi/2} \frac{d\phi}{1-\beta^2+\beta^2\sin^2\phi} = 2\int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{\left(1-\beta^2\right)\sec^2\theta+\beta^2\tan^2\theta}$$

$$= 2\int_0^{\pi/2} \frac{d\tan\theta}{1-\beta^2+\tan^2\theta} = \frac{2}{1-\beta^2}\int_0^{\pi/2} \frac{d\tan\theta}{1+\dfrac{\tan^2\theta}{1-\beta^2}}$$

$$= \frac{2}{\sqrt{1-\beta^2}}\int_0^{\infty} \frac{dx}{1+x^2} = \frac{2}{\sqrt{1-\beta^2}} \tan^{-1} x \Big|_0^{\infty} = \frac{\pi}{\sqrt{1-\beta^2}} \quad \Leftarrow \quad x = \frac{\tan\theta}{\sqrt{1-\beta^2}}$$

$$\Rightarrow \quad \int_0^{2\pi} \frac{d\phi}{1+\beta\cos\phi} = 2\int_0^{\pi} \frac{d\phi}{1+\beta\cos\phi} = \frac{2\pi}{\sqrt{1-\beta^2}}$$

$$\int_0^{\pi} \frac{\sin^2\phi \, d\phi}{1+\beta\cos\phi} = \int_0^{\pi/2} \frac{\sin^2\phi \, d\phi}{1+\beta\cos\phi} + \int_{\pi/2}^{\pi} \frac{\sin^2\phi \, d\phi}{1+\beta\cos\phi} \quad \Leftarrow \quad |\beta|<1$$

$$= \int_0^{\pi/2} \frac{\sin^2\phi \, d\phi}{1+\beta\cos\phi} + \int_0^{\pi/2} \frac{\sin^2\phi \, d\phi}{1-\beta\cos\phi} = 2\int_0^{\pi/2} \frac{\sin^2\phi \, d\phi}{1-\beta^2\cos^2\phi}$$

$$= 2\int_0^{\pi/2} \frac{\sin^2\phi \, d\phi}{1-\beta^2+\beta^2\sin^2\phi} = \frac{2}{\beta^2}\int_0^{\pi/2} \left( 1 - \frac{1-\beta^2}{1-\beta^2+\beta^2\sin^2\phi} \right) d\phi$$

$$= \frac{\pi}{\beta^2} - 2\,\frac{1-\beta^2}{\beta^2}\int_0^{\pi/2} \frac{\sec^2\theta \, d\theta}{(1-\beta^2)\sec^2\theta + \beta^2\tan^2\theta} = \frac{\pi}{\beta^2} - \frac{2}{\beta^2}\int_0^{\pi/2} \frac{d\tan\theta}{1+\dfrac{\tan^2\theta}{1-\beta^2}}$$

$$= \frac{\pi}{\beta^2} - \frac{2}{\beta^2}\sqrt{1-\beta^2}\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{\beta^2} - \frac{2}{\beta^2}\sqrt{1-\beta^2}\,\tan^{-1} x\Big|_0^{\infty} \quad \Leftarrow \quad x = \frac{\tan\theta}{\sqrt{1-\beta^2}}$$

$$= \frac{\pi}{\beta^2}\left(1-\sqrt{1-\beta^2}\right)$$

Thus $$\int_0^{\pi} \frac{\cos^2\phi \, d\phi}{1+\beta\cos\phi} = \frac{\pi}{\sqrt{1-\beta^2}} - \frac{\pi}{\beta^2}\left(1-\sqrt{1-\beta^2}\right) = \frac{\pi}{\beta^2}\left( \frac{1}{\sqrt{1-\beta^2}} - 1 \right)$$

## Maxwell's Equations

### Electrodynamics Before Maxwell

- So far

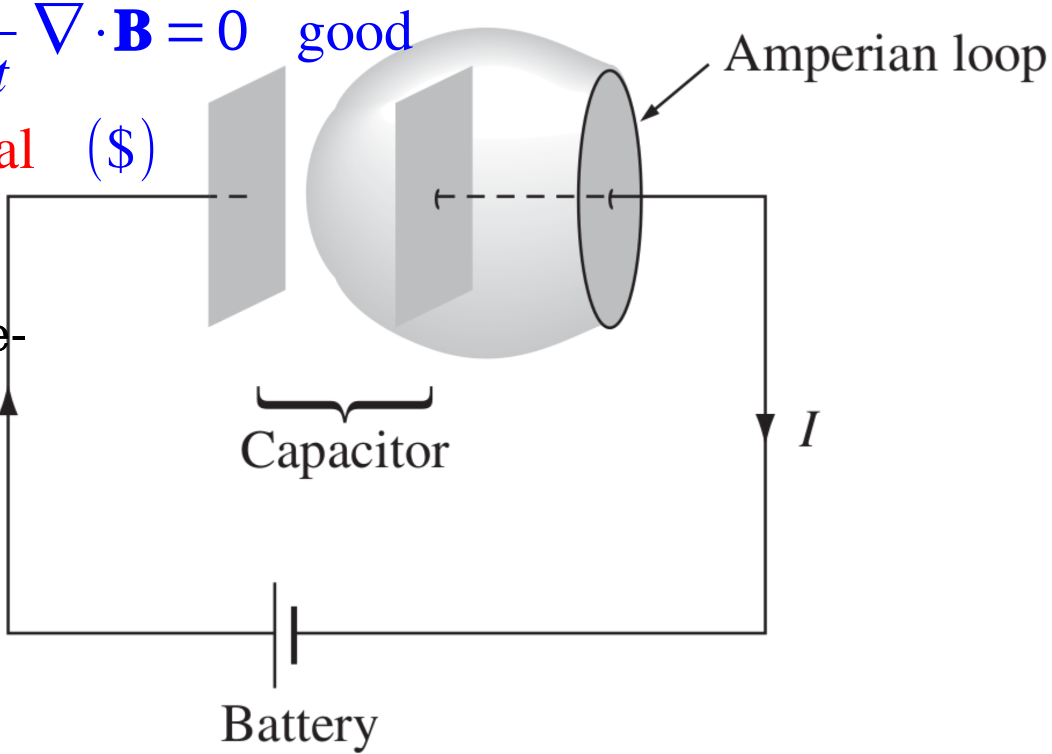
$$\text{(i)} \quad \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{Gauss's law} \qquad \text{(iii)} \quad \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{Faraday's law}$$

$$\text{(ii)} \quad \nabla\cdot\mathbf{B} = 0 \quad \text{no magnetic monopole} \qquad \text{(iv)} \quad \nabla\times\mathbf{B} = \mu_0 \boldsymbol{J} \quad \text{Ampere's law}$$

- These equations represent the state of EM theory in the mid-nineteenth century, when Maxwell began his work.

- There is a fatal inconsistency in these formulas. It has to do with the old rule that divergence of curl is always 0.

- $$0 = \nabla\cdot(\nabla\times\mathbf{E}) = \nabla\cdot\left(-\frac{\partial \mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\nabla\cdot\mathbf{B} = 0 \quad \text{good}$$
$$0 = \nabla\cdot(\nabla\times\mathbf{B}) = \mu_0 \nabla\cdot\boldsymbol{J} \neq 0 \quad \text{in general} \qquad (\$)$$




- For *steady* currents, the divergence of $\boldsymbol{J}$ is 0, but when we go beyond magnetostatics Ampère's law cannot be right.

- In the process of charging up a capacitor. In integral form, Ampère's law

$$\oint \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 \boldsymbol{I}_{\text{enc}}$$

- For the plane of the loop, $I_{enc}=I$; for the balloon-shaped surface, no current passes through this surface, so $I_{enc}=0$!

- The conflict arises only when charge is piling up (on the capacitor plates).

- For *nonsteady* currents “the current enclosed by the loop” is an ill-defined notion; it depends entirely on what surface you use.

- We had no right to expect Ampère’s law to hold outside of magnetostatics; after all, we derived it from the Biot-Savart law.

- The flaw was a purely theoretical one, and Maxwell fixed it by purely theoretical arguments.

## How Maxwell Fixed Ampère's Law

- The problem is on the right side of ($), which *should be* 0, but *isn't*.

- Applying the continuity equation and Gauss's law, the offending term becomes

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot \left( \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

- If we were to combine $\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ with $\boldsymbol{J}$, in Ampère's law, it would be just right to kill off the extra divergence:

$$\nabla \times \mathbf{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

- Such a modification changes nothing, as far as magnetostatics is concerned: when $\mathbf{E}$ is constant, we still have $\nabla \times \mathbf{B} = \mu_0 \boldsymbol{J}$. But the modification plays a crucial role in the propagation of EM waves.

- Apart from curing the defect in Ampère's law, Maxwell's term has a certain aesthetic appeal: Just as a changing magnetic field induces an electric field, so

A changing electric field induces a magnetic field.

- The confirmation of Maxwell's theory was Hertz's experiments on EM waves.

- Maxwell called his extra term the **displacement current**: $\boldsymbol{J}_d = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

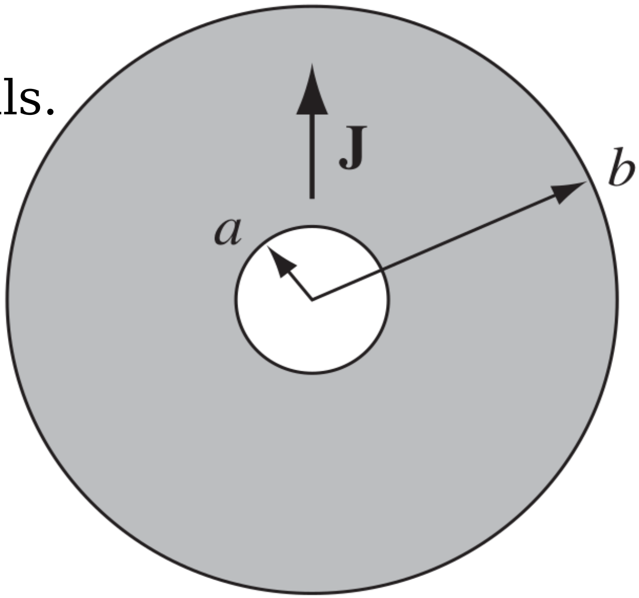
- If the capacitor plates are very close together, the electric field between them

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A} \quad \Rightarrow \quad \frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A} \frac{\mathrm{d} Q}{\mathrm{d} t} = \frac{I}{\epsilon_0 A}$$

$$\Rightarrow \quad \oint \mathbf{B} \cdot \mathrm{d} \boldsymbol{\ell} = \mu_0 I_{\mathrm{enc}} + \mu_0 \epsilon_0 \int \frac{\partial \mathbf{E}}{\partial t} \cdot \mathrm{d} \boldsymbol{a}$$

- If we choose the *flat* surface, then $E=0$ and $I_{\mathrm{enc}}=I$. If, on the other hand, we use the balloon-shaped surface, then $I_{\mathrm{enc}}=0$, but $\int \frac{\partial \mathbf{E}}{\partial t} \cdot \mathrm{d} \boldsymbol{a} = \frac{I}{\epsilon_0}$

- So we get the same answer for either surface, though in the 1st case it comes from the conduction current, and in the 2nd from the displacement current.




Example 7.14: Imagine 2 concentric metal spherical shells. The inner one carries a charge $Q(t)$, the outer one an opposite charge $-Q(t)$. The space between them is filled with Ohmic material of conductivity $\sigma$, so a radial current flows:

$$\boldsymbol{J} = \sigma\,\mathbf{E} = \frac{\sigma}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}\,,\quad I = -\dot{Q} = \int \boldsymbol{J}\cdot d\boldsymbol{a} = \frac{\sigma}{\epsilon_0}Q$$

This configuration is spherically symmetrical, so the magnetic field has to be 0. But currents produce magnetic fields! How can there be a $\boldsymbol{J}$ with no accompanying $\mathbf{B}$?

- This is not a static configuration: $Q$, $\mathbf{E}$, and $\boldsymbol{J}$ are all functions of time; Ampère and Biot-Savart do not apply.
- The displacement current $\boldsymbol{J}_d = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{4\pi}\frac{\dot{Q}}{r^2}\hat{\mathbf{r}} = -\frac{\sigma Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$ exactly cancels the conduction current, and the magnetic field (determined by $\nabla\cdot\mathbf{B}=0$, $\nabla\times\mathbf{B}=0$) is indeed 0.

## Maxwell's Equations

- Maxwell's equations:

(i) $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ Gauss's law

(iii) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ Faraday's law

(ii) $\nabla \cdot \mathbf{B} = 0$ no magnetic monopole

(iv) $\nabla \times \mathbf{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ Ampere's law with Maxwell's correction

together with the Lorentz force law, $\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$ summarize the entire theoretical content of classical electrodynamics.

- Even the continuity equation, $\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$ , the expression of conservation of charge, can be derived from Maxwell's eqns by applying the divergence to (iv).

- This form of Maxwell's equations reinforces the notion that electric fields can be produced either by charges $\rho$ or by changing magnetic fields $\frac{\partial \mathbf{B}}{\partial t}$ , magnetic fields can be produced either by currents $\boldsymbol{J}$ or by changing electric fields $\frac{\partial \mathbf{E}}{\partial t}$.

- It is misleading because $\frac{\partial \mathbf{B}}{\partial t}$ & $\frac{\partial \mathbf{E}}{\partial t}$ are themselves due to charge & current.

- It is logically better to write

$$(\text{i})\quad \nabla\cdot\mathbf{E}=\frac{\rho}{\epsilon_0}\qquad (\text{iii})\quad \nabla\times\mathbf{E}+\frac{\partial\,\mathbf{B}}{\partial\,t}=0$$

$$(\text{ii})\quad \nabla\cdot\mathbf{B}=0\qquad (\text{iv})\quad \nabla\times\mathbf{B}-\mu_0\,\epsilon_0\frac{\partial\,\mathbf{E}}{\partial\,t}=\mu_0\,\boldsymbol{J}$$

- This notation emphasizes that all EM fields are attributable to charges and currents. Maxwell’s equations tell you how charges produce fields; reciprocally, the Lorentz force law tells you how fields affect charges.

## Magnetic Charge

- There is a pleasing symmetry to Maxwell's equations; it is particularly striking in free space, where $\rho$ and $\boldsymbol{J}$ vanish:

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

- If you replace $\mathbf{E}$ by $\mathbf{B}$ and $\mathbf{B}$ by $-\mu_0 \epsilon_0 \mathbf{E}$, the 1st pair of equations turns into the 2nd, and vice versa.

- This symmetry between $\mathbf{E}$ and $\mathbf{B}$ is spoiled by the charge term in Gauss's law and the current term in Ampère's law.

- What are the corresponding quantities "missing" from $\nabla \cdot \mathbf{B} = 0$ & $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

- What if we had

$$\text{(i)}\ \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho_e \quad \text{(iii)}\ \nabla \times \mathbf{E} = -\mu_0 \boldsymbol{J}_m - \frac{\partial \mathbf{B}}{\partial t}$$

$$\text{(ii)}\ \nabla \cdot \mathbf{B} = \mu_0 \rho_m \quad \text{(iv)}\ \nabla \times \mathbf{B} = \mu_0 \boldsymbol{J}_e + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$\rho_m$ represents the density of magnetic "charge," and $\rho_e$ the density of electric charge; $\boldsymbol{J}_m$ is the current of magnetic charge, $\boldsymbol{J}_e$ the current of electric charge.

- Both charges are conserved: $\nabla \cdot \boldsymbol{J}_m = -\frac{\partial \rho_m}{\partial t}, \quad \nabla \cdot \boldsymbol{J}_e = -\frac{\partial \rho_e}{\partial t}$, by application of the divergence to (iii), the latter by taking the divergence of (iv).

- In a sense, Maxwell's equations *beg* for magnetic charge to exist—it would fit in so nicely.

- As far as we know, $\rho_m$ is 0 everywhere, and so is $\boldsymbol{J}_m$; $\mathbf{B}$ is not on equal footing with $\mathbf{E}$: there exist stationary sources for $\mathbf{E}$ (electric charges) but none for $\mathbf{B}$.

- This is reflected in the fact that magnetic multipole expansions have no monopole term, and magnetic dipoles consist of current loops, not separated north and south "poles."

- In *quantum* electrodynamics, it's a more than merely aesthetic shame that *magnetic* charge does not seem to exist: Dirac showed that the existence of magnetic charge would explain why *electric* charge is *quantized*.

Problem 7, 11, 18, 22, 26, 33, 37, 43, 54, 63

## Maxwell's Equations in Matter

- When you are working with materials that are subject to electric and magnetic polarization there is a more convenient way to *write* Maxwell's equations.

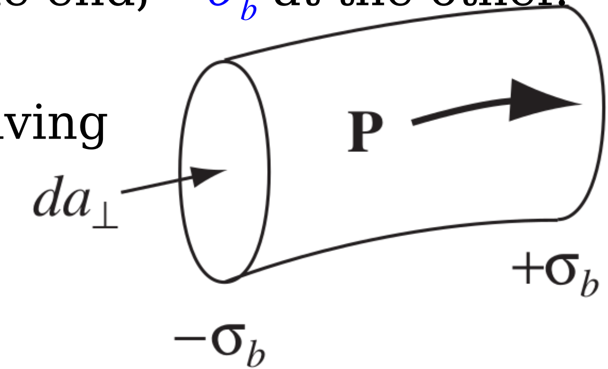
- For inside polarized matter there will be accumulations of "bound" charge and current, over which you exert no direct control. It would be nice to reformulate Maxwell's equations so as to make explicit reference only to the "free" charges and currents.

- From the static case, an electric polarization $\mathbf{P}$ produces a bound charge density, $\rho_b = -\nabla\cdot\mathbf{P}$; a magnetic polarization ("magnetization") $\mathbf{M}$ results in a bound current, $\boldsymbol{J}_b = \nabla\times\mathbf{M}$.

- One new feature to consider in the *non*static case: Any *change* in the electric polarization involves a flow of (bound) charge (call it $\boldsymbol{J}_p$), which must be included in the total current.

- The polarization introduces a charge density $\sigma_b = P$ at one end, $-\sigma_b$ at the other.

- If $P$ *increases* a bit, the charge on each end increases, giving a net current $\mathrm{d}I = \frac{\partial\sigma_b}{\partial t}\mathrm{d}a_\perp = \frac{\partial P}{\partial t}\mathrm{d}a_\perp \quad\Rightarrow\quad \boldsymbol{J}_p = \frac{\partial\mathbf{P}}{\partial t}$



- This **polarization current** has nothing to do with the bound current $\boldsymbol{J}_b$. The latter is associated with magnetization of the material and involves the spin and orbital motion of electrons; $\boldsymbol{J}_p$ is the result of the linear motion of charge when the electric polarization changes.

- If $\mathbf{P}$ points to the right increasingly, then each +charge moves a bit to the right and each −charge to the left; the cumulative effect is the polarization current $\boldsymbol{J}_p$.

- Its consistency with the continuity eqn: $\nabla \cdot \boldsymbol{J}_p = \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} = \frac{\partial}{\partial t} \nabla \cdot \mathbf{P} = -\frac{\partial \rho_b}{\partial t}$

- The continuity eqn is satisfied; in fact, $\boldsymbol{J}_p$ is essential to ensure the conservation of bound charge.

- A changing magnetization does not lead to any analogous accumulation of charge or current. Only the bound current $\boldsymbol{J}_b = \nabla \times \mathbf{M}$ varies to changes in $\mathbf{M}$.

- The total charge density can be separated into 2 parts: $\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \mathbf{P}$

- The current density has 3 parts: $\boldsymbol{J} = \boldsymbol{J}_f + \boldsymbol{J}_b + \boldsymbol{J}_p = \boldsymbol{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}$

- Gauss's law is written as $\nabla \cdot \mathbf{E} = \frac{\rho_f - \nabla \cdot \mathbf{P}}{\epsilon_0} \quad \Rightarrow \quad \nabla \cdot \mathbf{D} = \rho_f \quad \Leftarrow \quad \mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$

- Ampère’s law (with Maxwell’s term) becomes

$$\nabla\times\mathbf{B}=\mu_0\left(\boldsymbol{J}_f+\nabla\times\mathbf{M}+\frac{\partial\mathbf{P}}{\partial t}\right)+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$$\Rightarrow\quad\nabla\times\mathbf{H}=\boldsymbol{J}_f+\frac{\partial\mathbf{D}}{\partial t}\quad\Leftarrow\quad\mathbf{H}\equiv\frac{\mathbf{B}}{\mu_0}-\mathbf{M}$$

- Faraday’s law and $\nabla\cdot\mathbf{B}=0$ are not affected by our separation of charge and current into free and bound parts, since they do not involve $\rho$ or $\boldsymbol{J}$.

- In terms of *free* charges and currents, then, Maxwell’s equations read

$$\text{(i)}\quad\nabla\cdot\mathbf{D}=\rho_f\quad\Leftrightarrow\quad\oint\mathbf{D}\cdot\mathrm{d}\boldsymbol{a}=Q$$

$$\text{(ii)}\quad\nabla\cdot\mathbf{B}=0\quad\Leftrightarrow\quad\oint\mathbf{B}\cdot\mathrm{d}\boldsymbol{a}=0$$

$$\text{(iii)}\quad\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\quad\Leftrightarrow\quad\oint\mathbf{E}\cdot\mathrm{d}\boldsymbol{\ell}=-\frac{\mathrm{d}}{\mathrm{d}t}\int\mathbf{B}\cdot\mathrm{d}\boldsymbol{a}$$

$$\text{(iv)}\quad\nabla\times\mathbf{H}=\boldsymbol{J}_f+\frac{\partial\mathbf{D}}{\partial t}\quad\Leftrightarrow\quad\oint\mathbf{H}\cdot\mathrm{d}\boldsymbol{\ell}=I+\frac{\mathrm{d}}{\mathrm{d}t}\int\mathbf{D}\cdot\mathrm{d}\boldsymbol{a}$$

- These eqns are in *no way* more “general” than the earlier ones; they simply reflect a convenient division of charge and current into free and nonfree parts.

- They have the disadvantage of hybrid notation, since they contain both $\mathbf{E}$ and $\mathbf{D}$, both $\mathbf{B}$ and $\mathbf{H}$.

- They must be supplemented by appropriate **constitutive relations**, giving $\mathbf{D}$ and $\mathbf{H}$ in terms of $\mathbf{E}$ and $\mathbf{B}$.

- These depend on the nature of the material; for linear media

$$\begin{aligned}\mathbf{P} &= \epsilon_0 \chi_e \mathbf{E} \\ \mathbf{M} &= \chi_m \mathbf{H}\end{aligned} \quad \Rightarrow \quad \begin{aligned}\mathbf{D} &= \epsilon \mathbf{E} \\ \mathbf{H} &= \frac{1}{\mu} \mathbf{B}\end{aligned} \quad \Leftarrow \quad \begin{aligned}\epsilon &= \epsilon_0 (1 + \chi_e) \\ \mu &= \mu_0 (1 + \chi_m)\end{aligned}$$

- $\mathbf{D}$ is called the electric “displacement”; and the 2nd term in the Ampère/Maxwell equation (iv), $\boldsymbol{J}_d \equiv \dfrac{\partial \mathbf{D}}{\partial t}$, is called the **displacement current**.

# Boundary Conditions

- The fields **E**, **B**, **D**, and **H** are discontinuous at a boundary between 2 different media, or at a surface that carries a charge density $\sigma$ or a current density $\boldsymbol{K}$.

- The explicit form of these discontinuities can be deduced from Maxwell's equations in their integral form

$$\text{(i)} \quad \oint_{\mathcal{S}} \mathbf{D}\cdot d\boldsymbol{a} = Q_{f\,\text{enc}}$$

$$\text{(ii)} \quad \oint_{\mathcal{S}} \mathbf{B}\cdot d\boldsymbol{a} = 0$$

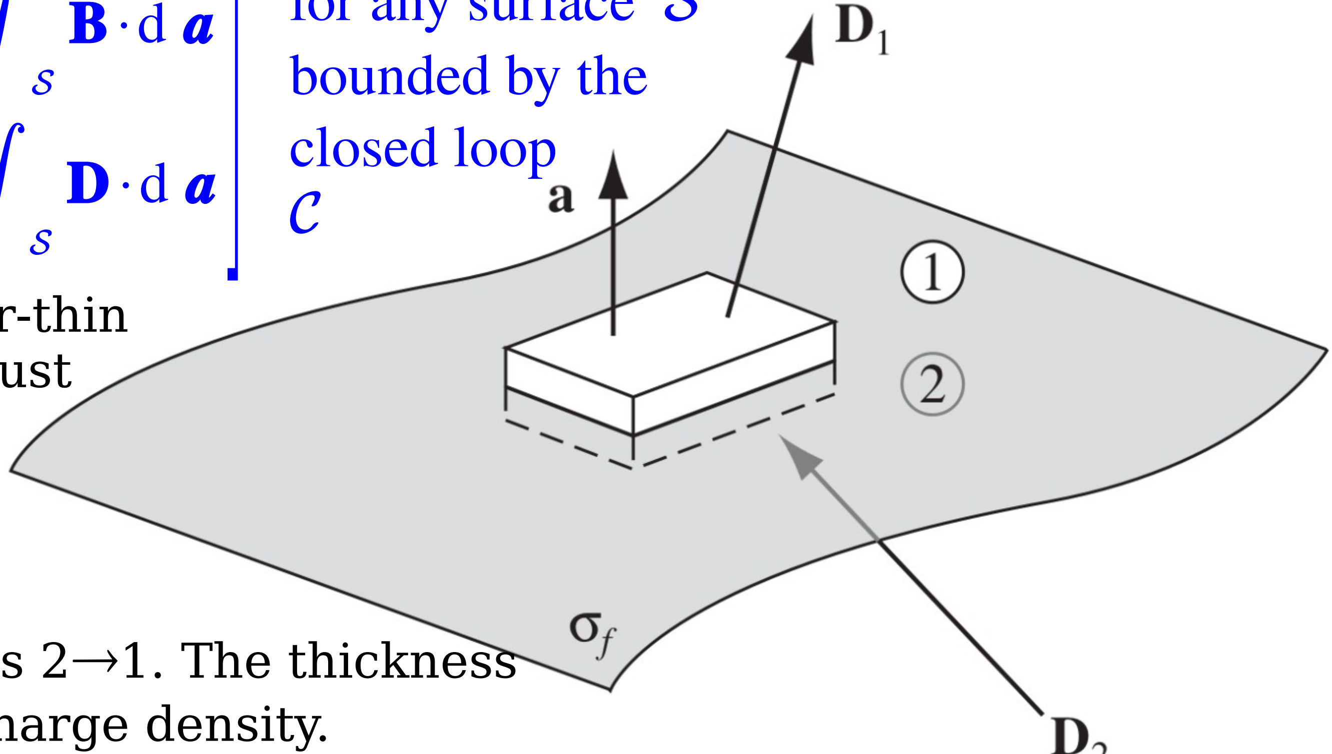
over any closed surface $\mathcal{S}$

$$\text{(iii)} \quad \oint_{\mathcal{C}} \mathbf{E}\cdot d\boldsymbol{\ell} = -\frac{d}{dt}\int_{\mathcal{S}} \mathbf{B}\cdot d\boldsymbol{a}$$

$$\text{(iv)} \quad \oint_{\mathcal{C}} \mathbf{H}\cdot d\boldsymbol{\ell} = I_{f\,\text{enc}} + \frac{d}{dt}\int_{\mathcal{S}} \mathbf{D}\cdot d\boldsymbol{a}$$

for any surface $\mathcal{S}$ bounded by the closed loop $\mathcal{C}$

- Applying (i) to a tiny, wafer-thin Gaussian pillbox extending just slightly into the material on either side of the boundary:




$$\mathbf{D}_1\cdot\boldsymbol{a} - \mathbf{D}_2\cdot\boldsymbol{a} = \sigma_f\, a$$

The positive direction for $\boldsymbol{a}$ is 2→1. The thickness of the edge→0; no volume charge density.

- The component of **D** ⊥ the interface is discontinuous as: $D_1^{\perp} - D_2^{\perp} = \sigma_f$
- Identical reasoning, applied to equation (ii), yields $B_1^{\perp} - B_2^{\perp} = 0$
- Turning to (iii), a very thin Amperian loop straddling the surface gives

$$\mathbf{E}_1 \cdot \boldsymbol{\ell} - \mathbf{E}_2 \cdot \boldsymbol{\ell} = -\frac{d}{dt}\int \mathbf{B} \cdot d\boldsymbol{a} \to 0 \text{ as the loop's width} \to 0 \quad \Rightarrow \quad \mathbf{E}_1^{\parallel} - \mathbf{E}_1^{\parallel} = 0$$

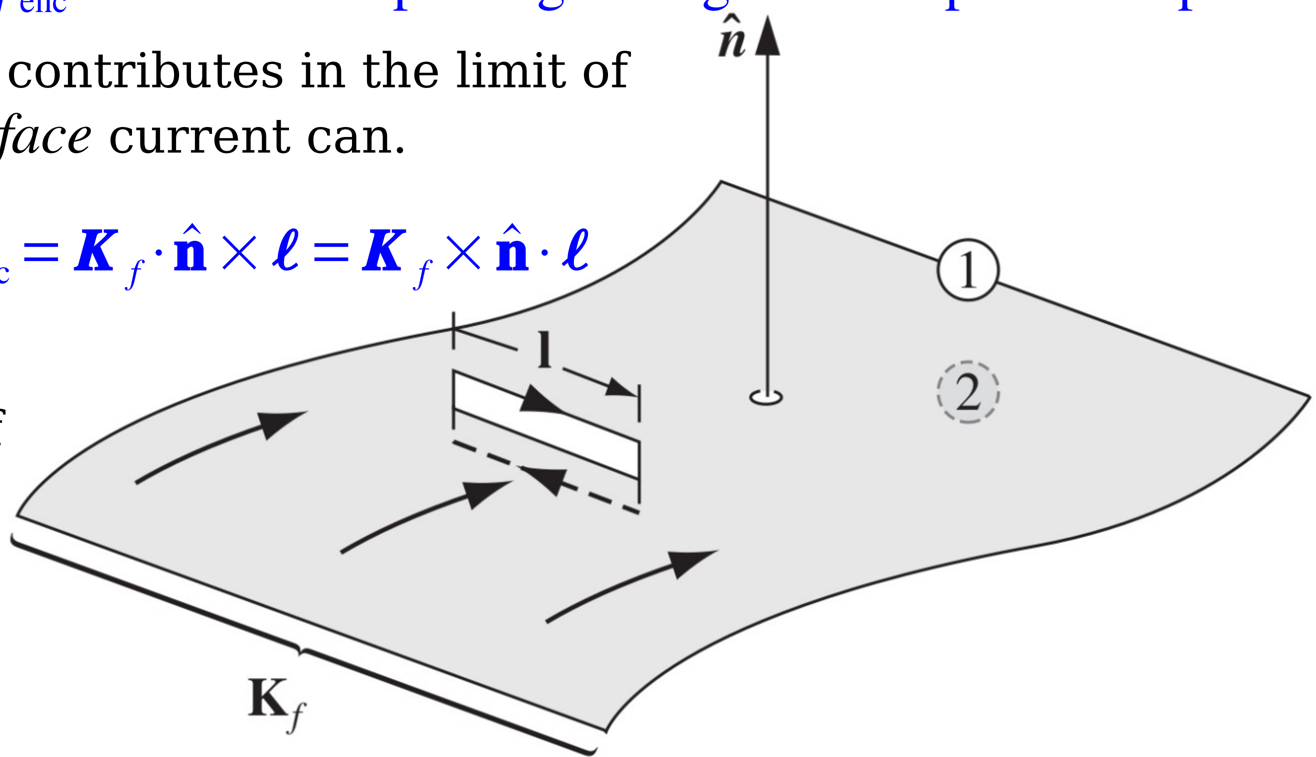
- The components of **E** ∥ the interface are continuous across the boundary.
- By the same token, (iv) implies

$$\mathbf{H}_1 \cdot \boldsymbol{\ell} - \mathbf{H}_2 \cdot \boldsymbol{\ell} = I_{f\,\text{enc}} \quad \Leftarrow \quad I_{f\,\text{enc}} : \text{free current passing through the Amperian loop}$$

- No volume current density contributes in the limit of infinitesimal width, but a *surface* current can.
- $(\hat{\mathbf{n}} \times \boldsymbol{\ell}) \perp$ the loop $\quad \Rightarrow \quad I_{f\,\text{enc}} = \boldsymbol{K}_f \cdot \hat{\mathbf{n}} \times \boldsymbol{\ell} = \boldsymbol{K}_f \times \hat{\mathbf{n}} \cdot \boldsymbol{\ell}$

$$\Rightarrow \quad \mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \boldsymbol{K}_f \times \hat{\mathbf{n}}$$

- The *parallel* components of **H** are discontinuous by an amount proportional to the free surface current density.



- The general statements about the EM boundary conditions:

  **1**. The tangential component of an $\mathbf{E}$ field is continuous across an interface.

  **2**. The tangential component of an $\mathbf{H}$ field is discontinuous across an interface where a surface current exists by $\mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \boldsymbol{K}_f \times \hat{\mathbf{n}}$ .

  **3**. The normal component of a $\mathbf{D}$ field is discontinuous across an interface where a surface charge exists by $D_1^{\perp} - D_2^{\perp} = \sigma_f$ .

  **4**. The normal component of a $\mathbf{B}$ field is continuous across an interface.

- In the case of linear media, they can be expressed in terms of $\mathbf{E}$ and $\mathbf{B}$ alone:

$$(\mathrm{i})\quad \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f \qquad (\mathrm{iii})\quad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0$$

$$(\mathrm{ii})\quad B_1^{\perp} - B_2^{\perp} = 0 \qquad (\mathrm{iv})\quad \frac{\mathbf{B}_1^{\parallel}}{\mu_1} - \frac{\mathbf{B}_2^{\parallel}}{\mu_2} = \boldsymbol{K}_f \times \hat{\mathbf{n}}$$

- If there is *no* free charge or free current at the interface, then

$$(\mathrm{i})\quad \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0 \qquad (\mathrm{iii})\quad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0$$

$$(\mathrm{ii})\quad B_1^{\perp} - B_2^{\perp} = 0 \qquad (\mathrm{iv})\quad \frac{\mathbf{B}_1^{\parallel}}{\mu_1} - \frac{\mathbf{B}_2^{\parallel}}{\mu_2} = 0$$

- These equations are the basis for the theory of reflection and refraction.

- In the case of perfect conductor (for Medium 2)

$$\epsilon_1 E_1^{\perp} = \sigma_f \,, \qquad E_2^{\perp} = 0$$

$$\mathbf{E}_1^{\parallel} = 0 \,, \qquad \mathbf{E}_2^{\parallel} = 0$$

$$B_1^{\perp} = 0 \,, \qquad B_2^{\perp} = 0$$

$$\frac{\mathbf{B}_1^{\parallel}}{\mu_1} = \boldsymbol{K}_f \times \hat{\mathbf{n}} \,, \quad \mathbf{B}_2^{\parallel} = 0$$