Chapter 11 Special Theory of Relativity

- Beginning with Chapter 11 we employ Gaussian units for EM quantities.

- Special Relativity (SR) is believed to apply to all forms of interaction except large-scale gravitational phenomena.

The Situation Before 1900, Einstein's 2 Postulates

- Before SR, it had been believed that the laws of mechanics were the same in different coordinate systems moving uniformly relative to one another—invariant under Galilean transformations

$$\begin{aligned} \mathbf{r}' &= \mathbf{r} - \mathbf{v}\,t \\ t' &= t \end{aligned} \quad (\ast) \quad \Leftarrow \quad \text{Galilean relativity}$$

Ex: consider a group of particles interacting via two-body central potentials

$$m_i \frac{d\,\mathbf{v}_i'}{d\,t'} = -\nabla_i' \sum V_{ij}\left(\left|\mathbf{r}_i' - \mathbf{r}_j'\right|\right) \quad \text{in reference frame } K' \quad \Rightarrow \quad \begin{aligned} \mathbf{v}_i' &= \mathbf{v}_i - \mathbf{v} \\ \nabla_i' &= \nabla_i \end{aligned}, \quad \frac{d\,\mathbf{v}_i}{d\,t'} = \frac{d\,\mathbf{v}_i}{d\,t}$$

$$\Rightarrow \quad m_i \frac{d\,\mathbf{v}_i}{d\,t} = -\nabla_i \sum V_{ij}\left(\left|\mathbf{r}_i - \mathbf{r}_j\right|\right) \quad \text{in reference frame } K \quad \Leftarrow \quad \mathbf{r}_i' - \mathbf{r}_j' = \mathbf{r}_i - \mathbf{r}_j$$

- The *form* of the equations of classical mechanics under Galilean transformation is preserved, but the form of the wave equation is not.

$$\underbrace{\left(\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right)\psi = 0}_{\text{in } K'} \quad \Rightarrow \quad \underbrace{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{2}{c^2}\mathbf{v}\cdot\nabla\frac{\partial}{\partial t} - \frac{1}{c^2}\mathbf{v}\cdot\nabla\,\mathbf{v}\cdot\nabla\right)\psi = 0}_{\text{in } K}$$

- The form of the EM wave equations is not invariant under Galilean transformations.

- 3 possibilities:

(1) The Maxwell equations were incorrect. The proper theory of EM is invariant under Galilean transformations. [×]

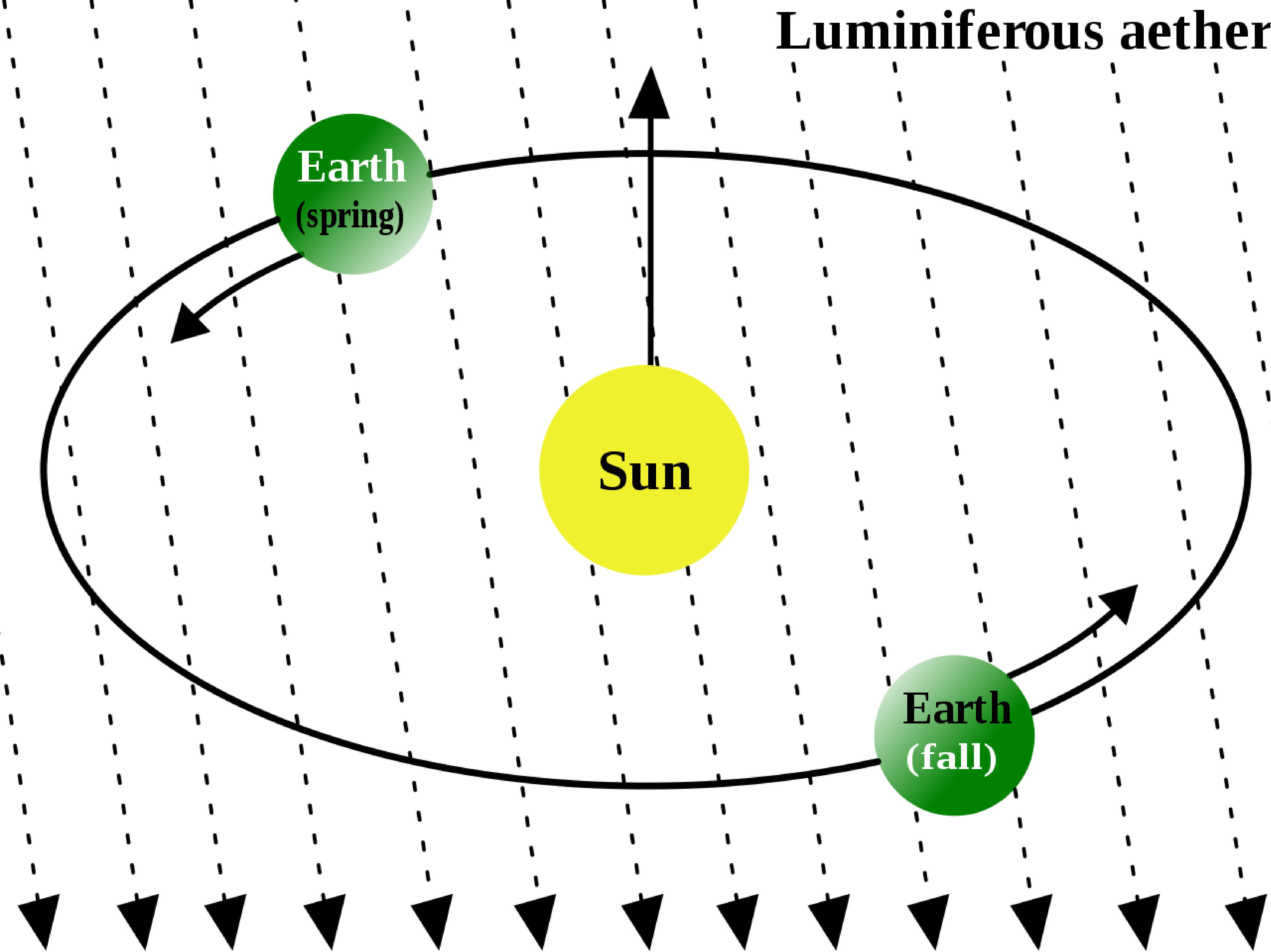
(2) Galilean relativity applied to classical mechanics, but EM had a preferred reference frame, the frame in which the luminiferous ether was at rest. [×]

(3) There is a relativity principle for classical mechanics & EM, but it was not Galilean relativity. This would imply that the laws of mechanics were in need of modification. [○]

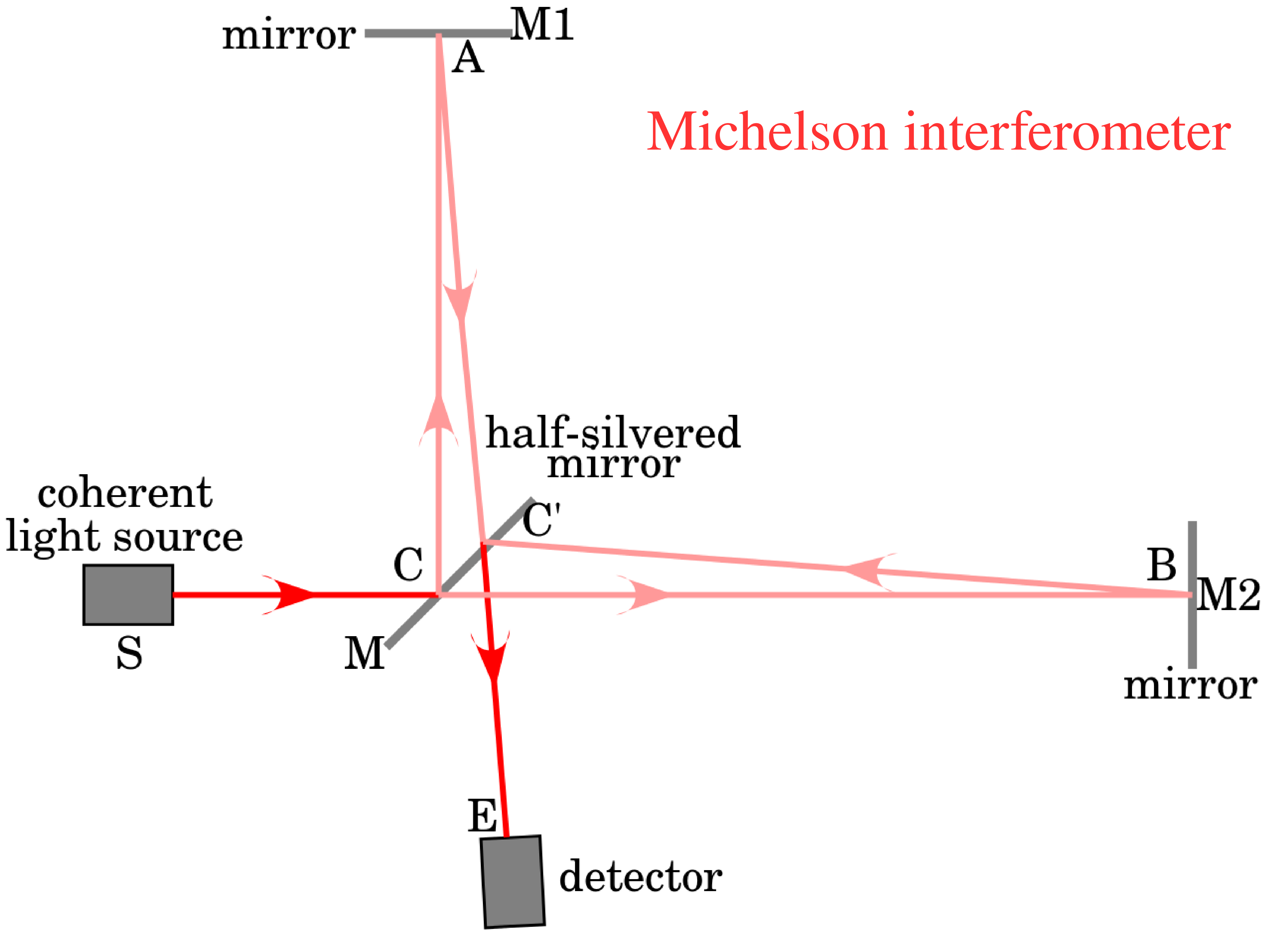
- Michelson-Morley's experiment fails to find ether, but leads to the FitzGerald-Lorentz contraction hypothesis: objects moving at a velocity $\mathbf{v}$ through the ether are contracted in the direction of motion $L(v) = L_0\sqrt{1-\frac{v^2}{c^2}} = L_0\sqrt{1-\beta^2} \Leftarrow \beta = \frac{v}{c}$

- Lorentz & Poincaré showed that the Maxwell equations are invariant in form under the Lorentz transformations and the contraction held for moving charge densities.

- The hypothesis of an ether was abandoned because of the implausibility of the explanation of experiments.



Luminiferous aether
Earth
(spring)
Sun
Earth
(fall)


Michelson interferometer
mirror
M1
A
half-silvered
mirror
coherent
light source
S
C
C'
M
B
M2
mirror
E
detector

- Einstein's special theory of relativity is based on 2 postulates:

1. POSTULATE OF RELATIVITY
The laws of nature & the results of all experiments performed in certain frames of reference are independent of the translational motion of the system as a whole.

2. POSTULATE OF THE CONSTANCY OF THE SPEED OF LIGHT
The speed of light is finite and independent of the motion of its source.

- These equivalent coordinate systems are called *inertial* reference frames.

- The 2nd postulate can be rephrased as

2′. POSTULATE OF A UNIVERSAL LIMITING SPEED
In every inertial frame, there is a finite universal limiting speed c for physical entities.

Some Recent Experiments

A. Ether Drift

- The null result of the Michelson-Morley experiment established that the velocity of the earth through the presumed ether was less than 1/3 of its orbital speed of approximately 3×10^4 m/s.

- These null results can be explained without abandoning the concept of an ether by the hypothesis of the FitzGerald-Lorentz contraction.

- **The Mössbauer effect**: the recoil momentum from the emission/absorption of a γ ray is taken up by the whole solid rather than by the emitting/absorbing nucleus $E_{\text{recoil}} \to 0 \;\Rightarrow\; E_\gamma = E_0 > E_0 - \dfrac{E_0^2}{2Mc^2} \;\Leftarrow\; M$: the mass of the nucleus

- With such recoilless transitions there are no thermal Doppler shifts. The γ-ray line approaches its natural shape with no broadening or shift in frequency.

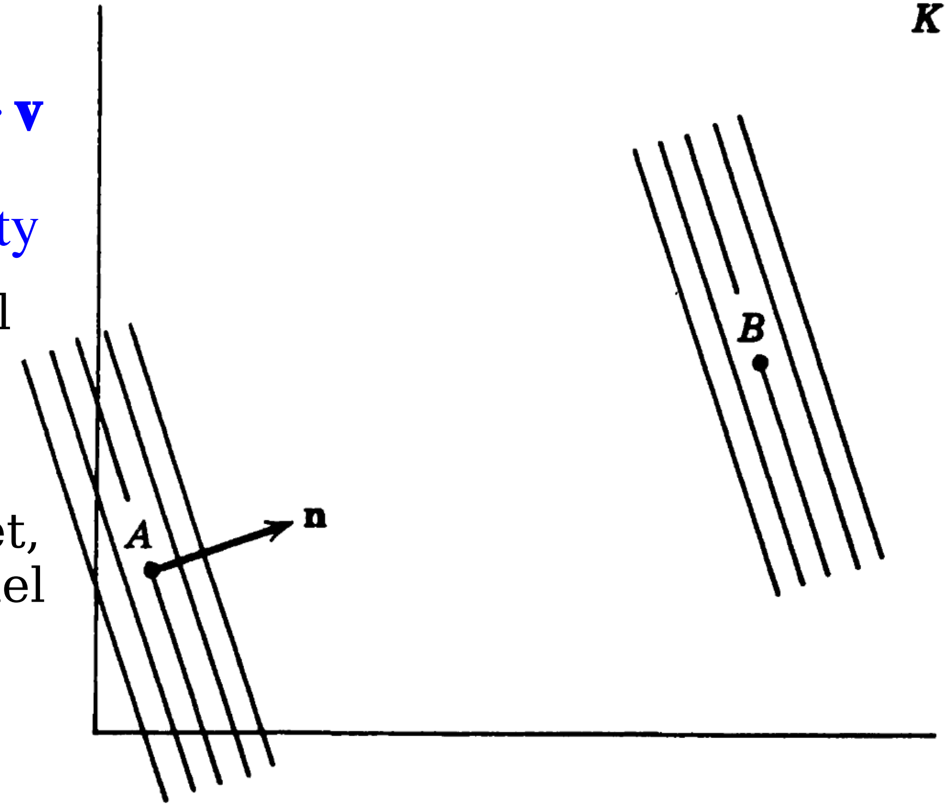
- Employing an absorber containing the same material as the emitter, one can study nuclear resonance absorption or use it as an instrument for the study of extremely small changes of frequency.

- The phase of a plane wave is an invariant quantity because the elapsed phase of a wave is proportional to the number of wave passing the observer. Since this is merely a counting operation, it must be independent of coordinate frame.

- $\omega\left[t'\left(1-\dfrac{\mathbf{n}\cdot\mathbf{v}}{c}\right)-\dfrac{\mathbf{n}\cdot\mathbf{r}'}{c}\right] \Leftarrow \omega\left(t-\dfrac{\mathbf{n}\cdot\mathbf{r}}{c}\right)=\phi=\omega'\left(t'-\dfrac{\mathbf{n}'\cdot\mathbf{r}'}{c'}\right) \;\Leftarrow\; (\star)$

$$\Rightarrow \quad \mathbf{n} = \mathbf{n}', \quad \omega' = \omega\left(1 - \frac{\mathbf{n}\cdot\mathbf{v}}{c}\right), \quad c' = c - \mathbf{n}\cdot\mathbf{v}$$

Doppler shift formulas of Galilean relativity

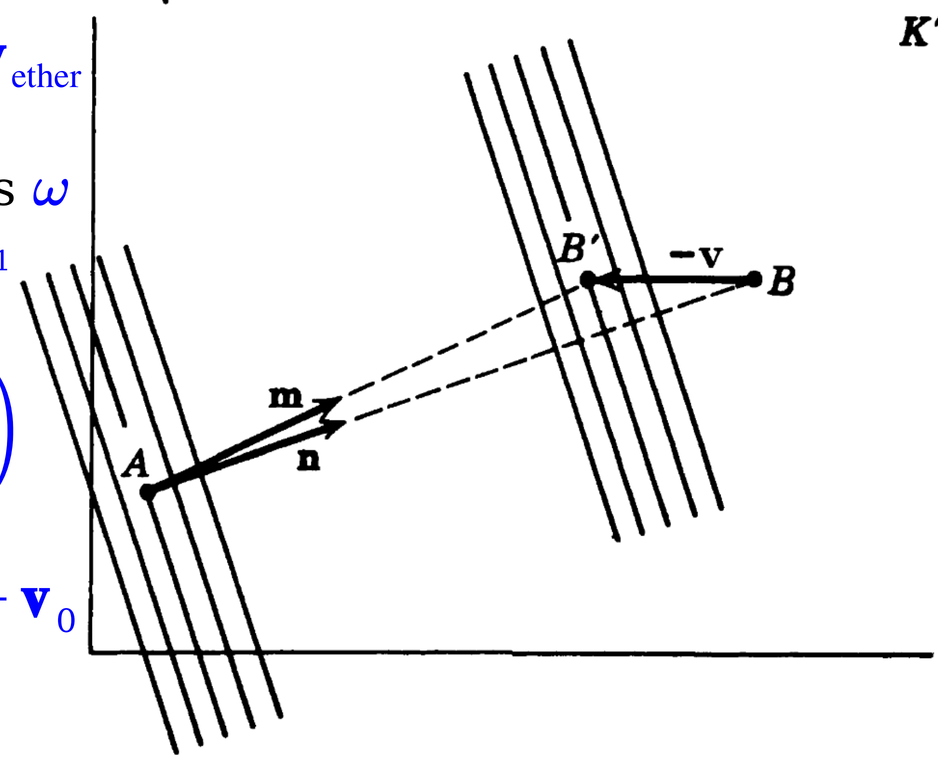


- The unit wave normal is an invariant in all inertial frames. But the direction of energy flow changes from frame to frame.

- The direction of motion of the wave packet, ie, the direction of energy flow, is not parallel to $\mathbf{n}$ in K'

$$\mathbf{m} = \frac{c\,\mathbf{n} - \mathbf{v}}{|c\,\mathbf{n} - \mathbf{v}|}$$

$$\Rightarrow \quad \mathbf{n} \simeq \left(1 - \frac{\mathbf{m}\cdot\mathbf{v}_0}{c}\right)\mathbf{m} + \frac{\mathbf{v}_0}{c} \quad \Leftarrow \quad \mathbf{v}_0 = \mathbf{v}_{\text{lab}} - \mathbf{v}_{\text{ether}}$$



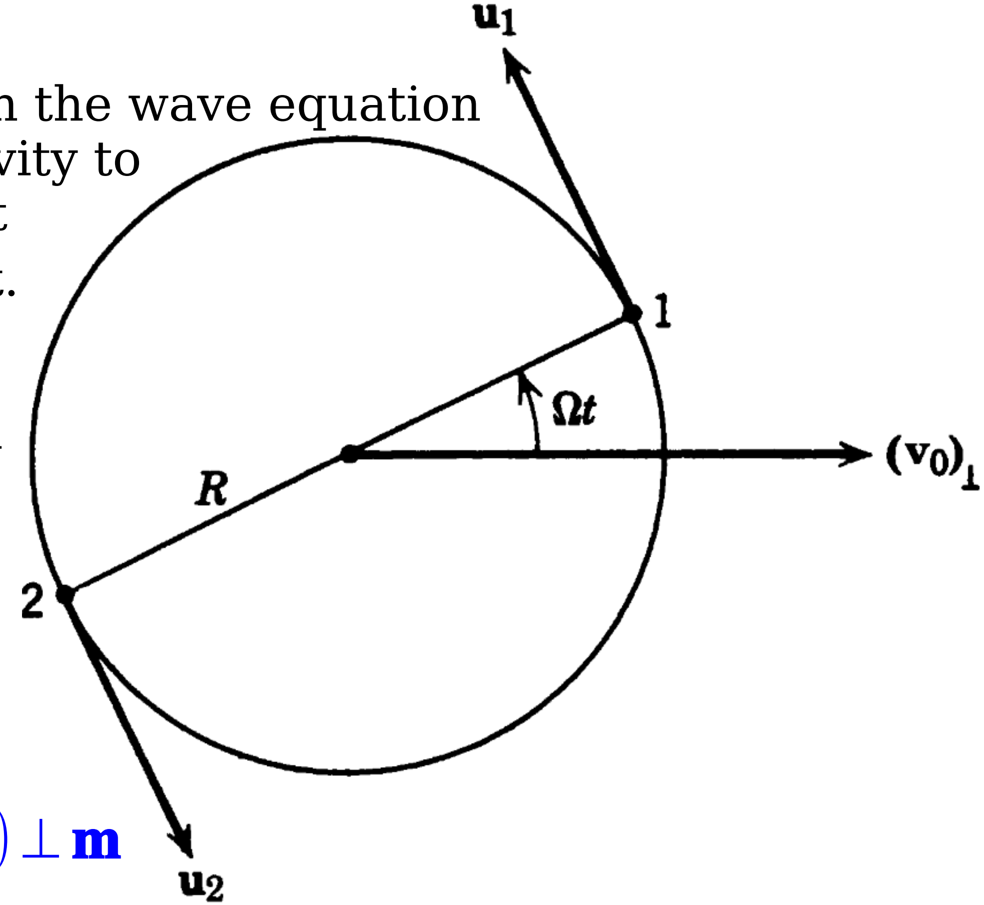
- Consider a plane wave whose frequency is ω in the ether rest frame, ω_0 in the lab, and ω_1 in an inertial frame K_1

$$\omega_1 = \omega\left(1 - \frac{\mathbf{n}\cdot\mathbf{v}_1}{c}\right), \quad \omega_0 = \omega\left(1 - \frac{\mathbf{n}\cdot\mathbf{v}_0}{c}\right)$$

$$\Rightarrow \quad \omega_1 \simeq \omega_0\left[1 - \frac{\mathbf{u}_1}{c}\cdot\left(\mathbf{m} + \frac{\mathbf{v}_0}{c}\right)\right] \quad \Leftarrow \quad \mathbf{u}_1 = \mathbf{v}_1 - \mathbf{v}_0$$

- It is a consequence of the validity of both the wave equation in the ether rest frame and Galilean relativity to transform to other inertial frames. Since it involves $\mathbf{v}_0$, it predicts an ether drift effect.

- Consider 2 Mössbauer systems, one an emitter and the other an absorber, moving with velocities $\mathbf{u}_1$ & $\mathbf{u}_2$ in the lab



$$\frac{\Delta\omega}{\omega_0} = \frac{\omega_1 - \omega_2}{\omega_0} = \frac{\mathbf{u}_2 - \mathbf{u}_1}{c} \cdot \left(\mathbf{m} + \frac{\mathbf{v}_0}{c} \right)$$

$$\Rightarrow \quad \frac{\Delta\omega}{\omega_0} = \frac{2\Omega R}{c^2} |\mathbf{v}_0^{\perp}| \sin\Omega t \quad \Leftarrow \quad (\mathbf{u}_2 - \mathbf{u}_1) \perp \mathbf{m}$$

- The experiment showed that the search for the ether velocity gives a null result.

- The Doppler shift experiments set observable ether drift speed limits 1000 times smaller than the speed of the earth and make the detection of any motion relative to some "absolute" reference frame quite implausible.

B. Speed of Light from a Moving Source

- The 2nd postulate of Einstein destroys the concept of time as a universal variable independent of the spatial coordinates.

- In the CERN experiment, the speed of 6 GeV photons produced in the decay of energetic neutral pions was measured by time of flight over paths up to 80m.

- The pions were produced by bombardment of a beryllium target by 19.2 GeV protons and had speeds of $0.99975c$.

- Within experimental error it was found that the speed of the photons emitted by the extremely rapidly moving source was equal to c.

C. Frequency Dependence of the Speed of Light in Vacuum

- One possible source of frequency dependence is photon mass $c(\omega) = c\sqrt{1 - \frac{\omega_0^2}{\omega^2}} \Leftarrow \hbar\omega_0$: photon rest energy

- The change in velocity of propagation from a photon mass is $\frac{\Delta c}{c} \sim 10^{-10}$

- Another source of frequency variation in the speed of light is dispersion of the vacuum, a concept occurring in models with a discrete space-time.

- The small time duration of the pulse from pulsars permits a simple estimate for the upper limit of variation on the speed of light for 2 frequencies

$$|c(\omega_1) - c(\omega_2)| = \frac{c^2 \Delta t}{D}$$

- Up to very high energies there is no evidence for dispersion of the vacuum. The speed of light is a universal constant, independent of frequency.

Lorentz Transformations and Basic Kinematic Results of Special Relativity

- The constancy of the velocity of light gives the relation between space and time coordinates in different inertial reference frames as Lorentz transformations.

A. Simple Lorentz Transformation of Coordinates

- Einstein's 2nd postulate implies that observers in *both* frame K and K′ will see a spherical shell of radiation expanding outward from the origins with speed c.

- $$\begin{array}{ll} -c^2t^2+x^2+y^2+z^2=0 & \text{in K} \\ -c^2t'^2+x'^2+y'^2+z'^2=0 & \text{in K}' \end{array} \Rightarrow \begin{array}{l} -c^2t'^2+x'^2+y'^2+z'^2 \\ =\lambda^2\left(-c^2t^2+x^2+y^2+z^2\right) \end{array} \Leftarrow \begin{array}{c}\text{where} \\ \lambda=\lambda\left(\mathbf{v}\right)\end{array}$$

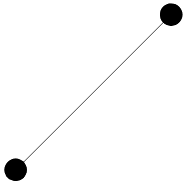
$$\Rightarrow \quad \lambda=1 \quad \Leftarrow \quad \text{inverse from K}' \text{ to K}$$

$$\Rightarrow \quad \textit{Lorentz transformation} \quad \begin{array}{l} x_0'=\gamma\left(x_0-\beta\, x_1\right) \\ x_1'=\gamma\left(x_1-\beta\, x_0\right) \\ x_2'=x_2 \\ x_3'=x_3 \end{array} \Leftarrow \begin{array}{l} x_0=c\,t \\ x_1=z \\ x_2=x \\ x_3=y \end{array}, \quad \boldsymbol{\beta}=\frac{\mathbf{v}}{c}, \quad \beta=|\boldsymbol{\beta}| \quad \gamma=\frac{1}{\sqrt{1-\beta^2}}$$

$$\Rightarrow \begin{array}{l} x_0=\gamma\left(x_0'+\beta\, x_1'\right) \\ x_1=\gamma\left(x_1'+\beta\, x_0'\right) \\ x_2=x_2' \\ x_3=x_3' \end{array} \Leftarrow \quad \text{inverse Lorentz transformation}$$

$$c = c' \quad \Rightarrow \quad 0 = c^2 (t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$

$$= c^2 (t'_2 - t'_1)^2 - (x'_2 - x'_1)^2 - (y'_2 - y'_1)^2 - (z'_2 - z'_1)^2 \quad \text{for light}$$



$\Rightarrow$ define an (infinitesimal) interval $d s^2 = c^2 d t^2 - d x^2 - d y^2 - d z^2$ for 2 events

$$\Rightarrow \quad d s^2 = d s'^2 \quad \Leftarrow \quad d s'^2 = a(v)\, d s^2, \quad d s^2 = a(v)\, d s'^2 \quad \Rightarrow \quad a = \pm 1 \quad \Rightarrow \quad a = 1$$

$$\Rightarrow \quad c^2 d t^2 - d x^2 = c^2 d t'^2 - d x'^2 \quad \Leftarrow \quad d y = d y', \quad d z = d z'$$

Let $\begin{cases} d x' = A(d x - v\, d t) \\ d t' = B\, d t + D\, d x \end{cases}$ $\Leftarrow$ generalized Galilean transformation

$$\Rightarrow \quad c^2 d t^2 - d x^2 = c^2 (B\, d t + D\, d x)^2 - A^2 (d x - v\, d t)^2$$

$$= \underline{(c^2 B^2 - A^2 v^2)}\, d t^2 + 2\underline{(c^2 B D + A^2 v)}\, d t\, d x - \underline{(A^2 - c^2 D^2)}\, d x^2$$

$$\Rightarrow \quad \begin{matrix} B^2 - \beta^2 A^2 = 1 \\ c B D + A^2 \beta = 0 \\ A^2 - c^2 D^2 = 1 \end{matrix} \quad \Rightarrow \quad A^2 = -\frac{c}{\beta} B D \quad \Rightarrow \quad \begin{matrix} B(B + c\beta D) = 1 \\ -\dfrac{c}{\beta} D(B + c\beta D) = 1 \end{matrix}$$

$$\Rightarrow \quad D = -\frac{\beta}{c} B \quad \Rightarrow \quad B^2 (1 - \beta^2) = 1 \quad \Rightarrow \quad B = \pm \gamma \quad \Rightarrow \quad A^2 = B^2 \quad \Rightarrow \quad A = \pm \gamma$$

Choose $+$ for v approaching 0 continuously.

$$\Rightarrow \begin{cases} c\,\mathrm{d}t' = \gamma(c\,\mathrm{d}t - \beta\,\mathrm{d}x) \\ \mathrm{d}x' = \gamma(\mathrm{d}x - \beta c\,\mathrm{d}t) \\ \mathrm{d}y' = \mathrm{d}y \\ \mathrm{d}z' = \mathrm{d}z \end{cases} \Rightarrow \begin{cases} c\,\mathrm{d}t = \gamma(c\,\mathrm{d}t' + \beta\,\mathrm{d}x') \\ \mathrm{d}x = \gamma(\mathrm{d}x' + \beta c\,\mathrm{d}t') \\ \mathrm{d}y = \mathrm{d}y' \\ \mathrm{d}z = \mathrm{d}z' \end{cases} \Leftarrow$$

inverse Lorentz transformation

- The coordinates $\perp$ to the direction of relative motion are unchanged while the $\parallel$ coordinate & the time are transformed, contrasted with Galilean transformation.

- If the axes in K and K$'$ remain $\parallel$, but the velocity $\mathbf{v}$ of frame K$'$ in frame K is in an arbitrary direction $x_0' = \gamma(x_0 - \boldsymbol{\beta}\cdot\mathbf{r}), \quad \mathbf{r}' = \mathbf{r} + (\gamma - 1)(\hat{\boldsymbol{\beta}}\cdot\mathbf{r})\hat{\boldsymbol{\beta}} - \gamma\boldsymbol{\beta} x_0 \qquad (0)$

- $\begin{matrix} 0 \le \beta < 1 \\ 1 \le \gamma \le \infty \end{matrix} \quad \Rightarrow \quad \tanh\xi \equiv \beta \quad \Leftarrow \quad \xi$: *boost parameter* or *rapidity* $\Rightarrow \quad \begin{matrix} \sinh\xi = \gamma\beta \\ \cosh\xi = \gamma \end{matrix}$

$$\Rightarrow \quad \begin{aligned} x_0' &= +x_0\cosh\xi - x_1\sinh\xi \\ x_1' &= -x_0\sinh\xi + x_1\cosh\xi \end{aligned} \quad (1) \quad \Rightarrow \quad \begin{bmatrix} x_0' \\ x_1' \end{bmatrix} = \begin{bmatrix} \cosh\xi & -\sinh\xi \\ -\sinh\xi & \cosh\xi \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

- The structure of the equations is a rotation of coordinates, but hyperbolically instead of circularly, because of the relative negative sign between the space and time.

B. 4-Vectors

- The Lorentz transformation describes the transformation of the coordinates of a point from one inertial frame to another.

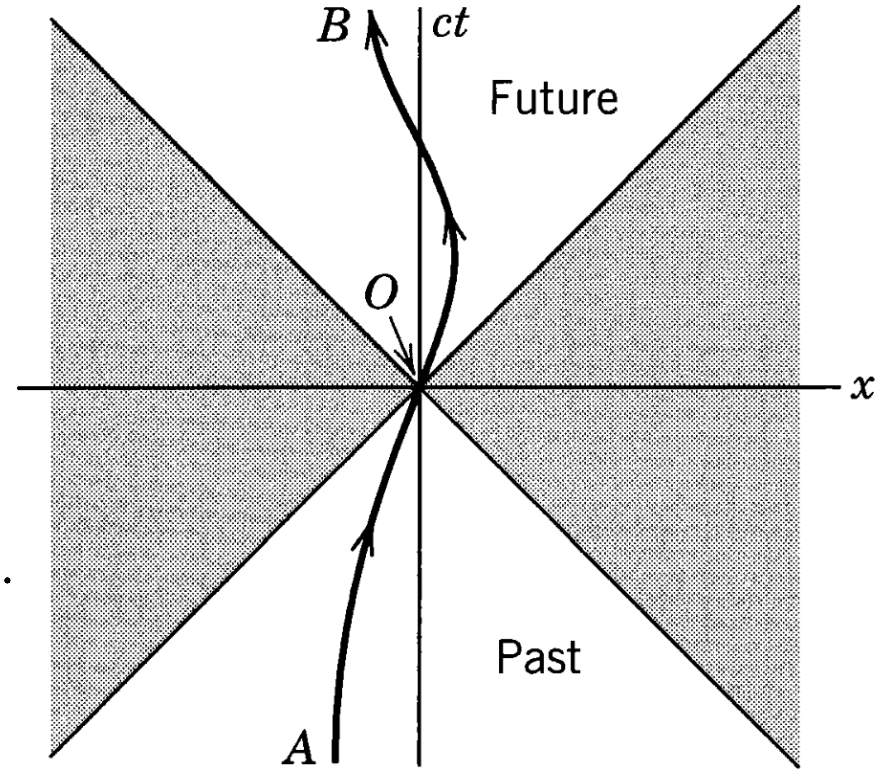
- Anticipate that there are numerous physical quantities that transform under Lorentz transformations in the same manner as a point — 4-vectors.

- $$\vec{A} = (A_0, \mathbf{A}) = (A_0, A_1, A_2, A_3) \quad \Rightarrow \quad \begin{aligned} A_0' &= \gamma\,(A_0 - \boldsymbol{\beta}\cdot\mathbf{A}) \\ A_\parallel' &= \gamma\,(A_\parallel - \beta\, A_0) \\ \mathbf{A}_\perp' &= \mathbf{A}_\perp \end{aligned} \quad \Rightarrow \quad {A_0'}^2 - |\mathbf{A}'|^2 = A_0^2 - |\mathbf{A}|^2$$

- $$\vec{B} = (B_0, \mathbf{B}) = (B_0, B_1, B_2, B_3)$$

$$\Rightarrow \quad \vec{A}\cdot\vec{B} = A_0' B_0' - \mathbf{A}'\cdot\mathbf{B}' = A_0 B_0 - \mathbf{A}\cdot\mathbf{B} \quad \Leftarrow \quad \text{the scalar product is an invariant}$$

C. Light Cone, Proper Time, and Time Dilatation



- The velocity of light is an upper bound on all velocities, thus the space-time domain can be divided into 3 regions by a light cone, ie, $x^2 + y^2 + z^2 = c^2 t^2$.

- As time goes on a particle traces out a path, called its *world line*, inside the upper half-cone.

- The upper half-cone ($t > 0$) is the *future*. The lower half-cone ($t < 0$) is the *past*.

- A system at O can never reach or come from a point outside the light cone.

- The square of the invariant interval $s_{12}^2 = c^2 (t_1 - t_2)^2 - |\mathbf{r}_1 - \mathbf{r}_2|^2 \Rightarrow$

$$\begin{aligned} s_{12}^2 &> 0, \quad \text{timelike} \\ s_{12}^2 &< 0, \quad \text{spacelike} \\ s_{12}^2 &= 0, \quad \text{lightlike} \end{aligned}$$

- For a timelike path, it is always possible to find a Lorentz transformation to a new coordinate frame such that $\mathbf{r}_1' = \mathbf{r}_2' \quad \Rightarrow \quad s_{12}^2 = c^2 (t_1' - t_2')^2 > 0$

ie, in the new frame the 2 events occur at the same space point, but are separated in time.

- For a spacelike path, it is possible to find a new inertial frame such that

$$t_1'' = t_2'' \quad \Rightarrow \quad s_{12}^2 = -\left|\mathbf{r}_1'' - \mathbf{r}_2''\right|^2 < 0$$

ie, in the new frame the 2 events occur at different space points at the same instant of time.

- $s_{12}^2 = 0$ implies *lightlike separation*. The events lie on the light cone with respect to each other and can be connected only by light signals.

- The division of the separation of 2 events in space-time is a Lorentz invariant one. 2 events within one separation in one coordinate system stay in the same separation in all coordinates.

- Events in timelike domains can be causally related, but not in spacelike domains.

- $d s^2 = c^2 d t^2 - |d\mathbf{r}|^2 = c^2 d t^2 \left(1 - \beta^2\right) \quad \Rightarrow$ find the coordinate where the system is instantaneously at rest $\Rightarrow \quad d t' \equiv d\tau, \quad d\mathbf{r}' = 0$

$$\Rightarrow \quad d s = c\, d\tau \quad \Rightarrow \quad d\tau = d t \sqrt{1 - \beta^2} = \frac{d t}{\gamma(t)} \quad \textit{Lorentz invariant quantity}$$

- The time τ is called the *proper time*, the time seen in the rest frame of a system.

- A certain proper time interval will be seen in the frame as a time interval

$$t_2 - t_1 = \int_{\tau_1}^{\tau_2} \frac{d\tau}{\sqrt{1-\beta^2(\tau)}} = \int_{\tau_1}^{\tau_2} \gamma(\tau)\, d\tau \quad \Leftarrow \quad \textit{time dilatation}$$

- A moving clock runs more slowly than a stationary clock.

- For equal time intervals in the clock's rest frame, the time intervals observed in the other frame are greater by a factor of $\gamma > 1$.

- This result is verified daily in high-energy physics labs where beams of particles of lifetimes $c\tau_0$ are transported before decay over distances much longer than $c\tau_0$.

- Time dilatation has also been verified with the comparison of the clock in an airplane with the one on the ground.

D. Relativistic Doppler Shift

- Consider a plane wave

$$\phi = \omega t - \boldsymbol{k}\cdot\mathbf{r} = \omega' t' - \boldsymbol{k}'\cdot\mathbf{r}' \quad \text{invariant} \quad \Rightarrow \quad \begin{aligned} k_0' &= \gamma\,(k_0 - \boldsymbol{\beta}\cdot\boldsymbol{k}) \\ k_\parallel' &= \gamma\,(k_\parallel - \beta\, k_0) \\ \boldsymbol{k}_\perp' &= \boldsymbol{k}_\perp \end{aligned} \quad \Leftarrow \quad \begin{aligned} \omega' &= c\, k_0' \\ \omega &= c\, k_0 \end{aligned}$$

- *The frequency and wave number of any plane wave form a 4-vector.*

- The invariance of phase is the invariance of the scalar product of 2 4-vectors.

- For light wave

$$|\boldsymbol{k}| = k_0\,, \quad |\boldsymbol{k}'| = k_0' \quad \Rightarrow \quad \begin{aligned} \omega' &= \gamma\,\omega\,(1 - \beta\cos\theta) \\ \tan\theta' &= \frac{\sin\theta}{\gamma\,(\cos\theta - \beta)} \end{aligned} \quad \Leftarrow \quad \textit{relativistic Doppler shift}$$

- There exists a relativistic *transverse* Doppler shift, observed with atoms in motion and in a precise resonance-absorption Mössbauer experiment.

Addition of Velocities, 4-Velocity

$$
\begin{aligned}
\mathrm{d}\,x_0 &= \gamma_v\left(\mathrm{d}\,x_0' + \beta\,\mathrm{d}\,x_1'\right)\\
\mathrm{d}\,x_1 &= \gamma_v\left(\mathrm{d}\,x_1' + \beta\,\mathrm{d}\,x_0'\right)\\
\mathrm{d}\,x_2 &= \mathrm{d}\,x_2'\\
\mathrm{d}\,x_3 &= \mathrm{d}\,x_3'
\end{aligned}
\quad\Rightarrow\quad
\begin{aligned}
\mathbf{u}' &= c\,\frac{\mathrm{d}\,\mathbf{x}'}{\mathrm{d}\,x_0'}, \quad \boldsymbol{\beta}_{u'} \equiv \frac{\mathbf{u}'}{c}\\
\mathbf{u} &= c\,\frac{\mathrm{d}\,\mathbf{x}}{\mathrm{d}\,x_0}, \quad \boldsymbol{\beta}_{u} \equiv \frac{\mathbf{u}}{c}
\end{aligned}
$$

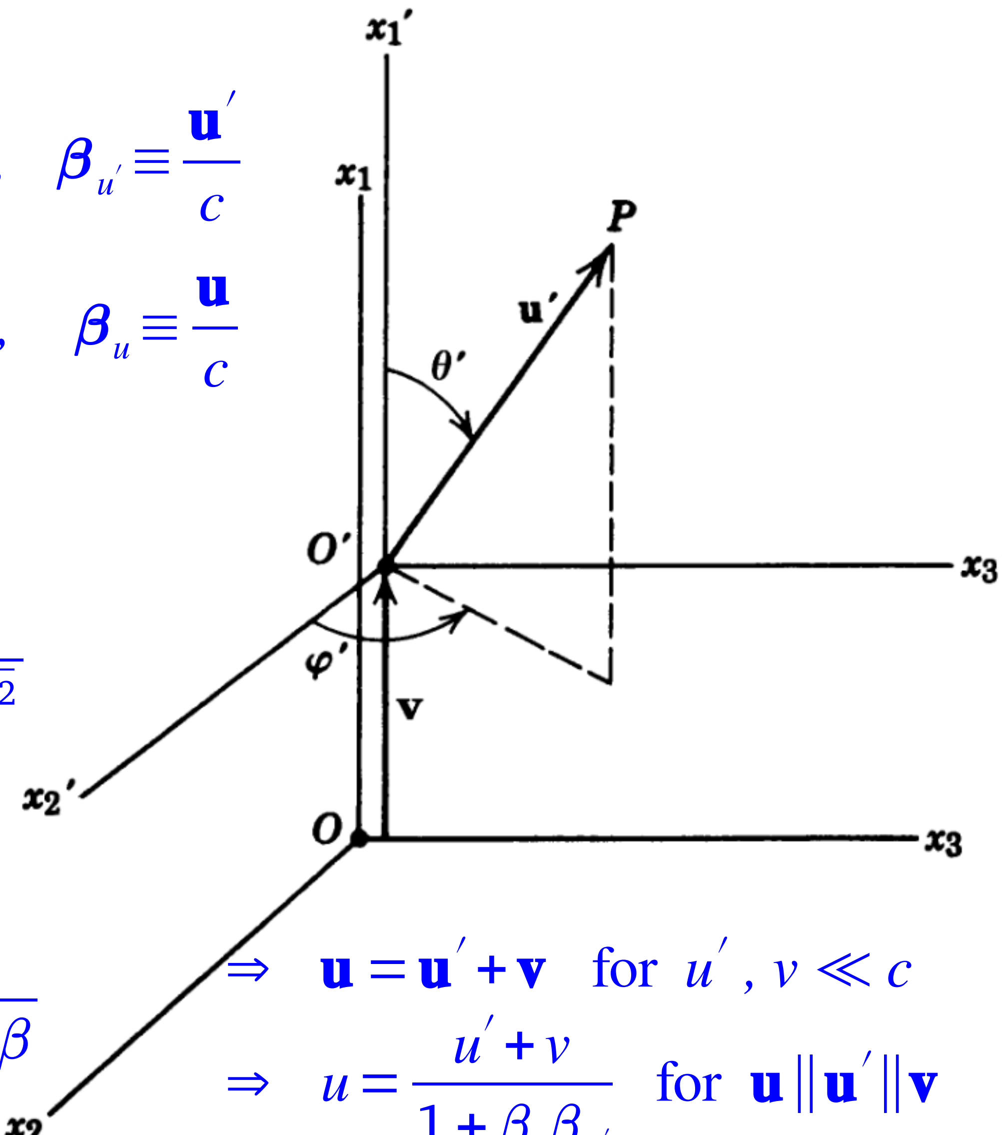
$$\Rightarrow \quad u_{\parallel} = \frac{u_{\parallel}' + v}{1 + \boldsymbol{\beta}\cdot\boldsymbol{\beta}_{u'}}, \quad \mathbf{u}_{\perp} = \frac{1}{\gamma}\,\frac{\mathbf{u}_{\perp}'}{1 + \boldsymbol{\beta}\cdot\boldsymbol{\beta}_{u'}}$$

where $\boldsymbol{\beta} = \boldsymbol{\beta}_v \equiv \dfrac{\mathbf{v}}{c}, \ \beta = |\boldsymbol{\beta}|, \ \gamma = \dfrac{1}{\sqrt{1-\beta^2}}$

$\dfrac{u_2}{u_2'} = \dfrac{u_3}{u_3'}$ the azimuthal angles in the 2 frames are equal

$$\Rightarrow \quad \tan\theta = \frac{1}{\gamma}\,\frac{u'\sin\theta'}{u'\cos\theta' + v} = \frac{1}{\gamma}\,\frac{\beta_{u'}\sin\theta'}{\beta_{u'}\cos\theta' + \beta}$$

$$u = \frac{\sqrt{(\mathbf{u}' + \mathbf{v})^2 - (\boldsymbol{\beta}\times\mathbf{u}')^2}}{1 + \boldsymbol{\beta}\cdot\boldsymbol{\beta}_{u'}} \qquad (\star')$$



$\Rightarrow \quad \mathbf{u} = \mathbf{u}' + \mathbf{v}$ for $u', v \ll c$

$\Rightarrow \quad u = \dfrac{u' + v}{1 + \beta\,\beta_{u'}}$ for $\mathbf{u} \parallel \mathbf{u}' \parallel \mathbf{v}$

$\Rightarrow$ if $u' = c \ \Rightarrow \ u = c$

- The formula for the addition of velocities is verified with the Fizeau experiments on the speed of light in moving liquids and the aberration of star positions from the motion of the earth in orbit.

- $(\ast') \Rightarrow 1-\beta_u^2 = \dfrac{1+2\,\boldsymbol{\beta}\cdot\boldsymbol{\beta}_{u'} + \beta^2\,\beta_{u'}^2\cos^2\theta' - \beta_{u'}^2 - 2\,\boldsymbol{\beta}\cdot\boldsymbol{\beta}_{u'} - \beta^2 + \beta^2\,\beta_{u'}^2\sin^2\theta'}{\left(1+\boldsymbol{\beta}\cdot\boldsymbol{\beta}_{u'}\right)^2}$

$$= \frac{1-\beta^2-\beta_{u'}^2+\beta^2\,\beta_{u'}^2}{\left(1+\boldsymbol{\beta}\cdot\boldsymbol{\beta}_{u'}\right)^2} = \frac{\left(1-\beta^2\right)\left(1-\beta_{u'}^2\right)}{\left(1+\boldsymbol{\beta}\cdot\boldsymbol{\beta}_{u'}\right)^2}$$

$\Rightarrow$ $\gamma_u = \gamma\,\gamma_{u'}\left(1+\boldsymbol{\beta}\cdot\boldsymbol{\beta}_{u'}\right)$ $\Leftarrow$ $\gamma_u = \dfrac{1}{\sqrt{1-\beta_u^2}}$, $\gamma_{u'} = \dfrac{1}{\sqrt{1-\beta_{u'}^2}}$ $\Leftarrow$ $\beta_u \equiv |\boldsymbol{\beta}_u|$, $\beta_{u'} \equiv |\boldsymbol{\beta}_{u'}|$

$\Rightarrow$ $\begin{aligned}\gamma_u\,u_\parallel &= \gamma_v\left(\gamma_{u'}\,u'_\parallel + \beta\,\gamma_{u'}\,c\right)\\ \gamma_u\,\mathbf{u}_\perp &= \gamma_{u'}\,\mathbf{u}'_\perp\end{aligned}$ $\Leftarrow$ Lorentz transformation of $\left(\gamma_u\,c\,,\,\gamma_u\,\mathbf{u}\right)$

$\Rightarrow$ time and space components of the 4-velocity $\vec{U} = \left(U_0\,,\,\mathbf{U}\right)$

- The addition law for velocities is not a 4-vector transformation law because time is not invariant under Lorentz transformations. The proper time τ is a Lorentz invariant. Thus define a 4-velocity by differentiation of the 4-position with respect to τ:

$$U_0 \equiv \frac{\mathrm{d}\,x_0}{\mathrm{d}\,\tau} = \frac{\mathrm{d}\,x_0}{\mathrm{d}\,t}\,\frac{\mathrm{d}\,t}{\mathrm{d}\,\tau} = \gamma_u\,c$$

$$\mathbf{U} \equiv \frac{\mathrm{d}\,\mathbf{x}}{\mathrm{d}\,\tau} = \frac{\mathrm{d}\,\mathbf{x}}{\mathrm{d}\,t}\,\frac{\mathrm{d}\,t}{\mathrm{d}\,\tau} = \gamma_u\,\mathbf{u}$$

$\Rightarrow$ $\vec{U} = \left(\gamma_u\,c\,,\,\gamma_u\,\mathbf{u}\right)$

Relativistic Momentum and Energy of a Particle

- For a particle with speed small compared to the speed of light

$$\mathbf{p} = m\,\mathbf{u}\,, \quad E = E(0) + \frac{1}{2} m\,u^2 \qquad (@)$$

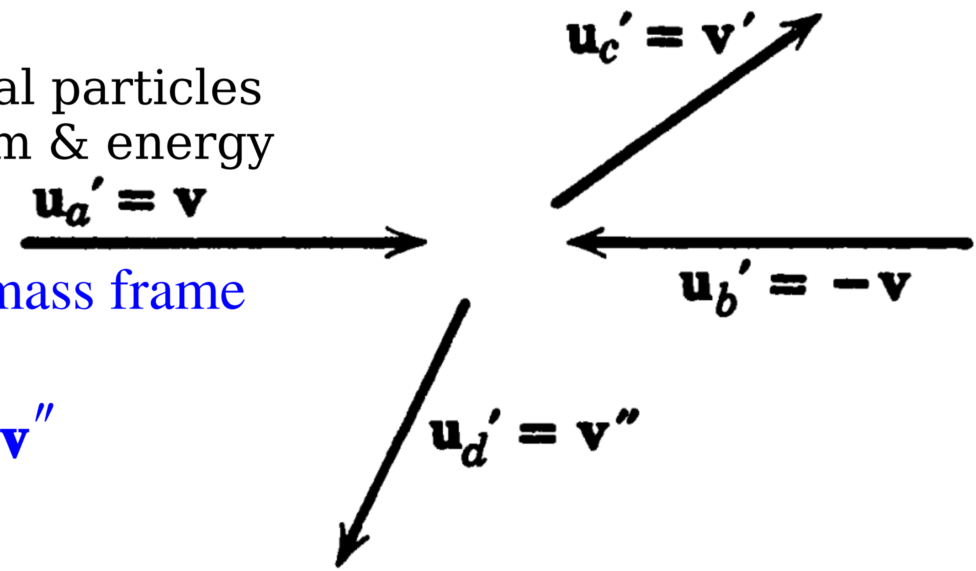
- Nonrelativistically, the rest energies can be ignored; it is only a constant.

- In SR, the rest energy cannot be ignored; it is the total energy that matters.

- Wish to find expressions for the momentum & energy consistent with the Lorentz transformation law of velocities and reducing to (@) nonrelativistically

$$\begin{aligned} \mathbf{p} &= \mathcal{M}(u)\,\mathbf{u} \\ E &= \mathcal{E}(u) \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\mathcal{M}(0) = m \\ &\frac{\partial \mathcal{E}}{\partial u^2}(0) = \frac{m}{2} \end{aligned} \quad (@') \quad \Rightarrow \quad \mathcal{M}(u)\,,\ \mathcal{E}(u) :$$ well-behaved monotonic functions of the argument

- Consider the elastic collision of 2 identical particles and require that conservation of momentum & energy hold in all equivalent inertial frames



$$\begin{aligned} \mathbf{p}_a' + \mathbf{p}_b' &= \mathbf{p}_c' + \mathbf{p}_d' \\ E_a' + E_b' &= E_c' + E_d' \end{aligned}$$ in K$'$ $\Leftarrow$ the center of mass frame

$$\Rightarrow \quad \begin{aligned} \mathcal{M}(v)\,\mathbf{v} - \mathcal{M}(v)\,\mathbf{v} &= \mathcal{M}(v')\,\mathbf{v}' + \mathcal{M}(v'')\,\mathbf{v}'' \\ \mathcal{E}(v) + \mathcal{E}(v) &= \mathcal{E}(v') + \mathcal{E}(v'') \end{aligned}$$

$\Rightarrow \quad \mathcal{E}(v') = \mathcal{E}(v'')$ for identical particles $\Rightarrow \quad v' = v'' = v \quad \Rightarrow \quad \mathbf{v}'' = -\mathbf{v}'$

In the lab frame K

$$\mathbf{u}_a = \frac{2\,\mathbf{v}}{1+v^2/c^2} = \frac{2\,c\,\boldsymbol{\beta}}{1+\beta^2} \quad \Rightarrow \quad \left(\mathbf{u}_c\right)_x = \frac{1}{\gamma}\,\frac{c\,\beta\,\sin\theta'}{1+\beta^2\cos\theta'}\,, \quad \left(\mathbf{u}_c\right)_z = \frac{c\,\beta\left(1+\cos\theta'\right)}{1+\beta^2\cos\theta'}$$

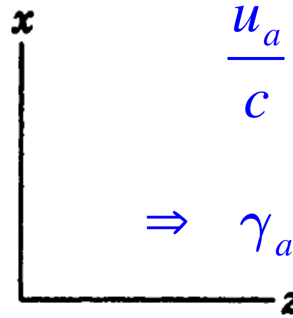
$$\left(\mathbf{u}_d\right)_x = -\frac{1}{\gamma}\,\frac{c\,\beta\,\sin\theta'}{1-\beta^2\cos\theta'}\,, \quad \left(\mathbf{u}_d\right)_z = \frac{c\,\beta\left(1-\cos\theta'\right)}{1-\beta^2\cos\theta'}$$

$$\mathcal{M}\left(u_a\right)\mathbf{u}_a + \mathcal{M}\left(u_b\right)\mathbf{u}_b = \mathcal{M}\left(u_c\right)\mathbf{u}_c + \mathcal{M}\left(u_d\right)\mathbf{u}_d \quad \Rightarrow \quad 0 = \frac{\mathcal{M}\left(u_c\right)\beta\sin\theta'}{1+\beta^2\cos\theta'} - \frac{\mathcal{M}\left(u_d\right)\beta\sin\theta'}{1-\beta^2\cos\theta'} \quad \text{the } \perp \text{ part}$$

$$\mathcal{E}\left(u_a\right) + \mathcal{E}\left(u_b\right) = \mathcal{E}\left(u_c\right) + \mathcal{E}\left(u_d\right)$$

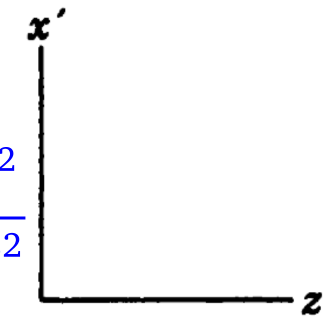
$$\Rightarrow \quad \mathcal{M}\left(u_c\right) = \frac{1+\beta^2\cos\theta'}{1-\beta^2\cos\theta'}\,\mathcal{M}\left(u_d\right)$$

$$\Rightarrow \quad \mathcal{M}\left(u_a\right) = \frac{1+\beta^2}{1-\beta^2}\,\mathcal{M}\left(0\right) \quad \Leftarrow \theta' = 0$$



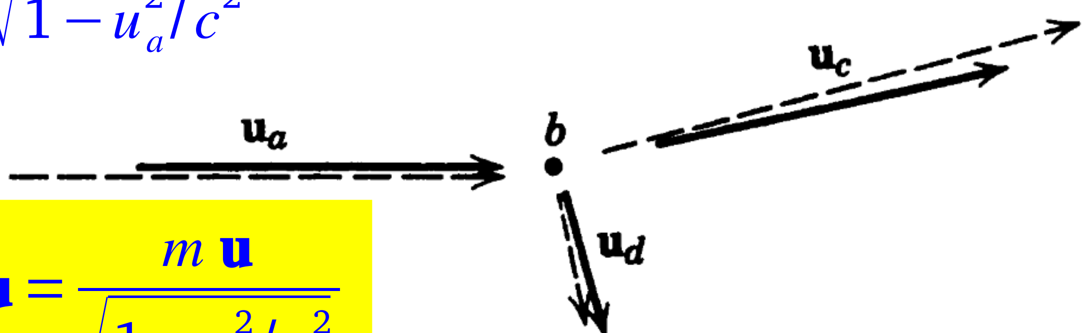
$$\frac{u_a}{c} = \frac{2\,\beta}{1+\beta^2}$$

$$\Rightarrow \quad \gamma_a \equiv \frac{1}{\sqrt{1-u_a^2/c^2}} = \frac{1+\beta^2}{1-\beta^2}$$

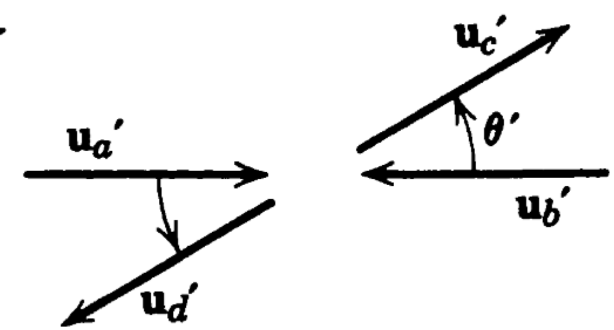


$$\Rightarrow \quad \mathcal{M}\left(u_a\right) = \frac{1}{\sqrt{1-u_a^2/c^2}}\,\mathcal{M}\left(0\right) = \gamma_a\,\mathcal{M}\left(0\right) = \gamma_a\,m$$

$$\Rightarrow \quad \mathbf{p} = \gamma_u\,m\,\mathbf{u} = \frac{m\,\mathbf{u}}{\sqrt{1-u^2/c^2}}$$



K



K'

- $u_c^2 = u_a^2 - \dfrac{\eta}{\gamma_a^3} + O(\eta^2)\,,\quad u_d^2 = \eta + O(\eta^2) \quad \Leftarrow \quad \eta = \dfrac{c^2 \beta^2 \theta'^2}{1-\beta^2}$

$$\Rightarrow \quad \mathcal{E}(u_a) + \mathcal{E}(0) = \mathcal{E}(u_c) + \mathcal{E}(u_d)$$

$$= \mathcal{E}(u_a) + \left(\frac{\mathrm{d}\,\mathcal{E}(u_c)}{\mathrm{d}\,u_c^2} \frac{\partial\, u_c^2}{\partial\, \eta} \right)_{\eta=0} \eta + \mathcal{E}(0) + \left(\frac{\mathrm{d}\,\mathcal{E}(u_d)}{\mathrm{d}\,u_d^2} \frac{\partial\, u_d^2}{\partial\, \eta} \right)_{\eta=0} \eta + \cdots$$

$$\Rightarrow \quad 0 = -\frac{1}{\gamma_a^3} \frac{\mathrm{d}\,\mathcal{E}(u_a)}{\mathrm{d}\,u_a^2} + \frac{\mathrm{d}\,\mathcal{E}(u_d)}{\mathrm{d}\,u_d^2}\bigg|_{u_d=0} \quad \Leftarrow \quad \text{the 1st-order terms}$$

$$\Rightarrow \quad \frac{\mathrm{d}\,\mathcal{E}(u_a)}{\mathrm{d}\,u_a^2} = \frac{m}{2}\gamma_a^3 = \frac{m}{2\sqrt{(1-u_a^2/c^2)^3}} \quad \Leftarrow (@') \quad \Rightarrow \quad \mathcal{E}(u) - \mathcal{E}(0) = \gamma_u\, m\, c^2 - m\, c^2$$

$$\Rightarrow \quad \text{the } \textit{kinetic} \text{ energy } T(u) = \mathcal{E}(u) - \mathcal{E}(0) = m\,c^2(\gamma_u - 1)$$

- $\mathcal{E}(0)$ cannot be determined from elastic scattering, but can be found from inelastic processes in which one particle is transformed into another or others of different masses. $\mathrm{K}^0 \to \pi^0\,\pi^0$ in K-meson's rest frame + energy conservation

$$\Rightarrow \quad T_\pi = \frac{1}{2}\mathcal{E}_\mathrm{K}(0) - \mathcal{E}_\pi(0) \quad \Rightarrow \quad \boxed{\mathcal{E}(0) = m\,c^2} \quad \Rightarrow \quad \boxed{E = \gamma\, m\, c^2 = \frac{m\,c^2}{\sqrt{1-u^2/c^2}}}$$

Einstein mass-energy relation

$$u_a = \frac{2\,c\,\beta}{1+\beta^2} \quad \Rightarrow \quad u_a^2 = \frac{4\,c^2\,\beta^2}{(1+\beta^2)^2}\,, \quad \gamma_a = \frac{1+\beta^2}{1-\beta^2}\,, \quad \cos\theta' \sim 1 - \frac{\theta'^2}{2}\,, \quad \sin\theta' \sim \theta'$$

$$u_{c\,x}^2 = \frac{c^2\,\beta^2 \sin^2\theta'}{\gamma^2 (1+\beta^2\cos\theta')^2} \approx \frac{u_a^2}{4}(1-\beta^2)\,\theta'^2 \quad \Leftarrow \quad \text{up to the order of } \theta'^2$$

$$u_{c\,z}^2 = \frac{c^2\,\beta^2 (1+\cos\theta')^2}{(1+\beta^2\cos\theta')^2} \approx \frac{u_a^2}{4}\left(\frac{1+\beta^2}{1+\beta^2\cos\theta'}\right)^2 \left(2-\frac{\theta'^2}{2}\right)^2$$

$$\approx u_a^2 \left(1+\frac{\beta^2\,\theta'^2}{2(1+\beta^2)}\right)^2 \left(1-\frac{\theta'^2}{2}\right) \approx u_a^2 \left(1+\frac{\beta^2\,\theta'^2}{1+\beta^2}\right)\left(1-\frac{\theta'^2}{2}\right)$$

$$\approx u_a^2 \left(1-\frac{1-\beta^2}{2(1+\beta^2)}\,\theta'^2\right) = u_a^2 - \frac{u_a^2}{2\,\gamma_a}\,\theta'^2$$

$$\Rightarrow \quad u_c^2 = u_{c\,x}^2 + u_{c\,z}^2 \approx u_a^2 + \frac{u_a^2}{4\,\gamma_a}\left[\gamma_a(1-\beta^2)-2\right]\theta'^2 = u_a^2 - \frac{1}{\gamma_a^3}\,\frac{c^2\,\beta^2\,\theta'^2}{1-\beta^2} = u_a^2 - \frac{\eta}{\gamma^3}$$

$$u_{d\,z}^2 = \frac{c^2\,\beta^2(1-\cos\theta')^2}{(1-\beta^2\cos\theta')^2} \approx \frac{c^2\,\beta^2}{(1-\beta^2\cos\theta')^2}\,\frac{\theta'^4}{4} \sim 0 \qquad \Uparrow \quad \eta \equiv \frac{c^2\,\beta^2\,\theta'^2}{1-\beta^2}$$

$$\Rightarrow \quad u_d^2 \approx u_{d\,x}^2 = \frac{c^2\,\beta^2\sin^2\theta'}{\gamma^2(1-\beta^2\cos\theta')^2} \approx \frac{c^2\,\beta^2\,\theta'^2}{1-\beta^2} = \eta$$

- Another way to find $\mathcal{E}(0)$: the conservation equations are 4 equations assumed to be valid in all equivalent inertial frames, thus are identified as relations among 4-vectors

$$\vec{p} = (p_0, \mathbf{p}) \Rightarrow \mathbf{p} = m\,\mathbf{U} \Rightarrow p_0 = m\,U_0 = m\,\gamma\,c \Leftarrow \vec{p} = m\,\vec{U}$$

$$\Rightarrow E = c\,p_0 + \mathcal{E}(0) - m\,c^2 \Rightarrow \sum_{\text{initial}} (p_0)_a - \sum_{\text{final}} (p_0)_b = \Delta_0 \qquad (\#)$$

$$0 = \sum_{\text{initial}} \mathbf{p}_a - \sum_{\text{final}} \mathbf{p}_b = \mathbf{\Delta}$$

$$\Rightarrow \begin{matrix} \vec{\Delta} = (\Delta_0, \mathbf{\Delta} = 0) \\ E_a = E_b \end{matrix} \Rightarrow c\,\Delta_0 = \sum_{\text{final}} [\mathcal{E}_b(0) - m_b\,c^2] - \sum_{\text{initial}} [\mathcal{E}_a(0) - m_a\,c^2]$$

From the 1st postulate, (#) must be valid in all inertial frames. If $\mathbf{\Delta} = 0$ in all inertial frames, $\Delta_0 = 0$ from the Lorentz transformation $\Rightarrow \mathcal{E}(0) = m\,c^2$

- The energy-momentum 4-vector $\vec{p} = \left(p_0 = \frac{E}{c}, \mathbf{p} \right) \Rightarrow \boxed{\mathbf{u} = \frac{c^2}{E}\mathbf{p}}$

$$\Rightarrow \boxed{\vec{p}^2 = p_0^2 - \mathbf{p}^2 = m^2 c^2} \Leftarrow \text{invariant} \Rightarrow E = c\sqrt{p^2 + m^2 c^2} \Leftarrow m:\ \text{rest mass}$$

The equation and the conservation equations form a powerful & elegant means of treating relativistic kinematics in collision and decay processes.

- $\vec{p} = m\,\vec{U} \Rightarrow \vec{p}^2 = m^2 c^2 = m^2 \vec{U}^2 \Rightarrow \boxed{\vec{U}^2 = c^2} \Leftarrow$ constraint

- It is sometimes convenient to use the 2 components of $\mathbf{p} \perp$ the z axis and a rapidity as kinematic variables.

$$\mathbf{p} = \mathbf{p}_\perp + p_\parallel \hat{\mathbf{z}} \quad \text{in K} \quad \Rightarrow \quad p'_\parallel = 0 \quad \text{in K}' \quad \Rightarrow \quad \mathbf{p}' = \mathbf{p}_\perp , \quad \frac{E'}{c} = \Omega = \sqrt{p_\perp^2 + m^2 c^2}$$

$$\Rightarrow \quad \mathbf{p}_\perp , \quad p_\parallel = \Omega \sinh \xi , \quad \frac{E}{c} = \Omega \cosh \xi \quad \text{in K} \quad \Leftarrow (1) \Rightarrow \frac{\Omega}{c} : \text{the} \begin{array}{c} \text{transverse} \\ \text{longitudinal} \end{array} \text{mass}$$

- At rest in K' $\quad \Rightarrow \quad \mathbf{p}_\perp = 0 \quad \Rightarrow \quad p = m\, c \sinh \xi , \quad E = m\, c^2 \cosh \xi \quad \Leftarrow \quad \begin{array}{l} \cosh \xi = \gamma \\ \sinh \xi = \gamma\, \beta \end{array}$

- One convenience of $\mathbf{p}_\perp$ & ξ as kinematic variables is a Lorentz transformation in the z direction shifts all rapidities by a constant amount, $\xi \to \xi - Z$.

- With these variables, the configuration of particles in a collision process viewed in the laboratory frame differs only by a trivial shift of the origin of rapidity from the same process viewed in the center of mass frame.

Mathematical Properties of the Space-Time of SR

- 3d rotations in classical & quantum mechanics can be discussed with the group of transformations of the coordinates that leave the norm of a 3-vector invariant.

- In SR, Lorentz transformations of the 4d coordinates follow from the invariance

$$s^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 \quad \text{interval}$$

We can rephrase the kinematics of special relativity as the consideration of the group of all transformations that leave s^2 invariant—*homogeneous Lorentz group*.

- *Inhomogeneous Lorentz group* or *Poincaré group*: the group of transformations that leave invariant $s^2(x, y) = (x_0 - y_0)^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2$

contains translations & reflections in space-time + transformations of the homogeneous Lorentz group.

- The mathematical equations of the laws of nature must be *covariant*, ie, invariant in form, under the transformations of the Lorentz group ⇒ need to study spacetime.

- The space-time continuum is defined in terms of a 4d space with coordinates. Suppose that there is a well-defined transformation that yields new coordinates

$$x'^{\mu} = x'^{\mu}(x^0, x^1, x^2, x^3) \qquad \mu = 0, 1, 2, 3$$

- A *scalar* (tensor of rank 0) is a single quantity whose value is not changed by the transformation. The interval is a Lorentz scalar.

- *Vectors* (tensors of rank 1) have 2 kinds

contravariant vector $A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{0}} A^{0} + \frac{\partial x'^{\alpha}}{\partial x^{1}} A^{1} + \frac{\partial x'^{\alpha}}{\partial x^{2}} A^{2} + \frac{\partial x'^{\alpha}}{\partial x^{3}} A^{3}$

covariant vector $B'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta} = \frac{\partial x^{0}}{\partial x'^{\alpha}} B_{0} + \frac{\partial x^{1}}{\partial x'^{\alpha}} B_{1} + \frac{\partial x^{2}}{\partial x'^{\alpha}} B_{2} + \frac{\partial x^{3}}{\partial x'^{\alpha}} B_{3}$

- *Summation convention*: a repeated index means a summation over all the range.

- Contravariant vectors & covariant vectors correspond to the presence of $\frac{\partial x'^{\alpha}}{\partial x^{\beta}}$ and its inverse in the rule of transformation.

- If the law of transformation is linear then the coordinates form the components of a contravariant position vector.

- $\mathbb{F}'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} \mathbb{F}^{\gamma\delta}, \quad \mathbb{G}'_{\alpha\beta} = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} \mathbb{G}_{\gamma\delta}, \quad \mathbb{H}'^{\alpha}{}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} \mathbb{H}^{\gamma}{}_{\delta}$

- The inner or scalar product of 2 vectors is the product of the components of a covariant and a contravariant vector $\vec{B} \cdot \vec{A} \equiv B_{\mu} A^{\mu} = B^{\mu} A_{\mu}$

- The scalar product is an invariant or scalar under the transformation

$$\vec{B}' \cdot \vec{A}' = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} B_{\beta} A^{\gamma} = \frac{\partial x^{\beta}}{\partial x^{\gamma}} B_{\beta} A^{\gamma} = \delta^{\beta}{}_{\gamma} B_{\beta} A^{\gamma} = \vec{B} \cdot \vec{A}$$

- The geometry of the space-time of SR is defined by the invariant interval

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \quad \Leftarrow \quad g_{\alpha\beta} = g_{\beta\alpha} : \text{ metric tensor}$$

- For the flat space-time of SR, the metric tensor is diagonal

$$g_{00} = 1\,, \quad g_{11} = g_{22} = g_{33} = -1 \quad \Rightarrow \quad g^{\alpha\beta} = g_{\alpha\beta} \quad \Leftarrow \quad g^{\alpha\gamma}\, g_{\gamma\beta} = \delta^\alpha{}_\beta$$

- $x_\alpha = g_{\alpha\beta}\, x^\beta\,, \quad x^\alpha = g^{\alpha\beta}\, x_\beta\,, \quad \mathbb{F}^{\cdot\,\cdot\,\alpha\,\cdot\,\cdot}_{\cdot\,\cdot} = g^{\alpha\beta}\,\mathbb{F}^{\cdot\,\cdot\,\cdot\,\cdot}_{\cdot\,\cdot\,\beta}\,, \quad \mathbb{G}^{\cdot\,\cdot\,\cdot}_{\cdot\,\cdot\,\cdot\,\alpha\,\cdot} = g_{\alpha\beta}\,\mathbb{G}^{\cdot\,\cdot\,\cdot\,\beta}_{\cdot\,\cdot\,\cdot\,\cdot}$

$$\Rightarrow \quad A^\alpha = (A^0, \mathbf{A})\,, \quad A_\alpha = (A^0, -\mathbf{A}) \quad \Rightarrow \quad \vec{B}\cdot\vec{A} = B_\alpha A^\alpha = B^0 A^0 - \mathbf{B}\cdot\mathbf{A}$$

- $\dfrac{\partial}{\partial x'^\alpha} = \dfrac{\partial x^\beta}{\partial x'^\alpha}\dfrac{\partial}{\partial x^\beta} \quad \Rightarrow$ differentiation wrt a contravariant component of the coordinate vector transforms as a covariant vector operator

$$\Rightarrow \quad \partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \nabla\right), \quad \partial^\alpha \equiv \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x^0}, -\nabla\right)$$

$$\Rightarrow \quad \partial_\alpha A^\alpha = \partial^\alpha A_\alpha = \frac{\partial A^0}{\partial x^0} + \nabla\cdot\mathbf{A}$$

familiar in form from continuity of charge and current density, the Lorentz condition on the scalar and vector potentials, etc.

- The 4d Laplacian operator is defined to be the invariant contraction, $\Box \equiv \partial_\mu \partial^\mu = \dfrac{\partial^2}{\partial x^{02}} - \nabla^2 = \partial_0^2 - \nabla^2$

Matrix Representation of Lorentz Transformations, Infinitesimal Generators

- The contravariant coordinate vector can be written as $\vec{x} = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$

- *Matrix scalar products* of 4-vectors $(\vec{a}\,,\vec{b}) \equiv \vec{a}^T\,\vec{b}$

- $$\mathrm{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \Rightarrow \quad \mathrm{g}^2 = \mathbb{I}_4 \quad \Rightarrow \quad \underline{x} = \mathrm{g}\,\vec{x} = \begin{bmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{bmatrix} \Leftarrow \text{ covariant coordinate vector}$$

$$\Rightarrow \quad \vec{a}\cdot\vec{b} = (\vec{a}\,,\mathrm{g}\,\vec{b}) = (\mathrm{g}\,\vec{a}\,,\vec{b}) = \vec{a}^{\mathrm{T}}\,\mathrm{g}\,\vec{b} \quad \Leftarrow \quad \text{scalar product}$$

$$\vec{x}\,' = \mathbf{\Lambda}\,\vec{x} \quad \Rightarrow \quad \vec{x}^{\mathrm{T}}\,\mathbf{\Lambda}^{\mathrm{T}}\,\mathrm{g}\,\mathbf{\Lambda}\,\vec{x} \Leftarrow \vec{x}\,'^{\mathrm{T}}\,\mathrm{g}\,\vec{x}\,' = \vec{x}^{\mathrm{T}}\,\mathrm{g}\,\vec{x} \Leftarrow \text{ invariant } \Rightarrow \mathbf{\Lambda}^{\mathrm{T}}\,\mathrm{g}\,\mathbf{\Lambda} = \mathrm{g} \quad (2)$$

$$\Rightarrow \quad \det\left(\mathbf{\Lambda}^{\mathrm{T}}\,\mathrm{g}\,\mathbf{\Lambda}\right) = \det \mathrm{g}\,\left(\det \mathbf{\Lambda}\right)^2 = \det \mathrm{g} \quad \Rightarrow \quad \det \mathbf{\Lambda} = \pm 1$$

$$\Rightarrow \begin{array}{ll} \det \mathbf{\Lambda} = \ \ 1 : \textit{proper Lorentz transformations}\,, & \text{continuous with the identity transformation} \\ \det \mathbf{\Lambda} = -1 : \textit{improper Lorentz transformations}\,, & \text{sufficient but not necessary} \end{array}$$

- 2 examples of improper transformations,

$$\begin{array}{llll} \mathbf{\Lambda} = \ \ \mathrm{g} & \text{space inversion} & \Rightarrow & \det \mathbf{\Lambda} = -1 \\ \mathbf{\Lambda} = -\mathbb{I}_4 & \text{space \& time inversion} & \Rightarrow & \det \mathbf{\Lambda} = +1 \end{array}$$

- The No. of free parameters: $4 \times 4 - 10\,(\text{No. of eqns}) = 6 = 3\,(\text{rotation}) + 3\,(\text{boost})$

$\Uparrow \quad (2)$

- Here we consider only proper Lorentz transformations.
- $\boldsymbol{\Lambda} = \lim_{n \to \infty} \left(\mathbb{I} + \frac{\mathbb{L}}{n} \right)^n = e^{\mathbb{L}} \quad \Rightarrow \quad 1 = \det \boldsymbol{\Lambda} = \det e^{\mathbb{L}} = e^{\mathrm{Tr}\,\mathbb{L}} \quad \Rightarrow \quad \mathbb{L} \in \mathbb{R} \quad \text{is traceless}$

$(2) \;\Rightarrow\; \mathbb{g}\,\boldsymbol{\Lambda}^T \mathbb{g} = \boldsymbol{\Lambda}^{-1} \;\Leftarrow\; \mathbb{g}^2 = \mathbb{I} \;\Rightarrow\; \boldsymbol{\Lambda}^T = e^{\mathbb{L}^T}, \quad \mathbb{g}\,\boldsymbol{\Lambda}^T \mathbb{g} = e^{\mathbb{g}\,\mathbb{L}^T \mathbb{g}}, \quad \boldsymbol{\Lambda}^{-1} = e^{-\mathbb{L}}$

$\Rightarrow \mathbb{g}\,\mathbb{L}^T \mathbb{g} = -\mathbb{L} \;\Rightarrow\; (\mathbb{g}\,\mathbb{L})^T = -\mathbb{g}\,\mathbb{L} \;\Rightarrow\; \mathbb{g}\,\mathbb{L} \text{ antisymmetric } (\mathbb{g}\,\mathbb{L})_{\mu\nu} = -(\mathbb{g}\,\mathbb{L})_{\nu\mu}$

$$\Rightarrow \quad \mathbb{L} = \left[\begin{array}{c|ccc} 0 & \mathbb{L}_{01} & \mathbb{L}_{02} & \mathbb{L}_{03} \\ \hline \mathbb{L}_{01} & 0 & \mathbb{L}_{12} & \mathbb{L}_{13} \\ \mathbb{L}_{02} & -\mathbb{L}_{12} & 0 & \mathbb{L}_{23} \\ \mathbb{L}_{03} & -\mathbb{L}_{13} & -\mathbb{L}_{23} & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & \textit{boosts} \\ \hline \textit{boosts} & \textit{rotations} \end{array}\right] \quad \Rightarrow \quad 6 \text{ fundamental matrices}$$

$$\mathbb{S}_1 = \left[\begin{array}{c|ccc} 0 & & \mathbf{0} & \\ \hline & 0 & 0 & 0 \\ \mathbf{0} & 0 & 0 & -1 \\ & 0 & 1 & 0 \end{array}\right], \quad \mathbb{S}_2 = \left[\begin{array}{c|ccc} 0 & & \mathbf{0} & \\ \hline & 0 & 0 & 1 \\ \mathbf{0} & 0 & 0 & 0 \\ & -1 & 0 & 0 \end{array}\right], \quad \mathbb{S}_3 = \left[\begin{array}{c|ccc} 0 & & \mathbf{0} & \\ \hline & 0 & -1 & 0 \\ \mathbf{0} & 1 & 0 & 0 \\ & 0 & 0 & 0 \end{array}\right] \quad \Leftarrow \quad \text{rotations}$$

$\Rightarrow$

$$\mathbb{K}_1 = \left[\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & & & \\ 0 & & \mathbf{0} & \\ 0 & & & \end{array}\right], \quad \mathbb{K}_2 = \left[\begin{array}{c|ccc} 0 & 0 & 1 & 0 \\ \hline 0 & & & \\ 1 & & \mathbf{0} & \\ 0 & & & \end{array}\right], \quad \mathbb{K}_3 = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & & & \\ 0 & & \mathbf{0} & \\ 1 & & & \end{array}\right] \quad \Leftarrow \quad \text{boosts}$$

$$\mathbb{S}_i = -\bar{\epsilon}_{0i}{}^{\mu\nu}\,\hat{\mathbf{e}}_\mu \otimes \hat{\mathbf{e}}_\nu \quad \Leftarrow \quad \begin{matrix} \bar{\epsilon}^{0123} = [0123] = 1 \\ \bar{\epsilon}_{0123} \qquad\qquad = 1 \end{matrix}, \quad \begin{matrix} \mu, \nu, \sigma, \cdots : \text{from } 0 \text{ to } 3 \\ i, j, k, \cdots \; : \text{from } 1 \text{ to } 3 \end{matrix}$$

$$\Rightarrow \quad \mathbb{S}_i \mathbb{S}_j = \bar{\epsilon}_{0i}{}^{kn}\,\bar{\epsilon}_{0jn}{}^{m}\,\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_m = \left(\delta_i^m \delta_j^k - \delta_{ij}\,\delta^{km}\right)\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_m = \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j - \delta_{ij}\,\mathbb{I}_3$$

$$\Rightarrow \quad [\mathbb{S}_i, \mathbb{S}_j] = \mathbb{S}_i \mathbb{S}_j - \mathbb{S}_j \mathbb{S}_i = \left(\delta_i^m \delta_j^k - \delta_i^k \delta_j^m\right)\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_m = -\bar{\epsilon}_{0ijn}\,\bar{\epsilon}_0{}^{nkm}\,\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_m$$

$$= \bar{\epsilon}_{0ij}{}^{k}\,\mathbb{S}_k = \epsilon_{ij}{}^{k}\,\mathbb{S}_k \quad \Leftarrow \quad \bar{\epsilon}_{0ijn}\,\bar{\epsilon}_{0km}{}^{n} = \epsilon_{ijn}\,\epsilon_{km}{}^{n} = \delta_{ik}\,\delta_{jm} - \delta_{im}\,\delta_{jk}$$

$$\mathbb{K}_i = \hat{\mathbf{e}}_0 \otimes \hat{\mathbf{e}}_i + \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_0 = \left(\delta_0^\mu \delta_i^\nu + \delta_0^\nu \delta_i^\mu\right)\hat{\mathbf{e}}_\mu \otimes \hat{\mathbf{e}}_\nu$$

$$\Rightarrow \quad \mathbb{K}_i \mathbb{K}_j = \left(\delta_0^\mu \delta_{i\sigma} + \delta_{0\sigma}\,\delta_i^\mu\right)\left(\delta_0^\sigma \delta_j^\nu + \delta_0^\nu \delta_j^\sigma\right)\hat{\mathbf{e}}_\mu \otimes \hat{\mathbf{e}}_\nu = \left(\delta_0^\mu \delta_0^\nu \delta_{ij} + \delta_{00}\,\delta_i^\mu \delta_j^\nu\right)\hat{\mathbf{e}}_\mu \otimes \hat{\mathbf{e}}_\nu$$

$$= \delta_{ij}\,\hat{\mathbf{e}}_0 \otimes \hat{\mathbf{e}}_0 + \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$$

$$\Rightarrow \quad [\mathbb{K}_i, \mathbb{K}_j] = \mathbb{K}_i \mathbb{K}_j - \mathbb{K}_j \mathbb{K}_i = \left(\delta_i^\mu \delta_j^\nu - \delta_j^\mu \delta_i^\nu\right)\hat{\mathbf{e}}_\mu \otimes \hat{\mathbf{e}}_\nu = -\epsilon_{ij}{}^{k}\,\mathbb{S}_k$$

$$\mathbb{S}_i \mathbb{K}_j = -\bar{\epsilon}_{0i}{}^{\mu\sigma}\left(\delta_{0\sigma}\,\delta_j^\nu + \delta_0^\nu \delta_{j\sigma}\right)\hat{\mathbf{e}}_\mu \otimes \hat{\mathbf{e}}_\nu = \bar{\epsilon}_{0ij}{}^{\mu}\,\delta_0^\nu\,\hat{\mathbf{e}}_\mu \otimes \hat{\mathbf{e}}_\nu = \bar{\epsilon}_{0ij}{}^{k}\,\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_0$$

$$\mathbb{K}_j \mathbb{S}_i = -\left(\delta_0^\mu \delta_{j\sigma} + \delta_{0\sigma}\,\delta_j^\mu\right)\bar{\epsilon}_{0i\sigma}{}^{\nu}\,\hat{\mathbf{e}}_\mu \otimes \hat{\mathbf{e}}_\nu = -\bar{\epsilon}_{0ij}{}^{\nu}\,\delta_0^\mu\,\hat{\mathbf{e}}_\mu \otimes \hat{\mathbf{e}}_\nu = -\bar{\epsilon}_{0ij}{}^{k}\,\hat{\mathbf{e}}_0 \otimes \hat{\mathbf{e}}_k$$

$$\Rightarrow \quad [\mathbb{S}_i, \mathbb{K}_j] = \left(\bar{\epsilon}_{0ij}{}^{\mu}\,\delta_0^\nu + \bar{\epsilon}_{0ij}{}^{\nu}\,\delta_0^\mu\right)\hat{\mathbf{e}}_\mu \otimes \hat{\mathbf{e}}_\nu = \bar{\epsilon}_{0ij}{}^{k}\left(\delta_k^\mu \delta_0^\nu + \delta_k^\nu \delta_0^\mu\right)\hat{\mathbf{e}}_\mu \otimes \hat{\mathbf{e}}_\nu = \epsilon_{ij}{}^{k}\,\mathbb{K}_k$$

- The squares of these 6 matrices are all diagonal

$$\mathbb{S}_1^2 = \begin{bmatrix} 0 & & \mathbf{0} & \\ & 0 & & \\ & & -1 & \\ & \mathbf{0} & & -1 \end{bmatrix}, \quad \mathbb{S}_2^2 = \begin{bmatrix} 0 & & \mathbf{0} & \\ & -1 & & \\ & & 0 & \\ & \mathbf{0} & & -1 \end{bmatrix}, \quad \mathbb{S}_3^2 = \begin{bmatrix} 0 & & \mathbf{0} & \\ & -1 & & \\ & & -1 & \\ & \mathbf{0} & & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbb{S}_i^2 = \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_i - \mathbb{I}_3$$

$$\mathbb{K}_1^2 = \begin{bmatrix} 1 & & \mathbf{0} & \\ & 1 & & \\ & & 0 & \\ & \mathbf{0} & & 0 \end{bmatrix}, \quad \mathbb{K}_2^2 = \begin{bmatrix} 1 & & \mathbf{0} & \\ & 0 & & \\ & & 1 & \\ & \mathbf{0} & & 0 \end{bmatrix}, \quad \mathbb{K}_3^2 = \begin{bmatrix} 1 & & \mathbf{0} & \\ & 0 & & \\ & & 0 & \\ & \mathbf{0} & & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \mathbb{K}_i^2 &= \hat{\mathbf{e}}_0 \otimes \hat{\mathbf{e}}_0 \\ &+ \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_i \end{aligned}$$

- $(\boldsymbol{\epsilon} \cdot \vec{\mathbb{S}})^3 = -\boldsymbol{\epsilon} \cdot \vec{\mathbb{S}}$, $(\boldsymbol{\epsilon}' \cdot \vec{\mathbb{K}})^3 = \boldsymbol{\epsilon}' \cdot \vec{\mathbb{K}} \Leftarrow \boldsymbol{\epsilon}, \boldsymbol{\epsilon}' \in \mathbb{R}$ are unit 3-vectors $\Rightarrow$ any power of one of the matrices can be expressed as a multiple of the matrix or its square. [Problem 11.10]

- $\mathbb{L} = -\boldsymbol{\omega} \cdot \vec{\mathbb{S}} - \boldsymbol{\xi} \cdot \vec{\mathbb{K}} \quad \Rightarrow \quad \boldsymbol{\Lambda} = e^{-\boldsymbol{\omega} \cdot \vec{\mathbb{S}} - \boldsymbol{\xi} \cdot \vec{\mathbb{K}}}$

For $\boldsymbol{\omega} = 0, \quad \boldsymbol{\xi} = \xi\, \hat{\mathbf{e}}_1 \quad \Rightarrow \quad \mathbb{L} = -\xi\, \mathbb{K}_1 \quad + \quad \mathbb{K}_1^3 = \mathbb{K}_1$

$$\Rightarrow \quad \boldsymbol{\Lambda} = e^{\mathbb{L}} = \mathbb{I} - \mathbb{K}_1 \sinh\xi + \mathbb{K}_1^2 (\cosh\xi - 1) = \begin{bmatrix} \cosh\xi & -\sinh\xi & 0 & 0 \\ -\sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = (1)$$

$$(\boldsymbol{\epsilon}'\cdot\vec{\mathbb{K}})^2=\epsilon'^{\,i}\,\epsilon'^{\,j}\,\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j+\hat{\mathbf{e}}_0\otimes\hat{\mathbf{e}}_0\,,\quad(\boldsymbol{\epsilon}'\cdot\vec{\mathbb{K}})^3=\boldsymbol{\epsilon}'\cdot\vec{\mathbb{K}}\quad\Leftarrow\quad|\boldsymbol{\epsilon}'|=1$$

$$e^{-\boldsymbol{\xi}\cdot\vec{\mathbb{K}}}=e^{-\xi\,\hat{\boldsymbol{\xi}}\cdot\vec{\mathbb{K}}}=\sum_{n=0}^{\infty}\frac{(-\xi\,\hat{\boldsymbol{\xi}}\cdot\vec{\mathbb{K}})^n}{n!}=\mathbb{I}-\sum_{\text{odd }n}\frac{\xi^n}{n!}(\hat{\boldsymbol{\xi}}\cdot\vec{\mathbb{K}})^n+\sum_{\text{even }n}\frac{\xi^n}{n!}(\hat{\boldsymbol{\xi}}\cdot\vec{\mathbb{K}})^n$$

$$=\mathbb{I}-(\hat{\boldsymbol{\xi}}\cdot\vec{\mathbb{K}})\sum_{m=1}\frac{\xi^{2m-1}}{(2m-1)!}+(\hat{\boldsymbol{\xi}}\cdot\vec{\mathbb{K}})^2\sum_{m=1}\frac{\xi^{2m}}{(2m)!}$$

$$=\mathbb{I}-(\hat{\boldsymbol{\xi}}\cdot\vec{\mathbb{K}})\sinh\xi+(\hat{\boldsymbol{\xi}}\cdot\vec{\mathbb{K}})^2(\cosh\xi-1)$$

Let $\boldsymbol{\xi}=\xi\,\hat{\mathbf{e}}_1\quad\Rightarrow\quad\hat{\boldsymbol{\xi}}\cdot\vec{\mathbb{K}}=\mathbb{K}_1\,,\quad(\hat{\boldsymbol{\xi}}\cdot\vec{\mathbb{K}})^2=\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_0\otimes\hat{\mathbf{e}}_0$

$$\Rightarrow\quad e^{-\boldsymbol{\xi}\cdot\vec{\mathbb{K}}}=e^{-\xi\,\mathbb{K}_1}=\mathbb{I}-\mathbb{K}_1\sinh\xi+\mathbb{K}_1^2(\cosh\xi-1)$$

$$(\boldsymbol{\epsilon}\cdot\vec{\mathbb{S}})^2=\epsilon^i\,\epsilon^j\,\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j-\mathbb{I}_3\,,\quad(\boldsymbol{\epsilon}\cdot\vec{\mathbb{S}})^3=-\boldsymbol{\epsilon}\cdot\vec{\mathbb{S}}\quad\Leftarrow\quad|\boldsymbol{\epsilon}|=1$$

$$e^{-\boldsymbol{\omega}\cdot\vec{\mathbb{S}}}=\sum_{n=0}^{\infty}\frac{(-\omega\,\hat{\boldsymbol{\omega}}\cdot\vec{\mathbb{S}})^n}{n!}=\mathbb{I}+(\hat{\boldsymbol{\omega}}\cdot\vec{\mathbb{S}})\sum_{m=1}\frac{(-1)^m\,\omega^{2m-1}}{(2m-1)!}-(\hat{\boldsymbol{\omega}}\cdot\vec{\mathbb{S}})^2\sum_{m=1}\frac{(-1)^m\,\omega^{2m}}{(2m)!}$$

$$=\mathbb{I}-(\hat{\boldsymbol{\omega}}\cdot\vec{\mathbb{S}})\sin\omega-(\hat{\boldsymbol{\omega}}\cdot\vec{\mathbb{S}})^2(\cos\omega-1)$$

Let $\boldsymbol{\omega}=\omega\,\hat{\mathbf{e}}_3\quad\Rightarrow\quad\hat{\boldsymbol{\omega}}\cdot\vec{\mathbb{S}}=\mathbb{S}_3\,,\quad(\hat{\boldsymbol{\omega}}\cdot\vec{\mathbb{S}})^2=-\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1-\hat{\mathbf{e}}_2\otimes\hat{\mathbf{e}}_2$

$$\Rightarrow\quad e^{-\boldsymbol{\omega}\cdot\vec{\mathbb{S}}}=e^{-\omega\,\mathbb{S}_3}=\mathbb{I}-\mathbb{S}_3\sin\omega-\mathbb{S}_3^2(\cos\omega-1)$$

- For $\boldsymbol{\xi}=0$, $\boldsymbol{\omega}=\omega\,\hat{\mathbf{e}}_3 \Rightarrow \boldsymbol{\Lambda}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Leftarrow$ clockwize rotation around z-axis

- $\boldsymbol{\Lambda}_{\text{boost}}(\boldsymbol{\beta})=e^{-\boldsymbol{\xi}\cdot\vec{\mathbb{K}}}=e^{-\hat{\boldsymbol{\beta}}\cdot\vec{\mathbb{K}}\tanh^{-1}\beta} \Leftarrow \boldsymbol{\xi}=\hat{\boldsymbol{\beta}}\tanh^{-1}\beta$

$$=\begin{bmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & (\gamma-1)\frac{\beta_1^2}{\beta^2}+1 & (\gamma-1)\frac{\beta_1\beta_2}{\beta^2} & (\gamma-1)\frac{\beta_1\beta_3}{\beta^2} \\ -\gamma\beta_2 & (\gamma-1)\frac{\beta_1\beta_2}{\beta^2} & (\gamma-1)\frac{\beta_2^2}{\beta^2}+1 & (\gamma-1)\frac{\beta_2\beta_3}{\beta^2} \\ -\gamma\beta_3 & (\gamma-1)\frac{\beta_1\beta_3}{\beta^2} & (\gamma-1)\frac{\beta_2\beta_3}{\beta^2} & (\gamma-1)\frac{\beta_3^2}{\beta^2}+1 \end{bmatrix} \Rightarrow \begin{array}{l} \vec{x}\,'=\boldsymbol{\Lambda}_{\text{boost}}(\boldsymbol{\beta})\,\vec{x} \\ =(0) \end{array}$$

- The 6 matrices are a representation of the *infinitesimal generators* of the Lorentz group. They satisfy the following commutation relations,

$[\mathbb{S}_i,\mathbb{S}_j]=\epsilon_{ijk}\,\mathbb{S}_k$ commutation relations for angular momentum

$[\mathbb{S}_i,\mathbb{K}_j]=\epsilon_{ijk}\,\mathbb{K}_k$ $\vec{\mathbb{K}}$ transforms as a vector under rotations $\Leftarrow$ $[A,B]=A\,B-B\,A$

$[\mathbb{K}_i,\mathbb{K}_j]=-\epsilon_{ijk}\,\mathbb{S}_k$ boosts do not in general commute

- The commutation relations, with the minus sign in the last one, specify the algebraic structure of the Lorentz group to be SL(2,C) or O(3,1).

Thomas Precession

- In general the result of successive Lorentz transformations depends on the order in which they are performed.

- The commutation relations imply that successive Lorentz transformations are equivalent to a single Lorentz transformation + a 3d rotation⇒*Thomas precession*

- Uhlenbeck & Goudsmit introduced the idea of electron spin and showed that, if the electron had a g factor of 2, the anomalous Zeeman effect could be explained, as well as the existence of multiplet splittings.

- The observed fine structure intervals were only half the theoretical values.

- Thomas showed that the origin of the discrepancy was a relativistic kinematic effect which gave both the anomalous Zeeman effect and the correct fine structure splittings with $g=2$.

- The *Thomas precession* also gives a qualitative explanation for a spin-orbit interaction in atomic nuclei and shows why the doublets are "inverted" in nuclei.

- The Uhlenbeck-Goudsmit hypothesis was that an electron possesses a spin angular momentum ($\pm\frac{\hbar}{2}$ along any axis) and a magnetic moment

$$\boldsymbol{\mu} = \frac{g\,e}{2\,m\,c}\mathbf{s} \quad \Leftarrow \quad g = 2 \quad \Leftarrow \quad \mu_L = \frac{e\,L}{2\,m}$$

- Suppose that an electron moves with a velocity in external fields $\mathbf{E}$ and $\mathbf{B}$. Then the equation of motion for its angular momentum in its rest frame

$$\left.\frac{d\mathbf{s}}{dt}\right|_{\text{rest}} = \mathbf{N} = \boldsymbol{\mu}\times\mathbf{B}' \simeq \boldsymbol{\mu}\times(\mathbf{B}-\boldsymbol{\beta}\times\mathbf{E}) \quad (b) \quad \Leftarrow \quad \mathbf{B}' \simeq \mathbf{B}-\boldsymbol{\beta}\times\mathbf{E}$$

$$e\mathbf{E} = -\hat{\mathbf{r}}\frac{d\Phi}{dr} \quad \Leftarrow \quad \Phi(r): \text{average potential energy} \quad + \quad \begin{array}{l}\mathbf{L} = \mathbf{r}\times m\mathbf{v} \\ \quad = c\,m\,\mathbf{r}\times\boldsymbol{\beta}\end{array}: \text{orbital angular momentum}$$

$$\Rightarrow \quad U' = -\boldsymbol{\mu}\cdot\mathbf{B}' \simeq -\boldsymbol{\mu}\cdot(\mathbf{B}-\boldsymbol{\beta}\times\mathbf{E}) = -\frac{ge}{2mc}\mathbf{s}\cdot\mathbf{B} + \frac{g}{2m^2c^2}\frac{\mathbf{s}\cdot\mathbf{L}}{r}\frac{d\Phi}{dr}$$

The interaction energy gives the anomalous Zeeman effect correctly, but has a spin-orbit interaction twice too large.

- The error comes from the incorrectness of (b) for the electron spin.

- If the coordinate system rotates $\left.\frac{d\mathbf{Q}}{dt}\right|_{\text{nonrot}} = \left.\frac{d\mathbf{Q}}{dt}\right|_{\text{rest}} + \boldsymbol{\omega}_T\times\mathbf{Q} \quad \Leftarrow \quad \boldsymbol{\omega}_T$: angular velocity of rotation

$$\Rightarrow \quad \left.\frac{d\mathbf{s}}{dt}\right|_{\text{nonrot}} = \mathbf{s}\times\left(\frac{ge}{2mc}\mathbf{B}' - \boldsymbol{\omega}_T\right) \quad \Rightarrow \quad U = U' + \mathbf{s}\cdot\boldsymbol{\omega}_T$$

- The origin of the Thomas precession frequency is the acceleration experienced by the electron as it moves under the action of external forces.

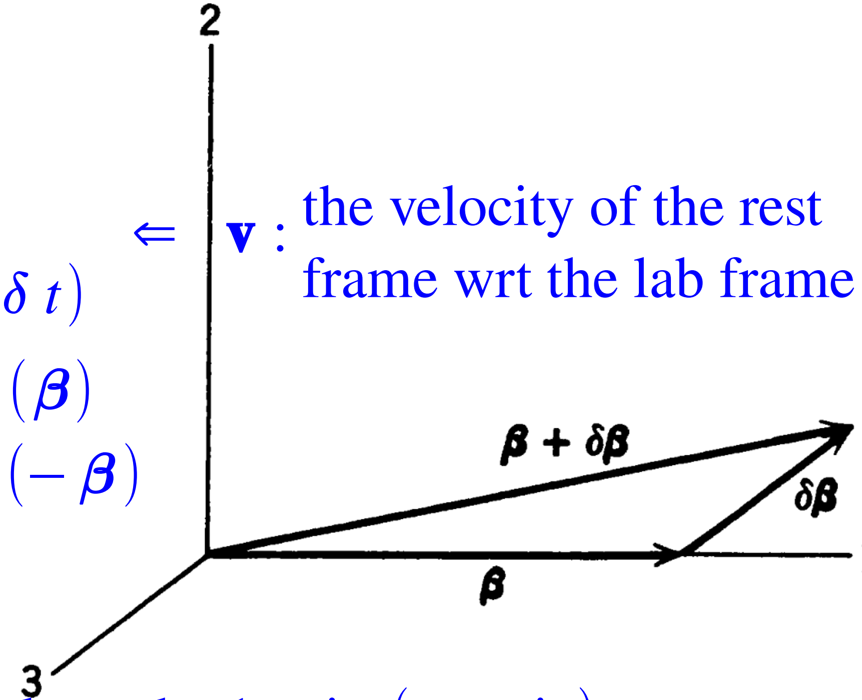
- The electron's rest frame: a co-moving sequence of inertial frames whose successive origins move at each instant with the velocity of the electron.

$$\vec{x}' = \mathbf{\Lambda}_{\text{boost}}(\boldsymbol{\beta})\,\vec{x} \quad \Leftarrow \quad c\,\boldsymbol{\beta} = \mathbf{v}(t)$$

$$\vec{x}'' = \mathbf{\Lambda}_{\text{boost}}(\boldsymbol{\beta}+\delta\boldsymbol{\beta})\,\vec{x} \quad \Leftarrow \quad c(\boldsymbol{\beta}+\delta\boldsymbol{\beta}) = \mathbf{v}(t+\delta t)$$

$\Leftarrow$ $\mathbf{v}$: the velocity of the rest frame wrt the lab frame

$$\Rightarrow \quad \vec{x}'' = \mathbf{\Lambda}_{\text{T}}\,\vec{x}' \quad \Leftarrow \quad \mathbf{\Lambda}_{\text{T}} = \mathbf{\Lambda}_{\text{boost}}(\boldsymbol{\beta}+\delta\boldsymbol{\beta})\,\mathbf{\Lambda}^{-1}_{\text{boost}}(\boldsymbol{\beta}) = \mathbf{\Lambda}_{\text{boost}}(\boldsymbol{\beta}+\delta\boldsymbol{\beta})\,\mathbf{\Lambda}_{\text{boost}}(-\boldsymbol{\beta})$$



$$\Rightarrow \quad \mathbf{\Lambda}_{\text{boost}}(-\boldsymbol{\beta}) = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \Leftarrow$$

Let $\boldsymbol{\beta}$ along the 1 axis (x-axis)
$\delta\boldsymbol{\beta}$ lies on the 1-2 plane (xy-plane)

$$\Rightarrow \quad \mathbf{\Lambda}_{\text{boost}}(\boldsymbol{\beta}+\delta\boldsymbol{\beta}) = \begin{bmatrix} \gamma(1+\gamma^2\beta\,\delta\beta_1) & -\gamma(\beta+\gamma^2\delta\beta_1) & -\gamma\,\delta\beta_2 & 0 \\ -\gamma(\beta+\gamma^2\delta\beta_1) & \gamma(1+\gamma^2\beta\,\delta\beta_1) & \dfrac{\gamma-1}{\beta}\delta\beta_2 & 0 \\ -\gamma\,\delta\beta_2 & \dfrac{\gamma-1}{\beta}\delta\beta_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{\Lambda}_T = \begin{bmatrix} 1 & -\gamma^2\,\delta\,\beta_1 & -\gamma\,\delta\,\beta_2 & 0 \\ -\gamma^2\,\delta\,\beta_1 & 1 & \dfrac{\gamma-1}{\beta}\,\delta\,\beta_2 & 0 \\ -\gamma\,\delta\,\beta_2 & -\dfrac{\gamma-1}{\beta}\,\delta\,\beta_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{\Lambda}_T = \mathbb{I} - (\gamma-1)\,\hat{\boldsymbol{\beta}} \times \frac{\delta\,\boldsymbol{\beta}}{\beta} \cdot \vec{\mathbb{S}} - \left(\gamma^2\,\delta\,\boldsymbol{\beta}_{\parallel} + \gamma\,\delta\,\boldsymbol{\beta}_{\perp}\right) \cdot \vec{\mathbb{K}}$$

$$= \mathbf{\Lambda}_{\text{boost}}(\Delta\,\boldsymbol{\beta})\,\mathbb{R}(\Delta\,\mathbf{\Omega}) = \mathbb{R}(\Delta\,\mathbf{\Omega})\,\mathbf{\Lambda}_{\text{boost}}(\Delta\,\boldsymbol{\beta}) \quad \text{to the 1st order}$$

$$\text{where} \quad \begin{array}{r} \mathbf{\Lambda}_{\text{boost}}(\Delta\,\boldsymbol{\beta}) = \mathbb{I} - \Delta\,\boldsymbol{\beta} \cdot \vec{\mathbb{K}} \\ \mathbb{R}(\Delta\,\mathbf{\Omega}) = \mathbb{I} - \Delta\,\mathbf{\Omega} \cdot \vec{\mathbb{S}} \end{array} \quad \Leftarrow \quad \begin{array}{l} \Delta\,\boldsymbol{\beta} = \gamma^2\,\delta\,\boldsymbol{\beta}_{\parallel} + \gamma\,\delta\,\boldsymbol{\beta}_{\perp} \\ \Delta\,\mathbf{\Omega} = (\gamma-1)\,\hat{\boldsymbol{\beta}} \times \dfrac{\delta\,\boldsymbol{\beta}}{\beta} = \dfrac{\gamma^2}{\gamma+1}\,\boldsymbol{\beta} \times \delta\,\boldsymbol{\beta} \end{array}$$

- Thus the pure Lorentz boost to the frame with velocity $c(\boldsymbol{\beta}+\delta\boldsymbol{\beta})$ is equivalent to a boost to a frame moving with velocity $c\boldsymbol{\beta}$, followed by an infinitesimal Lorentz transformation consisting of a boost with velocity $c\Delta\boldsymbol{\beta}$ and a rotation $\Delta\mathbf{\Omega}$.

- Consider the rest-frame coordinates at $t+\delta t$ that are given from those at t by

$$\vec{x}''' = \mathbf{\Lambda}_{\text{boost}}(\Delta\,\boldsymbol{\beta})\,\vec{x}\,' = \mathbb{R}(-\Delta\,\mathbf{\Omega})\,\mathbf{\Lambda}_{\text{boost}}(\boldsymbol{\beta}+\delta\,\boldsymbol{\beta})\,\vec{x}$$

The rest system of coordinates x''' is rotated by $-\Delta\Omega$ to the boosted lab axes x''.

- For the proper time rate of change of a physical vector in the rest frame, the precession of the rest-frame with respect to the lab makes the vector have a total time rate of change with respect to the lab axes

$$\left.\frac{\mathrm{d}\,\mathbf{Q}}{\mathrm{d}\,t}\right|_{\mathrm{nonrot}} = \left.\frac{\mathrm{d}\,\mathbf{Q}}{\mathrm{d}\,t}\right|_{\mathrm{rest}} + \boldsymbol{\omega}_T \times \mathbf{Q} = \frac{1}{\gamma}\left.\frac{\mathrm{d}\,\mathbf{Q}}{\mathrm{d}\,\tau}\right|_{\mathrm{rest}} + \boldsymbol{\omega}_T \times \mathbf{Q} \quad \Leftarrow \quad \boldsymbol{\omega}_T = -\lim_{\delta t \to 0} \frac{\delta\,\boldsymbol{\Omega}}{\delta\,t} = \gamma^2 \frac{\dot{\boldsymbol{\beta}} \times \boldsymbol{\beta}}{\gamma + 1}$$

- The Thomas precession is purely kinematical. If a component of acceleration exists $\perp\ \mathbf{v}$, then there is a Thomas precession, independent of other effects.

- For electrons in atoms the acceleration is caused by a screened Coulomb field

$$m\,\boldsymbol{a} = m\,c\,\dot{\boldsymbol{\beta}} = e\,\mathbf{E} = -\hat{\mathbf{r}}\,\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,r} \quad \Rightarrow \quad \boldsymbol{\omega}_T \simeq -\frac{1}{2\,c^2}\,\frac{\hat{\mathbf{r}} \times \mathbf{v}}{m}\,\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,r} = -\frac{1}{2\,m^2\,c^2}\,\frac{\mathbf{L}}{r}\,\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,r}$$

$$\Rightarrow \quad U = U' + \mathbf{s}\cdot\boldsymbol{\omega}_T = -\frac{g\,e}{2\,m\,c}\,\mathbf{s}\cdot\mathbf{B} + \frac{g-1}{2\,m^2\,c^2}\,\frac{\mathbf{s}\cdot\mathbf{L}}{r}\,\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,r}$$

- With $g=2$ the spin-orbit interaction is reduced by ½ (*Thomas factor*).

- In atomic nuclei one can treat the nucleons as moving separately in a short-range, spherically symmetric, attractive, potential well Φ_N. Then each nucleon will experience

$$U_N = U' + \mathbf{s}\cdot\boldsymbol{\omega}_T \simeq \mathbf{s}\cdot\boldsymbol{\omega}_T \simeq -\frac{1}{2\,M^2\,c^2}\,\frac{\mathbf{s}\cdot\mathbf{L}}{r}\,\frac{\mathrm{d}\,\Phi_N}{\mathrm{d}\,r} \quad \Leftarrow \quad \text{EM forces are comparatively weak}$$

- Since Φ and Φ_N are attractive, the signs of the spin-orbit energies are opposite. This means that in nuclei the single particle levels form "inverted" doublets.

Invariance of Electric Charge; Covariance of Electrodynamics

- The invariance of form or *covariance* of the Maxwell & Lorentz force equations implies that the various quantities ρ, $\boldsymbol{J}$, $\mathbf{E}$, $\mathbf{B}$ in these equations transform in well-defined ways under Lorentz transformations.

$$\frac{\mathrm{d}\,\mathbf{p}}{\mathrm{d}\,t}=q\left(\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B}\right) \;\Rightarrow\; \frac{\mathrm{d}\,\mathbf{p}}{\mathrm{d}\,\tau}=\frac{q}{c}\left(U_0\,\mathbf{E}+\mathbf{U}\times\mathbf{B}\right) \;\Leftarrow\; \begin{array}{l}\vec{p}=\left(p_0,\mathbf{p}\right)=m\left(U_0,\mathbf{U}\right)\\ c\,p_0=E\quad(\text{energy})\end{array}$$

To form $\dfrac{\mathrm{d}\,\vec{p}}{\mathrm{d}\,\tau}=\vec{f}\left(\rho,\vec{U},\mathbf{E},\mathbf{B}\right) \;\Rightarrow\; \dfrac{\mathrm{d}\,p_0}{\mathrm{d}\,\tau}=\dfrac{q}{c}\,\mathbf{U}\cdot\mathbf{E}$ the change rate of energy

- Experiments support the *invariance of electric charge* under Lorentz transformations, or the indep. of the observed charge of a particle on its speed.

- The continuity equation $\dfrac{\partial\,\rho}{\partial\,t}+\nabla\cdot\boldsymbol{J}=0 \;\Rightarrow\; \partial_\mu J^\mu=0 \;\Leftarrow\; \vec{J}=\left(c\,\rho,\boldsymbol{J}\right)$

- J^α is a legitimate 4-vector follows from the invariance of electric charge

$\delta\,q=\rho\,\mathrm{d}^3x=\rho'\,\mathrm{d}^3x' \;\Leftarrow\;$ invariance of electric charge

$$\mathrm{d}^4x'=\frac{\partial\left(x'^0,x'^1,x'^2,x'^3\right)}{\partial\left(x^0,x^1,x^2,x^3\right)}\,\mathrm{d}^4x=\det\Lambda\,\mathrm{d}^4x=\mathrm{d}^4x=\mathrm{d}\,x^0\,\mathrm{d}^3x \;\Leftarrow\;$$ 4d volume element is a Lorentz invariant

$\Rightarrow\; c\,\rho$ transforms like x^0

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{4\pi}{c}\boldsymbol{J}$$
$$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi\rho$$
$$\Leftarrow \frac{1}{c}\partial_t \Phi + \nabla\cdot\mathbf{A} = 0 \Rightarrow$$
$$\Box A^\mu = \frac{4\pi}{c}J^\mu \Leftarrow \vec{A} = (\Phi, \mathbf{A})$$
$$\partial_\mu A^\mu = 0$$

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi \Rightarrow E_x = -\frac{1}{c}\partial_t A_x - \partial_x \Phi = -(\partial^0 A^1 - \partial^1 A^0) \Leftarrow \partial^\mu = (\partial_0, -\nabla)$$
$$\mathbf{B} = \nabla\times\mathbf{A} \qquad B_x = \partial_y A_z - \partial_z A_y = -(\partial^2 A^3 - \partial^3 A^2)$$

- Define 2^{nd}-*rank antisymmetric field-strength tensor*

$$\mathbb{F}^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \Rightarrow \mathbb{F}_{\mu\nu} = g_{\mu\lambda}g_{\nu\sigma}\mathbb{F}^{\lambda\sigma} = \partial_\mu A_\nu - \partial_\nu A_\mu$$
$$= \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \qquad = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

dual field - strength tensor

$$\mathscr{F}^{\alpha\beta} \equiv \frac{1}{2}\epsilon^{\alpha\beta\gamma\sigma}\mathbb{F}_{\gamma\sigma} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix} \Leftarrow \epsilon^{\alpha\beta\gamma\sigma} = \begin{cases} +1, & [0123] \text{ \& even permutation} \\ -1, & \text{odd permutation} \\ 0, & \text{others} \end{cases}$$

$$\epsilon_{\alpha\beta\gamma\sigma} = -\epsilon^{\alpha\beta\gamma\sigma} \qquad \textit{pseudotensor}$$

$$\Rightarrow \quad \partial_\alpha \mathbb{F}^{\alpha\beta} = \frac{4\pi}{c} J^\beta \Leftarrow \quad \nabla \cdot \mathbf{E} = 4\pi\rho, \quad \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\boldsymbol{J}$$

$$\partial_\alpha \mathscr{F}^{\alpha\beta} = 0 \Leftarrow \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \quad \partial_\alpha \mathbb{F}_{\beta\gamma} + \partial_\beta \mathbb{F}_{\gamma\alpha} + \partial_\gamma \mathbb{F}_{\alpha\beta} = 0$$

$$\Rightarrow \quad \frac{d p^\alpha}{d\tau} = m\frac{d U^\alpha}{d\tau} = \frac{q}{c}\mathbb{F}^{\alpha\beta}U_\beta$$

covariant Lorentz force law

$$\Rightarrow \quad \frac{d\vec{p}}{d\tau} = m\frac{d\vec{U}}{d\tau} = \frac{q}{c}\mathbb{F}\cdot\vec{U}$$

For Macroscopic Maxwell equations

$$\partial_\alpha \mathbb{G}^{\alpha\beta} = \frac{4\pi}{c}J^\beta, \quad \partial_\alpha \mathscr{F}^{\alpha\beta} = 0 \quad \Leftarrow \quad \mathbb{G}^{\alpha\beta} = \begin{bmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & -H_z & H_y \\ D_y & H_z & 0 & -H_x \\ D_z & -H_y & H_x & 0 \end{bmatrix}$$

- With **D**, **H**, **P**, **M** as macroscopic averages of atomic properties *in the rest frame* of the medium, the electrodynamics of macroscopic matter in motion is specified.

Transformation of EM Fields

- $\mathbb{F}'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\sigma}} \mathbb{F}^{\gamma\sigma} \quad \Rightarrow$ $\quad E'_1 = E_1, \quad B'_1 = B_1$

  $E'_2 = \gamma\left(E_2 - \beta B_3\right), \quad B'_2 = \gamma\left(B_2 + \beta E_3\right)$

  $\Leftrightarrow \quad \mathbb{F}' = \mathbf{\Lambda}\, \mathbb{F}\, \mathbf{\Lambda}^{\mathrm{T}}$ $\quad E'_3 = \gamma\left(E_3 + \beta B_2\right), \quad B'_3 = \gamma\left(B_3 - \beta E_2\right)$

$$\Rightarrow \quad \begin{aligned} \mathbf{E}' &= \gamma\left(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}\right) - (\gamma-1)\left(\hat{\boldsymbol{\beta}}\cdot\mathbf{E}\right)\hat{\boldsymbol{\beta}} \\ \mathbf{B}' &= \gamma\left(\mathbf{B} - \boldsymbol{\beta}\times\mathbf{E}\right) - (\gamma-1)\left(\hat{\boldsymbol{\beta}}\cdot\mathbf{B}\right)\hat{\boldsymbol{\beta}} \end{aligned} \quad \Leftarrow \quad \frac{\gamma^2}{\gamma+1} = \frac{\gamma-1}{\beta^2}$$

- $\mathbf{E}$ and $\mathbf{B}$ have no independent existence. A purely electric or magnetic field in one coordinate system will appear as a mixture of both fields in another frame.

- A purely electrostatic field in one coordinate system cannot be transformed into a purely magnetostatic field in another.

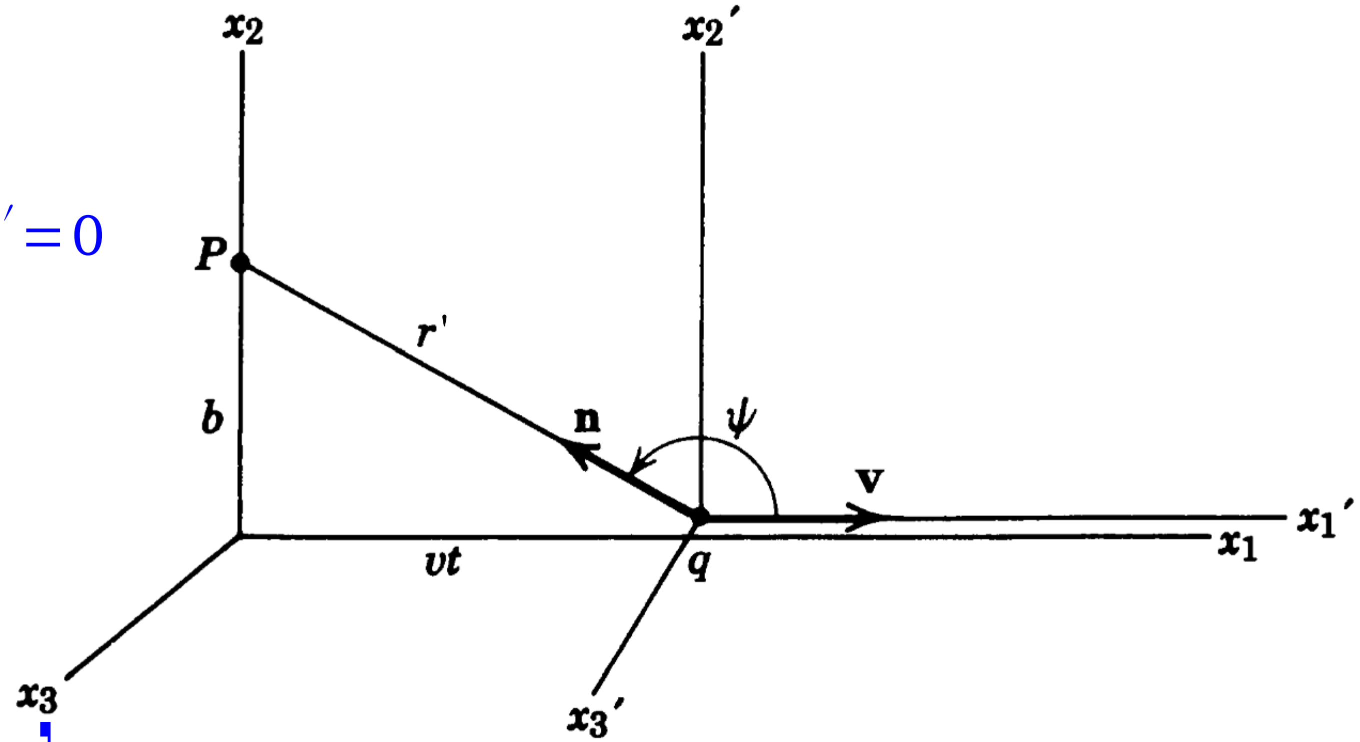
- One should properly speak of the EM field $\mathbf{F}$, rather than $\mathbf{E}$ or $\mathbf{B}$ separately.

- If $\mathbf{B}' = 0$ in K$'$ $\Rightarrow$ $\begin{aligned} \mathbf{E} &= \gamma\,\mathbf{E}' - (\gamma-1)\left(\hat{\boldsymbol{\beta}}\cdot\mathbf{E}'\right)\hat{\boldsymbol{\beta}} \\ \mathbf{B} &= \gamma\,\boldsymbol{\beta}\times\mathbf{E}' \end{aligned}$ $\Rightarrow$ $\mathbf{B} = \boldsymbol{\beta}\times\mathbf{E}$ in K

- For a moving point charge, the coordinates of the observer P are

$$\begin{aligned} &c\,t \\ &x_1 = 0 \\ &x_2 = b \\ &x_3 = 0 \end{aligned} \quad \text{in K} \quad \Rightarrow \quad \begin{aligned} c\,t' &= \gamma\left(c\,t - \beta\,x_1\right) = \gamma\,c\,t \\ x'_1 &= \beta\,c\,t' \\ x'_2 &= b \\ x'_3 &= 0 \end{aligned} \quad \text{in K}' \quad \Rightarrow \quad \begin{aligned} r' &= \sqrt{b^2 + \left(\beta\,c\,t'\right)^2} \\ &= \sqrt{b^2 + \gamma^2\beta^2 c^2 t^2} \end{aligned}$$

$$\Rightarrow \quad \mathbf{E}' = \begin{bmatrix} -\dfrac{q\,v\,t'}{r'^3} \\ \dfrac{q\,b}{r'^3} \\ 0 \end{bmatrix} \quad \& \quad \mathbf{B}' = 0$$



$$\Rightarrow \quad \mathbf{E} = \begin{bmatrix} E_1' \\ \gamma\,E_2' \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\dfrac{q\,\gamma\,v\,t}{\left(b^2+\gamma^2\,v^2\,t^2\right)^{3/2}} \\ \dfrac{\gamma\,q\,b}{\left(b^2+\gamma^2\,v^2\,t^2\right)^{3/2}} \\ 0 \end{bmatrix} \quad \& \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \gamma\,\beta\,E_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \beta\,E_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dfrac{\gamma\,\beta\,q\,b}{\left(b^2+\gamma^2\,v^2\,t^2\right)^{3/2}} \end{bmatrix}$$

$\beta \ll 1 \quad \Rightarrow \quad \gamma \simeq 1 \quad \Rightarrow \quad \mathbf{B} = \dfrac{q}{c}\,\dfrac{\mathbf{v}\times\mathbf{r}}{r^3}$ Ampere-Biot-Savart expression

$\beta \to 1 \quad \Rightarrow \quad \gamma \gg 1 \quad \Rightarrow \quad B_3 \to E_2\,, \quad E_2(t=0) = \gamma\,E_{2\,,\,\mathrm{nonrelativistic}}(t=0)$

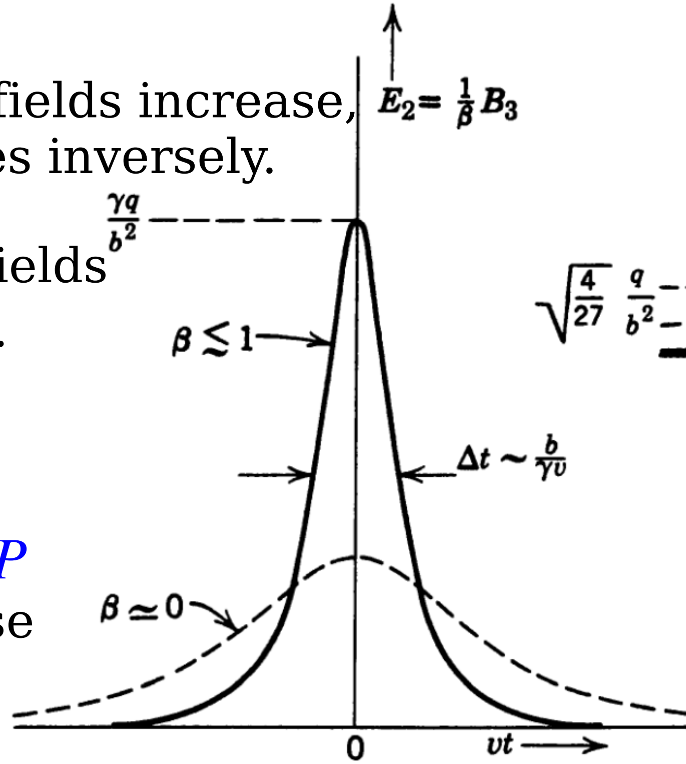
time interval for the fields being appreciable $\quad \Delta t \simeq (0.77)\,\dfrac{b}{\gamma\,v} \quad \Leftarrow \quad E_2(\Delta t) = \dfrac{1}{2}\,E_{2\,,\,\max} = \dfrac{1}{2}\,E_2(0) = \dfrac{\gamma\,q}{2\,b^2}$

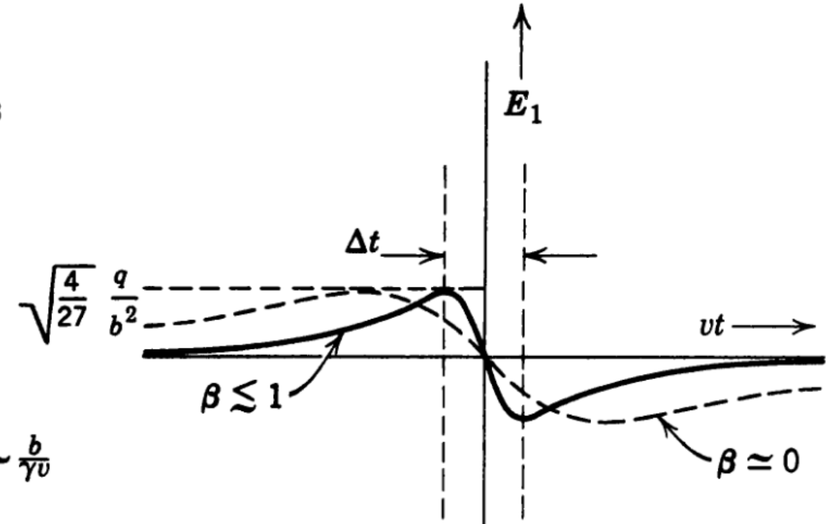
- As γ increases, the peak fields increase, but their duration decreases inversely.

- The time integral of the fields times v is indep. of velocity.

$$\int_{-\infty}^{\infty} E_2 \, \mathrm{d}\, t = \frac{2\, q}{v\, b}$$

- For $\beta \rightarrow 1$ the observer at P sees nearly equal transverse and mutually $\perp$ $\mathbf{E}$ & $\mathbf{B}$.





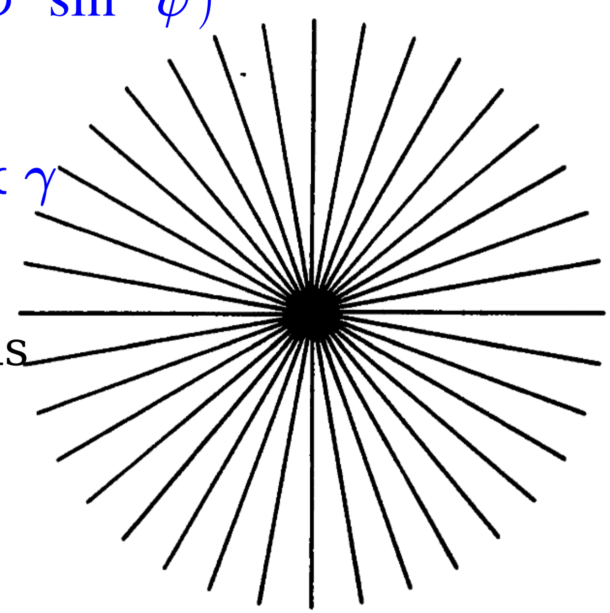
- In practice the observer will see only the transverse fields $\Leftarrow \int_{-\infty}^{+\infty} E_1 \, \mathrm{d}\, t = 0$

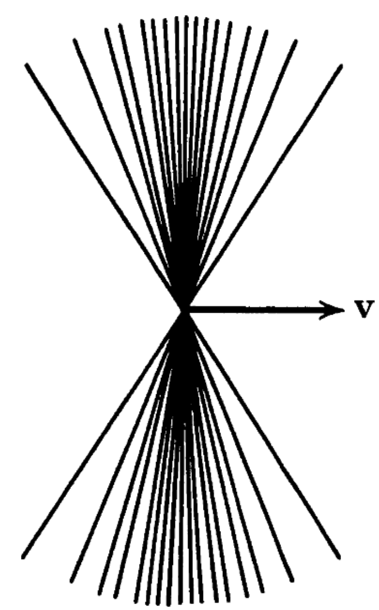
- $\frac{E_1}{E_2} = -\frac{v\, t}{b} \quad \Rightarrow \quad \mathbf{E} = \frac{q\, \hat{\mathbf{r}}}{r^2\, \gamma^2 \left(1 - \beta^2 \sin^2 \psi\right)^{3/2}}, \quad \mathbf{B} = \boldsymbol{\beta} \times \mathbf{E} \quad \Leftarrow \quad \psi = \cos^{-1}\left(\hat{\mathbf{r}} \cdot \hat{\mathbf{v}}\right)$

- $\mathbf{E}$ is radial, but not isotropic

$$E\left(\psi = 0\,,\, \pi\right) \propto \frac{1}{\gamma^2}\,, \quad E\left(\psi = \frac{\pi}{2}\right) \propto \gamma$$

- The compression of the lines of force in the transverse direction is a consequence of the FitzGerald-Lorentz contraction.





$$r^2 = v^2\, t^2 + b^2\,, \quad \sin\psi \equiv \frac{b}{r}\,, \quad \hat{\mathbf{r}} = \cos\psi\,\hat{\mathbf{x}} + \sin\psi\,\hat{\mathbf{y}} = -\frac{v\,t}{r}\,\hat{\mathbf{x}} + \frac{b}{r}\,\hat{\mathbf{y}}$$

$$\Rightarrow \quad r'^{\,2} = \gamma^2\, v^2\, t^2 + b^2 = \gamma^2\, r^2 + \left(1-\gamma^2\right) b^2 = \gamma^2\, r^2 \left(1 + \frac{1-\gamma^2}{\gamma^2}\,\frac{b^2}{r^2} \right)$$

$$= \gamma^2\, r^2 \left(1 - \beta^2 \sin^2\psi\right) \quad \Rightarrow \quad r'^{\,3} = \gamma^3\, r^3 \left(1 - \beta^2 \sin^2\psi\right)^{3/2}$$

$$\Rightarrow \quad \mathbf{E} = E_1\,\hat{\mathbf{x}} + E_2\,\hat{\mathbf{y}} = \frac{\gamma\, q\left(-v\,t\,\hat{\mathbf{x}} + b\,\hat{\mathbf{y}}\right)}{r'^{\,3}} = \frac{\gamma\, q\left(-v\,t\,\hat{\mathbf{x}} + b\,\hat{\mathbf{y}}\right)}{\gamma^3\, r^3 \left(1 - \beta^2 \sin^2\psi\right)^{3/2}}$$

$$= q\,\frac{-\dfrac{v\,t}{r}\,\hat{\mathbf{x}} + \dfrac{b}{r}\,\hat{\mathbf{y}}}{\gamma^2\, r^2 \left(1 - \beta^2 \sin^2\psi\right)^{3/2}} = \frac{q\,\hat{\mathbf{r}}}{\gamma^2\, r^2 \left(1 - \beta^2 \sin^2\psi\right)^{3/2}}$$

Note on Notation and Units in Relativistic Kinematics

- In doing relativistic kinematics, it is customary to suppress all factors of c by suitable choice of units.

- Adopt the convention that all momenta, energies, and masses are measured in energy units, while velocities are measured in units of the velocity of light

$$\begin{bmatrix} c\,p \\ E \\ m\,c^2 \\ \dfrac{v}{c} \end{bmatrix} \Rightarrow \begin{bmatrix} p \\ E \\ m \\ v \end{bmatrix} \Rightarrow \begin{array}{l} E^2 = p^2 + m^2 \\ \mathbf{v} = \dfrac{\mathbf{p}}{E} \end{array}$$

- As energy units, the electron volt (eV), the megaelectron volt (MeV), and the gigaelectron volt (GeV) are convenient

$$1\ \text{eV} = 1.602 \times 10^{-12}\ \text{erg} = 1.602 \times 10^{-19}\ \text{joule}$$

- $(\text{I})\quad \vec{a} \cdot \vec{b} = a_\mu b^\mu = a_0 b_0 - \boldsymbol{a} \cdot \boldsymbol{b} \qquad (\text{II})\quad \vec{P} = \vec{p} + \vec{q} \;\Rightarrow\; P^\mu = p^\mu + q^\mu$

Selected problem: 3, 10, 14, 19, 23, 28