Chapter 6 Maxwell Equations, Macroscopic Electromagnetism, Conservation Laws

Maxwell's Displacement Current; Maxwell Equations

- 4 equations:

Coulomb's law $\nabla \cdot \mathbf{D} = \rho$

Ampere's law $(\nabla \cdot \boldsymbol{J} = 0)$ $\nabla \times \mathbf{H} = \boldsymbol{J}$

Faraday's law $\nabla \times \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0$

Absence of free magnetic poles $\nabla \cdot \mathbf{B} = 0$

- All but Faraday's law were derived from steady-state observations. Thus they are inconsistent.

- The faulty equation is Ampere's law. It is from $\nabla \cdot \boldsymbol{J} = 0 \Leftarrow \nabla \cdot (\nabla \times \mathbf{H} = \boldsymbol{J}) = 0$ is valid for steady-state problems.

- The continuity equation $\dfrac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0 \;\Rightarrow\; \nabla \cdot \left(\boldsymbol{J} + \dfrac{\partial \mathbf{D}}{\partial t} \right) = 0 \;\Leftarrow\; \nabla \cdot \mathbf{D} = \rho$

$$\Rightarrow\; \nabla \times \mathbf{H} = \boldsymbol{J} + \frac{\partial \mathbf{D}}{\partial t} \;\Leftarrow\; \boldsymbol{J} \rightarrow \boldsymbol{J} + \frac{\partial \mathbf{D}}{\partial t} \;\Leftarrow\; \frac{\partial \mathbf{D}}{\partial t} : \text{ displacement current}$$

- It means that a changing electric field causes a magnetic field, even without a current. The term is of crucial importance for rapidly fluctuating fields.

- **The Maxwell equations**:

$$\nabla \cdot \mathbf{D} = \rho\,, \quad \nabla \times \mathbf{H} - \frac{\partial\,\mathbf{D}}{\partial\,t} = \boldsymbol{J}$$

$$\nabla \cdot \mathbf{B} = 0\,, \quad \nabla \times \mathbf{E} + \frac{\partial\,\mathbf{B}}{\partial\,t} = 0$$

- When combined with the Lorentz force equation and Newton's 2[nd] law of motion, these equations provide a complete description of the classical dynamics of interacting charged particles and EM fields:

$$m\,\mathbf{a} \;\Leftarrow\; \mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \;\Rightarrow\; \int \left(\rho\,\mathbf{E} + \boldsymbol{J} \times \mathbf{B}\right) \mathrm{d}^3 x$$

Vector and Scalar Potentials

- It is often convenient to introduce potentials, obtaining a smaller number of 2nd-order equations, while satisfying some of the Maxwell equations identically.

$$\nabla\cdot\mathbf{B}=0 \;\Rightarrow\; \mathbf{B}=\nabla\times\mathbf{A} \;\Rightarrow\; \nabla\times\left(\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}\right)=0 \;\Rightarrow\; \mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{D}=\epsilon_0\mathbf{E},\;\nabla\cdot\mathbf{D}=\rho \;\Rightarrow\; \nabla^2\Phi+\frac{\partial}{\partial t}\nabla\cdot\mathbf{A}=-\frac{\rho}{\epsilon_0}$$

$$\mathbf{B}=\mu_0\mathbf{H},\;\nabla\times\mathbf{H}=\boldsymbol{J}+\frac{\partial\mathbf{D}}{\partial t} \;\Rightarrow\; \nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}-\nabla\left(\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right)=-\mu_0\boldsymbol{J}$$

Reduced 4 1st-order Maxwell equations to 2 2nd-order equations.

$$\begin{aligned}\mathbf{A}\to\mathbf{A}'&=\mathbf{A}+\nabla\Lambda\\ \Phi\to\Phi'&=\Phi-\frac{\partial\Lambda}{\partial t}\end{aligned} \;\Rightarrow\; \begin{aligned}\mathbf{B}'&=\mathbf{B}\\ \mathbf{E}'&=\mathbf{E}\end{aligned} \;\Rightarrow\; \text{choose } \boxed{\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}=0} \;\Leftarrow\; \text{Lorenz condition}$$

$$\Rightarrow\; \boxed{\nabla^2\Phi-\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}=-\frac{\rho}{\epsilon_0},\quad \nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu_0\boldsymbol{J}}$$

uncouple the 2nd-order equations

Gauge Transformations, Lorenz Gauge, Coulomb Gauge

- The transformation is called a *gauge transformation*, and the invariance of the fields under such transformations is called *gauge invariance*.

- $\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}\neq 0 \;\Rightarrow\; \mathbf{A}'=\mathbf{A}+\nabla\Lambda,\quad \Phi'=\Phi-\frac{\partial\Lambda}{\partial t}$

$$\Rightarrow\; \nabla\cdot\mathbf{A}'+\frac{1}{c^2}\frac{\partial\Phi'}{\partial t}=0=\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}+\nabla^2\Lambda-\frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}$$

$$\Rightarrow\; \nabla^2\Lambda-\frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}=-\left(\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right)\;\Leftarrow$$

the new potentials $\mathbf{A}', \Phi'$ will satisfy the Lorenz condition and the wave equations

- $\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}=0 \;\Rightarrow\; \mathbf{A}'=\mathbf{A}+\nabla\Lambda',\quad \Phi'=\Phi-\frac{\partial\Lambda'}{\partial t} \;\Rightarrow\; \nabla\cdot\mathbf{A}'+\frac{1}{c^2}\frac{\partial\Phi'}{\partial t}=0$

$$\Rightarrow\; \nabla^2\Lambda'-\frac{1}{c^2}\frac{\partial^2\Lambda'}{\partial t^2}=0\;\Leftarrow$$

restricted gauge transformation

- All potentials in this restricted class are said to belong to the *Lorenz gauge*.

- The Lorenz gauge is commonly used because
  (1) it leads to the wave eqs, which treat the potentials on equivalent footings;
  (2) a concept indep. of the coordinate system and so fits into special relativity.

- Coulomb, radiation, or transverse gauge $\nabla \cdot \mathbf{A} = 0 \Rightarrow \nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \Leftarrow$ Poisson equation

$$\Rightarrow \Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}^3 x'$$ *instantaneous* Coulomb potential (1)

$$\Rightarrow \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \boldsymbol{J} + \frac{1}{c^2} \nabla \frac{\partial \Phi}{\partial t} \Rightarrow \nabla \times \nabla \frac{\partial \Phi}{\partial t} = 0 \Leftarrow \text{ irrotational}$$

$\boldsymbol{J} = \boldsymbol{J}_\ell + \boldsymbol{J}_t \Leftarrow$ $\nabla \times \boldsymbol{J}_\ell = 0$ longitudinal, irrotational; $\nabla \cdot \boldsymbol{J}_t = 0$ transverse, solenoidal

$+ \quad \nabla \times (\nabla \times \boldsymbol{J}) = \nabla(\nabla \cdot \boldsymbol{J}) - \nabla^2 \boldsymbol{J}$

$$\Rightarrow \begin{aligned} \nabla^2 \boldsymbol{J}_\ell &= +\nabla(\nabla \cdot \boldsymbol{J}) \\ \nabla^2 \boldsymbol{J}_t &= -\nabla \times (\nabla \times \boldsymbol{J}) \end{aligned} \quad + \quad \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -4\pi\delta(\mathbf{r} - \mathbf{r}')$$

$$\Rightarrow \begin{aligned} \boldsymbol{J}_\ell &= -\frac{1}{4\pi} \nabla \int \frac{\nabla' \cdot \boldsymbol{J}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}^3 x' \\ \boldsymbol{J}_t &= \frac{1}{4\pi} \nabla \times \nabla \times \int \frac{\boldsymbol{J}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}^3 x' \end{aligned} \quad + \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0 \quad (1) \quad \Rightarrow \frac{1}{c^2} \nabla \frac{\partial \Phi}{\partial t} = \mu_0 \boldsymbol{J}_\ell$$

$$\Rightarrow \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \boldsymbol{J}_t$$

the source for the wave equation for **A** can be expressed entirely in terms of the transverse current.

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \boldsymbol{J} + \frac{1}{c^2}\nabla \frac{\partial \Phi}{\partial t} \quad \Leftarrow \quad \boldsymbol{J} = \boldsymbol{J}_\ell + \boldsymbol{J}_t \quad \Leftarrow \quad \begin{array}{l} \nabla \times \boldsymbol{J}_\ell = 0 \\ \nabla \cdot \boldsymbol{J}_t = 0 \end{array}$$

$$\Rightarrow \quad \nabla \cdot \left(\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \nabla^2 (\nabla \cdot \mathbf{A}) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\nabla \cdot \mathbf{A}) = 0$$

$$= \nabla \cdot \left(-\mu_0 \boldsymbol{J} + \frac{1}{c^2}\nabla \frac{\partial \Phi}{\partial t} \right) = \nabla \cdot \left(-\mu_0 \boldsymbol{J}_\ell + \frac{1}{c^2}\nabla \frac{\partial \Phi}{\partial t} \right)$$

$$\Rightarrow \quad \mu_0 \boldsymbol{J}_\ell - \frac{1}{c^2}\nabla \frac{\partial \Phi}{\partial t} = \text{const} \Rightarrow 0 \quad \Rightarrow \quad \boldsymbol{J}_\ell = \epsilon_0 \frac{\partial}{\partial t}\nabla \Phi$$

$$\Rightarrow \quad \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \boldsymbol{J}_t$$

- The radiation gauge stems from that transverse radiation fields are given by the vector potential alone, the Coulomb potential contributing only to the near fields.

- This gauge is useful in quantum electrodynamics. A quantum-mechanical description of photons necessitates quantization of only the vector potential.

- The Coulomb or transverse gauge is often used with no source

$$\rho = 0\,, \quad \boldsymbol{J} = 0 \quad \Rightarrow \quad \Phi = 0\,, \quad \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \quad \Rightarrow \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}\,, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

- In the Coulomb gauge, the scalar potential "propagates" instantaneously, but the vector potential propagates in finite speed of propagation c.

- It is the fields, not the potentials, that matter. Another is that the transverse current extends over all space, even if $\boldsymbol{J}$ is localized.

Green Functions for the Wave Equation

- Form of the wave equation: $\nabla^2 \Psi - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = -4\pi f(\mathbf{r}, t) \Leftarrow f:$ source function

Fourier transform

$$\Psi(\mathbf{r}, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Psi(\mathbf{r}, \omega)\, e^{-i\omega t}\, \mathrm{d}\omega \qquad f(\mathbf{r}, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\mathbf{r}, \omega)\, e^{-i\omega t}\, \mathrm{d}\omega$$

$$\Rightarrow \quad \Psi(\mathbf{r}, \omega) = \int_{-\infty}^{+\infty} \Psi(\mathbf{r}, t)\, e^{i\omega t}\, \mathrm{d}t \qquad f(\mathbf{r}, \omega) = \int_{-\infty}^{+\infty} f(\mathbf{r}, t)\, e^{i\omega t}\, \mathrm{d}t$$

$$\Rightarrow \quad (\nabla^2 + k^2)\,\Psi(\mathbf{r}, \omega) = -4\pi f(\mathbf{r}, \omega) \quad \Leftarrow$$

inhomogeneous Helmholtz wave equation $\quad k = \frac{\omega}{c}$

- The Helmholtz wave equation is an elliptic partial differential equation similar to the Poisson equation to which it reduces for $k=0$.

- The Green function satisfies the inhomogeneous equation $\quad (\nabla^2 + k^2)\, G_k(\mathbf{r}, \mathbf{r}') = -4\pi\,\delta(\mathbf{r} - \mathbf{r}')$

- If there are no boundary surfaces, the Green function depend only on $\vec{\mathbb{r}} = \mathbf{r} - \mathbf{r}'$, and $\mathbb{r} = |\vec{\mathbb{r}}|$, and must in fact be spherically symmetric.

$$\Rightarrow \quad \frac{1}{\mathbb{r}}\frac{\mathrm{d}^2}{\mathrm{d}\,\mathbb{r}^2}(\mathbb{r}\, G_k) + k^2 G_k = -4\pi\,\delta(\vec{\mathbb{r}}) \quad \Rightarrow \quad \frac{\mathrm{d}^2}{\mathrm{d}\,\mathbb{r}^2}(\mathbb{r}\, G_k) + k^2(\mathbb{r}\, G_k) = 0 \ \text{ for } \mathbb{r} \neq 0$$

$$\Rightarrow \quad \mathbb{r}\, G_k(\vec{\mathbb{r}}) = A\, e^{ik\mathbb{r}} + B\, e^{-ik\mathbb{r}}$$

- The delta function has influence only at $\mathbb{r} \to 0$. In that limit the equation reduces to the Poisson equation, since $k\mathbb{r} \ll 1$

$$\Rightarrow \lim_{k\mathbb{r}\to 0} G_k(\mathbb{r}) = \frac{1}{\mathbb{r}} \quad \Rightarrow \quad G_k(\mathbb{r}) = A\,G_k^+(\mathbb{r}) + B\,G_k^-(\mathbb{r}) \quad \Leftarrow \quad G_k^\pm = \frac{e^{\pm ik\mathbb{r}}}{\mathbb{r}}, \quad A + B = 1$$

- The 1st term represents a diverging spherical wave propagating from the origin, while the 2nd represents a converging spherical wave. The choice of A and B depends on the boundary conditions in time.

- To understand their different time behaviors, construct the corresponding time-dependent Green functions

$$\left(\nabla_x^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G^\pm(\mathbf{r}, t; \mathbf{r}', t') = -4\pi\,\delta(\mathbf{r} - \mathbf{r}')\,\delta(t - t') \quad \Rightarrow \quad \begin{aligned} \vec{\mathbb{r}} &= \mathbf{r} - \mathbf{r}' \\ \mathbb{r} &= |\mathbf{r} - \mathbf{r}'| \end{aligned}$$

$$\Rightarrow \quad (\nabla^2 + k^2)\,G^\pm(\mathbf{r}, \omega; \mathbf{r}', t') = -4\pi\,\delta(\vec{\mathbb{r}})\,e^{i\omega t'}$$

$$\Rightarrow \quad G^\pm(\mathbf{r}, \omega; \mathbf{r}', t') = G^\pm(\mathbb{r})\,e^{i\omega t'} = \frac{e^{\pm ik\mathbb{r}}}{\mathbb{r}}\,e^{i\omega t'}, \quad \text{let } \tau \equiv t - t'$$

$$\Rightarrow \quad G^\pm(\mathbb{r}, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm ik\mathbb{r}}}{\mathbb{r}}\,e^{-i\omega\tau}\,d\omega = \frac{1}{\mathbb{r}}\,\delta\left(\tau \mp \frac{\mathbb{r}}{c}\right) \quad \text{for} \quad k = \frac{\omega}{c} \quad \text{nondispersive}$$

$$\Rightarrow \boxed{G^\pm(\mathbf{r}, t; \mathbf{r}', t') = \frac{\delta\left(t' - (t \mp \mathbb{r}/c)\right)}{\mathbb{r}}} = \frac{\delta(\tau \mp \mathbb{r}/c)}{\mathbb{r}} \quad \Leftarrow \quad \begin{aligned} &G^+: \ \text{retarded Green function} \\ &G^-: \text{advanced Green function} \end{aligned}$$

- The retarded Green function has a causal behavior: an effect observed at the point $\mathbf{r}$ at time t is caused by the action of a source a distance away at an earlier or retarded time $t' = t_r = t - \frac{\mathbb{r}}{c}$. Similar with the advanced Green function.

- $\Psi^{\pm}(\mathbf{r}, t) = \iint G^{\pm}(\mathbf{r}, t; \mathbf{r}', t') f(\mathbf{r}', t') \, d^3x' \, dt'$ + homogeneous solution

$$t \to -\infty \quad \Rightarrow \quad \left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Psi_{\text{in}}(\mathbf{r}, t) = 0$$

$$\Rightarrow \quad \Psi(\mathbf{r}, t) = \Psi_{\text{in}}(\mathbf{r}, t) + \iint G^{+}(\mathbf{r}, t; \mathbf{r}', t') f(\mathbf{r}', t') \, d^3x' \, dt'$$

$$t \to \infty \quad \Rightarrow \quad \left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Psi_{\text{out}}(\mathbf{r}, t) = 0$$

$$\Rightarrow \quad \Psi(\mathbf{r}, t) = \Psi_{\text{out}}(\mathbf{r}, t) + \iint G^{-}(\mathbf{r}, t; \mathbf{r}', t') f(\mathbf{r}', t') \, d^3x' \, dt'$$

- G^{+} guarantees at remotely early times before the source has been activated, there is no contribution from the integral. G^{-} assures that no signal from the source exists after the source shuts off.

- Usually $\Psi_{\text{in}} = 0 \;\Rightarrow\; \Psi(\mathbf{r}, t) = \int \frac{f(\mathbf{r}', t_r)}{\mathbb{r}} \, d^3x' \;\Leftarrow$ evaluated at the retarded time $t_r = t - \frac{\mathbb{r}}{c}, \quad \mathbb{r} = |\mathbf{r} - \mathbf{r}'|$

Green's 2nd identiy Green theorem

$$\int_{\mathcal{V}} (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, \mathrm{d}^3 x = \oint_{\mathcal{S}} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) \mathrm{d} a$$

$$\text{Let } \begin{array}{l} \psi \to \Psi \\ \phi \to G \end{array} \;\Rightarrow \int_{\mathcal{V}} (G \nabla^2 \Psi - \Psi \nabla^2 G) \, \mathrm{d}^3 x' = \oint_{\mathcal{S}} \left(G \frac{\partial \Psi}{\partial n'} - \Psi \frac{\partial G}{\partial n'} \right) \mathrm{d} a'$$

$$\Rightarrow \int [G (\nabla^2 \Psi + k^2 \Psi) - \Psi (\nabla^2 G + k^2 G)] \, \mathrm{d}^3 x'$$

$$= \oint_{r' \to \infty} \left(G \frac{\partial \Psi}{\partial n'} - \Psi \frac{\partial G}{\partial n'} \right) \mathrm{d} a' \to 0$$

$$\Rightarrow \Psi(\mathbf{r}) \Leftarrow \int \Psi(\mathbf{r}') \, \delta(\mathbf{r} - \mathbf{r}') \, \mathrm{d}^3 x' = \int G(\mathbf{r}, \mathbf{r}') \, f(\mathbf{r}') \, \mathrm{d}^3 x'$$

$$\text{where } \nabla^2 \Psi + k^2 \Psi = -4 \pi f, \quad \nabla^2 G + k^2 G = -4 \pi \delta$$

One can extend the argument to the space-time domain.

The homogeneous diffusion equation (5.160) for the vector potential for quasi-static fields in unbounded conducting media has a solution to the initial value problem of the form,

$$\mathbf{A}(\mathbf{x}, t) = \int d^3x' \; G(\mathbf{x} - \mathbf{x}', t)\mathbf{A}(\mathbf{x}', 0)$$

where $\mathbf{A}(\mathbf{x}', 0)$ describes the initial field configuration and G is an appropriate kernel.

(a) Solve the initial value problem by use of a three-dimensional Fourier transform in space for $\mathbf{A}(\mathbf{x}, t)$. With the usual assumptions on interchange of orders of integration, show that the Green function has the Fourier representation,

$$G(\mathbf{x} - \mathbf{x}', t) = \frac{1}{(2\pi)^3} \int d^3k \; e^{-k^2 t/\mu\sigma} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}$$

and it is assumed that $t > 0$. [Problem 6.3a]

The homogeneous diffusion equation is $\nabla^2 \mathbf{A} = \mu\sigma \dfrac{\partial \mathbf{A}}{\partial t}$. And the 3d Fourier transform in space for $\mathbf{A}$ is $\mathbf{A}(\mathbf{x}, t) = \dfrac{1}{(2\pi)^3} \displaystyle\int \mathcal{A}(\boldsymbol{k}, t) e^{-i\boldsymbol{k}\cdot\mathbf{x}} \mathrm{d}^3k$.

(a) Substitute the Fourier transform into to diffusion equation will get

$$\nabla^2 \int \mathcal{A}(\boldsymbol{k}, t) e^{-i\boldsymbol{k}\cdot\mathbf{x}} \mathrm{d}^3k = \mu\sigma \frac{\partial}{\partial t} \int \mathcal{A}(\boldsymbol{k}, t) e^{-i\boldsymbol{k}\cdot\mathbf{x}} \mathrm{d}^3k.$$

Therefore $\frac{\partial \mathcal{A}}{\partial t} + \frac{k^2}{\mu\sigma}\mathcal{A} = 0$. It leads to $\mathcal{A}(\boldsymbol{k}, t) = \mathcal{A}(\boldsymbol{k}, 0)e^{-k^2 t/\mu\sigma}$ with $t > 0$, here $\mathcal{A}(\boldsymbol{k}, 0) = \int \mathbf{A}(\mathbf{x}', 0)e^{i\boldsymbol{k}\cdot\mathbf{x}'}\mathrm{d}^3x'$. Thus

$$
\begin{aligned}
\mathbf{A}(\mathbf{x}, t) &= \frac{1}{(2\pi)^3}\int \mathcal{A}(\boldsymbol{k}, t)e^{-i\boldsymbol{k}\cdot\mathbf{x}}\mathrm{d}^3k = \frac{1}{(2\pi)^3}\int \mathcal{A}(\boldsymbol{k}, 0)e^{-k^2 t/\mu\sigma}e^{-i\boldsymbol{k}\cdot\mathbf{x}}\mathrm{d}^3k \\
&= \frac{1}{(2\pi)^3}\int \left(\int \mathbf{A}(\mathbf{x}', 0)e^{i\boldsymbol{k}\cdot\mathbf{x}'}\mathrm{d}^3x'\right) e^{-k^2 t/\mu\sigma}e^{-i\boldsymbol{k}\cdot\mathbf{x}}\mathrm{d}^3k \\
&= \int \left(\frac{1}{(2\pi)^3}\int e^{-k^2 t/\mu\sigma}e^{-i\boldsymbol{k}\cdot(\mathbf{x}-\mathbf{x}')}\mathrm{d}^3k\right) \mathbf{A}(\mathbf{x}', 0)\mathrm{d}^3x' \\
&= \int G(\mathbf{x} - \mathbf{x}', t)\mathbf{A}(\mathbf{x}', 0)\mathrm{d}^3x',
\end{aligned}
$$

where $G(\mathbf{x} - \mathbf{x}', t) = \frac{1}{(2\pi)^3}\int e^{-k^2 t/\mu\sigma}e^{-i\boldsymbol{k}\cdot(\mathbf{x}-\mathbf{x}')}\mathrm{d}^3k$, with $t > 0$.

Retarded Solutions for the Fields: Jefimenko's Generalizations of the Coulomb and Biot-Savart Laws; Heaviside-Feynman Expressions for Fields of Point Charge

- Use the retarded solution for the wave equations with $t_r = t - \frac{\mathbb{r}}{c}$

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{\mathbb{r}} \mathrm{d}^3 x', \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\mathbf{r}', t_r)}{\mathbb{r}} \mathrm{d}^3 x'$$

- From the 2 equations the $\mathbf{E}$ & $\mathbf{B}$ fields can be computed, but it is often useful to have retarded integral solutions for the fields in terms of the sources.

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\epsilon_0}\left(\nabla \rho + \frac{1}{c^2}\frac{\partial \boldsymbol{J}}{\partial t}\right), \quad \nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = -\mu_0 \nabla \times \boldsymbol{J}$$

$$\Rightarrow \quad \mathbf{E}(\mathbf{r}, t) = -\frac{1}{4\pi\epsilon_0} \int \frac{1}{\mathbb{r}} \left(\nabla' \rho + \frac{1}{c^2}\frac{\partial \boldsymbol{J}}{\partial t'}\right)_{t'=t_r} \mathrm{d}^3 x' \qquad \Leftarrow \quad t_r = t_r(t, \mathbb{r})$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{(\nabla' \times \boldsymbol{J})_{t'=t_r}}{\mathbb{r}} \mathrm{d}^3 x'$$

- $\nabla' f(t' = t_r) \neq (\nabla' f)_{t'=t_r}$: ∇' in the retarded bracket is a spatial gradient in $\mathbf{r}'$ with t_r fixed; ∇' outside the retarded bracket is a spatial gradient in $\mathbf{r}'$ with $\mathbf{r}$ and t fixed.

$$f(\mathbf{r}', t_r) = f\left(\mathbf{r}', t - \frac{\mathscr{r}(t_r)}{c}\right) = \int f(\mathbf{r}', t')\,\delta\left(t' - t + \frac{\mathscr{r}(t')}{c}\right) \mathrm{d}\,t' \qquad \Leftarrow \hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}}$$

$$\nabla' \rho\,(t' = t_r) = (\nabla' \rho)_{t' = t_r} + \frac{\partial \rho}{\partial t_r} \nabla' t_r = (\nabla' \rho)_{t' = t_r} + \frac{\partial \rho}{\partial t_r} \frac{\hat{\mathscr{r}}}{c} \quad \Leftarrow \quad \nabla' t_r = \frac{\hat{\mathscr{r}}}{c}$$

$\Rightarrow$

$$\nabla' \times \boldsymbol{J}\,(t' = t_r) = (\nabla' \times \boldsymbol{J})_{t' = t_r} + \nabla' t_r \times \frac{\partial \boldsymbol{J}}{\partial t_r} = (\nabla' \times \boldsymbol{J})_{t' = t_r} - \frac{\partial \boldsymbol{J}}{\partial t_r} \times \frac{\hat{\mathscr{r}}}{c}$$

$$\frac{(\nabla' \rho)_{t_r}}{\mathscr{r}} = \frac{\nabla' \rho(t_r)}{\mathscr{r}} - \frac{\partial \rho(t_r)}{\partial t_r} \frac{\hat{\mathscr{r}}}{c\,\mathscr{r}} = \nabla' \frac{\rho(t_r)}{\mathscr{r}} - \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \rho(t_r) - \frac{\partial \rho(t_r)}{\partial t_r} \frac{\hat{\mathscr{r}}}{c\,\mathscr{r}}$$

$\Rightarrow$

$$\frac{(\nabla' \times \boldsymbol{J})_{t_r}}{\mathscr{r}} = \frac{\nabla' \times \boldsymbol{J}(t_r)}{\mathscr{r}} + \frac{\partial \boldsymbol{J}(t_r)}{\partial t_r} \times \frac{\hat{\mathscr{r}}}{c\,\mathscr{r}} = \nabla' \times \frac{\boldsymbol{J}(t_r)}{\mathscr{r}} - \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \times \boldsymbol{J}(t_r) + \frac{\partial \boldsymbol{J}(t_r)}{\partial t_r} \times \frac{\hat{\mathscr{r}}}{c\,\mathscr{r}}$$

$\Rightarrow$

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \left(\rho(\mathbf{r}', t_r) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} + \frac{\partial \rho(\mathbf{r}', t_r)}{\partial t_r} \frac{\hat{\mathscr{r}}}{c\,\mathscr{r}} - \frac{1}{c^2\,\mathscr{r}} \frac{\partial \boldsymbol{J}(\mathbf{r}', t_r)}{\partial t_r} \right) \mathrm{d}^3 x'$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \left(\boldsymbol{J}(\mathbf{r}', t_r) \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} + \frac{\partial \boldsymbol{J}(\mathbf{r}', t_r)}{\partial t_r} \times \frac{\hat{\mathscr{r}}}{c\,\mathscr{r}} \right) \mathrm{d}^3 x'$$

Jefimenko's generalizations of the Coulomb and Biot-Savart laws

$$\Rightarrow \quad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}') \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \mathrm{d}^3 x', \quad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \boldsymbol{J}(\mathbf{r}') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \mathrm{d}^3 x'$$

for ρ, $\boldsymbol{J}$ are time-independent

$$\frac{\partial}{\partial t_r} f(\mathbf{r}', t_r) = \frac{\partial}{\partial t} f(\mathbf{r}', t_r) \quad \Leftarrow \quad t_r = t - \frac{\mathbb{r}}{c} \quad \Leftarrow \quad \mathbb{r} = |\mathbf{r} - \mathbf{r}'|$$

$$\rho(\mathbf{r}', t_r) = q\,\delta(\mathbf{r}' - \vec{\mathbb{x}}(t_r)), \quad \boldsymbol{J}(\mathbf{r}', t_r) = \rho\,\mathbf{v}(t_r) \quad \text{for a point charge} \quad \Leftarrow \quad \mathbf{v} = \frac{d\,\vec{\mathbb{x}}}{d\,t_r}$$

$$\Rightarrow \quad \begin{aligned} \mathbf{E} &= \frac{q}{4\pi\epsilon_0}\left(\frac{\hat{\mathbb{r}}}{\mathbb{r}^2(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})} + \frac{1}{c}\frac{\partial}{\partial t}\frac{\hat{\mathbb{r}}-\boldsymbol{\beta}}{\mathbb{r}(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})}\right) \\ \mathbf{B} &= \frac{\mu_0 q c}{4\pi}\left(\frac{\boldsymbol{\beta}\times\hat{\mathbb{r}}}{\mathbb{r}^2(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})} + \frac{1}{c}\frac{\partial}{\partial t}\frac{\boldsymbol{\beta}\times\hat{\mathbb{r}}}{\mathbb{r}(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})}\right) \end{aligned} \quad (2) \quad \Leftarrow \quad \text{Heaviside-Feynman expressions}$$

where $1 - \hat{\mathbb{r}}\cdot\boldsymbol{\beta}$ is the retardation factor, $\boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c}$, $\delta\left(t - t' - \frac{\mathbb{r}(t')}{c}\right) = \frac{\delta(t'-t)}{1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta}}$

- Because $\mathbf{r}' \to \vec{\mathbb{x}}(t_r)$, the fields are functions of $\mathbf{r}$ and t, with $t_r = t - \frac{|\mathbf{r} - \vec{\mathbb{x}}(t_r)|}{c}$

$$\frac{d\,t}{d\,t_r} = \frac{d}{d\,t_r}\left(t_r + \frac{\mathbb{r}(t_r)}{c}\right) = 1 - \hat{\mathbb{r}}\cdot\boldsymbol{\beta} \quad \Rightarrow \quad \frac{d\,t_r}{d\,t} = \frac{1}{1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta}} \quad \Leftarrow \quad \vec{\mathbb{r}} = \mathbf{r} - \vec{\mathbb{x}}(t_r) \quad \text{now}$$

$$\Rightarrow \quad (2) \quad \Rightarrow \quad \begin{aligned} \mathbf{E} &= \frac{q}{4\pi\epsilon_0}\left(\frac{\hat{\mathbb{r}}}{\mathbb{r}^2} + \frac{\mathbb{r}}{c}\frac{\partial}{\partial t}\frac{\hat{\mathbb{r}}}{\mathbb{r}^2} + \frac{1}{c^2}\frac{\partial^2\hat{\mathbb{r}}}{\partial t^2}\right) && \Leftarrow \text{Feynman's expression} \\ \mathbf{B} &= \frac{\mu_0 q}{4\pi}\left(\frac{c\,\boldsymbol{\beta}\times\hat{\mathbb{r}}}{\mathbb{r}^2(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})^2} + \frac{1}{\mathbb{r}}\frac{\partial}{\partial t}\frac{\boldsymbol{\beta}\times\hat{\mathbb{r}}}{1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta}}\right) && \Leftarrow \text{Heaviside's expression} \end{aligned}$$

- The sets are equivalent. [Problem 6.2]

Derivation of the Equations of Macroscopic Electromagnetism

- Present a serious derivation of them from a microscopic starting point.

- The derivation remains within a classical framework.

- For dimensions large compared to 10^{-14} m, the nuclei and electrons can be treated as point systems,

$$\nabla \cdot \mathbf{b} = 0\,, \qquad \nabla \times \mathbf{e} + \frac{\partial \mathbf{b}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{e} = \frac{\eta}{\epsilon_0}\,, \qquad \nabla \times \mathbf{b} - \frac{1}{c^2}\frac{\partial \mathbf{e}}{\partial t} = \mu_0 \boldsymbol{j}$$

$\Leftarrow$ *microscopic* Maxwell equations

no corresponding fields $\mathbf{d}$ and $\mathbf{h}$ because all the charges are included in η and $\boldsymbol{j}$.

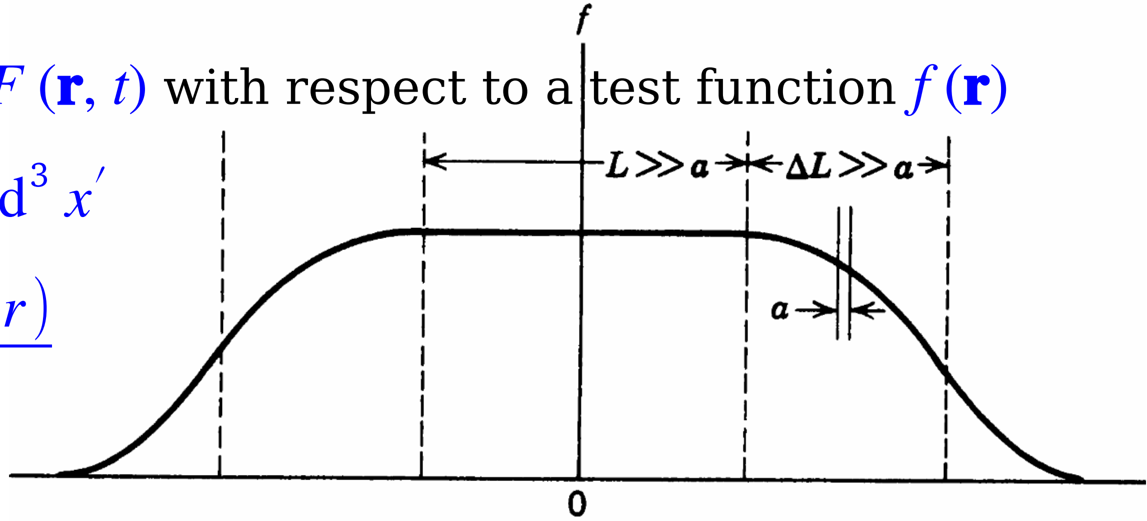
- The microscopic EM fields produced by these charges vary extremely rapidly. The spatial variations occur over distances $< 10^{-10}$ m, and the temporal fluctuations occur from 10^{-13} s for nuclear vibrations to 10^{-17} s for electronic orbital motion.

- All the microscopic fluctuations are averaged out, giving smooth and slowly varying macroscopic quantities, to appear in the macroscopic Maxwell equations.

- Only a spatial averaging is necessary because in a characteristic length there are still many charges, but the associated characteristic time is about the range of atomic & molecular motions, not appropriate to average out in time.

- The spatial average of a function $F(\mathbf{r}, t)$ with respect to a test function $f(\mathbf{r})$

$$\langle F(\mathbf{r}, t)\rangle = \int f(\mathbf{r}')\, F(\mathbf{r}-\mathbf{r}', t)\, d^3x'$$



$\Rightarrow$ 2 examples (isotropic)

$$f(\mathbf{r}) = \frac{3\,\Theta(R-r)}{4\pi R^3}$$

$$f(\mathbf{r}) = \frac{e^{-\frac{r^2}{R^2}}}{\pi^{3/2} R^3}$$

- A test function only needs to have general continuity & smoothness properties that permit a rapidly converging expansion over distances of atomic dimensions.

$$\nabla\langle F(\mathbf{r}, t)\rangle = \int f(\mathbf{r}')\,\nabla F(\mathbf{r}-\mathbf{r}', t)\, d^3x' = \langle \nabla F\rangle, \quad \frac{\partial}{\partial t}\langle F(\mathbf{r}, t)\rangle = \left\langle \frac{\partial F}{\partial t}\right\rangle$$

- The space & time differentiation commute with the averaging operation.

- $\mathbf{E}(\mathbf{r}, t) = \langle \mathbf{e}(\mathbf{r}, t)\rangle, \quad \mathbf{B}(\mathbf{r}, t) = \langle \mathbf{b}(\mathbf{r}, t)\rangle$

$$\langle \nabla\cdot\mathbf{b}\rangle = 0 \rightarrow \nabla\cdot\mathbf{B} = 0 \qquad\qquad \epsilon_0\nabla\cdot\mathbf{E} = \langle \eta(\mathbf{r}, t)\rangle$$

$$\Rightarrow \left\langle \nabla\times\mathbf{e} + \frac{\partial \mathbf{b}}{\partial t}\right\rangle = 0 \rightarrow \nabla\times\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad + \quad \nabla\times\frac{\mathbf{B}}{\mu_0} - \epsilon_0\frac{\partial \mathbf{E}}{\partial t} = \langle \mathbf{j}(\mathbf{r}, t)\rangle$$

homogeneous — inhomogeneous

- **D** and **H** are introduced by the extraction from η and $\boldsymbol{j}$ of certain contributions identified with the bulk properties of the medium.

- Consider a medium made up of molecules composed of nuclei and electrons and, in addition, "free" charges not localized around any particular molecule:

$$\eta_{\text{free}} = \sum_{j\,(\text{free})} q_j\, \delta(\mathbf{r} - \mathbf{r}_j), \quad \eta_{\text{bound}} = \sum_{n\,(\text{molecues})} \eta_n(\mathbf{r}, t) \quad \Leftarrow \quad \eta_n(\mathbf{r}, t) = \sum_{j\,(n)} q_j\, \delta(\mathbf{r} - \mathbf{r}_j)$$

$$\Rightarrow \quad \eta(\mathbf{r}, t) = \sum q_j\, \delta(\mathbf{r} - \mathbf{r}_j(t)) = \eta_{\text{free}} + \eta_{\text{bound}}$$

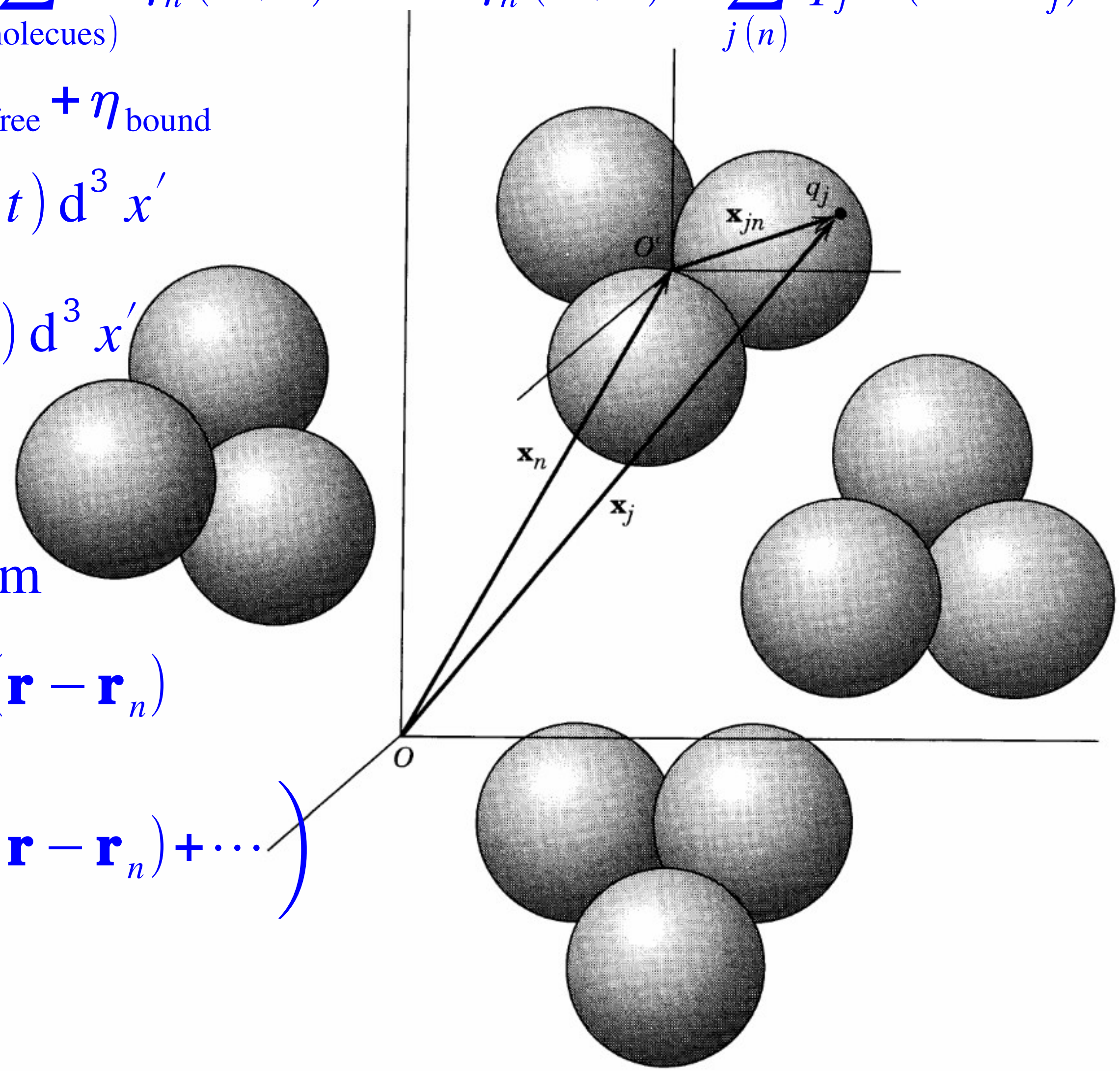
$$\Rightarrow \quad \langle \eta_n(\mathbf{r}, t) \rangle = \int f(\mathbf{r}')\, \eta_n(\mathbf{r} - \mathbf{r}', t)\, \mathrm{d}^3 x'$$

$$= \sum_{j\,(n)} q_j \int f(\mathbf{r}')\, \delta(\mathbf{r} - \mathbf{r}' - \mathbf{r}_{jn} - \mathbf{r}_n)\, \mathrm{d}^3 x'$$

$$= \sum_{j\,(n)} q_j\, f(\mathbf{r} - \mathbf{r}_{jn} - \mathbf{r}_n)$$

where $\mathbf{r}_{jn}$: order of atom dim

$$= \sum_{j\,(n)} q_j \left(f(\mathbf{r} - \mathbf{r}_n) - \mathbf{r}_{jn} \cdot \nabla f(\mathbf{r} - \mathbf{r}_n) + \frac{1}{2} \sum_{\alpha\beta} (\mathbf{r}_{jn})_\alpha (\mathbf{r}_{jn})_\beta \frac{\partial^2 f}{\partial x_\alpha \partial x_\beta}(\mathbf{r} - \mathbf{r}_n) + \cdots \right)$$



$$\Rightarrow \quad \textit{molecular multipole moments}: \begin{cases} \text{molecular charge} & q_n = \sum_{j(n)} q_j \\ \text{molecular dipole moment} & \mathbf{p}_n = \sum_{j(n)} q_j \mathbf{r}_{jn} \\ \text{molecular quadrupole moment:} & \mathbf{Q}'_n = 3 \sum_{j(n)} q_j \mathbf{r}_{jn} \otimes \mathbf{r}_{jn} \end{cases}$$

$$\Rightarrow \quad \langle \eta_n(\mathbf{r},t) \rangle = q_n f(\mathbf{r}-\mathbf{r}_n) - \mathbf{p}_n \cdot \nabla f(\mathbf{r}-\mathbf{r}_n) + \frac{1}{6} \sum_{\alpha\beta} (Q'_n)_{\alpha\beta} \frac{\partial^2 f}{\partial x_\alpha \partial x_\beta}(\mathbf{r}-\mathbf{r}_n) + \cdots$$

$$= \langle q_n \delta(\mathbf{r}-\mathbf{r}_n) \rangle - \nabla \cdot \langle \mathbf{p}_n \delta(\mathbf{r}-\mathbf{r}_n) \rangle + \frac{1}{6} \sum_{\alpha\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \langle (Q'_n)_{\alpha\beta} \delta(\mathbf{r}-\mathbf{r}_n) \rangle + \cdots$$

- *We can view the molecule as a collection of point multipoles.*

- The details of the molecular charge distribution is replaced in its effect by a sum of multipoles for macroscopic phenomena.

- Fourier transforms is an alternative approach to the spatial averaging

$$g(\mathbf{r},t) = \frac{1}{(2\pi)^3} \int \tilde{g}(\boldsymbol{k},t)\, e^{i\boldsymbol{k}\cdot\mathbf{r}}\, \mathrm{d}^3 k \quad \text{and} \quad \tilde{g}(\boldsymbol{k},t) = \int g(\mathbf{r},t)\, e^{-i\boldsymbol{k}\cdot\mathbf{r}}\, \mathrm{d}^3 x$$

$$\Rightarrow \quad \begin{matrix}\text{convolution} \\ (\text{faltung})\end{matrix} \quad f * F \equiv \int_0^\infty f(y)\, F(x-y)\, \mathrm{d}y \quad \Rightarrow \quad \begin{aligned} \mathscr{F}[f*F] &= \mathscr{F}[f]\,\mathscr{F}[F] \\ \mathscr{F}[f\,F] &= \frac{\mathscr{F}[f] * \mathscr{F}[F]}{2\pi} \end{aligned}$$

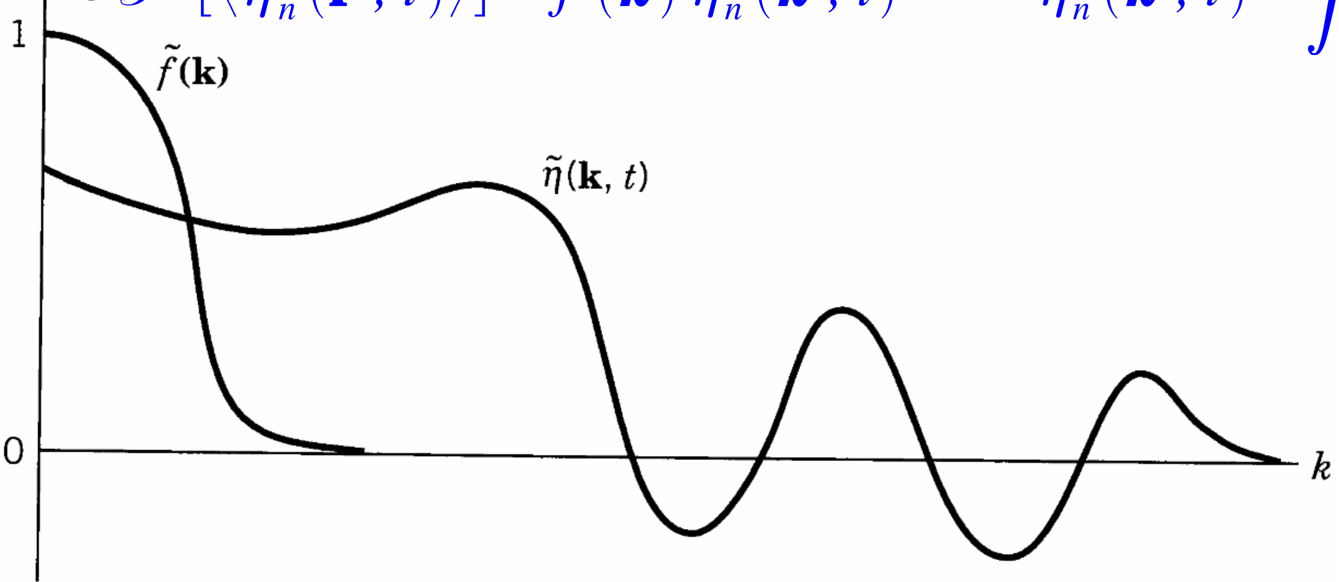
$$\Rightarrow \langle F(\mathbf{r},t)\rangle = \int \frac{\tilde{f}(\boldsymbol{k})\tilde{F}(\boldsymbol{k},t)}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\mathbf{r}} \mathrm{d}^3 k \quad \Rightarrow \quad \mathscr{F}[\langle F(\mathbf{r},t)\rangle] = \tilde{f}(\boldsymbol{k})\tilde{F}(\boldsymbol{k},t)$$

$$\int f(\mathbf{r})\,\mathrm{d}^3 x = 1 \quad \Rightarrow \quad \tilde{f}(0) = 1 = \int f(\mathbf{r})\, e^{-i\,0\cdot\mathbf{r}}\,\mathrm{d}^3 x$$

$$\Rightarrow$$

$$f = \frac{e^{-r^2/R^2}}{\sqrt{\pi^3}\,R^3} \quad \text{Gaussian test function} \quad \Rightarrow \quad \tilde{f}(\boldsymbol{k}) = \mathscr{F}[f(\mathbf{r})] = e^{-\frac{k^2 R^2}{4}}$$

- The Fourier transform of the averaged quantity contains only low wave numbers up to $k_{\max} = O\left(\frac{1}{R}\right)$, the inverse of the length scale of the averaging volume.
- $\tilde{f}(k \ll k_{\max}) \to 1 \quad \Rightarrow \quad \mathscr{F}[F(\mathbf{x},t)]$ gives a true representation of the long-wavelength aspects of $F(\mathbf{x}, t)$.
- $\mathscr{F}[\langle \eta_n(\mathbf{r},t)\rangle] = \tilde{f}(\boldsymbol{k})\,\tilde{\eta}_n(\boldsymbol{k},t) \quad \Leftarrow \quad \tilde{\eta}_n(\boldsymbol{k},t) = \int \eta_n(\mathbf{r}',t)\, e^{-i\boldsymbol{k}\cdot(\mathbf{r}'-\mathbf{r}_n)}\,\mathrm{d}^3 x'$



- The support for the product is confined to comparatively small wave numbers

$$\Rightarrow \quad \tilde{\eta}_n(\boldsymbol{k}, t) = \tilde{\eta}_n(0, t) + \boldsymbol{k}\cdot\nabla_k \tilde{\eta}_n(0, t) + \cdots$$

$$= \int \eta_n(\mathbf{r}', t)\left[1 - i\,\boldsymbol{k}\cdot(\mathbf{r}' - \mathbf{r}_n) + \cdots\right] e^{-i\,0\cdot(\mathbf{r}'-\mathbf{r}_n)}\, \mathrm{d}^3 x'$$

$$\Rightarrow \quad \tilde{\eta}_n(\boldsymbol{k}, t) \approx q_n - i\,\boldsymbol{k}\cdot\mathbf{p}_n + \text{quadrupole \& higher}$$

$$\Rightarrow \quad \langle \eta_n(\mathbf{r}, t)\rangle = \frac{1}{(2\pi)^3}\int \tilde{f}(\boldsymbol{k})\left[q_n - i\,\boldsymbol{k}\cdot\mathbf{p}_n + \cdots\right] e^{i\boldsymbol{k}\cdot(\mathbf{r}-\mathbf{r}_n)}\, \mathrm{d}^3 k$$

$$= q_n f(\mathbf{r}-\mathbf{r}_n) - \mathbf{p}_n\cdot\nabla f(\mathbf{r}-\mathbf{r}_n) + \cdots = \langle q_n \delta(\mathbf{r}-\mathbf{r}_n)\rangle - \nabla\cdot\langle \mathbf{p}_n \delta(\mathbf{r}-\mathbf{r}_n)\rangle + \cdots$$

- The Fourier transform has the advantage of giving a complementary view of the averaging as a cutoff in wave number space.

- $\langle \eta(\mathbf{r}, t)\rangle = \rho(\mathbf{r}, t) - \nabla\cdot\mathbf{P}(\mathbf{r}, t) + \nabla\cdot\left[\nabla\cdot\mathbf{Q}'(\mathbf{r}, t)\right] + \cdots$

$$\rho(\mathbf{r}, t) = \Big\langle \sum_{j\,(\text{free})} q_j \delta(\mathbf{r}-\mathbf{r}_j) + \sum_{n\,(\text{molecules})} q_n \delta(\mathbf{r}-\mathbf{r}_n)\Big\rangle \quad \Leftarrow \quad \text{macroscopic charge density}$$

where
$$\mathbf{P}(\mathbf{r}, t) = \Big\langle \sum_{n\,(\text{molecules})} \mathbf{p}_n \delta(\mathbf{r}-\mathbf{r}_n)\Big\rangle \quad \Leftarrow \quad \text{macroscopic polarization}$$

$$\mathbf{Q}'(\mathbf{r}, t) = \frac{1}{6}\Big\langle \sum_{n\,(\text{molecules})} \mathbf{Q}'_n \delta(\mathbf{r}-\mathbf{r}_n)\Big\rangle \quad \Leftarrow \quad \text{macroscopic quadrupole density}$$

$$\Rightarrow \quad \nabla\cdot(\epsilon_0 \mathbf{E} + \mathbf{P} - \nabla\cdot\mathbf{Q}') + \cdots = \rho \quad \Leftarrow \quad \epsilon_0 \nabla\cdot\mathbf{E} = \langle \eta(\mathbf{r}, t)\rangle$$

$$\Rightarrow \quad \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} - \nabla\cdot\mathbf{Q}' + \cdots \quad \Leftarrow \quad \text{macroscopic displacement vector}$$

$$\boldsymbol{j}(\mathbf{r},t)=\sum q_j\mathbf{v}_j\delta(\mathbf{r}-\mathbf{r}_j(t))=\boldsymbol{j}_{\text{free}}+\boldsymbol{j}_{\text{bound}} \quad\Rightarrow\quad \boldsymbol{j}_{\text{free}}=\sum_{j\,(\text{free})} q_j\mathbf{v}_j\delta(\mathbf{r}-\mathbf{r}_j)$$

$$\boldsymbol{j}_{\text{bound}}=\sum_{n\,(\text{molecues})}\boldsymbol{j}_n(\mathbf{r},t) \quad\Leftarrow\quad \boldsymbol{j}_n(\mathbf{r},t)=\sum_{j\,(n)} q_j\mathbf{v}_j\delta(\mathbf{r}-\mathbf{r}_j) \qquad \nwarrow \quad \mathbf{v}_j\equiv\frac{d\,\mathbf{r}_j}{d\,t}$$

$$\Rightarrow\quad \langle\boldsymbol{j}_n(\mathbf{r},t)\rangle=\sum_{j\,(n)} q_j(\mathbf{v}_{jn}+\mathbf{v}_n)\,f(\mathbf{r}-\mathbf{r}_n-\mathbf{r}_{jn}) \quad\Leftarrow\quad \mathbf{v}_n\equiv\frac{d\,\mathbf{r}_n}{d\,t},\quad \mathbf{v}_{jn}\equiv\frac{d\,\mathbf{r}_{jn}}{d\,t}$$

- The final result for the averaged microscopic current density is

$$\langle\boldsymbol{j}(\mathbf{r},t)\rangle=\boldsymbol{J}(\mathbf{r},t)+\frac{\partial}{\partial t}[\mathbf{D}(\mathbf{r},t)-\epsilon_0\mathbf{E}(\mathbf{r},t)]+\nabla\times\mathbf{M}(\mathbf{r},t)$$

$$+\sum_{\beta}\frac{\partial}{\partial x_\beta}\langle\sum_{n\,(\text{molecules})}[\mathbf{p}_n(\mathbf{v}_n)_\beta-(\mathbf{p}_n)_\beta\mathbf{v}_n]\,\delta(\mathbf{r}-\mathbf{r}_n)\rangle$$

$$-\frac{1}{6}\sum_{\alpha\beta\gamma}\frac{\partial^2}{\partial x_\beta\,\partial x_\gamma}\langle\sum_{n\,(\text{molecules})}[\hat{\mathbf{x}}_\alpha(Q_n')_{\alpha\beta}(\mathbf{v}_n)_\gamma-(Q_n')_{\gamma\beta}\mathbf{v}_n]\,\delta(\mathbf{r}-\mathbf{r}_n)\rangle+\cdots$$

$$\boldsymbol{J}(\mathbf{r},t)=\langle\sum_{j\,(\text{free})} q_j\mathbf{v}_j\delta(\mathbf{r}-\mathbf{r}_j)+\sum_{n\,(\text{molecules})} q_n\mathbf{v}_n\delta(\mathbf{r}-\mathbf{r}_n)\rangle \quad\Leftarrow\quad \text{macroscopic current density}$$

where $\mathbf{M}(\mathbf{r},t)=\langle\sum_{n\,(\text{molecules})}\mathbf{m}_n\delta(\mathbf{r}-\mathbf{r}_n)\rangle \quad\Leftarrow\quad$ macroscopic magnetization

where $\mathbf{m}_n=\sum_{j\,(n)}\frac{q_j}{2}\mathbf{r}_{jn}\times\mathbf{v}_{jn} \quad\Leftarrow\quad$ molecular magnetic moment

- $\nabla\times\mathbf{H}-\dfrac{\partial\,\mathbf{D}}{\partial\,t}=\boldsymbol{J}\,,\quad \dfrac{1}{\mu_0}\nabla\times\mathbf{B}-\epsilon_0\dfrac{\partial\,\mathbf{E}}{\partial\,t}=\langle\,\boldsymbol{j}\,\rangle$

$$\Rightarrow\quad \frac{\mathbf{B}}{\mu_0}-\mathbf{H}=\mathbf{M}+\langle\sum_{n\,(\text{molecules})}\mathbf{p}_n\times\mathbf{v}_n\,\delta\left(\mathbf{r}-\mathbf{r}_n\right)\rangle$$

$$-\frac{1}{6}\nabla\cdot\langle\sum_{n\,(\text{molecules})}\mathbf{Q}'_n\times\mathbf{v}_n\,\delta\left(\mathbf{r}-\mathbf{r}_n\right)\rangle+\cdots$$

- Except the 1st term of RHS, the other terms are small since $\mathbf{v}'$s are small.

- But if the medium as a whole has a translational velocity $\mathbf{v}$ $\quad \mathbf{v}_n=\mathbf{v}\quad\Rightarrow\quad\dfrac{\mathbf{B}}{\mu_0}-\mathbf{H}=\mathbf{M}+\left(\mathbf{D}-\epsilon_0\,\mathbf{E}\right)\times\mathbf{v}$

- For a medium in motion the electric polarization (and quadrupole density) enter the effective magnetization.

- $\mathbf{Q}\equiv\displaystyle\int\left(3\,\mathbf{r}\otimes\mathbf{r}-r^2\,\mathbb{I}\right)\mathrm{d}^3x$ traceless molecular quadrupole moment

$$\Rightarrow\quad \mathbf{Q}'_n=\mathbf{Q}_n+e\,r_n^2\,\mathbb{I}\quad\Leftarrow\quad e\,r_n^2=\sum_{j\,(n)}q_j\,x_{j\,n}^2$$

$$\Rightarrow\quad \mathbf{Q}'=\mathbf{Q}+\frac{1}{6}\langle\sum_{n\,(\text{molecule})}e\,r_n^2\,\mathbb{I}\,\delta\left(\mathbf{r}-\mathbf{r}_n\right)\rangle\quad\Leftarrow\quad\text{macroscopic quadrupole density}$$

$$\Rightarrow\quad \rho\to\rho_{\text{free}}+\langle\sum_{n\,(\text{molecule})}q_n\,\delta\left(\mathbf{r}-\mathbf{r}_n\right)\rangle+\frac{1}{6}\nabla^2\langle\sum_{n\,(\text{molecule})}e\,r_n^2\,\delta\left(\mathbf{r}-\mathbf{r}_n\right)\rangle\qquad(\ast\ast)$$

- The traceless quadrupole density replaces the quadrupole density and the charge density is augmented by an additional term.

- The trace of the tensor $\mathbf{Q}'$ is exhibited with the charge density because it is an $\ell=0$ contribution in terms of the multipole expansion.

- The molecular charge & mean square radius terms represent the first 2 terms in an expansion of the $\ell=0$ molecular multipole as we go beyond the static limit.

$$F\left(k^2\right) \equiv \int \rho\left(\mathbf{r}\right) \left\langle e^{i\boldsymbol{k}\cdot\mathbf{r}} \right\rangle_{\ell=0\text{ part}} \mathrm{d}^3x = \int \rho\left(\mathbf{r}\right) \mathrm{d}^3x - \frac{k^2}{6} \int r^2 \rho\left(\mathbf{r}\right) \mathrm{d}^3x + \cdots$$

$$= \int \rho\left(\mathbf{r}\right) \frac{\sin kr}{kr} \mathrm{d}^3x$$

$$\Rightarrow \quad \boldsymbol{k} \leftrightarrow -i\nabla \quad \Rightarrow \quad \left(\ast\ast\right)$$

Poynting's Theorem and Conservation of Energy & Momentum for a System of Charged Particles and Electromagnetic Fields

- For a single charge the rate of doing work by external EM fields is $q\,\mathbf{v}\cdot\mathbf{E}$. The magnetic field does no work, since the magnetic force is $\perp \mathbf{v}$.

- The total rate of doing work by the fields in a finite volume is $\int_{\mathcal{V}} \boldsymbol{J}\cdot\mathbf{E}\,\mathrm{d}^3x$

- It represents a conversion of EM energy into mechanical or thermal energy. It must be balanced by a corresponding rate of decrease of energy in the EM field.

- $$\int_{\mathcal{V}} \boldsymbol{J}\cdot\mathbf{E}\,\mathrm{d}^3x = \int_{\mathcal{V}} \left(\mathbf{E}\cdot\nabla\times\mathbf{H} - \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} \right) \mathrm{d}^3x \Leftarrow$$ Ampere-Maxwell law ⇙ Faraday's law

$$= -\int_{\mathcal{V}} \left(\nabla\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{E}\cdot\frac{\partial \mathbf{B}}{\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} \right) \mathrm{d}^3x \Leftarrow \nabla\cdot(\mathbf{E}\times\mathbf{H}) = \mathbf{H}\cdot\nabla\times\mathbf{E} - \mathbf{E}\cdot\nabla\times\mathbf{H}$$

$$\Rightarrow -\int_{\mathcal{V}} \boldsymbol{J}\cdot\mathbf{E}\,\mathrm{d}^3x = \int_{\mathcal{V}} \left(\frac{\partial u}{\partial t} + \nabla\cdot(\mathbf{E}\times\mathbf{H}) \right) \mathrm{d}^3x \Leftarrow$$ (1) the medium is linear $u \equiv \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$

$$\Rightarrow \frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = -\boldsymbol{J}\cdot\mathbf{E}$$ Poynting theorem $\Leftarrow$ $\mathbf{S} \equiv \mathbf{E}\times\mathbf{H}$ Poynting vector

- Since only its divergence appears in the conservation law, the Poynting vector seems arbitrary to the extent that the curl of any vector field can be added to it. Relativistic considerations show that it is unique.

- The time rate of change of EM energy in a volume, plus the energy flowing out through the boundary surfaces of the volume per unit time, is equal to the negative of the total work done by the fields on the sources within the volume.
- $\frac{dE_{\text{mech}}}{dt} = \int_{\mathcal{V}} \boldsymbol{J}\cdot\mathbf{E}\, d^3x, \quad E_{\text{field}} = \int_{\mathcal{V}} u\, d^3x = \frac{\epsilon_0}{2}\int_{\mathcal{V}} \left(E^2 + c^2 B^2\right) d^3x$

$$\Rightarrow \frac{dE}{dt} = \frac{dE_{\text{mech}}}{dt} + \frac{dE_{\text{field}}}{dt} = -\oint_{\mathcal{S}} \mathbf{S}\cdot d\boldsymbol{a} \quad \text{Poynting's theorem}$$

- The EM force on a charged particle $\mathbf{F} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)$

$$\Rightarrow \frac{d\mathbf{P}_{\text{mech}}}{dt} = \int_{\mathcal{V}} \left(\rho\mathbf{E} + \boldsymbol{J}\times\mathbf{B}\right) d^3x \quad + \quad \rho = \epsilon_0 \nabla\cdot\mathbf{E}, \quad \boldsymbol{J} = \frac{\nabla\times\mathbf{B}}{\mu_0} - \epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$$\Rightarrow \rho\mathbf{E} + \boldsymbol{J}\times\mathbf{B} = \epsilon_0\left(\left(\nabla\cdot\mathbf{E}\right)\mathbf{E} + \mathbf{B}\times\frac{\partial\mathbf{E}}{\partial t} - c^2\mathbf{B}\times\left(\nabla\times\mathbf{B}\right)\right)$$

$$= \epsilon_0\left(\mathbf{E}\left(\nabla\cdot\mathbf{E}\right) + c^2\mathbf{B}\left(\nabla\cdot\mathbf{B}\right) - \mathbf{E}\times\left(\nabla\times\mathbf{E}\right) - c^2\mathbf{B}\times\left(\nabla\times\mathbf{B}\right) - \frac{\partial}{\partial t}\left(\mathbf{E}\times\mathbf{B}\right)\right)$$

$$\Rightarrow \mathbf{F}_{\text{total}} = \frac{d\mathbf{P}_{\text{mech}}}{dt} + \frac{d\mathbf{P}_{\text{field}}}{dt} \quad \Leftarrow \quad \mathbf{P}_{\text{field}} \equiv \epsilon_0\int_{\mathcal{V}} \mathbf{E}\times\mathbf{B}\, d^3x = \epsilon_0\mu_0\int_{\mathcal{V}} \mathbf{E}\times\mathbf{H}\, d^3x$$

$$= \epsilon_0\int_{\mathcal{V}} \left[\mathbf{E}\left(\nabla\cdot\mathbf{E}\right) + c^2\mathbf{B}\left(\nabla\cdot\mathbf{B}\right) - \mathbf{E}\times\left(\nabla\times\mathbf{E}\right) - c^2\mathbf{B}\times\left(\nabla\times\mathbf{B}\right)\right] d^3x$$

- The density of EM momentum $\boldsymbol{g} \equiv \frac{\mathbf{E}\times\mathbf{H}}{c^2} = \epsilon_0\,\mu_0\,\mathbf{E}\times\mathbf{H} = \frac{\mathbf{S}}{c^2}$

$\boldsymbol{g}$ is the EM momentum associated with the fields.

- $[\mathbf{E}\,\nabla\cdot\mathbf{E} - \mathbf{E}\times(\nabla\times\mathbf{E})]_1 = E_1\left(\frac{\partial E_1}{\partial x_1}+\frac{\partial E_2}{\partial x_2}+\frac{\partial E_3}{\partial x_3}\right) - E_2\left(\frac{\partial E_2}{\partial x_1}-\frac{\partial E_1}{\partial x_2}\right)$

$$+E_3\left(\frac{\partial E_1}{\partial x_3}-\frac{\partial E_3}{\partial x_1}\right) = \frac{\partial E_1^2}{\partial x_1}+\frac{\partial}{\partial x_2}(E_1\,E_2)+\frac{\partial}{\partial x_3}(E_1\,E_3)-\frac{1}{2}\frac{\partial E^2}{\partial x_1}$$

$$\Rightarrow\quad \mathbf{E}(\nabla\cdot\mathbf{E})-\mathbf{E}\times(\nabla\times\mathbf{E}) = \sum_{\alpha\,\beta}\hat{\mathbf{x}}^{\alpha}\frac{\partial}{\partial x_\beta}\left(E_\alpha\,E_\beta-\frac{E^2}{2}\delta_{\alpha\beta}\right)$$

$$\Rightarrow\quad \mathbb{T}\equiv\epsilon_0\left(\mathbf{E}\otimes\mathbf{E}+c^2\,\mathbf{B}\otimes\mathbf{B}-\frac{E^2+c^2\,B^2}{2}\,\mathbb{I}\right)$$

$\Uparrow\quad E^2 = E_1^2+E_2^2+E_3^2$

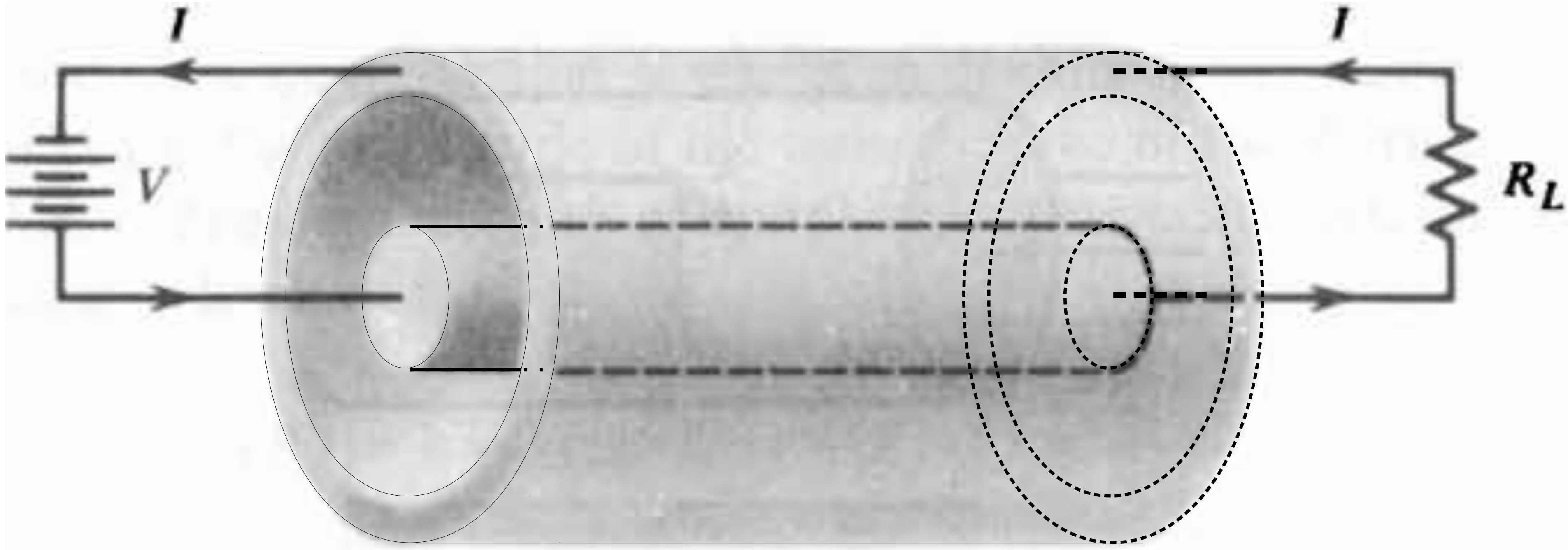
$\nwarrow$ Maxwell stress tensor

$$\Rightarrow\quad \mathbf{F}_{\text{total}} = \frac{\mathrm{d}}{\mathrm{d}\,t}(\mathbf{P}_{\text{mech}}+\mathbf{P}_{\text{field}}) = \int_{\mathcal{V}}\nabla\cdot\mathbb{T}\,\mathrm{d}^3x = \oint_{\mathcal{S}}\mathbb{T}\cdot\mathrm{d}\,\boldsymbol{a}$$

$\Leftarrow$ conservation of linear momentum

- $(\mathbb{T}\cdot\mathrm{d}\,\boldsymbol{a})_\alpha = \sum\mathbb{T}_{\alpha\beta}\,\mathrm{d}\,a_\beta$ is the α^{th} component of the flow of momentum across the surface into the volume, ie, it is the force transmitted across the surface to act on the combined system of particles and fields inside the volume.

- The conservation of angular momentum of the system of particles & fields can be treated in the same way as we have handled energy & linear momentum.



<u>Example</u>: Consider that a coaxial cable, of radii *a* (inner) & *b* (outer), is inserted between a source of constant emf and some load, a steady current *I* flows down the cable. If the emf provides a constant potential difference *V*, it will supply power to the cable of magnitude *VI*. Calculate the rate at which energy passes down the cable.

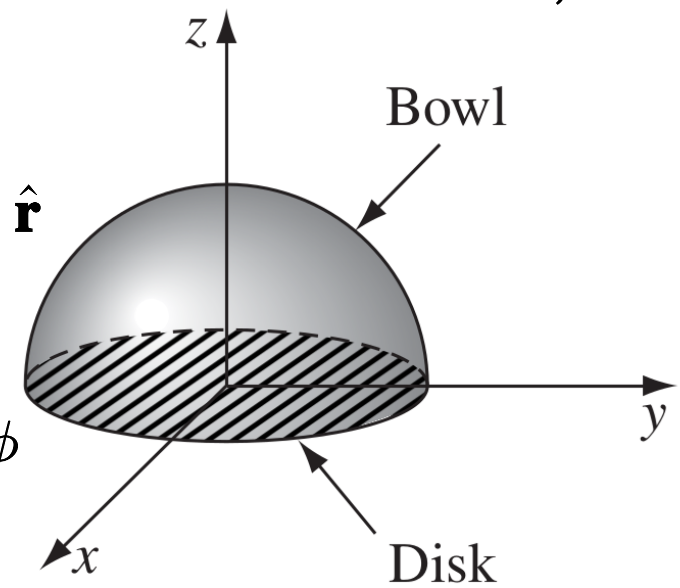
$$\mathbf{E}=\frac{V}{\ln(b/a)}\frac{\hat{\mathbf{s}}}{s}, \quad \mathbf{B}=\frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}, \quad a\le s\le b \quad \Rightarrow \quad \mathbf{S}=\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}=\frac{V I}{2\pi\ln(b/a)}\frac{\hat{\mathbf{z}}}{s^2}$$

$$\Rightarrow \int_{\mathcal{S}} \mathbf{S}\cdot \mathrm{d}\boldsymbol{a} = \int_0^{2\pi}\mathrm{d}\phi\int_a^b \frac{V I}{2\pi\ln(b/a)}\frac{s\,\mathrm{d}s}{s^2} = V I$$

- In practice, the conductors of the cable will have a finite resistance, so that energy will also be dissipated as heat in them.

Example: Determine the net force on the “northern” hemisphere of a uniformly charged solid sphere of radius R and charge Q.

- The boundary surface consists of 2 parts—a hemispherical bowl at radius R, and a circular disk at $\theta = \dfrac{\pi}{2}$.



- For the bowl, $\mathrm{d}\,\boldsymbol{a} = R^2 \sin\theta\,\mathrm{d}\,\phi\,\mathrm{d}\,\theta\,\hat{\mathbf{r}}$, $\mathbf{E} = \dfrac{Q}{4\,\pi\,\epsilon_0\,R^2}\,\hat{\mathbf{r}}$
- $\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$

$$\mathbb{T}_{z\,x} = \epsilon_0\,E_z\,E_x = \epsilon_0\left(\frac{Q}{4\,\pi\,\epsilon_0\,R^2}\right)^2 \sin\theta\cos\theta\cos\phi$$

$$\Rightarrow\quad \mathbb{T}_{z\,y} = \epsilon_0\,E_z\,E_y = \epsilon_0\left(\frac{Q}{4\,\pi\,\epsilon_0\,R^2}\right)^2 \sin\theta\cos\theta\sin\phi$$

$$\mathbb{T}_{z\,z} = \frac{\epsilon_0}{2}\left(E_z^2 - E_x^2 - E_y^2\right) = \frac{\epsilon_0}{2}\left(\frac{Q}{4\,\pi\,\epsilon_0\,R^2}\right)^2 \left(\cos^2\theta - \sin^2\theta\right)$$

$$\Rightarrow\quad \left(\mathbb{T}\cdot\mathrm{d}\,\boldsymbol{a}\right)_z = \mathbb{T}_{z\,x}\,\mathrm{d}\,a_x + \mathbb{T}_{z\,y}\,\mathrm{d}\,a_y + \mathbb{T}_{z\,z}\,\mathrm{d}\,a_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\,\pi\,\epsilon_0\,R}\right)^2 \sin\theta\cos\theta\,\mathrm{d}\,\phi\,\mathrm{d}\,\theta$$

$$\Rightarrow\quad F_{\text{bowl}} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\,\pi\,\epsilon_0\,R}\right)^2 2\,\pi \int_0^{\pi/2} \sin\theta\cos\theta\,\mathrm{d}\,\theta = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{Q^2}{8\,R^2}$$

- For the equatorial disk, *inside* the sphere, $\mathbf{r} = r\,\hat{\mathbf{r}}$

$$\theta = \frac{\pi}{2} \;\Rightarrow\; \mathrm{d}\boldsymbol{a} = -r\,\mathrm{d}\phi\,\mathrm{d}r\,\hat{\mathbf{z}} \;\Rightarrow\; \mathbf{E} = \frac{Q}{4\pi\epsilon_0 R^3}\mathbf{r} = \frac{Q\,r}{4\pi\epsilon_0 R^3}(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}})$$

$$\Rightarrow\; \mathbb{T}_{zz} = -\frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 r^2 \;\Rightarrow\; (\mathbb{T}\cdot\mathrm{d}\boldsymbol{a})_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 r^3\,\mathrm{d}r\,\mathrm{d}\phi$$

$$\Rightarrow\; F_{\text{disk}} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 2\pi\int_0^R r^3\,\mathrm{d}r = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{16R^2} \;\Rightarrow\; F = \frac{1}{4\pi\epsilon_0}\frac{3Q^2}{16R^2}$$

repulsive

- In applying the conservation formula, *any* volume that encloses all of the charge in question (and no *other* charge) will do the job.

- In this case we could use the whole region $z>0$. Then the boundary surface consists of the entire xy plane (+ a hemisphere at $r=\infty \Rightarrow E=0 \Rightarrow F_{\text{bowl}}=0$ there).

- We now have the outer portion of the plane ($r>R$), $\mathrm{d}\boldsymbol{a} = -r\,\mathrm{d}\phi\,\mathrm{d}r\,\hat{\mathbf{z}}$

$$\mathbb{T}_{zz} = -\frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 \frac{1}{r^4} \;\Rightarrow\; (\mathbb{T}\cdot\mathrm{d}\boldsymbol{a})_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 \frac{1}{r^3}\,\mathrm{d}r\,\mathrm{d}\phi$$

$$\Rightarrow\; F_{r>R} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 2\pi\int_R^\infty \frac{\mathrm{d}r}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{8R^2} \;\Rightarrow\; F = \frac{1}{4\pi\epsilon_0}\frac{3Q^2}{16R^2}$$

the same

Poynting's Theorem in Linear Dispersive Media with Losses

- Poynting's theorem was derived with linear media with no dispersion or losses, with ϵ and μ real and frequency independent. Actual materials exhibit dispersion and losses.

$$\mathbf{E}(\mathbf{r},t)=\int_{-\infty}^{+\infty}\mathbf{E}(\mathbf{r},\omega)\,e^{-i\omega t}\,\mathrm{d}\omega$$

$$\mathbf{D}(\mathbf{r},t)=\int_{-\infty}^{+\infty}\mathbf{D}(\mathbf{r},\omega)\,e^{-i\omega t}\,\mathrm{d}\omega$$

with the linearity

$$\mathbf{D}(\mathbf{r},\omega)=\epsilon(\omega)\,\mathbf{E}(\mathbf{r},\omega)$$

$$\mathbf{B}(\mathbf{r},\omega)=\mu(\omega)\,\mathbf{H}(\mathbf{r},\omega)$$

reality of field $\Rightarrow$ $\mathbf{E}(\mathbf{r},-\omega)=\mathbf{E}^*(\mathbf{r},\omega)$, $\mathbf{D}(\mathbf{r},-\omega)=\mathbf{D}^*(\mathbf{r},\omega)$, $\quad \epsilon(-\omega)=\epsilon^*(\omega) \quad \Leftarrow \quad \mathbf{E}\cdot\dfrac{\partial\mathbf{D}}{\partial t}\neq\dfrac{\partial}{\partial t}\dfrac{\mathbf{E}\cdot\mathbf{D}}{2}$

for $\mathbf{E}^*(\mathbf{r},t)=\displaystyle\int_{-\infty}^{+\infty}\mathbf{E}^*(\mathbf{r},\omega)\,e^{i\omega t}\,\mathrm{d}\omega=\int_{-\infty}^{+\infty}\mathbf{E}(\mathbf{r},-\omega)\,e^{-i(-\omega)t}\,\mathrm{d}(-\omega)$

$$=\int_{-\infty}^{+\infty}\mathbf{E}(\mathbf{r},\omega')\,e^{-i\omega' t}\,\mathrm{d}\omega'=\mathbf{E}(\mathbf{r},t) \quad \Leftarrow \quad \omega'\equiv-\omega,$$

$\mathbf{D}$ is similar.

$$\Rightarrow \quad \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}=\iint\mathbf{E}^*(\omega')\cdot\mathbf{E}(\omega)\left[-i\,\omega\,\epsilon(\omega)\right]e^{-i(\omega-\omega')t}\,\mathrm{d}\omega'\,\mathrm{d}\omega \qquad ⇦ \quad \omega'\leftrightarrow-\omega$$

$$=\frac{1}{2}\iint\mathbf{E}^*(\omega')\cdot\mathbf{E}(\omega)\left[i\,\omega'\,\epsilon^*(\omega')-i\,\omega\,\epsilon(\omega)\right]e^{-i(\omega-\omega')t}\,\mathrm{d}\omega'\,\mathrm{d}\omega$$

Assume $\mathbf{E}$ is dominated by frequency around $\omega'\approx\omega \quad \Rightarrow \quad \omega'=\omega+\delta\omega$

$$\epsilon = \Re[\epsilon] + i\,\Im[\epsilon], \quad \epsilon^* = \Re[\epsilon] - i\,\Im[\epsilon] \;\Rightarrow\; 2\,\Re[\epsilon] = \epsilon + \epsilon^*, \quad 2\,i\,\Im[\epsilon] = \epsilon - \epsilon^*$$

$$\begin{aligned}
\Rightarrow\quad i\,\omega'\,\epsilon^*(\omega') - i\,\omega\,\epsilon(\omega) &= i\,[(\omega + \delta\omega)\,\epsilon^*(\omega + \delta\omega) - i\,\omega\,\epsilon(\omega)] \\
&= i\,\omega\,[\epsilon^*(\omega) - \epsilon(\omega)] + i\,\delta\omega\,\epsilon^*(\omega) + i\,\omega\,\delta\omega\,\frac{\mathrm{d}\,\epsilon^*}{\mathrm{d}\,\omega} + \cdots \\
&= 2\,\omega\,\Im[\epsilon] + i\,\delta\omega\,\frac{\mathrm{d}}{\mathrm{d}\,\omega}(\omega\,\epsilon^*) + \cdots \\
&= 2\,\omega\,\Im[\epsilon(\omega)] - i\,(\omega - \omega')\,\frac{\mathrm{d}}{\mathrm{d}\,\omega}[\omega\,\epsilon^*(\omega)] + \cdots
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad \mathbf{E}\cdot\frac{\partial\,\mathbf{D}}{\partial\,t} &\approx \iint \mathbf{E}^*(\omega')\cdot\mathbf{E}(\omega)\,\omega\,\Im[\epsilon(\omega)]\,e^{-i(\omega-\omega')t}\,\mathrm{d}\,\omega'\,\mathrm{d}\,\omega \\
&\quad + \frac{1}{2}\,\frac{\partial}{\partial\,t}\iint \mathbf{E}^*(\omega')\cdot\mathbf{E}(\omega)\,\frac{\mathrm{d}}{\mathrm{d}\,\omega}[\omega\,\epsilon^*(\omega)]\,e^{-i(\omega-\omega')t}\,\mathrm{d}\,\omega'\,\mathrm{d}\,\omega \qquad (\#)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad \mathbf{H}\cdot\frac{\partial\,\mathbf{B}}{\partial\,t} &\approx \iint \mathbf{H}^*(\omega')\cdot\mathbf{H}(\omega)\,\omega\,\Im[\mu(\omega)]\,e^{-i(\omega-\omega')t}\,\mathrm{d}\,\omega'\,\mathrm{d}\,\omega \\
&\quad + \frac{1}{2}\,\frac{\partial}{\partial\,t}\iint \mathbf{H}^*(\omega')\cdot\mathbf{H}(\omega)\,\frac{\mathrm{d}}{\mathrm{d}\,\omega}[\omega\,\mu^*(\omega)]\,e^{-i(\omega-\omega')t}\,\mathrm{d}\,\omega'\,\mathrm{d}\,\omega \qquad (@)
\end{aligned}$$

- If ϵ and μ are real and frequency independent we go back the the last section.

$$\mathbf{E}\cdot\frac{\partial\,\mathbf{D}}{\partial\,t}=\iint\mathbf{E}^{*}\left(\omega'\right)\cdot\mathbf{E}\left(\omega\right)\omega\left(\Im\left[\epsilon\left(\omega\right)\right]-i\,\Re\left[\epsilon\left(\omega\right)\right]\right)e^{-i\left(\omega-\omega'\right)t}\,\mathrm{d}\,\omega'\,\mathrm{d}\,\omega$$

$$\int\mathbf{E}^{*}\left(\omega'\right)e^{i\,\omega'\,t}\,\mathrm{d}\,\omega'=\int\mathbf{E}\left(\phi\right)e^{-i\,\phi\,t}\,\mathrm{d}\,\phi$$

$$\int\mathbf{E}\left(\omega\right)\left[-i\,\omega\,\epsilon\left(\omega\right)\right]e^{-i\,\omega\,t}\,\mathrm{d}\,\omega=\int\mathbf{E}\left(\omega\right)\left[-i\,\omega\,\epsilon^{\star}\left(-\omega\right)\right]e^{-i\,\omega\,t}\,\mathrm{d}\,\omega$$

$$=\int\mathbf{E}\left(-\phi'\right)i\,\phi'\,\epsilon^{\star}\left(\phi'\right)e^{i\,\phi'\,t}\,\mathrm{d}\,\phi'=\int\mathbf{E}^{\star}\left(\phi'\right)i\,\phi'\,\epsilon^{\star}\left(\phi'\right)e^{i\,\phi'\,t}\,\mathrm{d}\,\phi'$$

$$\Rightarrow\quad\iint\mathbf{E}^{*}\left(\omega'\right)\cdot\mathbf{E}\left(\omega\right)\left[-i\,\omega\,\epsilon\left(\omega\right)\right]e^{-i\left(\omega-\omega'\right)t}\,\mathrm{d}\,\omega'\,\mathrm{d}\,\omega$$

$$=\iint\mathbf{E}\left(\phi\right)\cdot\mathbf{E}^{\star}\left(\phi'\right)i\,\phi'\,\epsilon^{\star}\left(\phi'\right)e^{-i\left(\phi-\phi'\right)t}\,\mathrm{d}\,\phi'\,\mathrm{d}\,\phi\quad\Rightarrow\quad\phi\,,\ \phi'\rightarrow\omega\,,\ \omega'$$

$$\Rightarrow\quad\iint\mathbf{E}^{*}\left(\omega'\right)\cdot\mathbf{E}\left(\omega\right)\left(-i\,\omega\,\Re\left[\epsilon\left(\omega\right)\right]\right)e^{-i\left(\omega-\omega'\right)t}\,\mathrm{d}\,\omega'\,\mathrm{d}\,\omega$$

$$=\frac{i}{2}\iint\mathbf{E}^{*}\left(\omega'\right)\cdot\mathbf{E}\left(\omega\right)\left[\omega'\,\epsilon^{\star}\left(\omega'\right)-\omega\,\epsilon^{\star}\left(\omega\right)\right]e^{-i\left(\omega-\omega'\right)t}\,\mathrm{d}\,\omega'\,\mathrm{d}\,\omega$$

$$=\frac{1}{2}\,\frac{\partial}{\partial\,t}\iint\mathbf{E}^{\star}\left(\omega'\right)\cdot\mathbf{E}\left(\omega\right)\frac{\mathrm{d}}{\mathrm{d}\,\omega}\left[\omega\,\epsilon^{\star}\left(\omega\right)\right]e^{-i\left(\omega-\omega'\right)t}\,\mathrm{d}\,\omega'\,\mathrm{d}\,\omega$$

since $\omega'\,\epsilon^{*}\left(\omega'\right)-\omega\,\epsilon^{*}\left(\omega\right)=\left(\omega+\delta\,\omega\right)\epsilon^{*}\left(\omega+\delta\,\omega\right)-\omega\,\epsilon^{*}\left(\omega\right)\approx\delta\,\omega\,\frac{\mathrm{d}\,\omega\,\epsilon^{\star}}{\mathrm{d}\,\omega}$

- The 1st terms in (#) & (@) evidently represent the conversion of electrical energy into heat, while the 2nd terms must be an effective energy density.

- Let $\begin{matrix} \mathbf{E} = \widetilde{\mathbf{E}}(t)\cos(\omega_0 t + \alpha) \\ \mathbf{H} = \widetilde{\mathbf{H}}(t)\cos(\omega_0 t + \beta) \end{matrix}$, and $\begin{matrix} \widetilde{\mathbf{E}}(t) \\ \widetilde{\mathbf{H}}(t) \end{matrix}$ slowly varies relative to $\frac{1}{\omega_0}$

$$\Rightarrow \quad \left\langle \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} \right\rangle = \omega_0 \Big(\Im[\epsilon(\omega_0)]\langle \mathbf{E}^2(\mathbf{r},t)\rangle + \Im[\mu(\omega_0)]\langle \mathbf{H}^2(\mathbf{r},t)\rangle \Big) + \frac{\partial u_{\text{eff}}}{\partial t}$$

$$\text{where} \quad u_{\text{eff}} = \frac{1}{2}\Re\left[\frac{d\,\omega\,\epsilon}{d\,\omega}(\omega_0)\right]\langle \mathbf{E}^2(\mathbf{r},t)\rangle + \frac{1}{2}\Re\left[\frac{d\,\omega\,\mu}{d\,\omega}(\omega_0)\right]\langle \mathbf{H}^2(\mathbf{r},t)\rangle$$

- Poynting's theorem in these circumstances reads

$$\frac{\partial u_{\text{eff}}}{\partial t} + \nabla\cdot\mathbf{S} = -\boldsymbol{J}\cdot\mathbf{E} - \omega_0 \Big(\Im[\epsilon(\omega_0)]\langle \mathbf{E}^2(\mathbf{r},t)\rangle + \Im[\mu(\omega_0)]\langle \mathbf{H}^2(\mathbf{r},t)\rangle \Big)$$

- The 1st term in RHS describes the explicit ohmic losses, while the next terms represent the absorptive dissipation in the medium, not counting conduction loss.

- It shows in realistic situations where, as energy flow out of the locality, there may be losses from heating of the medium, leading to a slow decay of the energy.

Poynting's Theorem for Harmonic Fields; Field Definitions of Impedance and Admittance

- Assume that all fields and sources have a time dependence $e^{-i\omega t}$

$$\mathbf{E}(\mathbf{r},t) = \Re\left[\mathbf{E}(\mathbf{r})e^{-i\omega t}\right] \equiv \frac{\mathbf{E}(\mathbf{r})e^{-i\omega t} + \mathbf{E}^*(\mathbf{r})e^{i\omega t}}{2}$$

$$\Rightarrow \quad \boldsymbol{J}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t) = \frac{\boldsymbol{J}(\mathbf{r})e^{-i\omega t} + \boldsymbol{J}^*(\mathbf{r})e^{i\omega t}}{2}\cdot\frac{\mathbf{E}(\mathbf{r})e^{-i\omega t} + \mathbf{E}^*(\mathbf{r})e^{i\omega t}}{2}$$

$$= \frac{1}{2}\Re\left[\boldsymbol{J}^*(\mathbf{r})\cdot\mathbf{E}(\mathbf{r}) + \boldsymbol{J}(\mathbf{r})\cdot\mathbf{E}(\mathbf{r})e^{-2i\omega t}\right]$$

- For time averages of products, the convention is to take ½ of the real part of the product of one complex quantity with the complex conjugate of the other.

- For harmonic fields the Maxwell equations

$$\nabla\cdot\mathbf{B} = 0\,, \quad \nabla\times\mathbf{E} - i\omega\mathbf{B} = 0$$
$$\nabla\cdot\mathbf{D} = \rho\,, \quad \nabla\times\mathbf{H} + i\omega\mathbf{D} = \boldsymbol{J}$$

- The time-averaged rate of work done by the fields

$$\left\langle\frac{\mathrm{d}W}{\mathrm{d}t}\right\rangle = \Re\left[\int_{\mathcal{V}}\frac{\boldsymbol{J}^*\cdot\mathbf{E}}{2}\,\mathrm{d}^3x\right]$$

$$\int_{\mathcal{V}}\boldsymbol{J}^*\cdot\mathbf{E}\,\mathrm{d}^3x = \int_{\mathcal{V}}\mathbf{E}\cdot(\nabla\times\mathbf{H}^* - i\omega\mathbf{D}^*)\,\mathrm{d}^3x \quad \Leftarrow \quad \nabla\cdot(\mathbf{E}\times\mathbf{H}^*) = \mathbf{H}^*\cdot\nabla\times\mathbf{E} - \mathbf{E}\cdot\nabla\times\mathbf{H}^*$$

$$= \int_{\mathcal{V}}\left[i\omega(\mathbf{B}\cdot\mathbf{H}^* - \mathbf{E}\cdot\mathbf{D}^*) - \nabla\cdot(\mathbf{E}\times\mathbf{H}^*)\right]\mathrm{d}^3x \quad \Leftarrow \quad \nabla\times\mathbf{E} = i\omega\mathbf{B}$$

$$\mathbf{S} \equiv \frac{\mathbf{E}\times\mathbf{H}^*}{2}, \quad w_e = \frac{\mathbf{E}\cdot\mathbf{D}^*}{4} \quad \text{electric energy density}, \quad w_m = \frac{\mathbf{B}\cdot\mathbf{H}^*}{4} \quad \text{magnetic energy density}$$

$$\Rightarrow \frac{1}{2}\int_{\mathcal{V}} \boldsymbol{J}^*\cdot\mathbf{E}\,\mathrm{d}^3x + 2i\omega\int_{\mathcal{V}}(w_e - w_m)\,\mathrm{d}^3x + \oint_{\mathcal{S}} \mathbf{S}\cdot\mathrm{d}\boldsymbol{a} = 0 \quad \text{Poynting theorem}$$

- The real part of the equation gives the conservation of energy for the time-averaged quantities and the imaginary part relates to the reactive or stored energy and its alternating flow.

- If $\int_{\mathcal{V}}(w_e - w_m)\,\mathrm{d}^3x$ is real $\Rightarrow \frac{1}{2}\int_{\mathcal{V}} \Re[\boldsymbol{J}^*\cdot\mathbf{E}]\,\mathrm{d}^3x = -\oint_{\mathcal{S}} \Re[\mathbf{S}\cdot\mathrm{d}\boldsymbol{a}]$

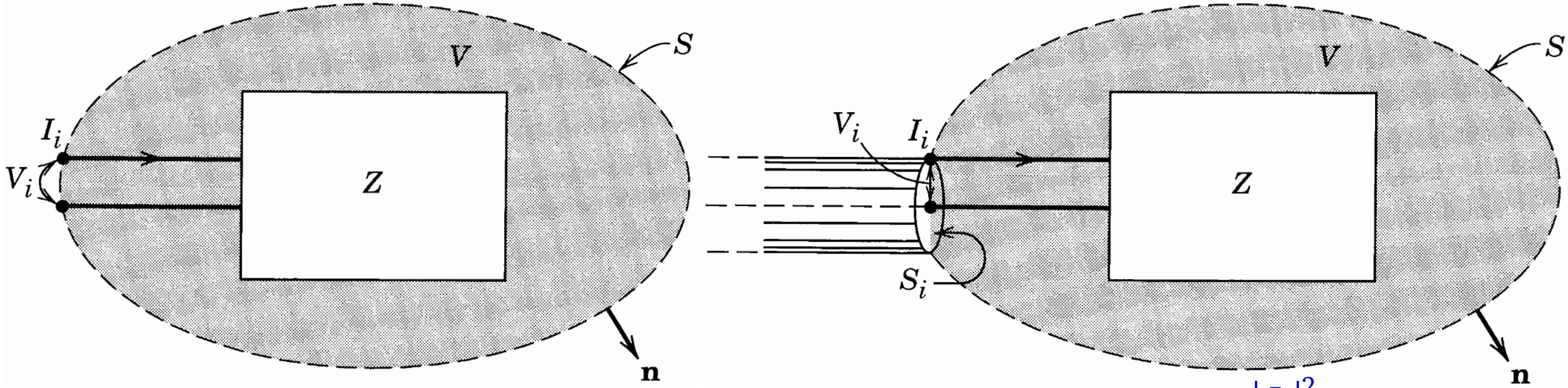
with lossless dielectrics and perfect conductors, the steady-state, time-averaged rate of doing work on the sources by the fields is equal to the average flow of power into the volume through the boundary surface.

- The complex power input

$$\frac{1}{2}I_i^* V_i = -\oint_{\mathcal{S}_i} \mathbf{S}\cdot\mathrm{d}\boldsymbol{a} = \frac{1}{2}\int_{\mathcal{V}} \boldsymbol{J}^*\cdot\mathbf{E}\,\mathrm{d}^3x + 2i\omega\int_{\mathcal{V}}(w_e - w_m)\,\mathrm{d}^3x + \oint_{\mathcal{S}-\mathcal{S}_i} \mathbf{S}\cdot\mathrm{d}\boldsymbol{a}$$

- If $\mathcal{S}-\mathcal{S}_i$ is taken to ∞, the surface integral is real and represents escaping radiation.

- At low frequencies the radiation is generally negligible. No distinction need be made between $\mathcal{S}_i$ and $\mathcal{S}$.



- The input impedance $Z = R - i\,X \;\Rightarrow\; V_i = Z\,I_i \;\Rightarrow\; \frac{1}{2} I_i^{\star} V_i = \frac{|I_i|^2}{2}(R - i\,X)$
- Assume the power flow out through $\mathcal{S}$ is *real*

$$\Rightarrow \begin{array}{l} R = \frac{1}{|I_i|^2}\left(\Re\left[\int_{\mathcal{V}} \boldsymbol{J}^* \cdot \mathbf{E}\, \mathrm{d}^3 x\right] + 2\oint_{\mathcal{S}-\mathcal{S}_i} \mathbf{S}\cdot \mathrm{d}\,\boldsymbol{a} + 4\,\omega\,\Im\left[\int_{\mathcal{V}} (w_m - w_e)\,\mathrm{d}^3 x\right]\right) \\ X = \frac{1}{|I_i|^2}\left(4\,\omega\,\Re\left[\int_{\mathcal{V}} (w_m - w_e)\,\mathrm{d}^3 x\right] - \Im\left[\int_{\mathcal{V}} \boldsymbol{J}^* \cdot \mathbf{E}\, \mathrm{d}^3 x\right]\right) \end{array}$$

- $\frac{2}{|I_i|^2}\oint_{\mathcal{S}-\mathcal{S}_i} \mathbf{S}\cdot \mathrm{d}\,\boldsymbol{a}$ is the "radiation resistance," important at high frequencies.
- At low frequencies, ohmic losses are the only appreciable source of dissipation.

$$\Rightarrow \quad R \simeq \frac{1}{|I_i|^2}\int_{\mathcal{V}} \sigma\,|\mathbf{E}|^2\,\mathrm{d}^3 x\,, \quad X \simeq \frac{4\,\omega}{|I_i|^2}\int_{\mathcal{V}} (w_m - w_e)\,\mathrm{d}^3 x$$

here σ is the real conductivity, the energy densities are also real essentially.

- The different frequency dependences of the low-frequency reactance for inductances ($X = \omega L$) and capacitances ($X = -\frac{1}{\omega C}$) can be traced to the definition of L in terms of current and voltage ($V = L \frac{\mathrm{d} I}{\mathrm{d} t}$) and of C in terms of charge and voltage ($V = \frac{Q}{C}$).

Transformation Properties of Electromagnetic Fields and Sources Under Rotations, Spatial Reflections, and Time Reversal

A. Rotations

- A rotation in 3d is a linear transformation of the coordinates of a point and the sum of the coordinates' squares remains invariant — orthogonal transformation.

- $x'_\alpha = \sum_\beta a_{\alpha\beta} x_\beta$ vector $\Rightarrow \sum_\alpha a_{\alpha\beta} a_{\alpha\gamma} = \delta_{\beta\gamma} \Leftarrow \mathbf{r}'^2 = \mathbf{r}^2 \Rightarrow$ $\boldsymbol{a}^{-1} = \boldsymbol{a}^{\mathrm{T}}$, $a^2 = 1 \Leftarrow a \equiv \det \boldsymbol{a}$

$\Rightarrow$ $a = +1$: proper rotation, obtainable from the original configuration by a sequence of infinitesimal steps

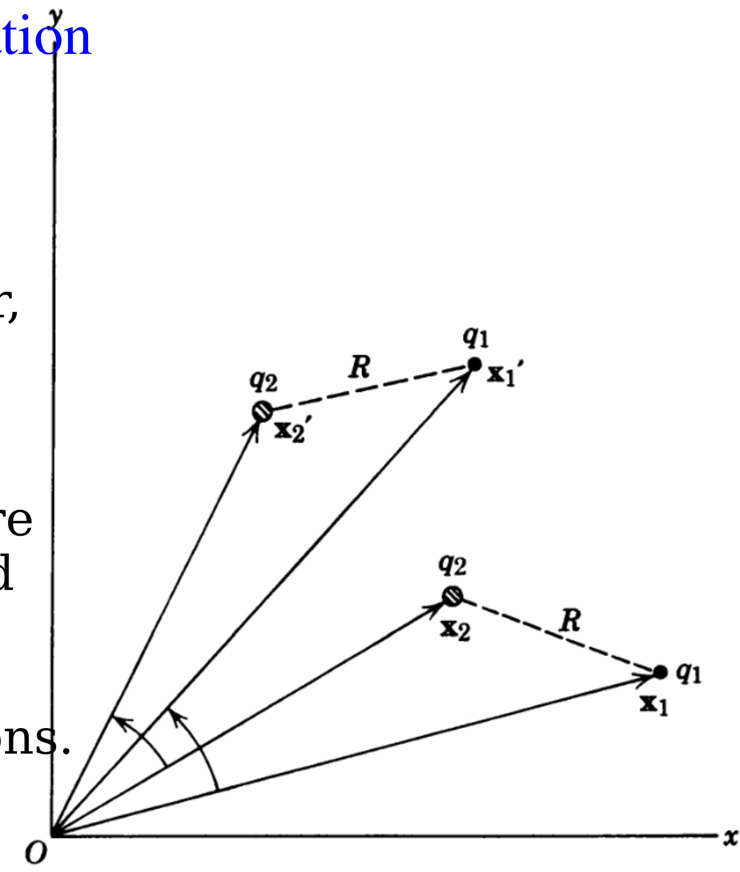
$a = -1$: improper rotation, a reflection plus a rotation

- Scalars are invariant under rotations and so are tensors of rank 0.

- $B'_{\alpha\beta} = \sum_{\gamma,\delta} a_{\alpha\gamma} a_{\beta\delta} B_{\gamma\delta}$ are called 2nd-rank tensors or, commonly, tensors, like Maxwell stress tensor.

- The active view of rotation—the coordinate axes are considered fixed and the physical system is imagined to undergo a rotation.

- The electrostatic potential is a scalar under rotations.



- If a physical quantity is a function of coordinates and when the physical system is rotated, the quantity is unchanged, then it is a scalar function under rotations.

$$\mathbf{r}_i \rightarrow \mathbf{r}_i' \quad \Rightarrow \quad \phi'(\mathbf{r}') = \phi(\mathbf{r})$$

- If a set of physical quantities transform under rotation of the system according to $V_\alpha'(\mathbf{r}') = \sum_\beta a_{\alpha\beta} V_\beta(\mathbf{r})$ then they form the components of a vector.

- The gradient of a scalar transforms as a vector, the divergence of a vector is a scalar, and the Laplacian operator is a scalar.

- $\mathbf{A} = \mathbf{B} \times \mathbf{C} \quad \Rightarrow \quad A_\alpha = \sum_{\beta\gamma} \epsilon_{\alpha\beta\gamma} B_\beta C_\gamma \quad \Rightarrow \quad A_\alpha' = a \sum_\beta a_{\alpha\beta} A_\beta \quad \Leftarrow \quad a \equiv \det \|a_{\alpha\beta}\|$

Under proper rotations, the cross product transforms as a vector.

B. Spatial Reflection or Inversion

- Spatial reflection in a plane is to change the signs of the $\perp$ components of the vectors of all points and to leaving the components $\parallel$ to the plane unchanged.

$$\text{For } xy \text{ plane: } \mathbf{r} = (x, y, z) \to \mathbf{r}' = (x, y, -z)$$

- Space inversion corresponds to reflection of all 3 components of every vector through the origin, $\mathbf{r} \to \mathbf{r}' = -\mathbf{r}$

- Spatial inversion or reflection is a discrete transformation that cannot in general be accomplished by proper rotations.

$$a = -1 \quad \Rightarrow \quad a_{\alpha\beta} = -\delta_{\alpha\beta} \text{ for inversion operation}$$

- *Polar vectors*: $V'_\alpha(\mathbf{r}') = \sum_\beta a_{\alpha\beta} V_\beta(\mathbf{r}) \ \ \& \ \ \mathbf{r} \to \mathbf{r}' = -\mathbf{r} \quad \Rightarrow \quad \mathbf{V} \to \mathbf{V}' = -\mathbf{V}$

- *Axial or pseudovectors*: $A'_\alpha = a \sum_\beta a_{\alpha\beta} A_\beta \ \ \& \ \ \mathbf{r} \to \mathbf{r}' = -\mathbf{r} \quad \Rightarrow \quad \mathbf{A} \to \mathbf{A}' = \mathbf{A}$

- Speak of *scalars* or *pseudoscalars*, depending on whether the quantities do not or do change sign under spatial inversion.

- The triple scalar product $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$ is a pseudoscalar if $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are polar vectors.

- If a tensor of rank N transforms under spatial inversion with a factor $(-1)^N$, it is a tensor, while if the factor is $(-1)^{N+1}$ it is a pseudotensor of rank N.

C. Time Reversal

- The basic laws of physics are invariant to the sense of direction of time.

- Under the time reversal transformation, the related physical quantities transform consistently so that the form of the equation is the same as before.

$$t \rightarrow t' = -t \quad \Rightarrow \quad \frac{\mathrm{d}\,\mathbf{p}}{\mathrm{d}\,t} = -\nabla U(\mathbf{r}) \text{ invariant} \quad \Leftarrow \quad \mathbf{r} \rightarrow \mathbf{r}' = \mathbf{r}, \quad \mathbf{p} \rightarrow \mathbf{p}' = -\mathbf{p}$$

- If an initial configuration of a system evolves into a final configuration, a possible state of motion of the system is that the time-reversed final configuration will evolve over the reversed path to the time-reversed initial configuration.

Physical Quantity	Symbol	Rotation (rank of tensor)	Space Inversion (name)	Time Reversal
Coordinate	$\mathbf{r}$	1	Odd(vector)	Even
Velocity	$\mathbf{v}$	1	Odd(vector)	Odd
Momentum	$\mathbf{p}$	1	Odd(vector)	Odd
Angular Momentum	$\mathbf{L} = \mathbf{r} \times \mathbf{p}$	1	Even(pseudovector)	Odd
Force	$\mathbf{F}$	1	Odd(vector)	Even
Torque	$\mathbf{N} = \mathbf{r} \times \mathbf{F}$	1	Even(pseudovector)	Even
Kinetic energy	$p^2/2m$	0	Even(scalar)	Even
Potential energy	$U(\mathbf{r})$	0	Even(scalar)	Even
Charge density	ρ	0	Even(scalar)	Even
Current density	$\boldsymbol{J}$	1	Odd(vector)	Odd
Electric field Polarization Displacement	$\mathbf{E}$ $\mathbf{P}$ $\mathbf{D}$	1	Odd(vector)	Even
Magnetic induction Magnetization Magnetic field	$\mathbf{B}$ $\mathbf{M}$ $\mathbf{H}$	1	Even(pseudovector)	Odd
Poynting vector	$\mathbf{S} = \mathbf{E} \times \mathbf{H}$	1	Odd(vector)	Odd
Maxwell stress tensor	$\mathbb{T}$	2	Even(tensor)	Even

D. Electromagnetic Quantities

- The forms of the equations governing EM phenomena are invariant under rotations, space inversion, and time reversal.

- Electric charge is invariant under Galilean & Lorentz transformations and is a scalar under rotations. Thus people assume that charge is also a scalar under spatial inversion & time reversal.

- The transformation properties of fields like $\mathbf{E}$ & $\mathbf{B}$ depend on the convention chosen for the charge.

- Charge density is a scalar. And $\mathbf{E}$ is a polar vector, even under time reversal.

- $\nabla \times \mathbf{E}$: pseudovector, even under time reversal $\Leftarrow \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$ Faraday's law

  $\Rightarrow \quad \mathbf{B}$: pseudovector, odd under time reversal

- The Ampere-Maxwell equation $\frac{1}{\mu_0} \nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \boldsymbol{J}$ polar vector, odd under time reversal on both sides.

- Consider the structure of a constitutive relation specifying the polarization $\mathbf{P}$, assuming 1st order in $\mathbf{E}$,

$\mathbf{B}_0 : \quad \mathbf{E} \times \mathbf{B}_0 , \quad \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}_0 , \quad \frac{\partial^2 \mathbf{E}}{\partial t^2} \times \mathbf{B}_0 , \cdots$ odd time derivatives of $\mathbf{E}$

$\mathbf{B}_0^2 : \quad (\mathbf{B}_0 \cdot \mathbf{B}_0) \mathbf{E} , \quad (\mathbf{E} \cdot \mathbf{B}_0) \mathbf{B}_0 , \quad (\mathbf{B}_0 \cdot \mathbf{B}_0) \frac{\partial \mathbf{E}}{\partial t} , \cdots$ 0 or even time derivatives of $\mathbf{E}$

- Up to 2nd-order of $\mathbf{B}_0$

$$\Rightarrow \quad \frac{\mathbf{P}}{\epsilon_0} = \chi_0 \mathbf{E} + \chi_1 \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}_0 + \chi_2 (\mathbf{B}_0 \cdot \mathbf{B}_0) \mathbf{E} + \chi_3 (\mathbf{E} \cdot \mathbf{B}_0) \mathbf{B}_0 + \cdots$$

odd time derivatives of $\mathbf{E}$ for the terms linear in $\mathbf{B}_0$

even time derivatives of $\mathbf{E}$ for the 0^{th} and 2^{nd} powers of $\mathbf{B}_0$.

- At low frequencies, the response of all material systems is via electric force

$$\Rightarrow \frac{\mathbf{P}}{\epsilon_0} = \chi_0 \mathbf{E} + \chi_1 \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}_0 + \chi_2' (\mathbf{B}_0 \cdot \mathbf{B}_0) \frac{\partial^2 \mathbf{E}}{\partial t^2} + \chi_3' \left(\frac{\partial^2 \mathbf{E}}{\partial t^2} \cdot \mathbf{B}_0 \right) \mathbf{B}_0 + \cdots$$

more realistic.

- At optical frequencies this equation permits an understanding of the gyrotropic behavior of waves in an isotropic medium in a constant magnetic field.

- In certain circumstances the constraints of space-time symmetries must be relaxed in constitutive relations. The optical rotatory power of chiral molecules has the constitutive relations, $\mathbf{P} = \epsilon_0 \chi_0 \mathbf{E} + \xi \dfrac{\partial \mathbf{B}}{\partial t}$ and $\mu_0 \mathbf{M} = \chi_0' \mathbf{B} + \xi' \dfrac{\partial \mathbf{E}}{\partial t}$

- The terms involve pseudoscalar quantities ξ and ξ' that reflect the underlying lack of parity symmetry for chiral substances.

On the Question of Magnetic Monopoles

- No experimental evidence for the existence of magnetic charges or monopoles.

- Dirac's argument is that the existence of magnetic monopole would offer an explanation of the discrete nature of electric charge.

- If there exist magnetic charge & current densities, then Maxwell equations become

$$\nabla\cdot\mathbf{D}=\rho_e,\quad \nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t}+\boldsymbol{J}_e \qquad \Leftarrow \qquad \frac{\partial\rho_e}{\partial t}+\nabla\cdot\boldsymbol{J}_e=0$$

$$\nabla\cdot\mathbf{B}=\rho_m,\quad -\nabla\times\mathbf{E}=\frac{\partial\mathbf{B}}{\partial t}+\boldsymbol{J}_m \qquad \frac{\partial\rho_m}{\partial t}+\nabla\cdot\boldsymbol{J}_m=0$$

- Duality transformation:

$$\begin{bmatrix}\mathbf{E}\\ Z_0\mathbf{H}\end{bmatrix}=\begin{bmatrix}\cos\xi & \sin\xi\\ -\sin\xi & \cos\xi\end{bmatrix}\begin{bmatrix}\mathbf{E}'\\ Z_0\mathbf{H}'\end{bmatrix},\quad \begin{bmatrix}Z_0\mathbf{D}\\ \mathbf{B}\end{bmatrix}=\begin{bmatrix}\cos\xi & \sin\xi\\ -\sin\xi & \cos\xi\end{bmatrix}\begin{bmatrix}Z_0\mathbf{D}'\\ \mathbf{B}'\end{bmatrix} \quad \Leftarrow \quad \text{impedance } Z_0=\sqrt{\frac{\mu_0}{\epsilon_0}}$$

- For a real/pseudoscalar angle ξ, the transformation leaves quadratic forms like $\mathbf{E}\times\mathbf{H}$, $(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H})$, and the components of the Maxwell stress tensor invariant.

- Apply to the sources

$$\begin{bmatrix}Z_0\rho_e\\ \rho_m\end{bmatrix}=\begin{bmatrix}\cos\xi & \sin\xi\\ -\sin\xi & \cos\xi\end{bmatrix}\begin{bmatrix}Z_0\rho'_e\\ \rho'_m\end{bmatrix},\quad \begin{bmatrix}Z_0\boldsymbol{J}_e\\ \boldsymbol{J}_m\end{bmatrix}=\begin{bmatrix}\cos\xi & \sin\xi\\ -\sin\xi & \cos\xi\end{bmatrix}\begin{bmatrix}Z_0\boldsymbol{J}'_e\\ \boldsymbol{J}'_m\end{bmatrix}$$

$\Rightarrow$ the generalized Maxwell equations are invariant

$$\nabla\cdot\mathbf{D}'=\rho'_e,\quad \nabla\times\mathbf{H}'=\frac{\partial\mathbf{D}'}{\partial t}+\boldsymbol{J}'_e,\quad \nabla\cdot\mathbf{B}'=\rho'_m,\quad -\nabla\times\mathbf{E}'=\frac{\partial\mathbf{B}'}{\partial t}+\boldsymbol{J}'_m$$

- The invariance of the EM equations under duality transformations shows that it is only a convention to think of electric charge instead of magnetic charge.

- If *all* particles have the same ratio of magnetic to electric charge, one can make a duality transformation so that $\rho_m=0$, $\boldsymbol{J}_m=0$, so we have the usual the Maxwell equations.

- If we choose the electric and magnetic charges of an electron $q_e=-e$, $q_m=0$, it is known that for a proton, $q_e=+e$ and $|q_m(\text{nucleon})|<2\times10^{-24}\,Z_0\,e$.

- The conclusion is that the particles of ordinary matter possess only electric charge or they all have the same ratio of magnetic to electric charge.

- ρ_m *is a pseudoscalar density, odd under time reversal,* and $\boldsymbol{J}_m$ *is a pseudovector density, even under time reversal*.

- Since the symmetries of ρ_m under spatial inversion & time reversal are opposite to those of ρ_e, then a particle with both electric & magnetic charges that space inversion and time reversal are no longer valid symmetries of the laws of physics.

- The present evidence is that the symmetry violation is extremely small and associated with the weak interactions.

- Considering the quantum mechanics of an electron near a magnetic monopole, Dirac showed that consistency required the quantization condition,

$$\frac{e\,g}{4\pi\hbar} = \frac{\alpha\,g}{Z_0\,e} = \frac{n}{2}\,, \quad n = 0\,, \pm 1, \pm 2, \cdots \quad \Leftarrow$$

g : magnetic charge

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}$$ fine structure constant

- The discrete nature of electric charge thus follows from the existence of a monopole. The magnitude of e is determined in terms of the magnetic charge g.

- With the known value of the fine structure constant, we can also infer the existence of magnetic monopoles with the magnetic "fine structure" constant

$$\frac{g^2}{4\pi\mu_0\hbar c} = \frac{n^2}{4}\,\frac{4\pi\epsilon_0\hbar c}{e^2} = \frac{137}{4}\,n^2 \quad \Leftarrow \quad \text{Dirac monopoles}$$

- The coupling strength is enormous, making their extraction from matter with dc magnetic fields and their subsequent detection very simple in principle.

Selected problems: 1, 4, 9, 10, 14, 21