Chapter 5 Magnetostatics, Faraday's Law, Quasistatic Fields

- The radical difference between magnetostatics and electrostatics: *there are no free magnetic charges*.

- The basic entity in magnetic studies is a magnetic dipole.

- The definition of the magnetic-flux density (or magnetic induction): $\mathbf{N} = \boldsymbol{\mu} \times \mathbf{B}$

- The magnetic phenomena was clearly understood after the connection between currents and magnetic fields was established.

- Conservation of charge $\dfrac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$

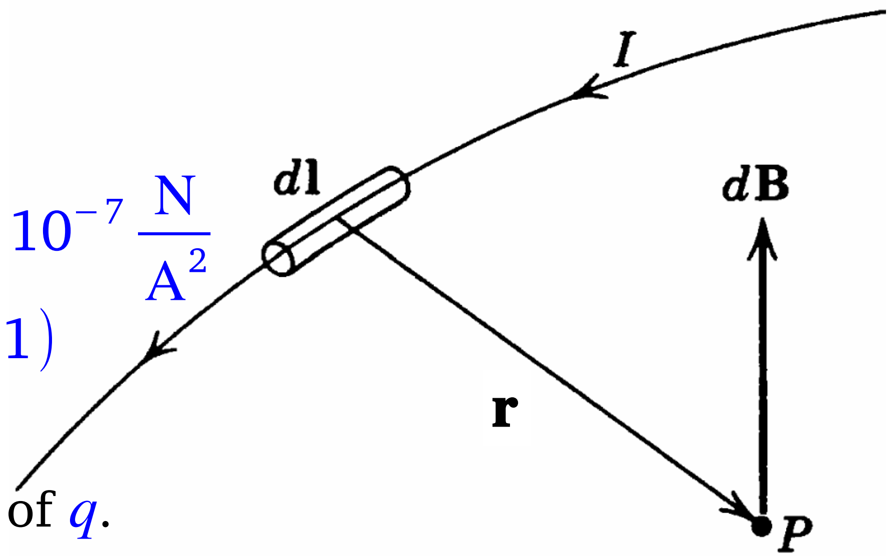
a decrease in charge inside a small volume with time must correspond to a flow of charge out through the surface of the small volume.

- In magnetostatics, no change in the net charge density anywhere in space

$$\Rightarrow \quad \nabla \cdot \boldsymbol{J} = 0$$

Biot & Savart Law

- $\mathrm{d}\,\mathbf{B} = k\, I\, \dfrac{\mathrm{d}\,\boldsymbol{\ell} \times \mathbf{r}}{r^3} = k\, I\, \dfrac{\mathrm{d}\,\boldsymbol{\ell} \times \hat{\mathbf{r}}}{r^2} \Leftarrow \quad k = \dfrac{\mu_0}{4\,\pi} = 10^{-7}\,\dfrac{\mathrm{N}}{\mathrm{A}^2} \qquad (1)$



- It is incorrect to think of the equation as the magnetic equivalent of the electric field of a point charge and to identify $I\,\mathrm{d}\,\ell$ as the analog of q.

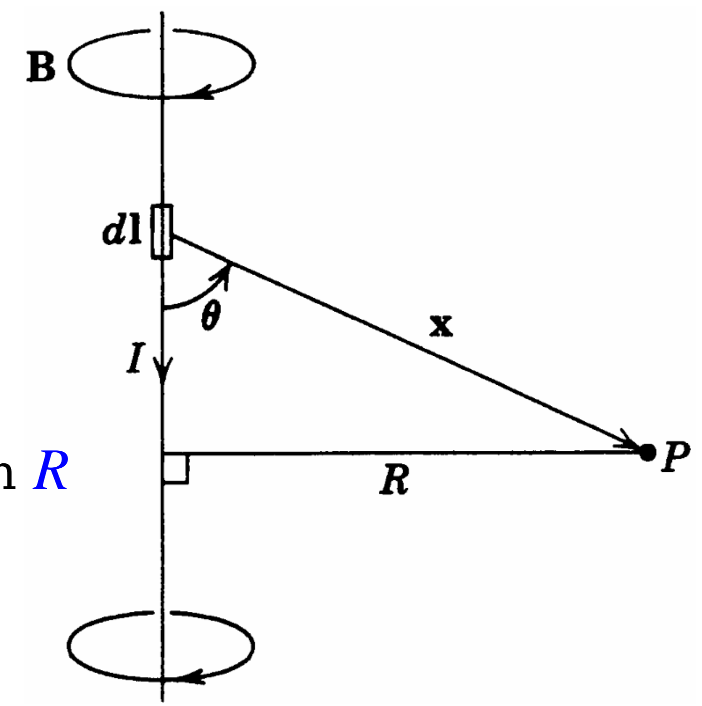
- The equation has meaning only as one element of a sum over a continuous set, the sum representing the magnetic induction of a current loop or circuit.

- One apparent way out of this difficulty is $\mathbf{B} = k\, q\, \dfrac{\mathbf{v} \times \mathbf{r}}{r^3} \quad \Leftarrow \quad I\,\mathrm{d}\,\boldsymbol{\ell} \to q\,\mathbf{v}$

- This expression is time dependent, and valid only for small velocities and negligible accelerations.

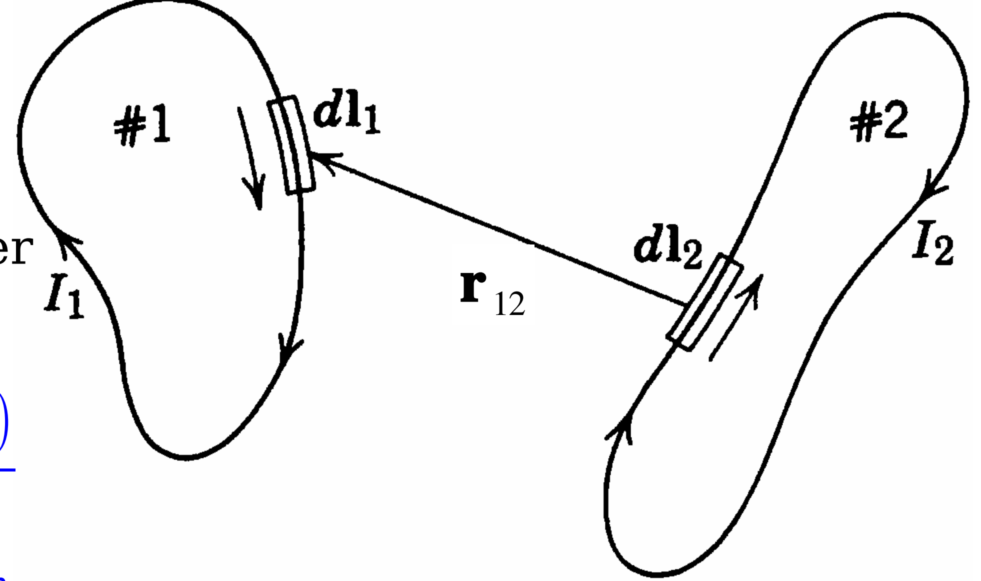
- For the magnetic induction of the long straight wire

$$B \equiv |\mathbf{B}| = \frac{\mu_0\, I\, R}{4\,\pi} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\,\ell}{(R^2 + \ell^2)^{3/2}} = \frac{\mu_0\, I}{2\,\pi\, R}$$



- The magnitude of the magnetic induction varies with R in the same way as the electric field due to a long line charge of uniform linear-charge density.

- Ampere's experiments were concerned with the force that one current-carrying wire experiences in the presence of another



$$d\mathbf{F} = I_1\, d\boldsymbol{\ell}_1 \times \mathbf{B}$$

$$\Rightarrow \quad \mathbf{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{d\boldsymbol{\ell}_1 \times (d\boldsymbol{\ell}_2 \times \mathbf{r}_{12})}{r_{12}^3}$$

$$\frac{d\boldsymbol{\ell}_1 \times (d\boldsymbol{\ell}_2 \times \mathbf{r}_{12})}{r_{12}^3} = -(d\boldsymbol{\ell}_1 \cdot d\boldsymbol{\ell}_2)\frac{\mathbf{r}_{12}}{r_{12}^3} + \frac{d\boldsymbol{\ell}_1 \cdot \mathbf{r}_{12}}{r_{12}^3} d\boldsymbol{\ell}_2$$

⇩ perfect differential

$$\Rightarrow \quad \mathbf{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{\mathbf{r}_{12}}{r_{12}^3} d\boldsymbol{\ell}_1 \cdot d\boldsymbol{\ell}_2 - \frac{\mu_0}{4\pi} I_1 I_2 \oint d\boldsymbol{\ell}_2 \oint \nabla \frac{1}{r_{12}} \cdot d\boldsymbol{\ell}_1$$

symmetric in $d\boldsymbol{\ell}_1$ and $d\boldsymbol{\ell}_2$ and satisfies Newton's 3rd law.

- For 2 long, parallel, straight wires $\frac{dF}{d\ell} = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}$. The force is attractive (repulsive) if the currents flow in the same (opposite) directions.

- If a current density is in an external magnetic-flux density, the total force and the total torque are $\mathbf{F} = \int \mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})\, d^3x, \quad \mathbf{N} = \int \mathbf{r} \times (\mathbf{J} \times \mathbf{B})\, d^3x$

Differential Equations of Magnetostatics and Ampere's Law

$$\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\int \boldsymbol{J}(\mathbf{r}')\times\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\,d^3x' \quad \Leftarrow \quad (1) \quad \text{vs} \quad \mathbf{E}=\frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\,d^3x'$$

$$\Rightarrow \quad \mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\nabla\times\int\frac{\boldsymbol{J}(\mathbf{r}')}{\mathfrak{r}}\,d^3x' \quad \Rightarrow \quad \nabla\cdot\mathbf{B}=0 \quad \text{vs} \quad \nabla\times\mathbf{E}=0$$

$$\Rightarrow \quad \nabla\times\mathbf{B}=\frac{\mu_0}{4\pi}\nabla\times\nabla\times\int\frac{\boldsymbol{J}(\mathbf{r}')}{\mathfrak{r}}\,d^3x' \quad \Leftarrow \quad \nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$$

$$=\frac{\mu_0}{4\pi}\left(\nabla\int\boldsymbol{J}(\mathbf{r}')\cdot\nabla\frac{1}{\mathfrak{r}}\,d^3x'-\int\boldsymbol{J}(\mathbf{r}')\,\nabla^2\frac{1}{\mathfrak{r}}\,d^3x'\right)$$

$$=-\frac{\mu_0}{4\pi}\nabla\int\boldsymbol{J}(\mathbf{r}')\cdot\nabla'\frac{1}{\mathfrak{r}}\,d^3x'+\mu_0\boldsymbol{J}(\mathbf{r}) \quad \Leftarrow \quad \begin{array}{l}\nabla\frac{1}{\mathfrak{r}}=-\nabla'\frac{1}{\mathfrak{r}}\\ \nabla^2\frac{1}{\mathfrak{r}}=-4\pi\,\delta(\vec{\mathfrak{r}})\end{array}$$

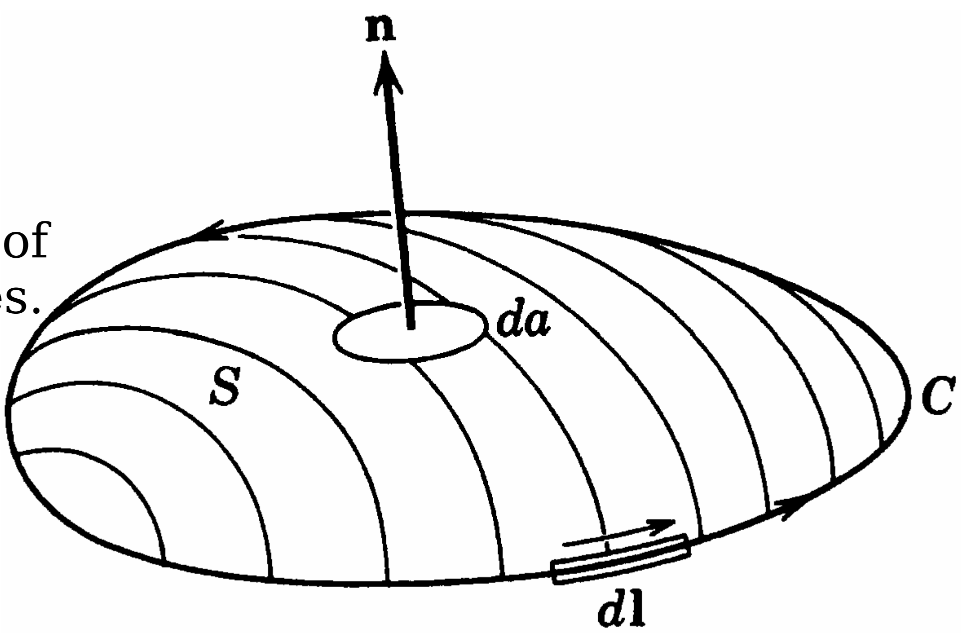
$$=\mu_0\boldsymbol{J}+\frac{\mu_0}{4\pi}\nabla\int\frac{\nabla'\cdot\boldsymbol{J}(\mathbf{r}')}{\mathfrak{r}}\,d^3x' \quad \Leftarrow \quad \text{integration by parts}$$

$$\Rightarrow \quad \nabla\times\mathbf{B}=\mu_0\boldsymbol{J} \quad \Leftarrow \quad \nabla\cdot\boldsymbol{J}=0$$ for steady-state magnetic phenomena $\quad$ vs $\quad \nabla\cdot\mathbf{E}=\frac{\rho}{\epsilon_0}$

where $\vec{\mathfrak{r}}\equiv\mathbf{r}-\mathbf{r}'$, $\quad \mathfrak{r}=|\vec{\mathfrak{r}}|=|\mathbf{r}-\mathbf{r}'|$, $\quad \hat{\mathfrak{r}}=\frac{\vec{\mathfrak{r}}}{\mathfrak{r}}=\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$

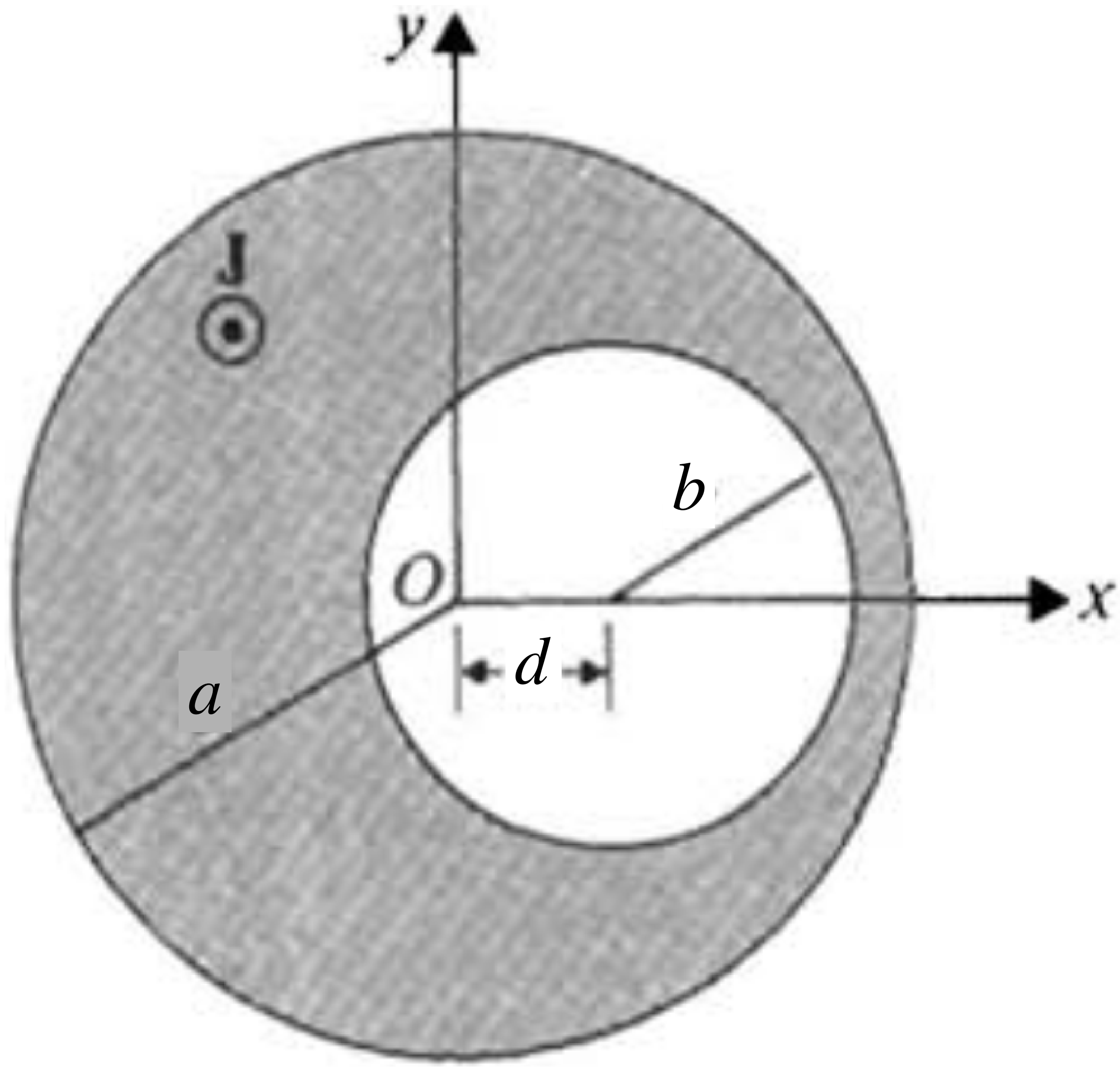
- $\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} \Leftarrow \int_S \nabla \times \mathbf{B} \cdot d\boldsymbol{a} = \mu_0 \int_S \boldsymbol{J} \cdot d\boldsymbol{a} = \mu_0 I \quad \Leftarrow \quad \text{Stokes's theorem}$

$\Rightarrow \quad \oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I \quad (\text{Ampere's law})$



- Ampere's law can be used for calculation of the magnetic field in highly symmetric cases.

A cylindrical conductor of radius a has a hole of radius b bored parallel to, and centered a distance d from, the cylinder axis ($d + b < a$). The current density is uniform throughout the remaining metal of the cylinder and is parallel to the axis. Use Ampère's law and principle of linear superposition to find the magnitude and the direction of the magnetic-flux density in the hole. [Problem 5.6]



Since the current density $\boldsymbol{J} = J\hat{\mathbf{z}}$ is uniform, the current in the bored cylinder can be considered as the one in a *complete* cylinder plus the current with the same uniform current density along the negative z-axis in the *hole* cylinder, i.e., $\boldsymbol{J}_{\text{hole}} = -\boldsymbol{J}$, as in the figure. For the position $\mathbf{r}$ being in the hole, by Ampere's law and the principle of superposition, the magnetic-flux density $\mathbf{B}(\mathbf{r})$ within the hole is $\mathbf{B} = \mathbf{B}_{\text{complete}} + \mathbf{B}_{\text{hole}}$, where

$$\oint_C \mathbf{B}_{\text{complete}} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \int_{\text{complete}} \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{a}, \quad \oint_{C'} \mathbf{B}_{\text{hole}} \cdot \mathrm{d}\boldsymbol{\ell}' = \mu_0 \int_{\text{hole}} (-\boldsymbol{J}) \cdot \mathrm{d}\boldsymbol{a}'.$$

Due to $\boldsymbol{J}$ being uniform and the cylindrical symmetry consideration,

$$\mathbf{B}_{\text{complete}}(\mathbf{r}) = \frac{\mu_0}{2} J r \hat{\boldsymbol{\phi}} = \frac{\mu_0}{2} J(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}),$$

$$\mathbf{B}_{\text{hole}}(\mathbf{r}') = \frac{\mu_0}{2}(-J) r' \hat{\boldsymbol{\phi}}' = \frac{\mu_0}{2} J(y'\hat{\mathbf{x}}' - x'\hat{\mathbf{y}}') = \frac{\mu_0}{2} J[y\hat{\mathbf{x}} - (x-d)\hat{\mathbf{y}}].$$

Thus $\mathbf{B} = \mathbf{B}_{\text{complete}} + \mathbf{B}_{\text{hole}} = \dfrac{\mu_0}{2} J d\hat{\mathbf{y}}$, a constant field. To be general and free of coordinate choice, $\mathbf{B} = \dfrac{\mu_0}{2}\boldsymbol{J} \times \mathbf{d}$.

Vector Potential

- The basic differential laws of magnetostatics $\begin{cases} \nabla \times \mathbf{B} = \mu_0 \boldsymbol{J} \\ \nabla \cdot \mathbf{B} = 0 \end{cases}$

- If $\boldsymbol{J} = 0 \quad \Rightarrow \quad \nabla \times \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = -\nabla \Phi_M$ magnetic scalar potential $\Rightarrow \quad \nabla^2 \Phi_M = 0$

- We can apply all the techniques for the electrostatic problems to it, but the boundary conditions are different from those in electrostatics and macroscopic magnetic properties are usually involved.

- $\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$ vector potential

$$\Rightarrow \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}^3 x' + \nabla \Psi(\mathbf{r}) \quad \Rightarrow \quad \mathbf{A} \to \mathbf{A} + \nabla \Psi \quad \text{gauge transformation}$$

- The freedom of gauge transformations allows us to make $\nabla \cdot \mathbf{A}$ have any convenient functional form we wish.

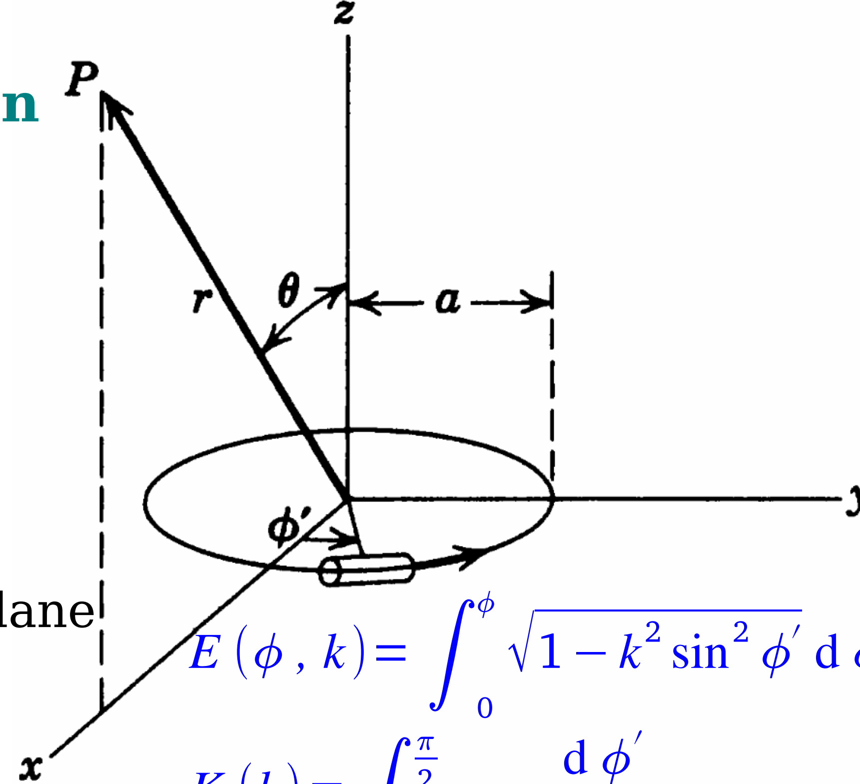
$$\Rightarrow \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \Leftarrow \nabla \times \nabla \times \mathbf{A} \Leftarrow \nabla \times \mathbf{B} = \mu_0 \boldsymbol{J} \quad \Rightarrow \quad -\nabla^2 \mathbf{A} = \mu_0 \boldsymbol{J} \Leftarrow \underset{\text{gauge choice}}{\nabla \cdot \mathbf{A} = 0}$$

$$\Rightarrow \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}^3 x' \quad \text{in unbounded space} \quad (2) \quad \Leftarrow \quad \Psi = \text{const}$$

- It can be understood as: $\nabla \cdot \mathbf{A} = 0$ Coulomb gauge $\Rightarrow \quad \nabla^2 \Psi = 0 \quad \Leftarrow \quad \nabla' \cdot \boldsymbol{J} = 0$

$\Rightarrow \quad \Psi = \text{const} \quad \Leftarrow \quad$ No current source at infinity

Vector Potential & Magnetic Induction for a Circular Current Loop



- $\boldsymbol{J} = J_\phi \hat{\boldsymbol{\phi}}' = I \dfrac{\delta(r'-a)}{r'} \delta\left(\theta' - \dfrac{\pi}{2}\right) \hat{\boldsymbol{\phi}}'$

$$= I \sin\theta' \, \delta(\cos\theta') \frac{\delta(r'-a)}{r'} \hat{\boldsymbol{\phi}}'$$

$$= -J_\phi \sin\phi' \, \hat{\mathbf{x}} + J_\phi \cos\phi' \, \hat{\mathbf{y}}$$

- Cylindrical symmetry ⇒ observe in the xz plane

$$E(\phi, k) = \int_0^\phi \sqrt{1 - k^2 \sin^2\phi'} \, d\phi'$$

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi'}{\sqrt{1 - k^2 \sin^2\phi'}}$$

⇒ Equation (2) is symmetric about $\phi' = 0$

⇒ J_x does not contribute

$$\Rightarrow \quad A_\phi = \frac{\mu_0 I}{4\pi} \int \frac{\sin\theta' \cos\phi'}{|\mathbf{r} - \mathbf{r}'|} \delta(\cos\theta') \frac{\delta(r'-a)}{r'} r'^2 \, d r' \, d\Omega'$$

$$= \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{\cos\phi' \, d\phi'}{\sqrt{a^2 + r^2 - 2 a r \sin\theta \cos\phi'}}$$

$$= \frac{\mu_0 I a}{\pi k^2} \frac{(2 - k^2) K(k) - 2 E(k)}{\sqrt{a^2 + r^2 + 2 a r \sin\theta}} \Leftarrow \text{elliptic integrals}, \quad k^2 = \frac{4 a r \sin\theta}{a^2 + r^2 + 2 a r \sin\theta}$$

$$\Rightarrow \quad B_r = \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (A_\phi \sin\theta), \quad B_\theta = -\frac{1}{r} \frac{\partial}{\partial r} (r A_\phi), \quad B_\phi = 0 \quad \Leftarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\int_0^{2\pi} \frac{\cos\phi \,\mathrm{d}\,\phi}{\sqrt{a^2+r^2-2\,a\,r\,\sin\theta\,\cos\phi}} = \int_{-\pi}^{+\pi} \frac{-\cos\psi\,\mathrm{d}\,\psi}{\sqrt{a^2+r^2+2\,a\,r\,\sin\theta\,\cos\psi}} \quad \Leftarrow \quad \psi=\phi-\pi$$

$$=2\int_0^{\pi} \frac{-\cos 2\,\varphi\,\mathrm{d}\,\varphi}{\sqrt{a^2+r^2+2\,a\,r\,\sin\theta\,\cos 2\,\varphi}} = 4\int_0^{\frac{\pi}{2}} \frac{\left(2\sin^2\varphi-1\right)\mathrm{d}\,\varphi}{\sqrt{a^2+r^2+2\,a\,r\,\sin\theta\left(1-2\sin^2\varphi\right)}}$$

Define $R^2 \equiv a^2+r^2+2\,a\,r\,\sin\theta\,, \quad k^2 \equiv \dfrac{4\,a\,r\,\sin\theta}{R^2}$

$$\Rightarrow \quad \int_0^{\frac{\pi}{2}} \frac{\left(2\sin^2\varphi-1\right)\mathrm{d}\,\varphi}{\sqrt{a^2+r^2+2\,a\,r\,\sin\theta\left(1-2\sin^2\varphi\right)}} = \frac{1}{R}\int_0^{\frac{\pi}{2}} \frac{2\sin^2\varphi-1}{\sqrt{1-k^2\sin^2\varphi}}\,\mathrm{d}\,\varphi$$

$$=\frac{1}{R}\left(2\int_0^{\frac{\pi}{2}} \frac{\sin^2\varphi}{\sqrt{1-k^2\sin^2\varphi}}\,\mathrm{d}\,\varphi - K(k)\right) \quad \Leftarrow \quad K(k)=\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\,\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

$$=\frac{1}{R}\left(\frac{2}{k^2}\int_0^{\frac{\pi}{2}} \frac{k^2\sin^2\varphi-1}{\sqrt{1-k^2\sin^2\varphi}}\,\mathrm{d}\,\varphi + \frac{2}{k^2}K(k) - K(k)\right)$$

$$=\frac{1}{R}\,\frac{\left(2-k^2\right)K(k)-2\,E(k)}{k^2} \quad \Leftarrow \quad E(k)=\int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\varphi}\,\mathrm{d}\,\varphi$$

- $A_\phi(r,\theta) = \dfrac{\mu_0 I}{4} \dfrac{a^2 r \sin\theta}{(a^2+r^2)^{3/2}} \left(1 + \dfrac{15}{8} \dfrac{a^2 r^2 \sin^2\theta}{(a^2+r^2)^2} + \cdots \right) \Leftarrow$ in powers of $\dfrac{a^2 r^2 \sin^2\theta}{(a^2+r^2)^2}$

$$\Rightarrow \quad B_r = \frac{\mu_0 I}{2} \frac{a^2 \cos\theta}{(a^2+r^2)^{3/2}} \left(1 + \frac{15}{4} \frac{a^2 r^2 \sin^2\theta}{(a^2+r^2)^2} + \cdots \right)$$

$$B_\theta = -\frac{\mu_0 I}{4} \frac{a^2 \sin\theta}{(a^2+r^2)^{5/2}} \left(2a^2 - r^2 + \frac{15(4a^2-3r^2)}{8} \frac{a^2 r^2 \sin^2\theta}{(a^2+r^2)^2} + \cdots \right)$$

- These can be specialized into 3 regions, near the axis ($\theta \ll 1$), near the center of the loop ($r \ll a$), and far from the loop ($r \gg a$).

$$r \gg a \quad \Rightarrow \quad B_r \approx \frac{\mu_0 m}{2\pi} \frac{\cos\theta}{r^3} \quad (3) \qquad B_\theta \approx \frac{\mu_0 m}{4\pi} \frac{\sin\theta}{r^3} \quad (4)$$

$\Leftarrow$ $m \equiv \pi I a^2$ magnetic dipole moment $\Leftarrow$ the magnetic fields far away from a circular current loop are dipole

- Use a spherical harmonic expansion to point out similarities and differences between the magnetostatic and electrostatic problems. Expand $|\mathbf{r}-\mathbf{r}'|^{-1}$,

$$A_\phi = \frac{\mu_0 I}{a} \Re \sum_{\ell, m} \frac{Y_\ell^m(\theta, 0)}{2\ell+1} \int e^{i\phi'} \frac{r_<^\ell}{r_>^{\ell+1}} Y_\ell^{*m}(\theta', \phi') \, \delta(\cos\theta') \, \delta(r'-a) \, r'^2 \, \mathrm{d}r' \, \mathrm{d}\Omega'$$

$$= 2\pi \mu_0 I a \sum_{\ell=1}^{\infty} \frac{Y_\ell^1(\theta, 0)}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_\ell^1\left(\frac{\pi}{2}, 0\right) \Leftarrow e^{i\phi'} \Rightarrow \text{ only } m=1 \text{ contributes}$$

$r_< = \min(a, r), \quad r_> = \max(a, r)$

$$Y_\ell^1\left(\frac{\pi}{2},0\right)=\sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}}\,P_\ell^1(0)=\begin{cases}0, & \ell \text{ even}\\ \sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}}\,\frac{(-1)^{n+1}\,\Gamma(n+3/2)}{\Gamma(n+1)\,\Gamma(3/2)}, & \ell=2n+1\end{cases}$$

$$\Rightarrow\quad A_\phi=-\frac{\mu_0 I a}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n(n+1)!}\frac{r_<^{2n+1}}{r_>^{2n+2}}P_{2n+1}^1(\cos\theta)$$

$$B_r=\frac{\mu_0 I a}{2r}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!!}{2^n n!}\frac{r_<^{2n+1}}{r_>^{2n+2}}P_{2n+1}(\cos\theta)\quad\Leftarrow\quad \frac{d}{dx}\left[\sqrt{1-x^2}\,P_\ell^1(x)\right]=\ell(\ell+1)P_\ell(x)$$

$$\Rightarrow\quad B_\theta=\begin{cases}+\frac{\mu_0 I a^2}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!!}{2^n(n+1)!}\frac{2n+2}{2n+1}\frac{1}{a^3}\left(\frac{r}{a}\right)^{2n}P_{2n+1}^1(\cos\theta) & \Leftarrow\ r<a\\ -\frac{\mu_0 I a^2}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!!}{2^n(n+1)!}\frac{1}{r^3}\left(\frac{a}{r}\right)^{2n}P_{2n+1}^1(\cos\theta) & \Leftarrow\ r>a\end{cases}$$

$r \gg a \quad \Rightarrow \quad \begin{matrix} B_r=(3) \\ B_\theta=(4)\end{matrix} \quad \Leftarrow \quad$ only the $n=0$ term matters & $P_1^1(\cos\theta)=-\sin\theta$

$r \ll a \quad \Rightarrow \quad$ only the $n=0$ term matters $\quad \Rightarrow \quad$ a magnetic induction $\dfrac{\mu_0 I}{2a}$ in the z-axis

- Associated Legendre polynomials appear as well as Legendre polynomials. This can be traced to the vector character of the current and vector potential, as opposed to the scalar properties of charge and electrostatic potential.

- Can also employ an expansion in cylindrical coordinates to attack this problem.

$$\int \nabla\cdot\mathbf{v}\,\mathrm{d}\tau = \oint \mathbf{v}\cdot\mathrm{d}\boldsymbol{a} \quad \Leftarrow \quad \text{divergence theorem}$$

Let $\mathbf{v}\to\mathbf{v}\times\mathbf{c}$ where $\mathbf{c}$ is a constant vector $\Rightarrow \quad \nabla\times\mathbf{c}=0$

$$\nabla\cdot(\mathbf{u}\times\mathbf{w}) = \mathbf{w}\cdot(\nabla\times\mathbf{u}) - \mathbf{u}\cdot(\nabla\times\mathbf{w})$$

$$= \sum_m \hat{\mathbf{x}}^m\,\partial_m \cdot \sum_{ijk} \epsilon^{ijk}\,\hat{\mathbf{x}}_i\,u_j\,w_k = \sum_{ijk}\epsilon^{ijk}\,\partial_i\left(u_j\,w_k\right) = \sum_{ijk}\epsilon^{ijk}\left(u_j\,\partial_i\,w_k + w_k\,\partial_i\,u_j\right)$$

$$= \sum_m w_m\,\hat{\mathbf{x}}^m \cdot \sum_{ijk}\epsilon^{ijk}\,\hat{\mathbf{x}}_i\,\partial_j\,u_k - \sum_m u_m\,\hat{\mathbf{x}}^m\cdot\sum_{ijk}\epsilon^{ijk}\,\hat{\mathbf{x}}_i\,\partial_j\,w_k$$

$$\Rightarrow \int \nabla\cdot(\mathbf{v}\times\mathbf{c})\,\mathrm{d}\tau = \int\left[\mathbf{c}\cdot(\nabla\times\mathbf{v}) - \mathbf{v}\cdot(\nabla\times\mathbf{c})\right]\mathrm{d}\tau = \mathbf{c}\cdot\int\nabla\times\mathbf{v}\,\mathrm{d}\tau$$

$$\oint(\mathbf{v}\times\mathbf{c})\cdot\mathrm{d}\boldsymbol{a} = \qquad \mathbf{c}\cdot\oint\mathrm{d}\boldsymbol{a}\times\mathbf{v}$$

$$\Rightarrow \int_{\mathcal{V}} \nabla\times\mathbf{v}\,\mathrm{d}\tau = \oint_{\mathcal{S}}\mathrm{d}\boldsymbol{a}\times\mathbf{v} = -\oint_{\mathcal{S}}\mathbf{v}\times\mathrm{d}\boldsymbol{a} \quad \Leftarrow \quad \mathbf{c} \text{ can be any constant.}$$

A sphere of radius a carries a uniform surface-charge distribution σ. The sphere is rotated about a diameter with constant angular velocity ω. Find the vector potential and magnetic-flux density both inside and outside the sphere. [Problem 5.13]

The current density is $\mathbf{J}(\mathbf{x}') = \rho(\mathbf{x}')\mathbf{v} = \sigma r' \omega \sin\theta \delta(r'-a)\hat{\boldsymbol{\phi}}' = a\sigma\omega \sin\theta' \delta(r'-a)(-\sin\phi'\hat{\mathbf{x}} + \cos\phi'\hat{\mathbf{y}})$. Then the vector potential is

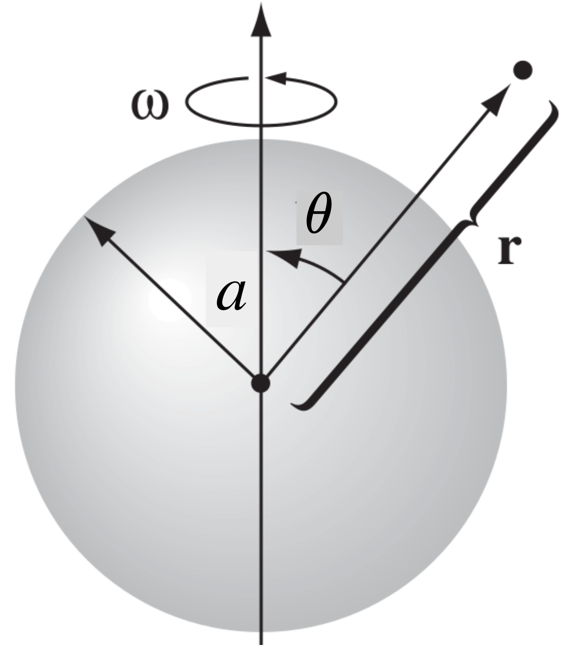
$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3x' = \frac{\mu_0\sigma\omega a^3}{4\pi}\int \frac{\sin^2\theta'}{|\mathbf{x}-\mathbf{x}'|}(\cos\phi'\hat{\mathbf{y}} - \sin\phi'\hat{\mathbf{x}})\mathrm{d}\theta'\mathrm{d}\phi'$$

$$= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{\mu_0\sigma\omega a^3}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}(\theta,\phi)\int Y_{\ell m}^*(\theta',\phi')\sin^2\theta' \frac{(\hat{\mathbf{y}}+i\hat{\mathbf{x}})e^{i\phi'} + (\hat{\mathbf{y}}-i\hat{\mathbf{x}})e^{-i\phi'}}{2}\mathrm{d}\theta'\mathrm{d}\phi'$$

$$= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{\mu_0\sigma\omega a^3}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}(\theta,\phi)\int \sqrt{\frac{2\pi}{3}} Y_{\ell m}^*[(\hat{\mathbf{y}}-i\hat{\mathbf{x}})Y_{1,-1} - (\hat{\mathbf{y}}+i\hat{\mathbf{x}})Y_{11}]\mathrm{d}\cos\theta'\mathrm{d}\phi'$$

$$= \frac{\mu_0\sigma\omega a^3}{3}\sqrt{\frac{2\pi}{3}}\frac{r_<}{r_>^2}[(\hat{\mathbf{y}}-i\hat{\mathbf{x}})Y_{1,-1}(\theta,\phi) - (\hat{\mathbf{y}}+i\hat{\mathbf{x}})Y_{11}(\theta,\phi)] = \frac{\mu_0}{3}\sigma\omega a^3 \sin\theta \frac{r_<}{r_>^2}\hat{\boldsymbol{\phi}}$$

$$= \frac{\mu_0}{3}\sigma\omega a \sin\theta\hat{\boldsymbol{\phi}} \begin{bmatrix} r \\ \dfrac{a^3}{r^2} \end{bmatrix}, \text{ for } \begin{bmatrix} r < a \text{ (interior)} \\ r > a \text{ (exterior)} \end{bmatrix},$$

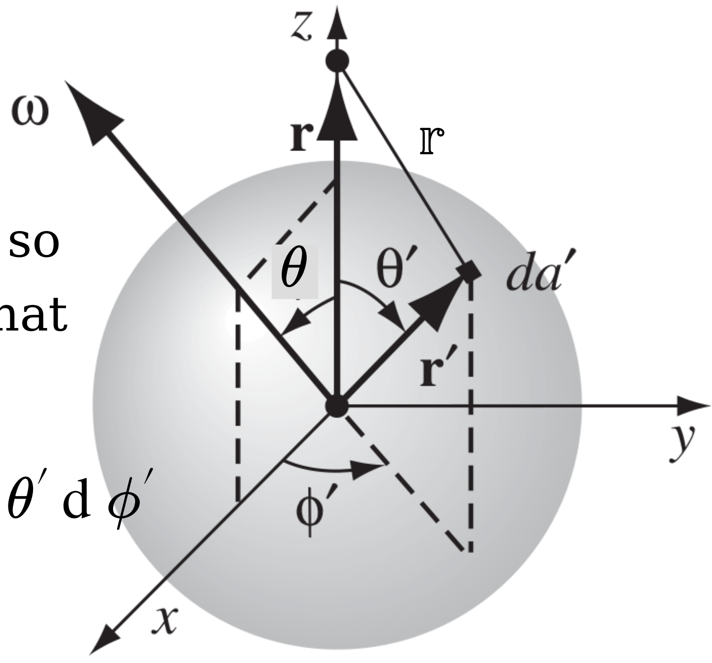


where $r_<(r_>)$ is the smaller (larger) of a and r, the spherical harmonic expansion of $\dfrac{1}{|\mathbf{x}-\mathbf{x}'|}$ is used, and $Y_{1,\pm1}(\theta,\phi)=\mp\sqrt{\dfrac{3}{8\pi}}\sin\theta e^{\pm i\phi}$. And the magnetic-flux density is

$$\mathbf{B}(\mathbf{x})=\nabla\times\mathbf{A}(\mathbf{x})=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}}\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi}\\ 0 & 0 & r\sin\theta A_\phi\end{vmatrix}$$

$$=\frac{\mu_0}{3}\sigma\omega\left[\begin{array}{cc} 2a\ \hat{\mathbf{z}} & \text{for uniform interior}\\ a^4\dfrac{3(\hat{\mathbf{r}}\cdot\hat{\mathbf{z}})\hat{\mathbf{r}}-\hat{\mathbf{z}}}{r^3} & \text{for dipole exterior}\end{array}\right. .$$

Alternative as in Ex 5.11 in Griffiths's EM book



- The integration is easier if we let **r** lie on the z axis, so that $\boldsymbol{\omega}$ is tilted at an angle θ. We orient the x axis so that $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}'$ lies in the xz plane.

- For $\mathbf{A}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\boldsymbol{K}(\mathbf{r}')}{\mathbb{r}}\,\mathrm{d}\,a' \quad \Leftarrow \quad \mathrm{d}\,a' = a^2 \sin\theta'\,\mathrm{d}\,\theta'\,\mathrm{d}\,\phi'$

$$\boldsymbol{K} = \sigma\,\mathbf{v}\,, \quad \mathbb{r} = \sqrt{a^2 + r^2 - 2\,a\,r\cos\theta'} \quad \Leftarrow \quad \mathbf{r} = r\,\hat{\mathbf{z}}$$

$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}' = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega\sin\theta & 0 & \omega\cos\theta \\ a\sin\theta'\cos\phi' & a\sin\theta'\sin\phi' & a\cos\theta' \end{vmatrix}$$

$$= a\,\omega\,[\,\sin\theta\sin\theta'\sin\phi'\,\hat{\mathbf{z}} - \cos\theta\sin\theta'\sin\phi'\,\hat{\mathbf{x}} + \left(\cos\theta\sin\theta'\cos\phi' - \sin\theta\cos\theta'\right)\hat{\mathbf{y}}\,]$$

- $\displaystyle\int_0^{2\pi} \sin\phi'\,\mathrm{d}\,\phi' = \int_0^{2\pi}\cos\phi'\,\mathrm{d}\,\phi' = 0$

$$\Rightarrow \quad \mathbf{A}(\mathbf{r}) = -\frac{\mu_0\,\sigma\,\omega\,a^3\sin\theta}{2}\int_0^{\pi}\frac{\cos\theta'\sin\theta'\,\mathrm{d}\,\theta'}{\sqrt{a^2 + r^2 - 2\,a\,r\cos\theta'}}\,\hat{\mathbf{y}}$$

$$\int_{-1}^{1}\frac{u\,\mathrm{d}\,u}{\sqrt{a^2+r^2-2\,a\,r\,u}}=-\frac{2}{2\,a\,r}\int_{-1}^{1}u\,\mathrm{d}\sqrt{a^2+r^2-2\,a\,r\,u}\quad\Leftarrow\quad u=\cos\theta'$$

$$=-\frac{u\sqrt{a^2+r^2-2\,a\,r\,u}}{a\,r}\Bigg|_{-1}^{1}+\frac{1}{a\,r}\int_{-1}^{1}\sqrt{a^2+r^2-2\,a\,r\,u}\,\mathrm{d}\,u$$

$$=-\frac{|a-r|+a+r}{a\,r}-\frac{1}{2\,a^2\,r^2}\,\frac{2}{3}\left(a^2+r^2-2\,a\,r\,u\right)^{\frac{3}{2}}\Bigg|_{-1}^{1}\qquad r_{\lessgtr}=\begin{matrix}\min\\ \max\end{matrix}\,(a\,,\,r)$$

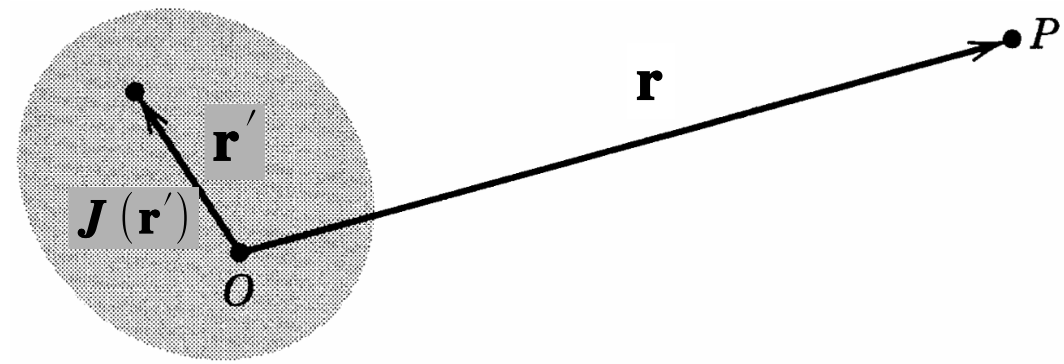
$$=\frac{(a+r)^3-|a-r|^3}{3\,a^2\,r^2}-\frac{|a-r|+a+r}{a\,r}=\frac{a^3+r^3-|a^3-r^3|}{3\,a^2\,r^2}=\frac{2}{3}\,\frac{r_<}{r_>^2}$$

$$\Rightarrow\quad \mathbf{A}(\mathbf{r})=-\frac{\mu_0\,\sigma\,\omega\,a^3\sin\theta}{2}\,\frac{2}{3}\,\frac{r_<}{r_>^2}\,\hat{\mathbf{y}}$$

$$=\frac{\mu_0\,\sigma\,a^3\,\omega}{3}\,\frac{r_<}{r_>^2}\sin\theta\,\hat{\boldsymbol{\phi}}\quad\Leftarrow\quad\text{revert the coordinates }\ \boldsymbol{\omega}\,\|\,\hat{\mathbf{z}}\,,\quad \mathbf{r}=(r\,,\,\theta\,,\,\phi)$$

$$\Rightarrow\quad \mathbf{B}=\nabla\times\mathbf{A}=\begin{cases}\dfrac{2}{3}\,\mu_0\,\sigma\,a\,\boldsymbol{\omega} & \textit{uniform}\ \text{ inside the spherical shell}\\[2ex] \dfrac{1}{3}\,\mu_0\,\sigma\,a^4\,\dfrac{3\,(\hat{\mathbf{r}}\cdot\boldsymbol{\omega})\,\hat{\mathbf{r}}-\boldsymbol{\omega}}{r^3} & \textit{dipole}\ \text{ outside the spherical shell}\end{cases}$$

Magnetic Fields of a Localized Current Distribution, Magnetic Moment



$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{r}+\frac{\mathbf{r}\cdot\mathbf{r}'}{r^3}+\cdots \quad \Leftarrow \quad r \gg r'$$

$$\Rightarrow \quad A_i(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{J_i(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3x'$$

$$=\frac{\mu_0}{4\pi}\left(\frac{1}{r}\int J_i(\mathbf{r}')\,\mathrm{d}^3x'+\frac{\mathbf{r}}{r^3}\cdot\int J_i(\mathbf{r}')\,\mathbf{r}'\,\mathrm{d}^3x'+\cdots\right) \qquad ⇍ \; \boldsymbol{J} \text{ localized}$$

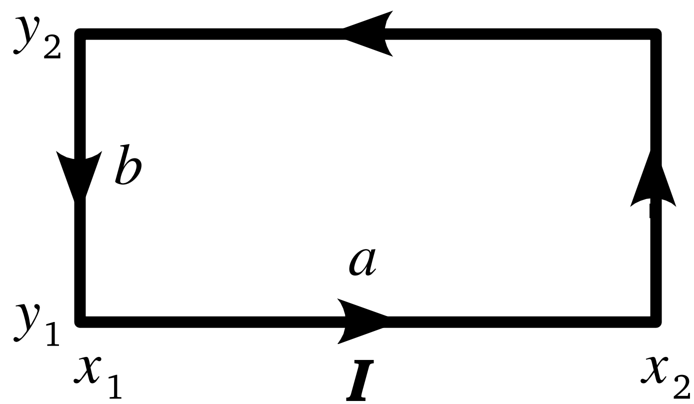
$$\int\nabla\cdot(f\,g\,\boldsymbol{J})\,\mathrm{d}^3x=\int(f\,\boldsymbol{J}\cdot\nabla g+g\,\boldsymbol{J}\cdot\nabla f+f\,g\,\nabla\cdot\boldsymbol{J})\,\mathrm{d}^3x=0 \quad (5)$$

$$\Rightarrow \left[\begin{array}{ll}\int J_i(\mathbf{r}')\,\mathrm{d}^3x' = 0 & \Leftarrow \quad f=1, \quad g=x_i', \\ \int(x_i'\,J_j+x_j'\,J_i)\,\mathrm{d}^3x'=0 & \Leftarrow \quad f=x_i', \quad g=x_j',\end{array}\right. \qquad \nabla'\cdot\boldsymbol{J}=0 \quad \text{divergenceless}$$

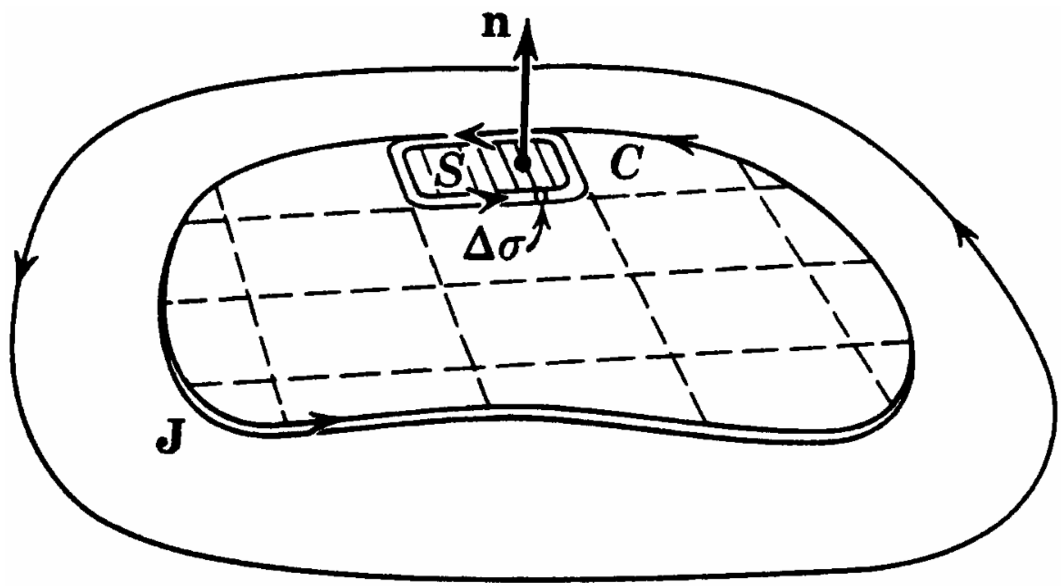
$$\Rightarrow \quad \mathbf{r}\cdot\int\mathbf{r}'\,J_i(\mathbf{r}')\,\mathrm{d}^3x'\equiv\sum_j x_j\int x_j'\,J_i\,\mathrm{d}^3x'$$

$$=-\frac{1}{2}\sum_j x_j\int(x_i'\,J_j-x_j'\,J_i)\,\mathrm{d}^3x' \quad \Leftarrow \quad \int(x_i'\,J_j+x_j'\,J_i)\,\mathrm{d}^3x'=0$$

$$=-\frac{1}{2}\sum_{j,k}\epsilon_{ijk}\,x_j\int(\mathbf{r}'\times\boldsymbol{J})_k\,\mathrm{d}^3x'=-\frac{1}{2}\left(\mathbf{r}\times\int\mathbf{r}'\times\boldsymbol{J}\,\mathrm{d}^3x'\right)_i$$



$$\int (x J_y + y J_x)\, d^3 x = \int (x I_y + y I_x)\, d\ell = y_1 I a + x_2 I b - y_2 I a - x_1 I b$$
$$= (x_2 - x_1) I b - (y_2 - y_1) I a = I a b - I a b = 0$$



An arbitrary-shaped current loop can be divided into as many rectangular loops as it takes. Since the integral on a rectangular loop vanishes, therefore the integral with an arbitrary-shaped loop vanishes.

$$\Rightarrow \quad \mathbf{A}(\mathbf{r}) \approx \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3}$$ lowest nonzero term $\Leftarrow$ $$\mathbf{m} \equiv \frac{1}{2}\int \mathbf{r}' \times \boldsymbol{J}(\mathbf{r}')\, d^3x'$$ magnetic moment

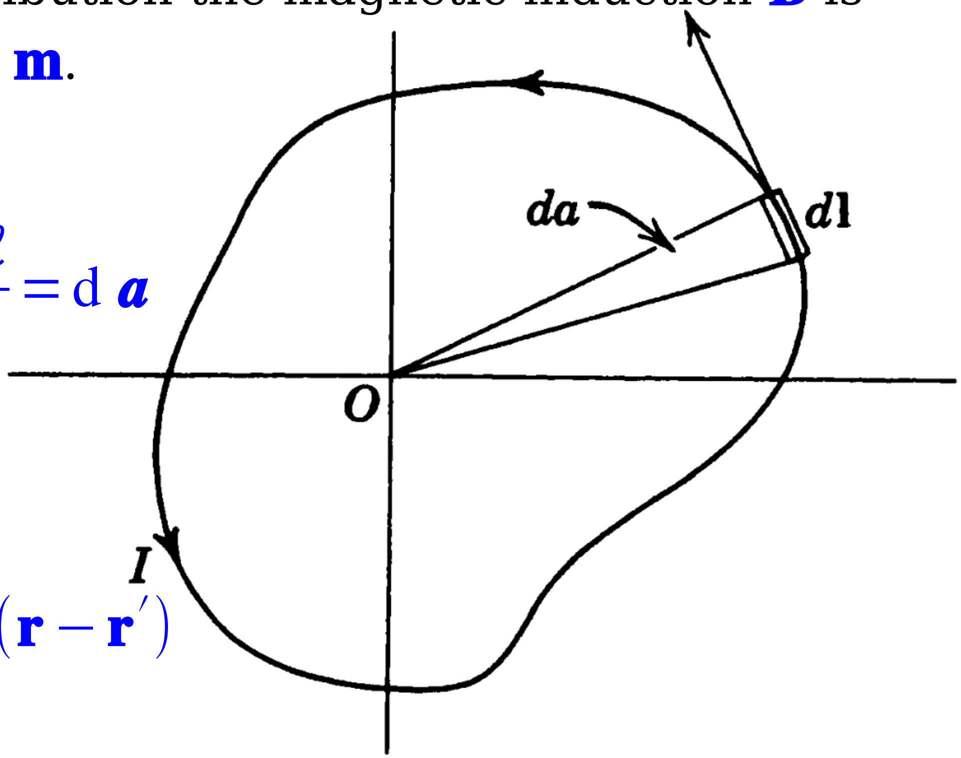
$$\Rightarrow \quad \mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m})-\mathbf{m}}{r^3} \quad \Leftarrow \quad \hat{\mathbf{r}} \equiv \frac{\mathbf{r}}{r} \quad \Leftarrow$$ the form of the field of a dipole

- Far away from *any* localized current distribution the magnetic induction $\mathbf{B}$ is that of a magnetic dipole of dipole moment $\mathbf{m}$.

- If the current is confined to a plane,

$$\mathbf{m} \equiv \frac{I}{2}\int \mathbf{r}\times d\boldsymbol{\ell} = I \times \text{Area}\,\hat{\mathbf{n}} \quad \Leftarrow \quad \frac{\mathbf{r}\times d\boldsymbol{\ell}}{2} = d\boldsymbol{a}$$

regardless of the shape of the circuit.



- For discrete charges

$$\mathbf{m} = \frac{1}{2}\sum_i q_i(\mathbf{r}_i\times\mathbf{v}_i) \quad \Leftarrow \quad \boldsymbol{J} = \sum_i q_i\mathbf{v}_i\,\delta(\mathbf{r}-\mathbf{r}')$$

$$\mathbf{m} = \sum_i \frac{q_i}{2M_i}\mathbf{L}_i \quad \Leftarrow \quad \mathbf{L}_i = M_i(\mathbf{r}_i\times\mathbf{v}_i)$$

$$= \frac{q}{2M}\sum_i \mathbf{L}_i = \frac{q}{2M}\mathbf{L} \quad \text{if} \quad \frac{q_i}{M_i} = \frac{q}{M} \quad \text{is the same}$$

- The classical connection between angular momentum and magnetic moment holds for orbital motion, but fails for the intrinsic moment.

- For electrons, the intrinsic moment is twice as large as the above. We speak of the electron having a g factor of 2.

- There are 2 limits, one is that the sphere of radius R contains all of the current and the other is that the current is completely external to the spherical volume.

- $$\int_{r<R} \mathbf{B}(\mathbf{r})\, d^3x = \int_{r<R} \nabla\times\mathbf{A}\, d^3x = R^2 \oint_{r=R} \hat{\mathbf{r}}\times\mathbf{A}\, d\Omega$$

$$= -\frac{\mu_0}{4\pi} R^2 \int \boldsymbol{J}(\mathbf{r}')\, d^3x' \times \oint \frac{\hat{\mathbf{r}}\, d\Omega}{|\mathbf{r}-\mathbf{r}'|}$$

$$\hat{\mathbf{r}} = \sin\theta\left(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}\right) + \cos\theta\,\hat{\mathbf{z}}$$

$$= \sqrt{\frac{2\pi}{3}}\left[\left(\hat{\mathbf{x}} + i\,\hat{\mathbf{y}}\right) Y_1^{-1}(\theta,\phi) - \left(\hat{\mathbf{x}} - i\,\hat{\mathbf{y}}\right) Y_1^{1}(\theta,\phi) + \sqrt{2}\, Y_1^0(\theta,\phi)\,\hat{\mathbf{z}}\right]$$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{\ell,m} \frac{4\pi}{2\ell+1} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y_\ell^{*m}(\theta,\phi)\, Y_\ell^{m}(\theta',\phi') \quad \Leftarrow \quad r_{\lessgtr} = \begin{matrix}\min\\ \max\end{matrix}(r', R)$$

$$\Rightarrow \oint_{r=R} \frac{\hat{\mathbf{r}}\, d\Omega}{|\mathbf{r}-\mathbf{r}'|} = \frac{4\pi}{3} \frac{r_<}{r_>^2}\hat{\mathbf{r}}' \quad \Leftarrow \quad \text{only the } \ell = 1 \text{ terms survive}$$

$$\Rightarrow \int_{r<R} \mathbf{B}(\mathbf{r})\, d^3x = \frac{\mu_0}{3} \int \frac{R^2}{r'} \frac{r_<}{r_>^2} \mathbf{r}' \times \boldsymbol{J}(\mathbf{r}')\, d^3x'$$

$$= \begin{cases} \dfrac{\mu_0}{3} \displaystyle\int \mathbf{r}' \times \boldsymbol{J}(\mathbf{r}')\, d^3x' & \text{for} \quad r' < R\,, \quad r_< = r'\,, \quad r_> = R \\ \dfrac{\mu_0 R^3}{3} \displaystyle\int \frac{\hat{\mathbf{r}}'}{r'^2} \times \boldsymbol{J}(\mathbf{r}')\, d^3x' & \text{for} \quad r' > R\,, \quad r_< = R\,, \quad r_> = r' \end{cases}$$

$$= \begin{cases} \dfrac{2\mu_0}{3} \mathbf{m} & \text{all the current density is inside the sphere} \\ \dfrac{4\pi R^3}{3} \mathbf{B}(0) & \text{all the current density is outside the sphere} \end{cases}$$

$$\Rightarrow \boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left(\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m}) - \mathbf{m}}{r^3} + \frac{8\pi}{3} \mathbf{m}\, \delta(\mathbf{r}) \right)}$$

$$\text{vs} \quad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{p}) - \mathbf{p}}{r^3} - \frac{4\pi}{3} \mathbf{p}\, \delta(\mathbf{r}) \right)$$

Force & Torque on and Energy of a Localized Current Distribution in an External Magnetic Induction

- $\mathbf{F} = \int \boldsymbol{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})\, \mathrm{d}^3 x \quad \Leftarrow \quad \mathbf{B}(\mathbf{r}) = \mathbf{B}(0) + \mathbf{r}\cdot\nabla\,\mathbf{B}(0) + \cdots$

$$= \sum_{ijk} \epsilon_{ijk}\, \hat{\mathbf{x}}_i \left(B_k(0) \cancel{\int J_j\, \mathrm{d}^3 y} + \int J_j(\mathbf{y})\, \mathbf{y}\cdot\nabla B_k(0)\, \mathrm{d}^3 y + \cdots \right)$$

$$\approx \sum_{ijk} \epsilon_{ijk}\, \hat{\mathbf{x}}_i (\mathbf{m}\times\nabla)_j B_k(0) \quad \Leftarrow \quad \int x_i J_j\, \mathrm{d}^3 x = -\int x_j J_i\, \mathrm{d}^3 x$$

$$\Rightarrow \quad \mathbf{F} \simeq (\mathbf{m}\times\nabla)\times\mathbf{B} = \nabla(\mathbf{m}\cdot\mathbf{B}) - \mathbf{m}\,\cancel{\nabla\cdot\mathbf{B}} \quad \Rightarrow \quad \boxed{\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})} \quad \Rightarrow \quad U = -\mathbf{m}\cdot\mathbf{B}$$

- The force is the change rate of the total mechanical momentum, including the "hidden mechanical momentum" associated with the EM momentum.

- $\mathbf{F}_{\text{effective}} = \text{mass} \times \ddot{\mathbf{r}} = \nabla(\mathbf{m}\cdot\mathbf{B}) + \dfrac{1}{c^2}\dfrac{\mathrm{d}}{\mathrm{d}t}(\mathbf{E}\times\mathbf{m})$ in Newton's equation of motion

$$\Rightarrow \left[\frac{B}{L}\right] + \left[\frac{E}{c\,\lambda\!\!\!^{-}}\right] \quad \Leftarrow \quad \begin{array}{l} L: \text{length scale over which } \mathbf{B} \text{ changes significantly} \\ \lambda\!\!\!^{-}: \text{wavelength of radiation at the typical frequencies of } \mathbf{E} \end{array}$$

- A charged particle in a uniform magnetic induction moves circularly $\perp$ the field and with constant velocity $\|$ the field, tracing out a helical path.

- If the field is not uniform but has a small gradient, the motion of the particle can be affected by the force on the equivalent magnetic moment.

- Charged particles will be repelled by regions of high flux density. This is the basis of the "magnetic mirrors," important in the confinement of plasmas.

$$\int x_i J_j \, d^3x = -\int x_j J_i \, d^3x$$

$$\Rightarrow \int J_j \mathbf{y} \, d^3y = \int \frac{J_j \mathbf{y} - \boldsymbol{J} y_j}{2} d^3y = \sum_{k\ell} \epsilon_{\ell jk} \hat{\mathbf{x}}_\ell \int \frac{(\mathbf{y} \times \boldsymbol{J})_k}{2} d^3y = \sum_{k\ell} \epsilon_{\ell jk} m_k \hat{\mathbf{x}}_\ell$$

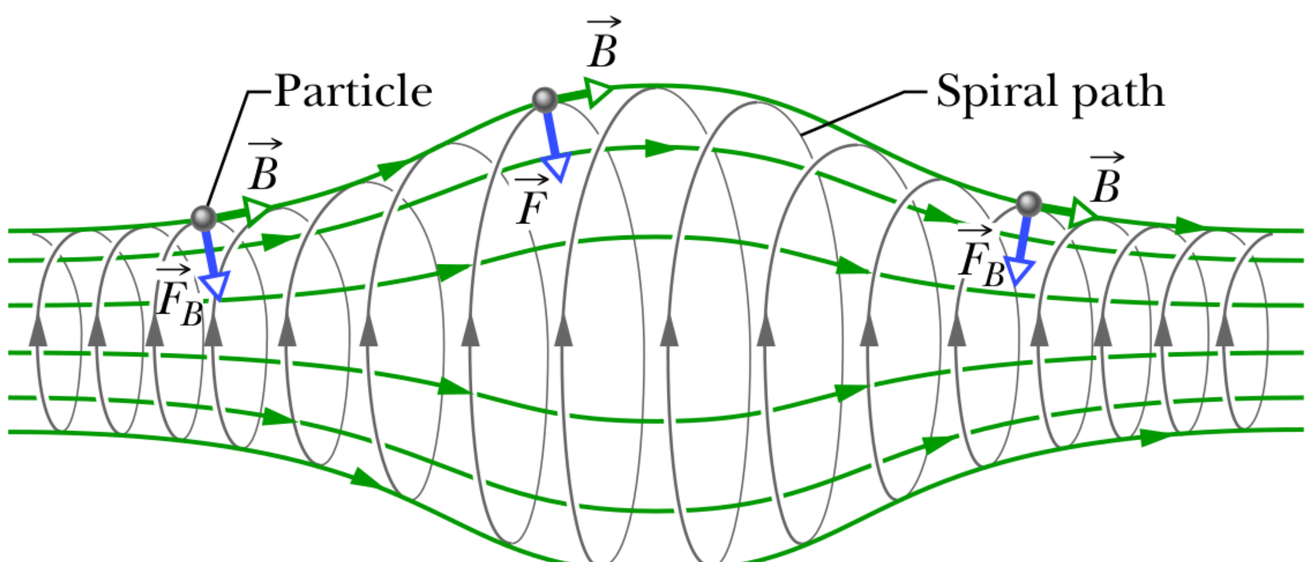
$$\Rightarrow \sum_{ijk} \epsilon_{ijk} \hat{\mathbf{x}}_i \int J_j(\mathbf{y}) \, \mathbf{y} \cdot \nabla B_k(0) \, d^3y = \sum_{ijk\ell} \epsilon_{ijk} \hat{\mathbf{x}}_i \left(\int J_j y_\ell \, d^3y \right) \partial_\ell B_k(0)$$

$$= \sum_{ijk\ell n} \epsilon_{ijk} \hat{\mathbf{x}}_i (\epsilon_{\ell jn} m_n \partial_\ell) B_k(0) = \sum_{ijk} \epsilon_{ijk} \hat{\mathbf{x}}_i (\mathbf{m} \times \nabla)_j B_k(0) = (\mathbf{m} \times \nabla) \times \mathbf{B}(0)$$

$$(\mathbf{m} \times \nabla) \times \mathbf{B} = \sum_{ijk\ell n} \epsilon_{ijk} \hat{\mathbf{x}}_i \epsilon_{\ell jn} m_n \partial_\ell B_k \quad \Leftarrow \quad \sum_j \epsilon_{jik} \epsilon_{j\ell n} = \delta_{i\ell} \delta_{kn} - \delta_{in} \delta_{k\ell}$$

$$= \sum_{ik} \hat{\mathbf{x}}_i (m_k \partial_i B_k - m_i \partial_k B_k) = \sum_i m_i \cdot \nabla B_i - \mathbf{m}(\nabla \cdot \mathbf{B})$$

$$= \nabla(\mathbf{m} \cdot \mathbf{B}) - \mathbf{m}(\nabla \cdot \mathbf{B}) \quad \text{for } \mathbf{m} \text{ is not a function of } \mathbf{r} \text{ here.}$$



- $\mathbf{N} = \int \mathbf{r}' \times (\boldsymbol{J} \times \mathbf{B})\, d^3x' \approx \int \mathbf{r}' \times (\boldsymbol{J} \times \mathbf{B}_0)\, d^3x' \quad \Leftarrow \quad \mathbf{B}_0 \equiv \mathbf{B}(0)$

$$= \int [(\mathbf{r}' \cdot \mathbf{B}_0)\boldsymbol{J} - (\mathbf{r}' \cdot \boldsymbol{J})\mathbf{B}_0]\, d^3x' \quad \Rightarrow \quad g = f = r \text{ in } (5) \quad \Rightarrow \quad \int \mathbf{r} \cdot \boldsymbol{J}\, d^3x = 0$$

$$\int (x_i J_j + x_j J_i)\, d^3x = 0 \quad \Rightarrow \quad 2\int [\mathbf{r} \cdot \mathbf{B}_0]\boldsymbol{J}\, d^3x = \int [(\mathbf{B}_0 \cdot \mathbf{r})\boldsymbol{J} - (\mathbf{B}_0 \cdot \boldsymbol{J})\mathbf{r}]\, d^3x$$

$$\Rightarrow \quad \mathbf{N} = \mathbf{m} \times \mathbf{B}(0)$$

- Interpret the force as the negative gradient of a potential energy $\Rightarrow$ $U = -\mathbf{m} \cdot \mathbf{B}$
A dipole tends to orient itself parallel to the field to have lowest potential energy.

- The potential energy is not the total energy of the magnetic moment in the external field. Work is needed to keep the current, producing $\mathbf{m}$, constant.

- The potential energy expression can be employed in the treatment of magnetic effects on atom, as in the Zeeman effect or for the fine and hyperfine structure.

- The fine structure comes from differences in energy of an electron's intrinsic magnetic moment in the magnetic field seen in its rest frame [chapter 11].

- The hyperfine interaction is that of the magnetic moment of the nucleus with the magnetic field produced by the electron.

$$H_{\text{HFS}} = -\boldsymbol{\mu}_N \cdot \mathbf{B}(0) = -\boldsymbol{\mu}_N \cdot [\mathbf{B}_{\text{dipole}}(0) + \mathbf{B}_{\text{orbit}}(0)] \quad \Leftarrow \quad \mathbf{B}_{\text{orbit}}(0) = \frac{\mu_0}{4\pi}\frac{e}{m_e}\frac{\mathbf{L}}{r^3}$$

$$\text{since } \; B_{\text{orbital}}(0) \sim \frac{\mu_0 I}{2r} \sim \frac{\mu_0 2\pi r^2 I}{4\pi r^3} \sim \frac{\mu_0}{4\pi}\frac{2m}{r^3} \sim \frac{\mu_0}{4\pi}\frac{e}{m_e}\frac{L}{r^3}$$

$$\Rightarrow \quad H_{\text{HFS}} = \frac{\mu_0}{4\pi r^3} \boldsymbol{\mu}_N \cdot \left(\boldsymbol{\mu}_e - 3\left(\boldsymbol{\mu}_e \cdot \hat{\mathbf{r}}\right)\hat{\mathbf{r}} - \frac{e}{m}\mathbf{L} \right) - \frac{2\mu_0}{3} \boldsymbol{\mu}_e \cdot \boldsymbol{\mu}_N \, \delta(\mathbf{r}) \qquad (6)$$

- The expectation values of the Hamiltonian in the various atomic (and nuclear spin) states yield the hyperfine energy shifts.

- For spherically symmetric s states only the 2nd term of (6) has value:

$$\Rightarrow \quad \Delta E = -\frac{\mu_0}{4\pi} \frac{8\pi}{3} \left|\psi_e(0)\right|^2 \langle \boldsymbol{\mu}_e \cdot \boldsymbol{\mu}_N \rangle$$

- For $\ell \neq 0$, the hyperfine energy comes entirely from the 1st term of (6) because the wave functions for $\ell \neq 0$ vanish at the origin.

- $\boldsymbol{\mu}_e$ points in the opposite direction to the electron's spin because e is negative.

- ΔE between the singlet and triplet states of the $1s$ state of atomic hydrogen is the source of the famous 21cm line in astrophysics.

- Comparing eqn (4.20) & (5.64), if the magnetic moments were caused by *magnetic charges*, the coefficient $\dfrac{8\pi}{3}$ in ΔE would be replaced by $-\dfrac{4\pi}{3}$!

- The astrophysical hyperfine line of hydrogen would be at 42cm wavelength, and the singlet and triplet states would be reversed.

Macroscopic Equations, Boundary Conditions on B and H

- In macroscopic problems the current density is not a known function of position. Only its average over a macroscopic volume is known or pertinent.

- $\langle \nabla \cdot \mathbf{B}_{\text{micro}} = 0 \rangle \;\Rightarrow\; \nabla \cdot \mathbf{B} = 0 \;\Rightarrow\; \mathbf{B} = \nabla \times \mathbf{A}$

- Average macroscopic *magnetization* $\mathbf{M}(\mathbf{r}) = \sum N_i \langle \mathbf{m}_i \rangle \;\Leftarrow\; \mathbf{m}_i :$ molecular magnetic moment

- Suppose there is also a macroscopic current density

$$\Delta \mathbf{A} = \frac{\mu_0}{4\pi} \left(\frac{\boldsymbol{J}(\mathbf{r}')}{\mathbb{r}} + \mathbf{M}(\mathbf{r}') \times \frac{\hat{\mathbb{r}}}{\mathbb{r}^2} \right) \Delta V' \;\Leftarrow\; \text{vs} \quad \text{chapter 4}$$

$$\Rightarrow \quad \mathbf{A} = \frac{\mu_0}{4\pi} \int \left(\frac{\boldsymbol{J}(\mathbf{r}')}{\mathbb{r}} + \mathbf{M}(\mathbf{r}') \times \frac{\hat{\mathbb{r}}}{\mathbb{r}^2} \right) d^3 x' \quad (7a)$$

⇙ integration by parts + $\mathbf{M}$ is localized

$$= \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')}{\mathbb{r}} d^3 x' \;\Leftarrow\; \mathbf{M}(\mathbf{r}') \times \frac{\hat{\mathbb{r}}}{\mathbb{r}^2} = \mathbf{M} \times \nabla' \frac{1}{\mathbb{r}}$$

$$\Rightarrow \quad \boldsymbol{J}_M \equiv \nabla \times \mathbf{M} \;\Leftarrow\; \text{effective current density from magnetization}$$

$$\Rightarrow \quad \nabla \times \mathbf{B} = \mu_0 (\boldsymbol{J} + \nabla \times \mathbf{M}) \;\Leftarrow\; \text{macroscopic equivalent} \quad \nabla \times \mathbf{B}_{\text{micro}} = \mu_0 \boldsymbol{J}_{\text{micro}}$$

$\Rightarrow$ magnetic field $\mathbf{H} \equiv \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \;\Rightarrow\; \begin{matrix} \nabla \times \mathbf{H} = \boldsymbol{J} \\ \nabla \cdot \mathbf{B} = 0 \end{matrix}$ vs $\begin{matrix} \nabla \cdot \mathbf{D} = \rho \\ \nabla \times \mathbf{E} = 0 \end{matrix}$

Bound Currents

- Let the magnetic dipole moment per unit volume is $\mathbf{M}$ in magnetized material.
- The vector potential of a single dipole $\mathbf{m}$ is $\mathbf{A}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\dfrac{\mathbf{m}\times\hat{\mathscr{r}}}{\mathscr{r}^2} = \dfrac{\mu_0}{4\pi}\mathbf{m}\times\nabla'\dfrac{1}{\mathscr{r}}$
- In the magnetized object, each volume element d^3x' carries a dipole moment $\mathbf{M}\,\mathrm{d}^3x'$, so the total vector potential is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{M}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,\mathrm{d}^3x' = \frac{\mu_0}{4\pi}\int\left(\mathbf{M}(\mathbf{r}')\times\nabla'\frac{1}{\mathscr{r}}\right)\mathrm{d}^3x' \quad \Leftarrow \quad \nabla'\frac{1}{\mathscr{r}} = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

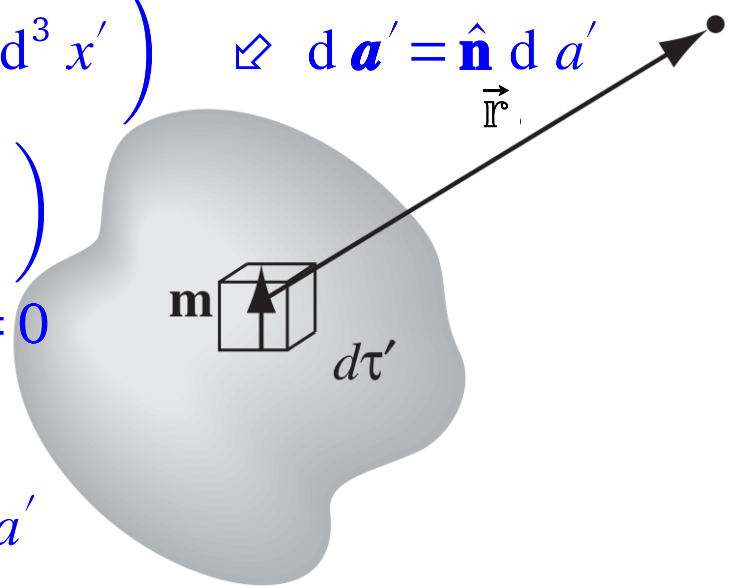
$$= \frac{\mu_0}{4\pi}\left(\int \frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{\mathscr{r}}\,\mathrm{d}^3x' - \int \nabla'\times\frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}}\,\mathrm{d}^3x'\right) \quad ⇦ \quad \mathrm{d}\boldsymbol{a}' = \hat{\mathbf{n}}\,\mathrm{d}a'$$

$$= \frac{\mu_0}{4\pi}\left(\int \frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{\mathscr{r}}\,\mathrm{d}^3x' + \oint \frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}}\times\mathrm{d}\boldsymbol{a}'\right)$$

$$\boldsymbol{J}_M \equiv \nabla\times\mathbf{M}, \qquad \boldsymbol{K}_M \equiv \mathbf{M}\times\hat{\mathbf{n}} \quad \Rightarrow \quad \nabla\cdot\boldsymbol{J}_M = 0$$

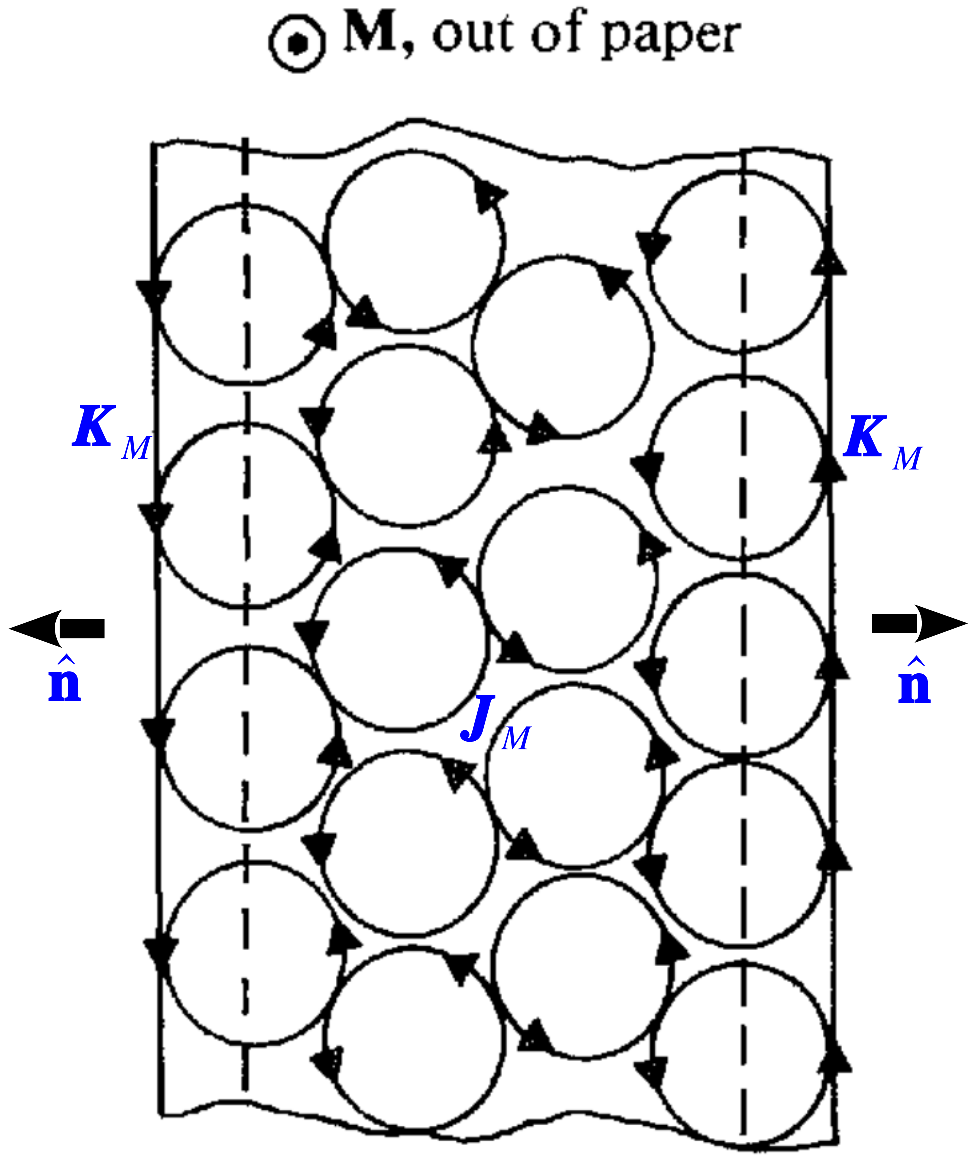
volume current   surface current

$$\Rightarrow \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\boldsymbol{J}_b(\mathbf{r}')}{\mathscr{r}}\,\mathrm{d}^3x' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\boldsymbol{K}_b(\mathbf{r}')}{\mathscr{r}}\,\mathrm{d}a'$$



- Instead of integrating the contributions of all the infinitesimal dipoles, we first determine the bound currents, and then find the field they produce.

$$\Rightarrow \quad \mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\boldsymbol{J}_M(\mathbf{r}') \times \hat{\mathfrak{r}}}{\mathfrak{r}^2}\, \mathrm{d}^3 x' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\boldsymbol{K}_M(\mathbf{r}') \times \hat{\mathfrak{r}}}{\mathfrak{r}^2}\, \mathrm{d}\, a'$$



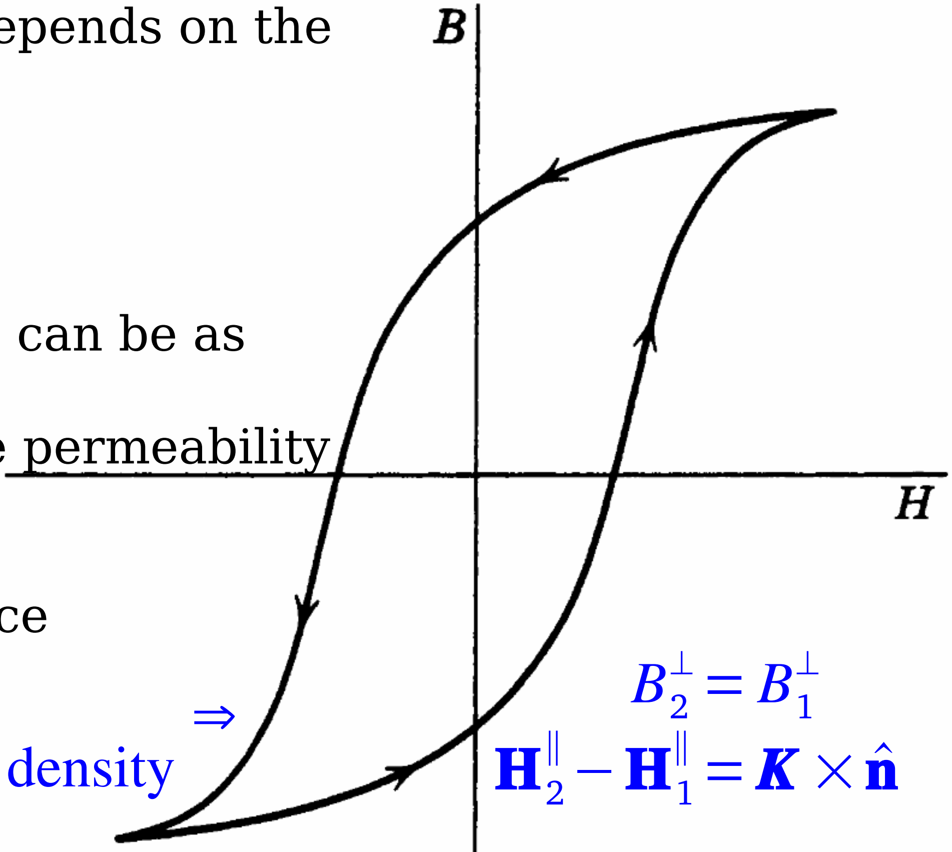
- The fundamental fields are $\mathbf{E}$ & $\mathbf{B}$. The derived fields $\mathbf{D}$ & $\mathbf{H}$ are introduced for convenience to take into account in an average way the contributions to ρ and $\boldsymbol{J}$ of the atomic charges and currents.

- To complete the description of macroscopic magnetostatics, there must be a constitutive relation between $\mathbf{H}$ and $\mathbf{B}$.

$\mathbf{B} = \mu \, \mathbf{H} \quad \Leftarrow \quad$ isotropic diamagnetic and paramagnetic substances, linear

$\mathbf{B} = \mathbf{F}(\mathbf{H}) \quad \Leftarrow \quad$ ferromagnetic substances, nonlinear $\qquad \mu$: magnetic permeability

- The phenomenon of hysteresis implies that $\mathbf{B}$ is not a single-valued function of $\mathbf{H}$. In fact, $\mathbf{F}(\mathbf{H})$ depends on the history of preparation of the material.



- Assuming $\mathbf{B} \parallel \mathbf{H} \quad \Rightarrow \quad \mu(\mathbf{H}) \equiv \dfrac{\mathrm{d}\,B}{\mathrm{d}\,H}$

- For high-permeability substances, $\dfrac{\mu(\mathbf{H})}{\mu_0}$ can be as high as 10^6. Typical values of initial relative permeability range from 10 to 10^4.

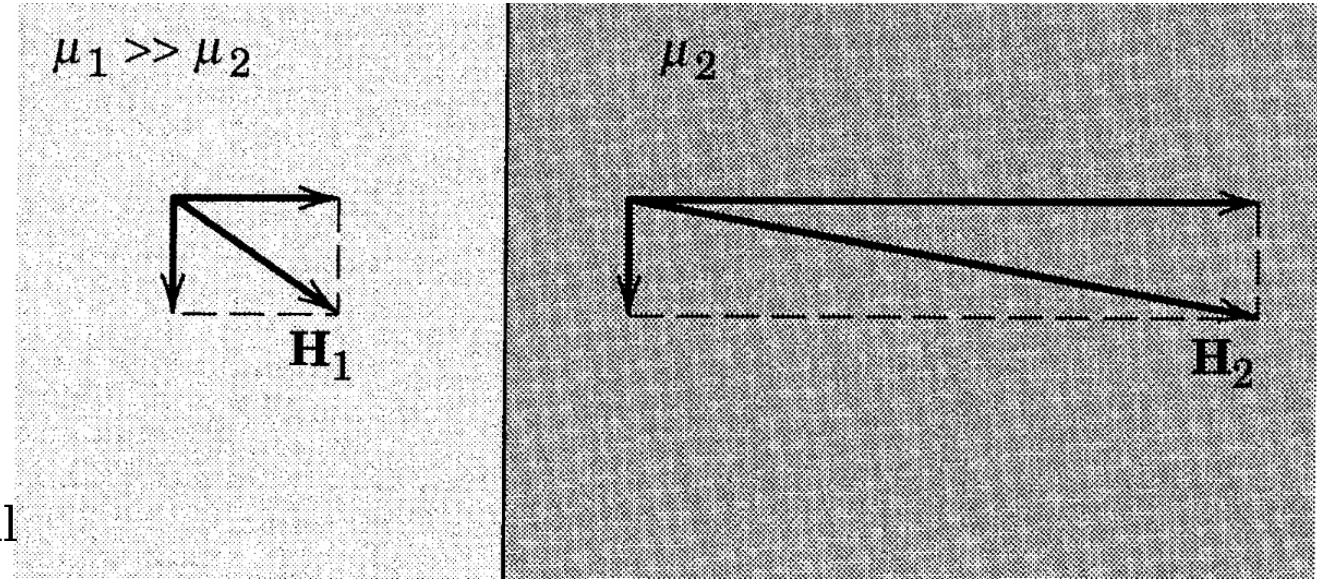
- For the boundary conditions at an interface

$$\begin{aligned} \hat{\mathbf{n}} \cdot (\mathbf{B}_2 - \mathbf{B}_1) &= 0 \\ \hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) &= \boldsymbol{K} \end{aligned} \quad \Leftarrow \quad \begin{aligned} &\hat{\mathbf{n}} : \text{normal vector} \\ &\boldsymbol{K} : \text{surface current density} \end{aligned} \quad \Rightarrow \quad \begin{aligned} B_2^{\perp} &= B_1^{\perp} \\ \mathbf{H}_2^{\parallel} - \mathbf{H}_1^{\parallel} &= \boldsymbol{K} \times \hat{\mathbf{n}} \end{aligned}$$

- For media satisfying linear relations $\begin{array}{c} B_2^{\perp} = B_1^{\perp} \\ \dfrac{\mathbf{B}_2^{\parallel}}{\mu_2} = \dfrac{\mathbf{B}_1^{\parallel}}{\mu_1} \end{array}$ or $\begin{array}{c} \mu_2 H_2^{\perp} = \mu_1 H_1^{\perp} \\ \mathbf{H}_2^{\parallel} = \mathbf{H}_1^{\parallel} \end{array}$

- $\mu_1 \gg \mu_2 \quad \Rightarrow \quad H_2^{\perp} \gg H_1^{\perp}$
- $\lim \dfrac{\mu_1}{\mu_2} \to \infty \quad \Rightarrow \quad \mathbf{H}_2 \propto \hat{\mathbf{n}}$

independent of the direction of $\mathbf{H}_1$ (except $\mathbf{H}_1 \perp \hat{\mathbf{n}}$).



- The boundary condition on $\mathbf{H}$ at the surface of a material of very high permeability is the same as for the electric field at the surface of a conductor.

- We may therefore use electrostatic potential theory for the magnetic field. The surfaces of the high-permeability material are approximately "equipotentials," and the lines of $\mathbf{H}$ are normal to these equipotentials.

Methods of Solving Boundary Value Problems in Magnetostatics

- The basic equations of magnetostatics $\nabla\cdot\mathbf{B}=0\,,\quad \nabla\times\mathbf{H}=\boldsymbol{J}\,,\quad \mathbf{B}=\mathbf{B}[\mathbf{H}]$

A. Generally Applicable Method of the Vector Potential

- $\nabla\cdot\mathbf{B}=0 \;\Rightarrow\; \mathbf{B}=\nabla\times\mathbf{A} \;\Rightarrow\; \nabla\times\mathbf{H}[\nabla\times\mathbf{A}]=\boldsymbol{J} \;\Leftarrow\; \mathbf{H}=\mathbf{H}[\mathbf{B}]$

  $\mathbf{B}=\mu\,\mathbf{H} \;\Rightarrow\; \nabla\times(\mu^{-1}\nabla\times\mathbf{A})=\boldsymbol{J} \;\Rightarrow\; \nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}=\mu\,\boldsymbol{J} \;\Leftarrow\; \mu=\text{const}$

  $\nabla\cdot\mathbf{A}=0 \;\;(\text{Coulomb gauge}) \;\Rightarrow\; \nabla^2\mathbf{A}=-\mu\,\boldsymbol{J}$

- Parallels the treatment of uniform isotropic dielectric media. The boundary conditions must be matched across the interface.

B. $\boldsymbol{J}=0$; Magnetic Scalar Potential

- $\boldsymbol{J}=0 \;\Rightarrow\; \nabla\times\mathbf{H}=0 \;\Rightarrow\; \mathbf{H}=-\nabla\Phi_M \;\Leftarrow\; \Phi_M:$ magnetic scalar potential vs $\mathbf{E}=-\nabla\Phi$

  $\mathbf{B}=\mathbf{B}[\mathbf{H}] \;\Rightarrow\; \nabla\cdot\mathbf{B}[-\nabla\Phi_M]=0 \;\Rightarrow\; \nabla\cdot(\mu\nabla\Phi_M)=0 \;\Leftarrow\; \mathbf{B}=\mu\,\mathbf{H}$

  For $\mu=\text{const} \;\Rightarrow\; \nabla^2\Phi_M=0$ + the boundary conditions for $\mathbf{H}$

  $\nabla^2\Psi_M=0 \;\Leftarrow\; \mathbf{B}=-\nabla\Psi_M$ + the boundary conditions for $\mathbf{B}$

- Φ_M can also be use for closed loops of current. Then Φ_M is proportional to the solid angle subtended by the boundary of the loop at the observation point [Problem 5.1]. Such a potential is evidently multiple-valued.

C. Hard Ferromagnets (M given and J=0)

- "Hard" ferromagnets has a magnetization that is independent of applied fields for moderate field strengths. Such materials can be treated as if they had a fixed, specified magnetization.

(a) Scalar Potential

- $\nabla\cdot\mathbf{B}=\mu_0\nabla\cdot(\mathbf{H}+\mathbf{M})=0 \;\Rightarrow\; \nabla^2\Phi_M=-\rho_M \;\Leftarrow\; \rho_M=-\nabla\cdot\mathbf{M}$, $\mathbf{H}=-\nabla\Phi_M \;\Leftarrow\; \boldsymbol{J}=0$

$$\Rightarrow\; \Phi_M(\mathbf{r})=\frac{1}{4\pi}\int\frac{\rho_M(\mathbf{r}')}{\mathbb{r}}\,\mathrm{d}^3x'=-\frac{1}{4\pi}\int\frac{\nabla'\cdot\mathbf{M}(\mathbf{r}')}{\mathbb{r}}\,\mathrm{d}^3x' \quad \text{if no boundary surface}$$

$$=\frac{1}{4\pi}\left(\int\mathbf{M}\cdot\nabla'\frac{1}{\mathbb{r}}\,\mathrm{d}^3x'-\oint_{r'\to\infty}\frac{\mathbf{M}}{\mathbb{r}}\cdot\mathrm{d}\boldsymbol{a}'\right) \;\Leftarrow\; \mathbf{M}\text{ well behaved \& localized}$$

$$\Rightarrow\; \Phi_M(\mathbf{r})=-\frac{1}{4\pi}\nabla\cdot\int\frac{\mathbf{M}(\mathbf{x}')}{\mathbb{r}}\,\mathrm{d}^3x' \quad (7) \;\Leftarrow\; \nabla'\frac{1}{\mathbb{r}}=-\nabla\frac{1}{\mathbb{r}}$$

$$\approx-\frac{1}{4\pi}\nabla\frac{1}{r}\cdot\int\mathbf{M}(\mathbf{r}')\,\mathrm{d}^3x'=\frac{\mathbf{m}\cdot\mathbf{r}}{4\pi r^3} \;\Leftarrow\; \mathbf{m}\equiv\int\mathbf{M}\,\mathrm{d}^3x,\quad r\gg0$$

- An arbitrary localized distribution of magnetization asymptotically has a dipole field with strength given by the total magnetic moment of the distribution.

- If a "hard" ferromagnet has a volume and surface, we specify **M** inside the volume and assume that it falls suddenly to 0 at the surface, and assign an *effective magnetic surface-charge density* $\sigma_M=\hat{\mathbf{n}}\cdot\mathbf{M}$

$$\Phi(\mathbf{r}) = -\frac{1}{4\pi}\nabla\cdot\int\frac{\mathbf{M}}{\mathbb{r}}\,d^3x' = \frac{1}{4\pi}\int\mathbf{M}\cdot\nabla'\frac{1}{\mathbb{r}}\,d^3x' = \frac{1}{4\pi}\int_{\mathcal{V}}\mathbf{M}\cdot\nabla'\frac{1}{\mathbb{r}}\,d^3x'$$

$$= \frac{1}{4\pi}\oint_{\mathcal{S}}\frac{\mathbf{M}}{\mathbb{r}}\cdot d\boldsymbol{a}' - \frac{1}{4\pi}\int_{\mathcal{V}}\frac{\nabla'\cdot\mathbf{M}}{\mathbb{r}}\,d^3x' \quad \Leftarrow \quad \mathbf{M}(\mathbf{r}'\notin\mathcal{V}) = 0, \quad d\boldsymbol{a}' = \hat{\mathbf{n}}\,da'$$

$$\Rightarrow \boxed{\Phi(\mathbf{r}) = \frac{1}{4\pi}\int_{\mathcal{V}}\frac{\rho_M}{\mathbb{r}}\,d^3x' + \frac{1}{4\pi}\oint_{\mathcal{S}}\frac{\sigma_M}{\mathbb{r}}\,da'} \quad \Leftarrow \quad \boxed{\rho_M = -\nabla\cdot\mathbf{M}, \quad \sigma_M = \mathbf{M}\cdot\hat{\mathbf{n}}}$$

$$\text{if } \mathbf{M} \text{ is uniform} \Rightarrow \quad \Phi(\mathbf{r}) = \frac{1}{4\pi}\oint_{\mathcal{S}}\frac{\mathbf{M}}{\mathbb{r}}\cdot d\boldsymbol{a}' \qquad (7b)$$

- (7) is generally applicable even for the limit of discontinuous distributions of **M**. *Never combine the surface integral of* σ_M *with* (7)!

(b) Vector Potential $\mathbf{B} = \nabla\times\mathbf{A} \Rightarrow \nabla\times\mathbf{H} = \nabla\times\left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = 0$

$$\Rightarrow \quad \nabla^2\mathbf{A} = -\mu_0\boldsymbol{J}_M \quad \Leftarrow \quad \text{Coulomb gauge} \;+\; \boldsymbol{J}_M = \nabla\times\mathbf{M}$$

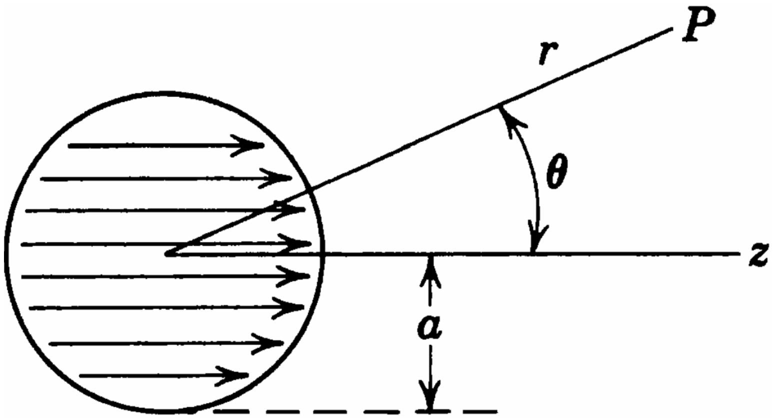
$$\Rightarrow \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}}\frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{\mathbb{r}}\,d^3x'$$

- If the distribution of **M** is discontinuous, a surface integral is needed.

$$\Rightarrow \boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}}\frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{\mathbb{r}}\,d^3x' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}}\frac{\mathbf{M}(\mathbf{r}')}{\mathbb{r}}\times d\boldsymbol{a}'} \quad \Leftarrow \quad (7a)$$

- If **M** is constant throughout the volume, only the surface integral survives.

Uniformly Magnetized Sphere



- Consider Eq. (7b) of the previous section,

$$\mathbf{M} = M_0\,\hat{\mathbf{z}} \quad \Rightarrow \quad \sigma_M = \hat{\mathbf{n}}\cdot\mathbf{M} = \hat{\mathbf{r}}\cdot\mathbf{M} = M_0 \cos\theta$$

$$\Rightarrow \quad \Phi_M(r,\theta) = \frac{M_0\,a^2}{4\pi}\int \frac{\cos\theta'}{\mathbb{r}}\,\mathrm{d}\Omega' \quad \Leftarrow \quad \rho_M = 0$$

$$= \frac{1}{3} M_0\,a^2 \frac{r_<}{r_>^2}\cos\theta \quad \Leftarrow \quad \begin{matrix}\text{only the } \ell = 1 \\ \text{term survives}\end{matrix} \quad \Leftarrow \quad r_{\lessgtr} = \begin{matrix}\min \\ \max\end{matrix}(a, r)$$

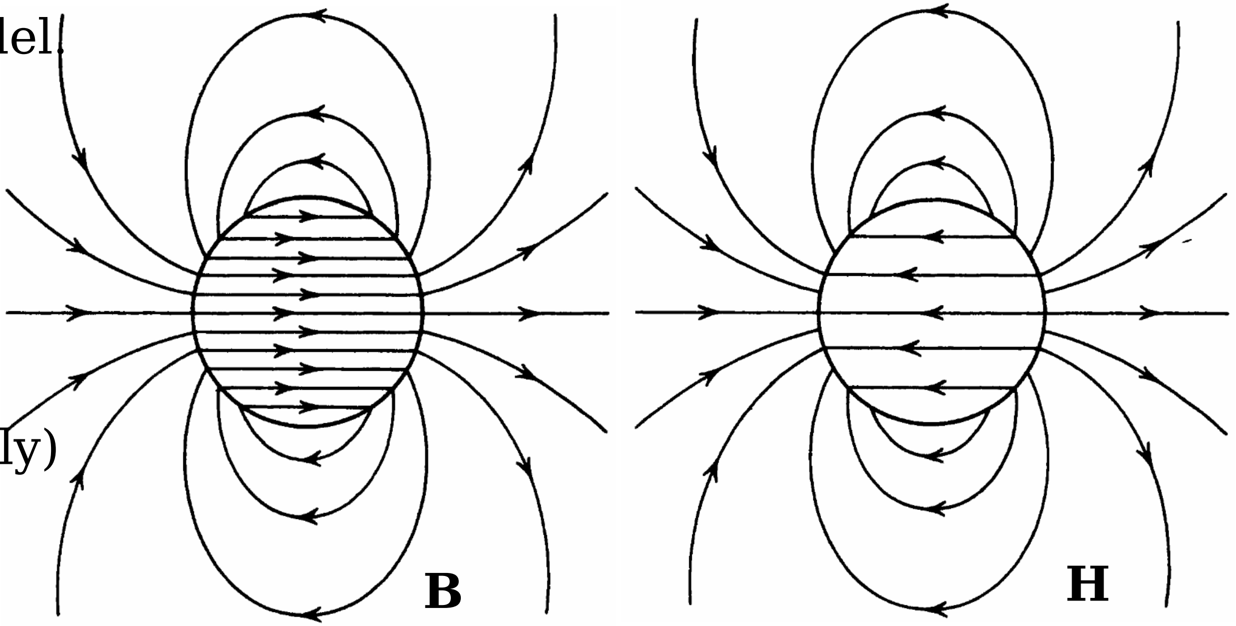
$$r < a \quad \Rightarrow \quad \Phi_M = \frac{1}{3} M_0\, r\cos\theta = \frac{1}{3} M_0\, z \quad \Rightarrow \quad \mathbf{H}_{\text{in}} = -\frac{1}{3}\mathbf{M}, \qquad \mathbf{B}_{\text{in}} = \frac{2}{3}\mu_0\mathbf{M}$$

$$r > a \quad \Rightarrow \quad \Phi_M = \frac{1}{3} M_0\, a^3 \frac{\cos\theta}{r^2} = \frac{m}{4\pi}\frac{\cos\theta}{r^2} \quad \Rightarrow \quad \mathbf{m} = \frac{4\pi a^3}{3}\mathbf{M}$$

- $\mathbf{B}_{\text{in}} \parallel \mathbf{M}$, while $\mathbf{H}_{\text{in}}$ is antiparallel.

- For the sphere with uniform $\mathbf{M}$, the fields are not only dipole asymptotically, but also close to the sphere.

- For this geometry (and this only) there are no higher multipoles.



- The lines of $\mathbf{B}$ are continuous closed paths, but those of $\mathbf{H}$ terminate on the surface because there is an effective surface-charge density σ_M .
- Use (7) $\Rightarrow$ $\Phi_M(r,\theta) = -\frac{1}{4\pi} M_0 \frac{\partial}{\partial z} \int_0^a r'^2 \, dr' \int \frac{d\Omega'}{\mathbb{r}}$

$$= -M_0 \cos\theta \frac{\partial}{\partial r} \int_0^a \frac{r'^2}{r_>} dr' = \frac{1}{3} M_0 a^2 \frac{r_<}{r_>^2} \cos\theta \quad \Leftarrow \quad \text{only } \ell = 0 \text{ term survives}$$

- Use part C(b) $\Rightarrow$ $\boldsymbol{J}_M = \nabla' \times \mathbf{M} = 0$

$$\boldsymbol{K}_M = \mathbf{M} \times \hat{\mathbf{r}}' = M_0 \sin\theta' \hat{\boldsymbol{\phi}}' = M_0 \sin\theta' (-\sin\phi' \hat{\mathbf{x}} + \cos\phi' \hat{\mathbf{y}}) \quad \Leftarrow \quad \mathbf{M} = M_0 \hat{\mathbf{z}}$$

$$\Rightarrow \quad \mathbf{A} = \frac{\mu_0}{4\pi} \oint_S \frac{\boldsymbol{K}_M}{\mathbb{r}} a^2 \, d\Omega = \frac{\mu_0}{4\pi} a^2 M_0 \oint \frac{\sin\theta'}{\mathbb{r}} (-\sin\phi' \hat{\mathbf{x}} + \cos\phi' \hat{\mathbf{y}}) \, d\Omega'$$

$$= \frac{\mu_0}{4\pi} a^2 M_0 \oint \sqrt{\frac{2\pi}{3}} \frac{(\hat{\mathbf{y}} - i\hat{\mathbf{x}}) Y_{1,-1} - (\hat{\mathbf{y}} + i\hat{\mathbf{x}}) Y_{11}}{\mathbb{r}} \, d\Omega'$$

$$= \frac{\mu_0}{3} M_0 a^2 \frac{r_<}{r_>^2} \sin\theta \, \hat{\boldsymbol{\phi}} \quad \Leftarrow \quad \text{with the expansion of } \frac{1}{\mathbb{r}}$$

only the $\ell = 1, m = \pm 1$ terms survive

$\Rightarrow$ $\mathbf{B} = \nabla \times \mathbf{A}$ $\Rightarrow$ give the same result

- One can also try the $\mathbf{r}$ as $\mathbf{z}$ method.

Using a scalar potential (7) as the alternative **1**:

$$\Phi_M = -\frac{1}{4\pi}\nabla\cdot\int\frac{\mathbf{M}}{\mathbb{r}}\,\mathrm{d}^3x'$$

$$\int\frac{\mathbf{M}}{\mathbb{r}}\,\mathrm{d}^3x' = \mathbf{M}\int\frac{\mathrm{d}^3x'}{\mathbb{r}} = \mathbf{M}\int\frac{r'^2\sin\theta'\,\mathrm{d}r'\,\mathrm{d}\theta'\,\mathrm{d}\phi'}{\sqrt{r^2+r'^2-2\,r\,r'\cos\theta'}} \quad \Leftarrow \quad \text{choose } \mathbf{r} = r\,\hat{\mathbf{z}} \text{ temporarily}$$

$$= \frac{2\pi\mathbf{M}}{r}\int_0^a r'\left(r+r'-|r-r'|\right)\mathrm{d}r' = 2\pi\mathbf{M}\begin{cases} a^2-\dfrac{r^2}{3} & \text{for } r<a \text{ with } \displaystyle\int_0^r+\int_r^a \\ \dfrac{2a^3}{3r} & \text{for } r>a \;\Rightarrow\; r>r' \end{cases}$$

$$\Phi_M = -\frac{1}{4\pi}\nabla\cdot\int\frac{\mathbf{M}}{\mathbb{r}}\,\mathrm{d}^3x' \;\Rightarrow\; \Phi_{\text{in}}(r,\theta) = \frac{M}{3}\,z\,, \quad \Phi_{\text{out}}(r,\theta) = \frac{M}{3}\,\frac{a^3}{r^2}\cos\theta$$

$$\mathbf{H} = -\nabla\Phi_M \;\Rightarrow\; \mathbf{H}_{\text{in}} = -\frac{1}{3}\mathbf{M}\,, \quad \mathbf{H}_{\text{out}} = \frac{M}{3}\,\frac{a^3}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

$$\mathbf{B} = \mu_0\left(\mathbf{H}+\mathbf{M}\right) \;\Rightarrow\; \mathbf{B}_{\text{in}} = \frac{2}{3}\mu_0\mathbf{M}\,, \quad \mathbf{B}_{\text{out}} = \mu_0\,\frac{M}{3}\,\frac{a^3}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

Using a scalar potential (7b) as the alternative **2**: $\Phi_M = \dfrac{1}{4\pi} \oint_S \dfrac{\sigma_M}{\mathscr{r}}\, \mathrm{d}\, a'$

Use the similar trick like the one with vector potential:

Choose $\mathbf{r} = r\,\hat{\mathbf{z}}$, put $\mathbf{M}$ in the xz-plane $\Rightarrow$ $\mathbf{M} = M \sin\theta\,\hat{\mathbf{x}} + M\cos\theta\,\hat{\mathbf{z}}$

And $\hat{\mathbf{r}}' = \sin\theta' \cos\phi'\,\hat{\mathbf{x}} + \sin\theta' \sin\phi'\,\hat{\mathbf{y}} + \cos\theta'\,\hat{\mathbf{z}}$

$$\Rightarrow \quad \sigma_M = \mathbf{M}\cdot\hat{\mathbf{r}}' = M\left(\sin\theta \sin\theta' \cos\phi' + \cos\theta\cos\theta'\right)$$

$$\Rightarrow \quad \oint \frac{\sigma_M}{\mathscr{r}}\,\mathrm{d}\,a' = M \oint \frac{\sin\theta\sin\theta' \cos\phi' + \cos\theta\cos\theta'}{\sqrt{r^2 + a^2 - 2\,a\,r\cos\theta'}}\, a^2 \sin\theta'\,\mathrm{d}\,\theta'\,\mathrm{d}\,\phi'$$

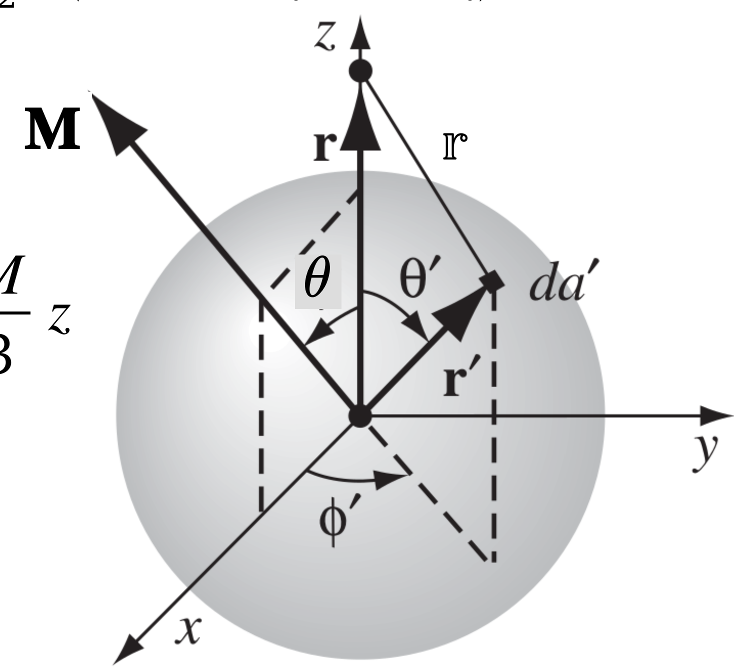
$$= 2\pi\, M\, a^2 \cos\theta \int_0^\pi \frac{\cos\theta' \sin\theta'\,\mathrm{d}\,\theta'}{\sqrt{r^2 + a^2 - 2\,a\,r\cos\theta'}} = \frac{2\pi M}{3\,r^2}\left(a^3 + r^3 - |a^3 - r^3|\right)\cos\theta$$

$$= \frac{4\pi M}{3}\,\frac{r_<^3}{r^2}\cos\theta \quad \Leftarrow \quad r_< = \min(r, a)$$

$$\Rightarrow \quad \Phi_M = \frac{M}{3}\,\frac{r_<^3}{r^2}\cos\theta \quad \Rightarrow \quad \Phi_{\text{in}}(r,\theta) = \frac{M}{3}\, r\cos\theta = \frac{M}{3}\, z$$

$$\Phi_{\text{out}}(r,\theta) = \frac{M}{3}\,\frac{a^3}{r^2}\cos\theta$$

$$\Rightarrow \quad \mathbf{H} = -\nabla\,\Phi_M, \quad \mathbf{B} = \mu_0\left(\mathbf{H} + \mathbf{M}\right)$$



Using a scalar potential as the alternative **3**:

$$\mathbf{H} = -\nabla \Phi_M \quad \Leftarrow \quad \nabla \times \mathbf{H} = \boldsymbol{J}_f = 0 \quad \Leftarrow \quad \text{no free current}$$

$$\nabla^2 \Phi_M = 0 \quad \Leftarrow \quad \nabla \cdot \mathbf{H} = \nabla \cdot \mathbf{M} = 0 \quad \Leftarrow \quad \mathbf{M} = M\,\hat{\mathbf{z}}$$

$$\Rightarrow \quad \Phi_{\text{in}}(r,\theta) = \sum_{\ell=0} C_\ell\, r^\ell\, P_\ell(\cos\theta), \quad \Phi_{\text{out}}(r,\theta) = \sum_{\ell=0} \frac{D_\ell}{r^{\ell+1}}\, P_\ell(\cos\theta)$$

Boundary conditions: $(1)\ \Phi_{\text{in}}(a) = \Phi_{\text{out}}(a), \quad (2)\ B_{\text{in},r}(a) = B_{\text{out},r}(a) \quad \Leftarrow \quad \mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$

$$\Rightarrow (1)\ C_\ell = \frac{D_\ell}{a^{2\ell+1}} \quad (2)\ \mu_0(C_1 + M) = -2\mu_0 \frac{D_1}{a^3}, \quad \mu_0\,\ell\, C_\ell = -\mu_0 \frac{\ell+1}{a^{2\ell+1}} D_\ell \ \text{ for } \ \ell \neq 1$$

$$\Rightarrow \quad C_1 = \frac{M}{3}, \quad D_1 = \frac{M}{3} a^3, \quad C_\ell = D_\ell = 0 \ \text{ for } \ \ell \neq 1$$

$$\Rightarrow \quad \Phi_{\text{in}}(r,\theta) = \frac{M}{3} z, \quad \Phi_{\text{out}}(r,\theta) = \frac{M}{3} \frac{a^3}{r^2} \cos\theta$$

$$\Rightarrow \quad \mathbf{H}_{\text{in}} = -\frac{1}{3}\mathbf{M}, \quad \mathbf{H}_{\text{out}} = \frac{1}{3} \frac{a^3}{r^3}\left[3(\hat{\mathbf{r}}\cdot\mathbf{M})\,\hat{\mathbf{r}} - \mathbf{M}\right]$$

$$\Rightarrow \quad \mathbf{B}_{\text{in}} = \frac{2}{3}\mu_0 \mathbf{M}, \quad \mathbf{B}_{\text{out}} = \frac{\mu_0}{3} \frac{a^3}{r^3}\left[3(\hat{\mathbf{r}}\cdot\mathbf{M})\,\hat{\mathbf{r}} - \mathbf{M}\right]$$

Magnetized Sphere in an External Field; Permanent Magnets

- Consider in the space

$$\mathbf{B}_0 = \mu_0 \mathbf{H}_0 \quad \Rightarrow \quad \mathbf{B}_{in} = \mathbf{B}_0 + \frac{2\mu_0}{3}\mathbf{M}, \quad \mathbf{H}_{in} = \frac{1}{\mu_0}\mathbf{B}_0 - \frac{1}{3}\mathbf{M} \qquad (8) \quad \Leftarrow \quad \text{inside the permanent magneitzed sphere}$$

- Consider a paramagnetic or diamagnetic sphere of permeability μ, $\mathbf{M}$ comes from the external field

$$\mathbf{B}_{in} = \mu \mathbf{H}_{in} \quad \Rightarrow \quad \mathbf{B}_0 + \frac{2\mu_0}{3}\mathbf{M} = \mu\left(\frac{1}{\mu_0}\mathbf{B}_0 - \frac{1}{3}\mathbf{M}\right) \quad \Rightarrow \quad \mathbf{M} = \frac{3}{\mu_0}\frac{\mu-\mu_0}{\mu+2\mu_0}\mathbf{B}_0$$

analogous to the polarization of a dielectric sphere in a uniform electric field.

- For a ferromagnetic substance, the above argument fails because the existence of permanent magnets contradicts this result.

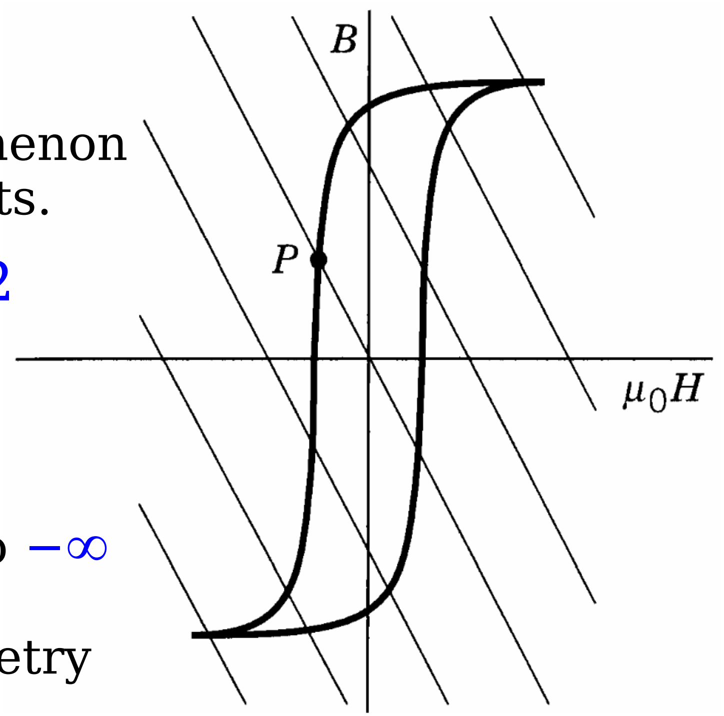
- The nonlinear constitutive relation and the phenomenon of hysteresis allow the creation of permanent magnets.

$$(8) \quad \Rightarrow \quad \mathbf{B}_{in} + 2\mu_0 \mathbf{H}_{in} = 3\mathbf{B}_0 \quad \Leftarrow \quad \text{line with slope } -2$$

- Increase $\mathbf{B}_0$ till saturation then decrease it to 0.

$$\mathbf{B}_0 = 0 \quad \text{gives} \quad \mathbf{M}$$

- The slope of the lines range from 0 for a flat disc to $-\infty$ for a long needle-like object. Thus a larger internal magnetic induction can be obtained with a rod geometry than with the other shapes.



Magnetic Shielding, Spherical Shell of Permeable Material in a Uniform Field

- Cosider $\mathbf{B}_0=\mu_0\mathbf{H}_0$ in an empty space. A permeable body is placed in the region.

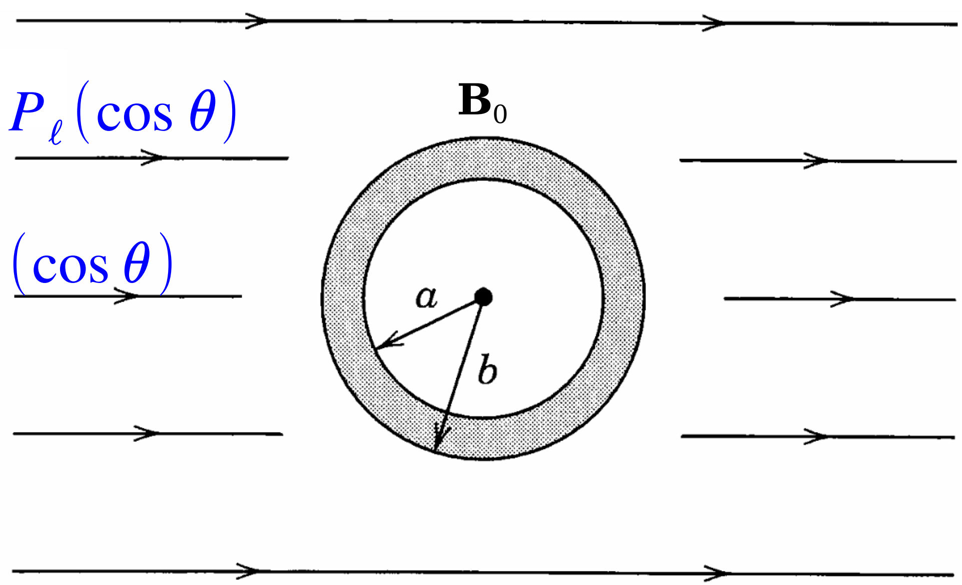
- For high permeability, the field lines should tend to be normal to the body's surface. If the body is hollow, the field in the cavity should be smaller than the external field, vanishing in the limit $\mu\to\infty$, ie, *magnetic shielding*.

- $$\boldsymbol{J}=0 \Rightarrow \mathbf{H}=-\nabla\Phi_M \Rightarrow \left[\begin{array}{l}\mathbf{B}=\mu\mathbf{H}\\ \nabla\cdot\mathbf{B}=0\end{array}\right. \Rightarrow \nabla\cdot\mathbf{H}=0 \Rightarrow \nabla^2\Phi_M=0$$

$$r>b \Rightarrow \Phi_M=-H_0 r\cos\theta+\sum\frac{\alpha_\ell}{r^{\ell+1}}P_\ell(\cos\theta)$$

$$a<r<b \Rightarrow \Phi_M=\sum\left(\beta_\ell r^\ell+\frac{\gamma_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

$$r<a \Rightarrow \Phi_M=\sum\lambda_\ell r^\ell P_\ell(\cos\theta)$$



$$\Rightarrow \begin{array}{l}\dfrac{\partial\Phi_M}{\partial\theta}\Big|_{r=b^+}=\dfrac{\partial\Phi_M}{\partial\theta}\Big|_{r=b^-}, \quad \dfrac{\partial\Phi_M}{\partial\theta}(a_+)=\dfrac{\partial\Phi_M}{\partial\theta}(a_-)\\ \mu_0\dfrac{\partial\Phi_M}{\partial r}(b_+)=\mu\dfrac{\partial\Phi_M}{\partial r}(b_-), \quad \mu\dfrac{\partial\Phi_M}{\partial r}(a_+)=\mu_0\dfrac{\partial\Phi_M}{\partial r}(a_-)\end{array} \Leftarrow$$

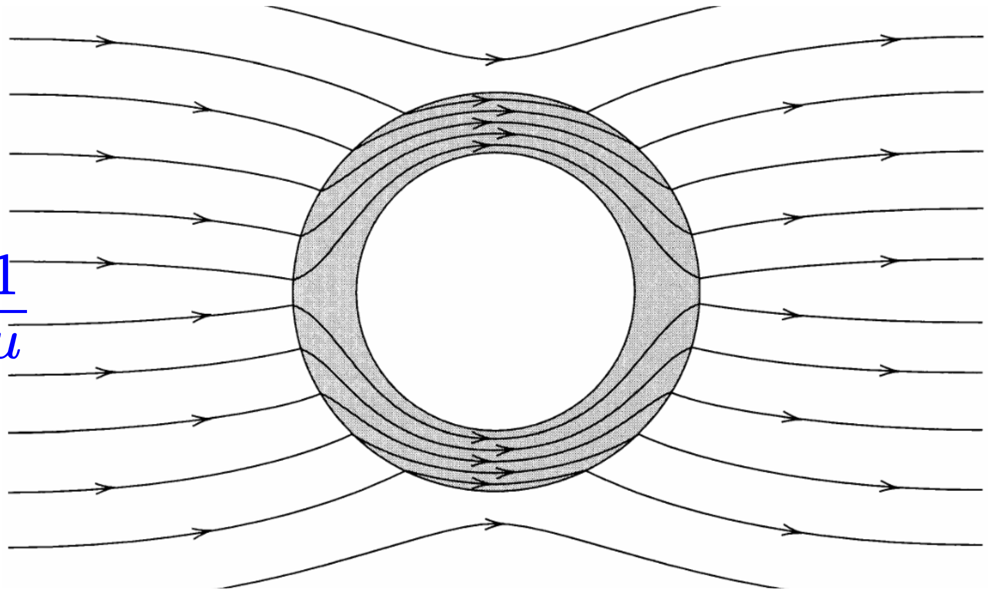
H_θ & B_r are continuous at $r=a$ & $r=b$

$$\begin{aligned} \alpha_1 - b^3\beta_1 - \gamma_1 &= b^3 H_0 \\ \Rightarrow \quad 2\alpha_1 + \mu_r b^3 \beta_1 - 2\mu_r\gamma_1 &= -b^3 H_0 \quad \Leftarrow \quad \mu_r \equiv \frac{\mu}{\mu_0} \\ a^3\beta_1 + \gamma_1 - a^3\lambda_1 &= 0 \\ \mu_r a^3 \beta_1 - 2\mu_r\gamma_1 - a^3\lambda_1 &= 0 \end{aligned}$$

all $\ell \neq 1$ terms vanish

$$\Rightarrow \quad \alpha_1 = \frac{(2\mu_r+1)(\mu_r-1)(b^3-a^3)H_0}{(2\mu_r+1)(\mu_r+2) - 2\frac{a^3}{b^3}(\mu_r-1)^2}, \quad \lambda_1 = \frac{-9\mu_r H_0}{(2\mu_r+1)(\mu_r+2) - 2\frac{a^3}{b^3}(\mu_r-1)^2}$$

- The potential outside the spherical shell corresponds to a uniform field $\mathbf{H}_0$ plus a dipole field with dipole moment α_1 oriented parallel to $\mathbf{H}_0$. Inside the cavity, there is a uniform magnetic field parallel to $\mathbf{H}_0$, equal to $-\lambda_1$.

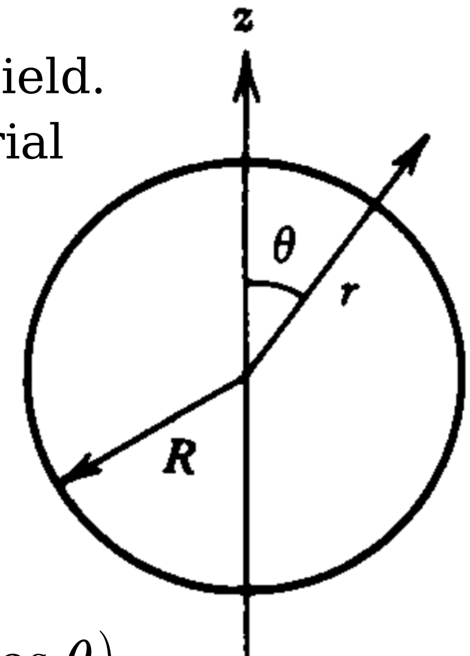


- $\mu \gg \mu_0 \quad \Rightarrow \quad \begin{aligned} &\alpha_1 \to b^3 H_0 \\ &-\lambda_1 \to \frac{9b^3}{2\mu_r(b^3-a^3)} H_0 \propto \frac{1}{\mu} \end{aligned}$

- with $\mu_r \sim 10^3$ to 10^6, a shield causes a great reduction in the field inside it, even with a relatively thin shell.

$$\beta_1 = \frac{-3(2\mu_r+1)b^3 H_0}{b^3(2\mu_r+1)(\mu_r+2) - 2a^3(\mu_r-1)^2}, \quad \gamma_1 = \frac{-3(\mu_r-1)a^3 b^3 H_0}{b^3(2\mu_r+1)(\mu_r+2) - 2a^3(\mu_r-1)^2}$$

Example: A Magnetic Sphere in a Uniform External Magnetic Field. Consider a sphere of radius R, made of a linear magnetic material of permeability μ_1, embedded in a medium of permeability μ_2. The sphere is placed in a magnetic field $\mathbf{H}_0$ which is initially uniform and pointing along the z direction.



$$\text{Current} = 0 \quad \Rightarrow \quad \mathbf{H} = -\nabla\Phi \quad \Leftarrow \quad \nabla\times\mathbf{H} = \boldsymbol{J}_f = 0\,, \quad \mathbf{B} = \mu\,\mathbf{H}$$

$$\Rightarrow \quad \Phi(r\to\infty) = -H_0\,z = -H_0\,r\cos\theta\,, \quad \text{choose} \;\; \Phi(r=0) = 0$$

$$\Rightarrow \quad \Phi_{\text{in}} = \sum C_\ell\, r^\ell\, P_\ell(\cos\theta)\,, \quad \Phi_{\text{out}} = -H_0\, r\cos\theta + \sum_{\ell=0} \frac{D_\ell}{r^{\ell+1}}\, P_\ell(\cos\theta)$$

Boundary conditions: $(1)\;\; \Phi_{\text{in}}(R) = \Phi_{\text{out}}(R)\,, \;\; (2)\;\; B_{\text{in}\,,\,r}(R) = B_{\text{out}\,,\,r}(R)$

$$\Rightarrow \quad C_1 = \frac{D_1}{R^3} - H_0\,, \qquad C_\ell = \frac{D_\ell}{R^{2\ell+1}} \;\; \text{for} \;\; \ell \neq 1 \quad \Leftarrow \quad (1)$$

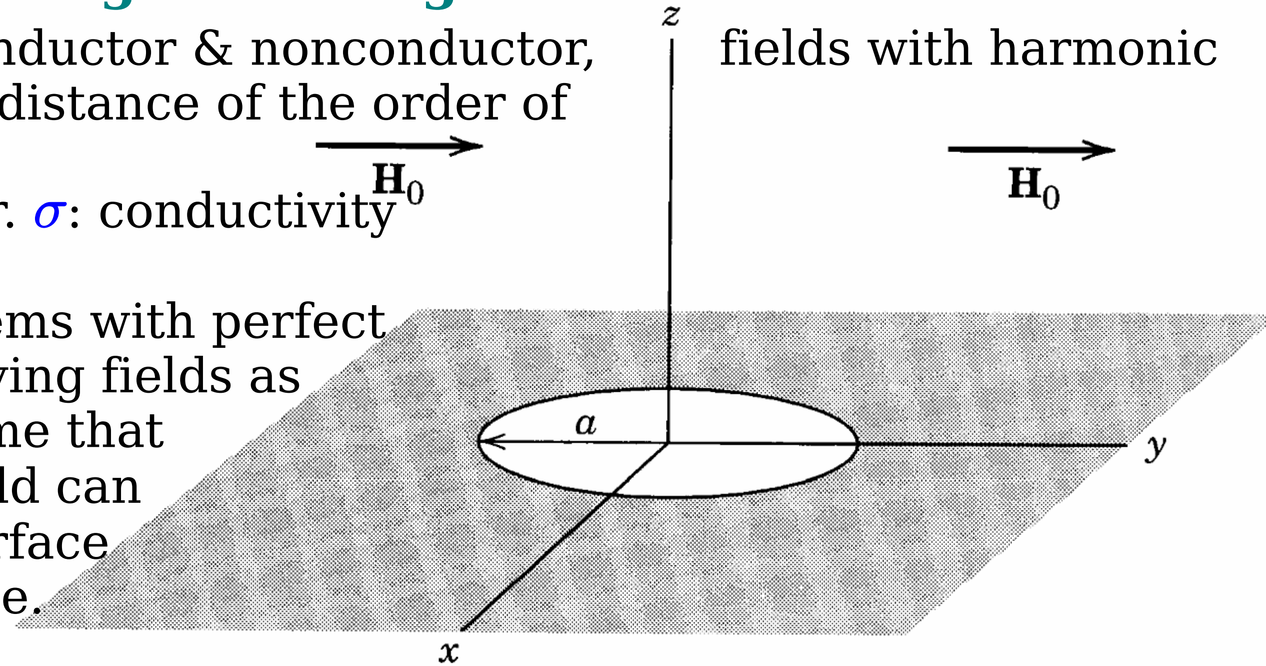
$$\mu_1\, C_1 = -\mu_2\left(2\,\frac{D_1}{R^3} + H_0\right), \quad \mu_1\,\ell\, C_\ell = -\mu_2(\ell+1)\,\frac{D_\ell}{R^{2\ell+1}} \;\; \text{for} \;\; \ell\neq 1 \quad \Leftarrow \quad (2)$$

$$\Rightarrow \quad C_1 = -\frac{3\,\mu_2}{\mu_1 + 2\,\mu_2}\, H_0\,, \quad D_1 = \frac{\mu_1 - \mu_2}{\mu_1 + 2\,\mu_2}\, H_0\, R^3\,, \quad C_\ell = D_\ell = 0 \;\; \text{for} \;\; \ell \neq 1$$

$$\Rightarrow \quad \mathbf{B}_{\text{in}} = \frac{3\,\mu_1\,\mu_2}{\mu_1 + 2\,\mu_2}\,\mathbf{H}_0\,, \quad \mathbf{B}_{\text{out}} = \mu_2\left(\mathbf{H}_0 + \frac{\mu_1 - \mu_2}{\mu_1 + 2\,\mu_2}\,\frac{R^3}{r^3}\left[3\,(\hat{\mathbf{r}}\cdot\mathbf{H}_0)\,\hat{\mathbf{r}} - \mathbf{H}_0\right]\right)$$

Effect of a Circular Hole in a Perfectly Conducting Plane with an Asymptotically Uniform Tangential Magnetic Field on One Side

- At the interface between conductor & nonconductor, fields with harmonic time dependence penetrate a distance of the order of $\delta = \sqrt{\dfrac{2}{\mu\,\omega\,\sigma}}$ into the conductor. σ: conductivity



- Define magnetostatic problems with perfect conductors as the limit of varying fields as $\omega \to 0$, provided at the same time that $\omega\sigma \to \infty$. Then the magnetic field can exist outside and up to the surface of the conductor, but not inside.

$$B_\perp = 0\,,\quad \hat{\mathbf{n}} \times \mathbf{H}_\parallel = \boldsymbol{K} \quad \text{vs} \quad \mathbf{E}_\parallel = 0\,,\quad D_\perp = \sigma \quad \Leftarrow \quad \sigma:\ \text{surface-charge density}$$

- No currents except on the surface $z = 0$
$$\Rightarrow \quad \mathbf{H} = -\nabla\,\Phi_M \quad \Rightarrow \quad \nabla^2 \Phi_M = 0 \quad \Leftarrow \quad \nabla \cdot \mathbf{B} = 0$$

$$\Rightarrow \ \Phi_M(\mathbf{r}) = \begin{cases} -H_0\,y + \Phi^{(1)}, & \text{for } z > 0 \\ \quad 0 \quad - \Phi^{(1)}, & \text{for } z < 0 \end{cases} \Leftarrow \begin{cases} H_x^{(1)} \ \&\ H_y^{(1)} \ \text{are odd in } z \\ H_z^{(1)} \ \&\ \Phi^{(1)} \ \text{are even in } z \end{cases} \Leftarrow \ \begin{array}{c}\text{the symmetry} \\ \text{properties of} \\ \text{the added fields}\end{array}$$

$$\Rightarrow \ \Phi^{(1)} = \int_0^\infty A(k)\, e^{-k|z|}\, J_1(k\,\rho)\, \sin\phi \,\mathrm{d}\,k \ \Leftarrow \ \begin{array}{c}\text{only} \\ m = 1\end{array} \ \Leftarrow \begin{array}{l}(3.106)\ \text{cylindrically symmetric} \\ \Phi_M(\mathbf{r} \to \infty) = \Phi_M(y = \rho \sin\phi)\end{array}$$

$$\boldsymbol{J}(\mathbf{r}') = \boldsymbol{J}(z'=0) = J_x\,\hat{\mathbf{x}} + J_y\,\hat{\mathbf{y}}$$

$$\Rightarrow \quad \mathbf{B}^{(1)}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \boldsymbol{J}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}^3x'$$

$$= \frac{\mu_0}{4\pi}\int (J_x\,\hat{\mathbf{x}} + J_y\,\hat{\mathbf{y}})\times\frac{(x-x')\,\hat{\mathbf{x}} + (y-y')\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}\,\mathrm{d}^3x'$$

$$= \frac{\mu_0}{4\pi}\int \frac{J_y\,z\,\hat{\mathbf{x}} - J_x\,z\,\hat{\mathbf{y}} + [J_x\,(y-y') - J_y\,(x-x')]\,\hat{\mathbf{z}}}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}\,\mathrm{d}^3x'$$

$$\mathbf{H}^{(1)} \propto \mathbf{B}^{(1)} \quad \Rightarrow \quad H_x^{(1)}(z^+) = -H_x^{(1)}(z^-), \quad H_y^{(1)}(z^+) = -H_y^{(1)}(z^-)$$

$$H_z^{(1)}(z^+) = +H_z^{(1)}(z^-) = -\partial_z\,\Phi^{(1)} \quad \Rightarrow \quad \Phi^{(1)}(z^+) = +\Phi^{(1)}(z^-)$$

$$-\nabla\,\Phi_M^{(1)} \Leftarrow \mathbf{H}^{(1)} = \frac{\mathbf{B}^{(1)}}{\mu_0} = \frac{1}{4\pi}\,\frac{3\,\hat{\mathbf{r}}\,(\hat{\mathbf{r}}\cdot\mathbf{m}) - \mathbf{m}}{r^3} \quad \Leftarrow \quad \Phi_M^{(1)} = \pm\,\Phi^{(1)} \rightarrow \frac{2\,H_0\,a^3}{3\,\pi}\,\frac{y}{r^3}$$

$$\Rightarrow \quad \mathbf{m} = \pm\frac{8\,a^3}{3}\,\mathbf{H}_0 \;\text{ for }\; z \gtrless 0$$

$$\Rightarrow \begin{cases} \Phi_M \text{ continuous across } z=0 \quad \text{for} \quad 0 \le \rho < a \\ \dfrac{\partial \Phi_M}{\partial z} = 0 \quad \text{at} \quad z=0 \quad \text{for} \quad a < \rho < \infty \end{cases} \Leftarrow \text{boundary conditions} \quad \begin{array}{c} \Phi_{|z=0^+} = \Phi_{|z=0^-} \\ B_\perp = 0 \end{array}$$

$$\Rightarrow \begin{cases} \displaystyle\int_0^\infty A(k)\, J_1(k\rho)\, \mathrm{d}k = \frac{H_0 \rho}{2} \quad \text{for} \quad 0 \le \rho < a \\ \displaystyle\int_0^\infty k\, A(k)\, J_1(k\rho)\, \mathrm{d}k = 0 \quad \text{for} \quad a < \rho < \infty \end{cases} \Leftarrow \text{dual integral eqns}$$

$$g(y) = \frac{2\,\Gamma(n+1)}{\sqrt{\pi}\,\Gamma(n+1/2)}\, j_n(y) = \frac{\Gamma(n+1)}{\Gamma(n+1/2)} \sqrt{\frac{2}{y}}\, J_{n+1/2}(y) \Leftarrow \begin{cases} \displaystyle\int_0^\infty g(y)\, J_n(yx)\, \mathrm{d}y = x^n \quad \text{for} \quad 0 \le x < 1 \\ \displaystyle\int_0^\infty y\, g(y)\, J_n(yx)\, \mathrm{d}y = 0 \quad \text{for} \quad 1 < x < \infty \end{cases}$$

$$\Rightarrow A(k) = \frac{2 H_0 a^2}{\pi}\, j_1(ka) \Leftarrow g = \frac{2A(k)}{H_0 a^2}, \quad n=1, \quad x = \frac{\rho}{a}, \quad y = ka$$

$$\Rightarrow \Phi^{(1)}(\mathbf{r}) = \frac{2 H_0 a^2}{\pi} \int_0^\infty j_1(ka)\, e^{-k|z|}\, J_1(k\rho) \sin\phi\, \mathrm{d}k \Rightarrow \Phi^{(1)}(\infty) \to \frac{2 H_0 a^3}{3\pi} \frac{y}{r^3}$$

the potential of a dipole aligned in the y direction, the direction of $\mathbf{H}_0$

- At large distances the circular hole is equivalent to a magnetic dipole with

$$\mathbf{m} = \pm \frac{8a^3}{3} \mathbf{H}_0 \quad \text{for} \quad z \gtrless 0$$

$$\int_0^\infty j_1(k\,a)\,e^{-k|z|}\,J_1(k\,\rho)\,\mathrm{d}\,k = \sqrt{\frac{\pi}{2\,a}}\int_0^\infty \frac{J_{3/2}(k\,a)\,J_1(k\,\rho)}{e^{k|z|}\sqrt{k}}\,\mathrm{d}\,k$$

$$= \frac{\rho}{a^2}\int_0^{a/R}\frac{x^2\,\mathrm{d}\,x}{\sqrt{1-x^2}} = \frac{\rho}{2\,a^2}\left(\sin^{-1}\frac{a}{R} - \frac{a\sqrt{R^2-a^2}}{R^2}\right)$$

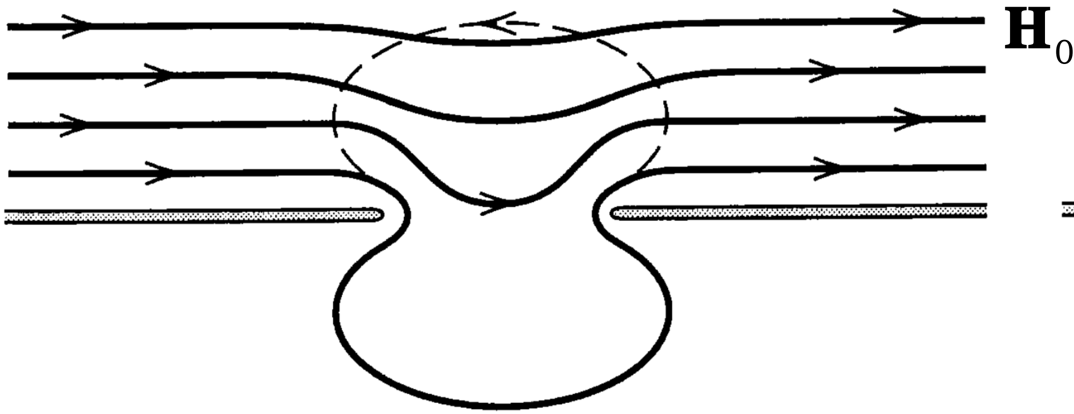
$$\text{where}\quad R = \frac{\sqrt{(a+\rho)^2+z^2}+\sqrt{(a-\rho)^2+z^2}}{2}$$

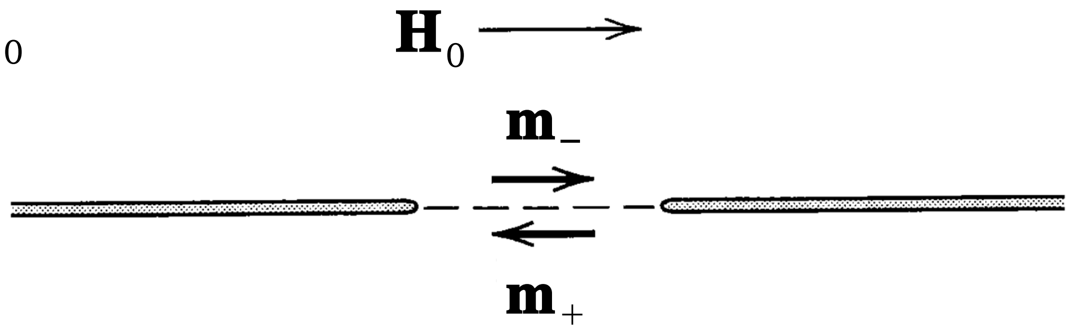
6.752 of Table of Integrals, Series, and Products, Gradshteyn & Ryzhik (2007)

$$\Rightarrow\quad \Phi^{(1)}(\mathbf{r}) = \frac{2\,H_0\,a^2}{\pi}\sin\phi\int_0^\infty j_1(k\,a)\,e^{-k|z|}\,J_1(k\,\rho)\,\mathrm{d}\,k$$

$$= \frac{H_0}{\pi}\rho\sin\phi\left(\sin^{-1}\frac{a}{R} - \frac{a}{R}\sqrt{1-\frac{a^2}{R^2}}\right)$$

$$\Rightarrow\quad \mathbf{H}^{(1)}(z \gtrless 0) \equiv \mp\nabla\,\Phi^{(1)} \quad\Rightarrow\quad \mathbf{H} = \Theta(z)\,\mathbf{H}_0 + \mathbf{H}^{(1)}$$





- In the opening

$$\left[\begin{array}{l} \mathbf{H}_{\parallel} = \frac{1}{2}\mathbf{H}_0 \\ H_z(\rho, 0) = \frac{2H_0}{\pi}\frac{\rho \sin\phi}{\sqrt{a^2-\rho^2}} \end{array}\right. \quad \text{for } z=0, \quad 0 \le \rho < a$$

- Comparing the magnetic problem with the similar electrostatic problem shows the roles of tangential and normal components of fields have been interchanged.

- The dipoles point is parallel to the asymptotic fields, but the magnetic moment is 2 times larger than the electrostatic moment for the same field strengths.

- For *arbitrarily shaped holes* the far field in the electrostatic case is that of a dipole ⊥ the plane, but the magnetic case has its effective dipole in the plane, the direction of the magnetic dipole depends on both the field direction and the orientation of the hole.

Selected problems: 3, 7, 14, 20, 21, 26, 27, 30

Numerical Methods for 2D Magnetic Fields

- Magnetic fields in the presence of highly permeable materials can be evaluated numerically in 2d by the relaxation method or by the finite element method.

- Consider the boundary conditions for the field components at the smooth interface of a highly permeable medium and a nonpermeable one.

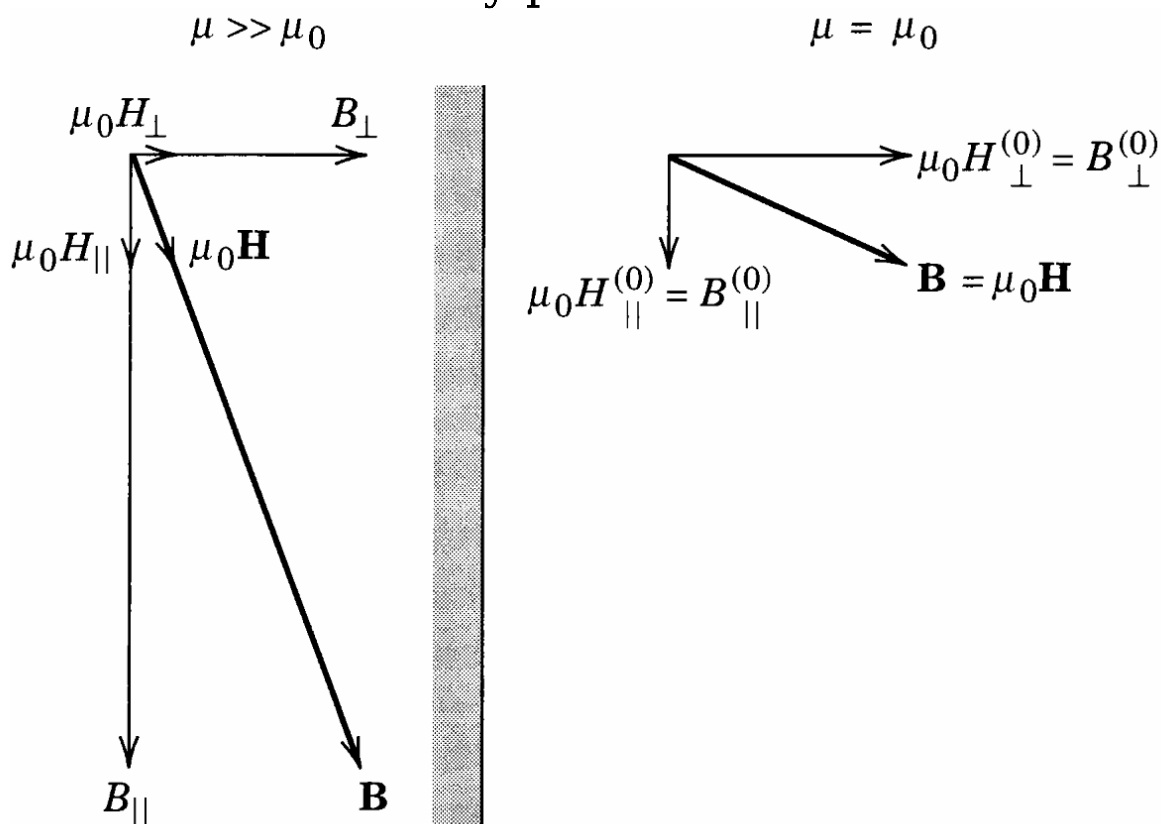
- The boundary conditions are that the tangential component of $\mathbf{H}$ and the normal component of $\mathbf{B}$ are continuous across the interface, if no surface currents.

- For a given external field $\mathbf{B}_{(0)}$ in the nonpermeable region, the components of $\mathbf{B}$ (& $\mathbf{H}$) in the highly permeable medium are more closely parallel to the interface.

$$|\mathbf{B}|^2 = B_{(0)}^{\perp\,2} + \frac{\mu^2}{\mu_0^2} B_{(0)}^{\|\,2}$$

$$\Rightarrow \quad \frac{|\mathbf{B}|^2}{2\mu} = \frac{B_{(0)}^{\perp\,2}}{2\mu} + \frac{\mu}{2\mu_0^2} B_{(0)}^{\|\,2}$$

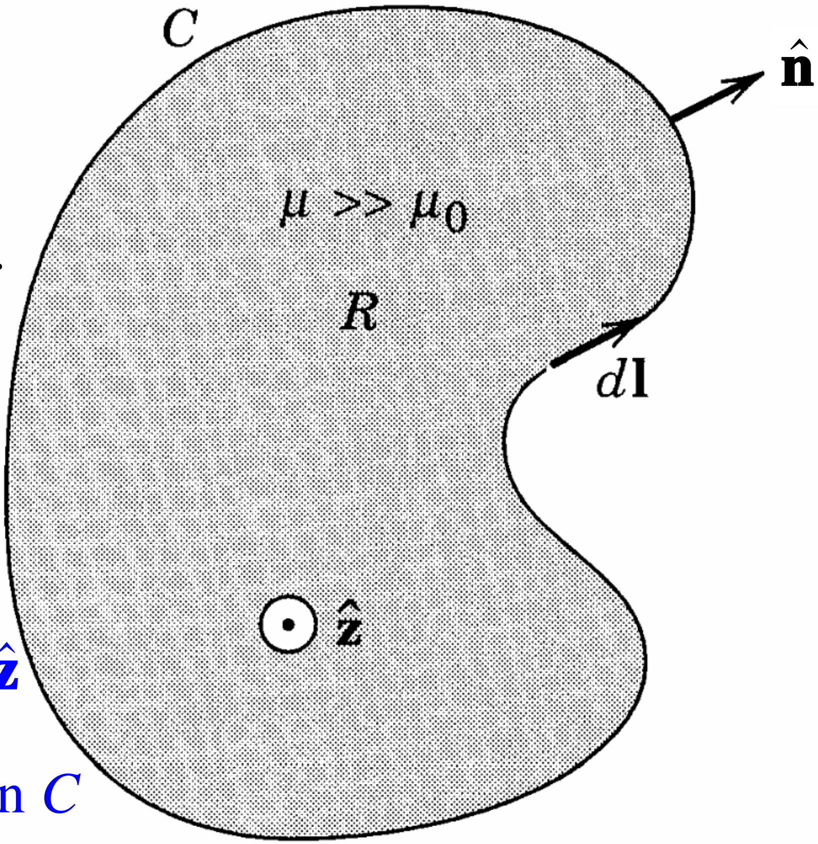
energy density



- These 2 relations are useful in learning the appropriate boundary conditions of exterior and interior problems in the limit $\dfrac{\mu}{\mu_0} \to \infty$.

- The most familiar static magnetic fields are those around a permanent magnet of high permeability excited by remote current-carrying windings.
- $\frac{\mu}{\mu_0} \to \infty \quad \Rightarrow \quad B^{\parallel}_{(0)} = 0 \quad \Leftarrow \quad \frac{|\mathbf{B}|^2}{2\mu} = \frac{B^{\perp 2}_{(0)}}{2\mu} + \frac{\mu}{2\mu_0^2} B^{\parallel 2}_{(0)} = \text{finite}, \quad \text{or} \quad \frac{B^{\parallel}_{(0)}}{\mu_0} = \frac{B^{\parallel}}{\mu}$

the "external" magnetic field at the surface is ⊥ the interface.
- $\boldsymbol{J} = 0 \quad \Rightarrow \quad \nabla \times \mathbf{H} = 0 \quad \Rightarrow \quad \mathbf{H} = -\nabla \Phi_M \quad \Rightarrow \quad \begin{array}{l} \nabla^2 \Phi_M = 0 \\ \Phi_M = \text{const at boundary} \quad \Leftarrow \quad B^{\parallel}_{(0)} = 0 \end{array}$
- Consider a 2d “interior” problems, with steady current in the 3rd direction in a uniform, highly permeable conducting medium. The current produces a magnetic induction both inside and outside the medium.



- The boundary conditions assure that $\mathbf{B}$ is ∥ the

surface just inside as $\frac{\mu}{\mu_0} \to \infty \quad \Rightarrow \quad \mu H^{\perp} = \mu_0 H^{\perp}_{(0)}$.
- $\boldsymbol{J} = J_z(x, y)\,\hat{\mathbf{z}} \;\Rightarrow\; \mathbf{A} = A_z\,\hat{\mathbf{z}} \;\Rightarrow\; \nabla^2 A_z = -\mu J_z$

$$\Rightarrow \quad B_x = \frac{\partial A_z}{\partial y}, \quad B_y = -\frac{\partial A_z}{\partial x}, \quad B_z = 0$$

- If the internal field is ∥ the boundary

$$\Rightarrow \quad B_{\parallel}\,\hat{\boldsymbol{\ell}} \Leftarrow \mathbf{B} = \nabla \times \mathbf{A} = \left(\hat{\mathbf{n}}\,\partial_{\perp} + \hat{\boldsymbol{\ell}}\,\partial_{\parallel} + \hat{\mathbf{z}}\,\partial_z\right) \times A_z\,\hat{\mathbf{z}}$$

$$= \frac{\partial A_z}{\partial \ell}\,\hat{\mathbf{n}} - \frac{\partial A_z}{\partial n}\,\hat{\boldsymbol{\ell}} \quad \Rightarrow \quad \frac{\partial A_z}{\partial \ell} = 0 \;\text{ on } C$$

- The vector potential is constant along the boundary curve. We can infer that in the interior region the magnetic field lines are $\parallel$ the contours of constant A_z.

- $\mathbf{B}=\nabla\times\mathbf{A}$ ⇒ the density of force lines is the derivative of $A_z \perp$ the surfaces of constant value; the spacing of contours of constant shows the intensity and the direction of the field.

- The constant value of A_z on the contour must be specified to solve the Poisson equation numerically.

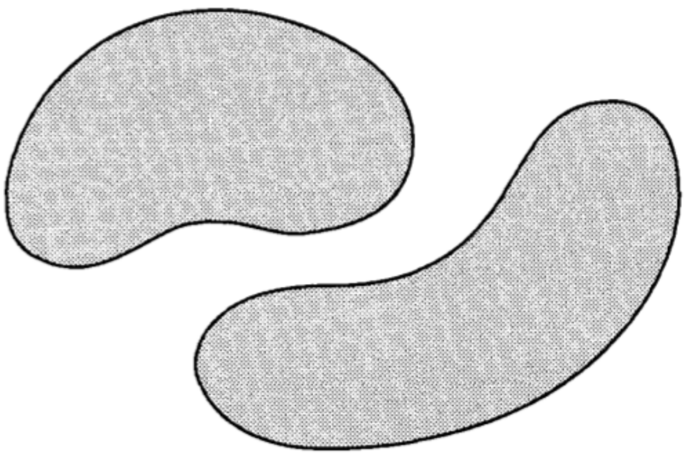
- The vector potential is arbitrary to the addition of the gradient of a scalar

$$\Rightarrow \quad \mathbf{A}'=\mathbf{A}+\nabla\chi \quad \Rightarrow \quad A_z'=A_z(x,y)-A_0 \quad \Leftarrow \quad \chi=-A_0 z$$
$$\Rightarrow \quad \nabla^2 A_z'=-\mu J_z \text{ in } R \quad + \quad A_z'=0 \text{ on } C$$

- The value of A_z on C is not physically meaningful and is not needed.

- Powerful numerical codes exist to solve more realistic magnetic field problems where the permeable materials have large, but not infinite, values of $\dfrac{\mu}{\mu_0}$.

The figure represents a transmission line consisting of two, parallel *perfect* conductors of arbitrary, but constant, cross section. Current flows down one conductor and returns via the other.



Problem 5.29

Show that the product of the inductance per unit length L and the capacitance per unit length C is

$$LC = \mu\epsilon$$

where μ and ϵ are the permeability and the permittivity of the medium surrounding the conductors. (See the discussion about magnetic fields near perfect conductors at the beginning of Section 5.13.)

For perfect conductors, the free charges inside them are pushed to their surfaces. And these surface charges flow to form the surface current. Let the two wires are parallel to the z-axis. Assume the surface charge density and the surface current density on $(\pm)$ wire are $\sigma_\pm(\rho', \phi')$ and $\boldsymbol{K}^\pm = K_z^\pm \hat{\mathbf{z}} = \sigma_\pm v_\pm \hat{\mathbf{z}}$. Then $\int_{S_+} \sigma_+(\rho', \phi\prime)\mathrm{d}a' + \int_{S_-} \sigma_-(\rho', \phi')\mathrm{d}a = 0$ because of the electrical neutrality. Due to their symmetry along the z-axis, it can be simplified as $\oint_{C_+} \sigma_+(\rho', \phi\prime)\mathrm{d}\ell' + \oint_{C_-} \sigma_-(\rho', \phi')\mathrm{d}\ell' = 0$, where $S_\pm = C_\pm \times Z$. Similarly, the charge conservtion demands

$$\oint_{C_+} \sigma_+ v_+(\rho', \phi\prime)\mathrm{d}\ell' + \oint_{C_-} \sigma_- V_-(\rho', \phi')\mathrm{d}\ell' = v_+ \oint_{C_+} \sigma_+(\rho', \phi\prime)\mathrm{d}\ell' + v_- \oint_{C_-} \sigma_-(\rho', \phi')\mathrm{d}\ell' = 0,$$

here $v_\pm$ has to be the same everwhere in $S_\pm$ for the stationary state. Thus $v_+ = v_- = v$ by combining the both conditions.

The scalar electrical potential and the magnetic vector potential with the charge and current distributions are

$$\Phi(\rho, \phi) = \frac{1}{4\pi\epsilon} \left(\oint_{S_+} \frac{\sigma_+(\rho', \phi\prime)}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}a' + \oint_{S_-} \frac{\sigma_-(\rho', \phi')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}a' \right)$$

$$\begin{aligned} \mathbf{A}(\rho, \phi) = A_z \hat{\mathbf{z}} &= \frac{\mu}{4\pi} \hat{\mathbf{z}} \left(\oint_{S_+} \frac{\sigma_+ v_+(\rho', \phi\prime)}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}a' + \oint_{S_-} \frac{\sigma_- v_-(\rho', \phi')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}a' \right) \\ &= \frac{\mu v}{4\pi} \hat{\mathbf{z}} \left(\oint_{S_+} \frac{\sigma_+(\rho', \phi\prime)}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}a' + \oint_{S_-} \frac{\sigma_-(\rho', \phi')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}a' \right) = \epsilon \mu v \Phi \hat{\mathbf{z}}. \end{aligned}$$

Thus $A_z = \epsilon\mu v\Phi$. Then

$$\mathbf{E} = -\nabla\Phi = -\left(\frac{\partial\Phi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\boldsymbol{\phi}}\right)$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{1}{\rho}\frac{\partial A_z}{\partial\phi}\hat{\boldsymbol{\rho}} - \frac{\partial Az}{\partial\rho}\hat{\boldsymbol{\phi}} = \epsilon\mu v\left(\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\boldsymbol{\rho}} - \frac{\partial\Phi}{\partial\rho}\hat{\boldsymbol{\phi}}\right).$$

The electric and magnetic energies per unit length are

$$w_e = \frac{W_e}{Z} = \frac{\epsilon}{2Z}\int E^2 \mathrm{d}^3x' = \frac{\epsilon}{2Z}\int\left[\left(\frac{\partial\Phi}{\partial\rho}\right)^2 + \left(\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\right)^2\right]\mathrm{d}^3x',$$

$$w_m = \frac{W_m}{Z} = \frac{1}{2\mu Z}\int B^2 \mathrm{d}^3x' = \frac{\epsilon^2\mu v^2}{2Z}\int\left[\left(\frac{\partial\Phi}{\partial\rho}\right)^2 + \left(\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\right)^2\right]\mathrm{d}^3x',$$

thus $w_m = \epsilon\mu v^2 w_e$. Conventionaly, $W_e = \dfrac{Q^2}{2C}$ and $W_m = \dfrac{L}{2}I^2$. So $w_m = LC\dfrac{I^2}{Q^2}w_e$. This leads to $LC = \epsilon\mu$ since $v = \dfrac{I}{Q}$.

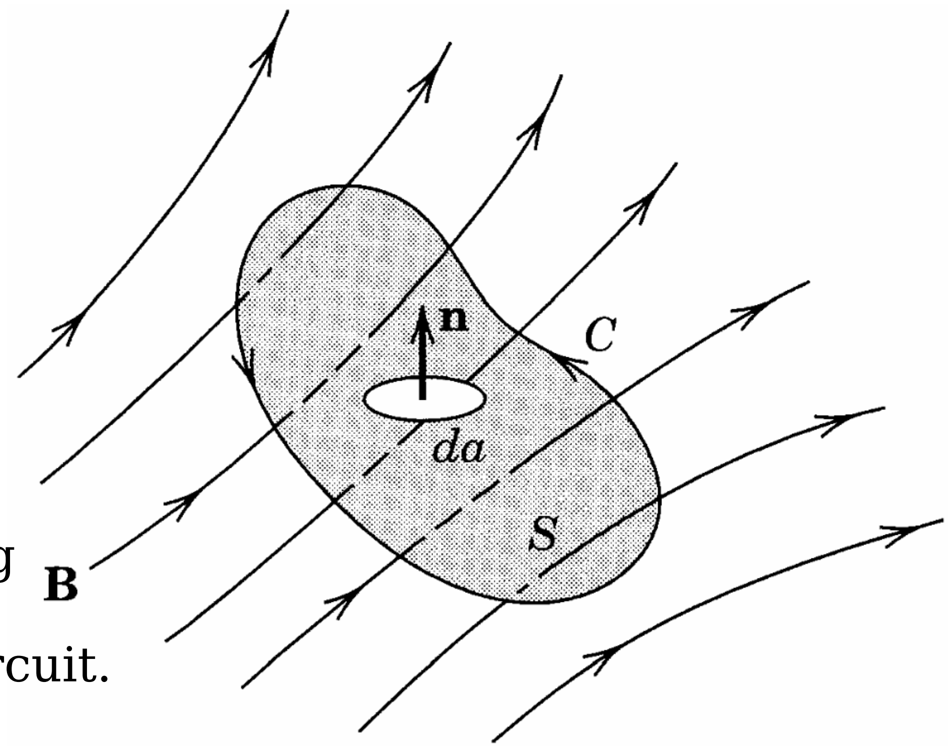
Faraday's Law of Induction

- Faraday (1831) observed a transient induced current in a circuit if
  (a) the steady current in an adjacent circuit is turned on or off,
  (b) the adjacent circuit with a steady current is moved relative to the 1^{st} circuit,
  (c) a permanent magnet is thrust into or out of the circuit.

- Faraday attributed the transient current to a changing magnetic flux. The changing flux induces an electric field around the circuit, the line integral of which is called the *electromotive force* (EMF). The EMF causes a current.

$$\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\boldsymbol{a} \quad \& \quad \mathscr{E} = \oint_C \mathbf{E}' \cdot \mathrm{d}\boldsymbol{\ell}$$

$$\Rightarrow \quad \mathscr{E} = -\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,t} \quad \Leftarrow \quad \text{by Faraday}$$



- The induced EMF around the circuit is proportional to the time rate of change of magnetic flux linking the circuit.

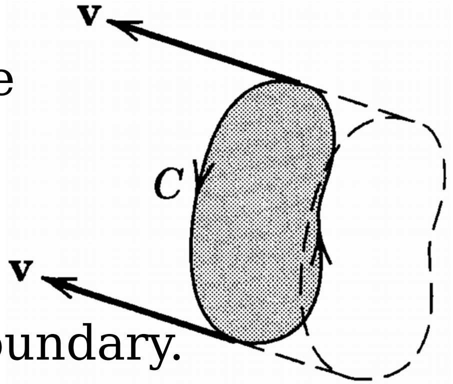
- The sign is specified by Lenz's law, stating that the induced current is in the direction to oppose the change of flux through the circuit.

- Before special relativity, physical laws are considered invariant under Galilean transformations. Physical phenomena are the same when viewed by 2 observers moving with a constant velocity relative to one another, provided the coordinates are related by the Galilean transformation, $\mathbf{r}' = \mathbf{r} - \mathbf{v}\,t, \;\; t' = t$.

- The same current is induced in a secondary circuit whether it is *moved* while the primary circuit through which current is flowing is stationary or it is held fixed while *the primary circuit is moved* in the same relative manner.

- $\oint_C \mathbf{E}' \cdot d\boldsymbol{\ell} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\boldsymbol{a}$ $\Rightarrow$ The EMF is proportional to the *total* time derivative of the flux—the flux can be changed by changing the magnetic induction or by changing the shape/orientation/position of the circuit.

- The circuit C can be thought of as any closed path in space, not necessarily an electric circuit. Then the equation becomes a relation between the EM fields.



- If the circuit is moving with a velocity, the total time derivative must take into account this motion.

- The flux through the circuit may change because
  (a) the flux changes with time at a point, or
  (b) the translation of the circuit changes the location of the boundary.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla \quad \Rightarrow \quad \frac{d\mathbf{B}}{dt} = \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{B}$$

$$= \frac{\partial \mathbf{B}}{\partial t} + \nabla\times(\mathbf{B}\times\mathbf{v}) + \cancel{(\nabla\cdot\mathbf{B})}\,\mathbf{v} \quad \Leftarrow \quad \mathbf{v} \text{ is not a field.}$$

$$\Rightarrow \frac{d}{dt}\int_S \mathbf{B}\cdot d\boldsymbol{a} = \int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\boldsymbol{a} + \oint_C \mathbf{B}\times\mathbf{v}\cdot d\boldsymbol{\ell} \quad \Leftarrow \text{(non-moving)} + \text{(moving)}$$

for the magnetic flux change

$$\Rightarrow \oint_{c} (\mathbf{E}' - \mathbf{v} \times \mathbf{B}) \cdot d\boldsymbol{\ell} = -\int_{s} \frac{\partial \mathbf{B}}{\partial t} \cdot d\boldsymbol{a}$$

- Think of the circuit and surface as instantaneously at a certain position in space in the laboratory $\oint_{c} \mathbf{E} \cdot d\boldsymbol{\ell} = -\int_{s} \frac{\partial \mathbf{B}}{\partial t} \cdot d\boldsymbol{a} \Rightarrow \mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$

- A charged particle co-moving with in a circuit experiences a force $q\mathbf{E}'$. When viewed from the laboratory, the charge experiences the Lorentz force $q\mathbf{v} \times \mathbf{B}$.

$$\Rightarrow \oint_{c} \mathbf{E}' \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int_{s} \mathbf{B} \cdot d\boldsymbol{a} \Leftarrow \mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$$

$\mathbf{E}'$ is in the rest frame of circuit, the time derivative is a *total* time derivative.

- In the same frame,

$$\oint_{c} \mathbf{E} \cdot d\boldsymbol{\ell} = -\int_{s} \frac{\partial \mathbf{B}}{\partial t} \cdot d\boldsymbol{a} \Rightarrow \int_{s} \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) \cdot d\boldsymbol{a} = 0$$

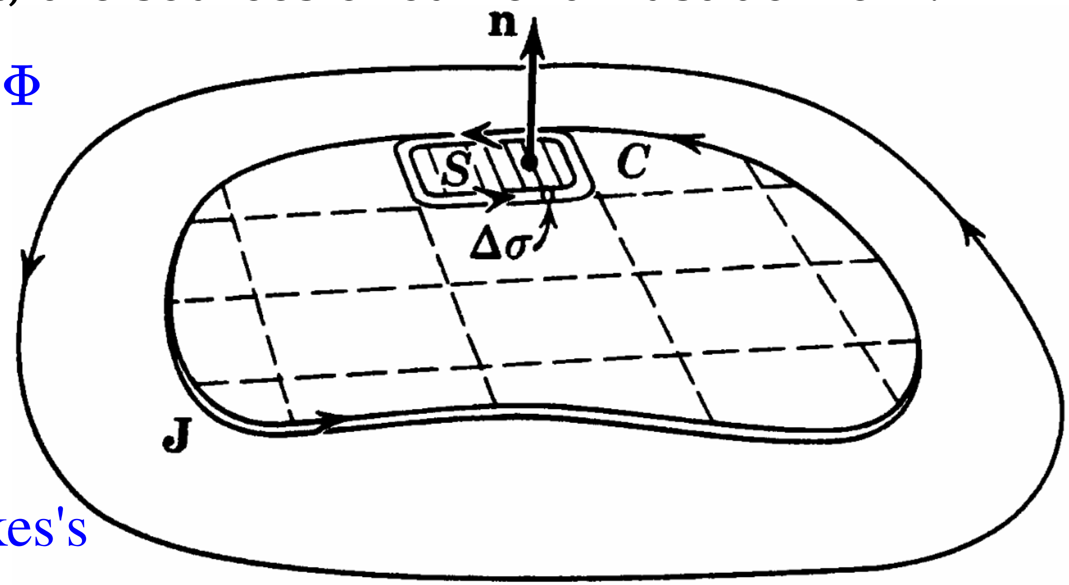
$$\Rightarrow \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \Rightarrow \nabla \times \mathbf{E} = 0 \quad \text{for electrostatics}$$

Energy in the Magnetic Field

- The creation of a steady-state configuration of currents and associated magnetic fields involves an initial transient period during which the currents and fields are brought from 0 to the final values.

- If the magnetic flux through a circuit changes, an electromotive force is induced around it. To keep the current constant, the sources of current must do work.

$$\frac{dW}{dt} = -I\mathscr{E} = I\frac{d\Phi}{dt} \quad \Rightarrow \quad \delta W = I\,\delta\Phi$$



- $$\Delta(\delta W) = J\,\Delta\sigma \int_S \delta\mathbf{B}\cdot d\boldsymbol{a}$$

$$= J\,\Delta\sigma \int_S \nabla\times\delta\mathbf{A}\cdot d\boldsymbol{a}$$

$$= J\,\Delta\sigma \oint_C \delta\mathbf{A}\cdot d\boldsymbol{\ell} \quad \Leftarrow \text{Stokes's theorem}$$

$$\Rightarrow \quad \delta W = \int \delta\mathbf{A}\cdot\boldsymbol{J}\,d^3x \quad \Leftarrow \quad \boldsymbol{J}\,d^3x = J\,\Delta\sigma\,d\boldsymbol{\ell}$$

$$= \int \delta\mathbf{A}\cdot\nabla\times\mathbf{H}\,d^3x \quad \Leftarrow \quad \nabla\times\mathbf{H} = \boldsymbol{J} \ \text{ Ampere's law} \quad \Leftarrow \quad \nabla\cdot\boldsymbol{J} = 0$$

$$= \int [\mathbf{H}\cdot\nabla\times\delta\mathbf{A} + \nabla\cdot(\mathbf{H}\times\cancel{\delta\mathbf{A}})]\,d^3x \quad \Leftarrow \nabla\cdot(\mathbf{P}\times\mathbf{Q}) = \mathbf{Q}\cdot\nabla\times\mathbf{P} - \mathbf{P}\cdot\nabla\times\mathbf{Q}$$

$$= \int \mathbf{H}\cdot\delta\mathbf{B}\,d^3x \qquad \Uparrow \quad \text{localized field}$$

$$\Rightarrow \quad W=\frac{1}{2}\int \mathbf{H}\cdot\mathbf{B}\,d^3x \quad (7c) \Leftarrow \quad \mathbf{H}\cdot\delta\mathbf{B}=\frac{1}{2}\delta(\mathbf{H}\cdot\mathbf{B}) \quad \Leftarrow \quad \mathbf{H}\propto\mathbf{B} \quad \Leftarrow$$

paramagnetic
diamagnetic

$$\Rightarrow \quad W=\frac{1}{2}\int \boldsymbol{J}\cdot\mathbf{A}\,d^3x \quad \Leftarrow \quad \mathbf{A}\propto\boldsymbol{J}$$

- The change in energy when an object of μ_1 is placed in a magnetic field with fixed current sources can be treated in analogy with the electrostatics [Sec. 4.7].

$$W=\int_{V_1}\frac{\mathbf{B}\cdot\mathbf{H}_0-\mathbf{H}\cdot\mathbf{B}_0}{2}\,d^3x=\int_{V_1}\frac{\mu_1-\mu_0}{2}\mathbf{H}\cdot\mathbf{H}_0\,d^3x=\frac{1}{2}\int_{V_1}\frac{\mu_1-\mu_0}{\mu_0\mu_1}\mathbf{B}\cdot\mathbf{B}_0\,d^3x$$

$$W=\frac{1}{2}\int_{V_1}\mathbf{M}\cdot\mathbf{B}_0\,d^3x \quad \Leftarrow \quad \mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M})=\mu_1\mathbf{H} \quad \text{vs} \quad W_E=-\frac{1}{2}\int_{V_1}\mathbf{P}\cdot\mathbf{E}_0\,d^3x$$

- This sign difference comes from the work done by the sources against the emf.

- The magnetic problem with fixed currents is analogous to the electrostatic problem with fixed potentials on the surfaces that determine the fields.

- For a small displacement the work done against the induced emf 's is twice as large as, and of the opposite sign to, the potential-energy change of the body.

- The force acting on the body $F_\xi=\left.\frac{\partial W}{\partial \xi}\right|_{J=\text{const}}$ vs $\mathbf{F}=-\nabla U \quad \Leftarrow \quad U=-\mathbf{m}\cdot\mathbf{B}$

- W is the total energy required to produce the configuration, whereas U includes only the work to establish the permanent magnetic moment in the field, not the work to create the magnetic moment and to keep it permanent.

Energy and Self- and Mutual Inductances

A. Coefficients of Self- and Mutual Inductance

- $W = \frac{1}{2}\int \boldsymbol{J}\cdot\mathbf{A}\,\mathrm{d}^3x = \frac{\mu_0}{8\pi}\iint \frac{\boldsymbol{J}(\mathbf{r})\cdot\boldsymbol{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3x\,\mathrm{d}^3x' \Leftarrow \mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\mathbf{r}')\,\mathrm{d}^3x'}{|\mathbf{r}-\mathbf{r}'|}$

$$= \frac{\mu_0}{8\pi}\sum_{i,j=1}^{N}\iint \frac{\boldsymbol{J}(\mathbf{r}_i)\cdot\boldsymbol{J}(\mathbf{r}'_j)}{|\mathbf{r}_i-\mathbf{r}'_j|}\,\mathrm{d}^3x_i\,\mathrm{d}^3x'_j \quad \Leftarrow$$

broken into sums of separate integrals over each circuit

$$= \frac{1}{2}\sum_{i=1}^{N} L_i I_i^2 + \sum_{i=1,\,j>i}^{N} M_{ij} I_i I_j \quad \Leftarrow$$

$$L_i = \frac{\mu_0}{4\pi I_i^2}\iint_{\mathcal{C}_i} \frac{\boldsymbol{J}(\mathbf{r}_i)\cdot\boldsymbol{J}(\mathbf{r}'_i)}{|\mathbf{r}_i-\mathbf{r}'_i|}\,\mathrm{d}^3x_i\,\mathrm{d}^3x'_i$$

$$M_{ij} = \frac{\mu_0}{4\pi I_i I_j}\int_{\mathcal{C}_i}\int_{\mathcal{C}_j} \frac{\boldsymbol{J}(\mathbf{r}_i)\cdot\boldsymbol{J}(\mathbf{r}'_j)}{|\mathbf{r}_i-\mathbf{r}'_j|}\,\mathrm{d}^3x'_j\,\mathrm{d}^3x_i$$

- To establish the connection between the current density and the flux linkage

$$\boldsymbol{J}\,\mathrm{d}^3x = \boldsymbol{J}_{\parallel}\,\mathrm{d}a\,\mathrm{d}\ell \;\Rightarrow\; \int \boldsymbol{J}\,\mathrm{d}^3x = I\oint_{\mathcal{C}} \mathrm{d}\boldsymbol{\ell}$$

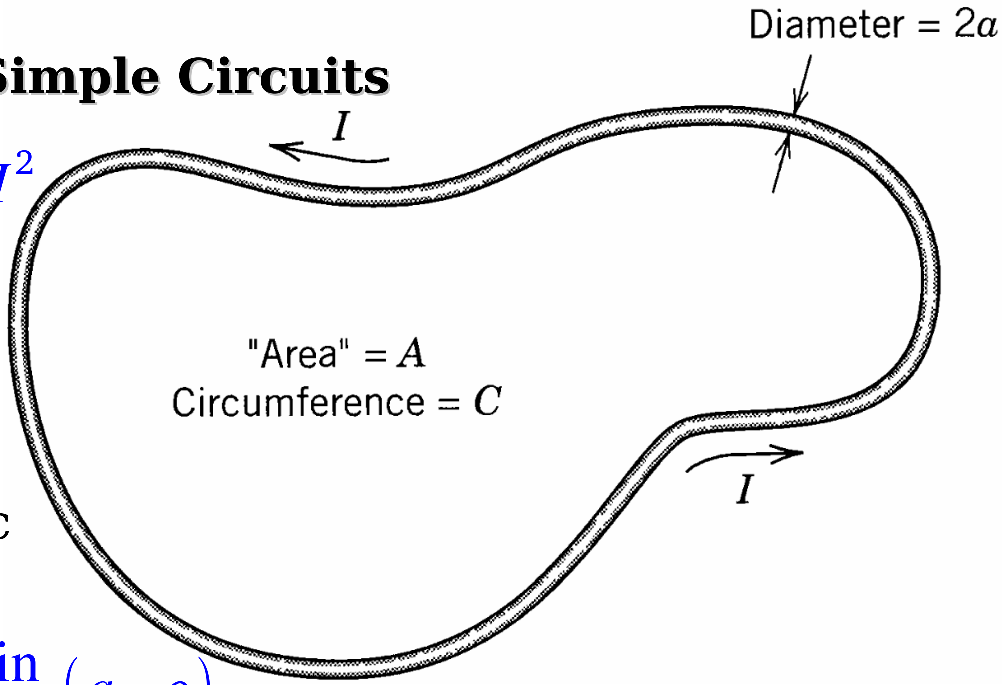
$$\Rightarrow\; M_{ij} = \frac{1}{I_i I_j}\int_{\mathcal{C}_i} \boldsymbol{J}(\mathbf{r}_i)\cdot\mathbf{A}_{ij}\,\mathrm{d}^3x_i = \frac{1}{I_i I_j} I_i \oint_{\mathcal{C}_i}\mathbf{A}_{ij}\cdot\mathrm{d}\boldsymbol{\ell} = \frac{1}{I_j}\int_{\mathcal{S}_i}\nabla\times\mathbf{A}_{ij}\cdot\mathrm{d}\boldsymbol{a}$$

$$= \frac{1}{I_j}\int_{\mathcal{S}_i}\mathbf{B}_{ij}\cdot\mathrm{d}\boldsymbol{a} \;\Rightarrow\; I_j M_{ij} = \Phi_{ij} =$$

magnetic flux from circuit j linked within circuit i

- For self-inductance, the physical argument is the same.
- For current in a medium of $\mu \neq \mu_0$, it is the best to use (7c) for magnetic energy.
- $\mathscr{E} = -\frac{d\,\Phi}{d\,t} = -L\frac{d\,I}{d\,t} - \sum M_{1i}\frac{d\,I_i}{d\,t}$

B. Estimation of Self-Inductance for Simple Circuits



- $W = \int \frac{\mathbf{H}\cdot\mathbf{B}}{2} d^3x = \int \frac{\mathbf{B}\cdot\mathbf{B}}{2\mu} d^3x = \frac{1}{2} L I^2$

$$\Rightarrow \quad L = \frac{1}{I^2} \int \frac{B^2}{\mu} d^3x$$

- If the current density is uniform, from symmetry and Ampere's law the magnetic induction is azimuthal

$$\mathbf{B} = B_\phi \hat{\boldsymbol{\phi}} \quad \Leftarrow \quad B_\phi = \frac{\mu_0 I}{2\pi a} \frac{\rho_<}{\rho_>} \quad \Leftarrow \quad \rho_{\lessgtr} = \begin{matrix}\min \\ \max\end{matrix} (a, \rho)$$

- Assume the wire and the medium are nonpermeable

$$L_{in} \equiv L(\rho = a) = \frac{\mu_0}{8\pi} \int d\ell$$

$$L_{out} \equiv L(\rho_{max} > a) = \frac{\mu_0}{2\pi} \ln\frac{\rho_{max}}{a} \int d\ell$$

$$\Rightarrow \quad \frac{dL_{in}}{d\ell} = \frac{\mu_0}{8\pi}$$

$$\frac{dL_{out}}{d\ell} = \frac{\mu_0}{2\pi} \ln\frac{\rho_{max}}{a} \quad \Leftarrow \quad \rho_{max} = O\left(\sqrt{A}\right) = O\left(\frac{C}{2\pi}\right)$$

- At distances large compared to $\sqrt{A}$, the falloff of the magnetic induction as $\frac{1}{\rho}$ is replaced by a dipole field $|\mathbf{B}| = O\left(\frac{\mu_0 m}{4\pi r^3}\right) \quad \Leftarrow \quad m = O(I A)$ magnetic moment of the loop

$$\Rightarrow \quad \frac{d L_{\text{dipole}}}{d\ell} = O\left(\frac{4\pi}{\mu_0 I^2 C} \int_{\rho_{\max}}^{\infty} \left(\frac{\mu_0 I A}{4\pi r^3} \right)^2 r^2 \, dr \right) = O\left(\frac{\mu_0 A^2}{4\pi \rho_{\max}^3 C} \right)$$

$$= O\left(\frac{\mu_0 \sqrt{A}}{4\pi C} \right) = O\left(\frac{\mu_0}{4\pi} \right) \quad \text{for} \quad \rho_{\max} = \sqrt{\xi' A} \quad \Leftarrow \quad \xi' \sim 1$$

- $L = L_{\text{in}} + L_{\text{out}} + L_{\text{dipole}} \approx \frac{\mu_0}{4\pi} C \left(\ln \frac{\xi A}{a^2} + \frac{1}{2} \right) \quad \Leftarrow \quad \xi \sim 1, \quad C \gg 1$

4 comments:

(1) $\mu_0 \to \mu \quad \Rightarrow \quad \frac{1}{2} \to \frac{\mu}{2\mu_0}$

(2) $\xi = \frac{64}{\pi e^4} \approx 0.373$ for a thin wire bent into a circle [Problem 5.32]

(3) High frequency can get rid of the interior contribution because the current will be confined to near the surface of the wire.

(4) $1 \text{ turn} \to N \text{ turns} \quad \Rightarrow \quad L_N = N^2 L_1$

Exercise $\frac{d L_{\text{tot}}}{d\ell} = \frac{d L_1}{d\ell} + \frac{d L_2}{d\ell} = \frac{\mu_0}{8\pi} \left(1 + 4 \ln \frac{\rho_{\max}}{a_1} + 1 + 4 \ln \frac{\rho_{\max}}{a_2} \right) \quad \Leftarrow \quad \rho_{\max} = \frac{d}{2}$

$$= \frac{\mu_0}{\pi} \left(\frac{1}{4} + \ln \frac{d/2}{\sqrt{a_1 a_2}} \right) \quad \Rightarrow \quad \xi = \frac{1}{2}$$

compare with Problem 5.26

Example: Determine the mutual inductance between a conducting triangular loop and a very long straight wire.

Apply Ampere's law and write the expression for $\mathbf{B}_2$, caused by a current I_2 in the long straight wire:

$$\oint \mathbf{B}_2 \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 I_2 \quad \Rightarrow \quad \mathbf{B}_2 = \frac{\mu_0 I_2}{2\pi r}\hat{\boldsymbol{\phi}} \quad \Rightarrow \quad \Phi_1 = \int \mathbf{B}_2 \cdot \mathrm{d}\boldsymbol{a}_1$$

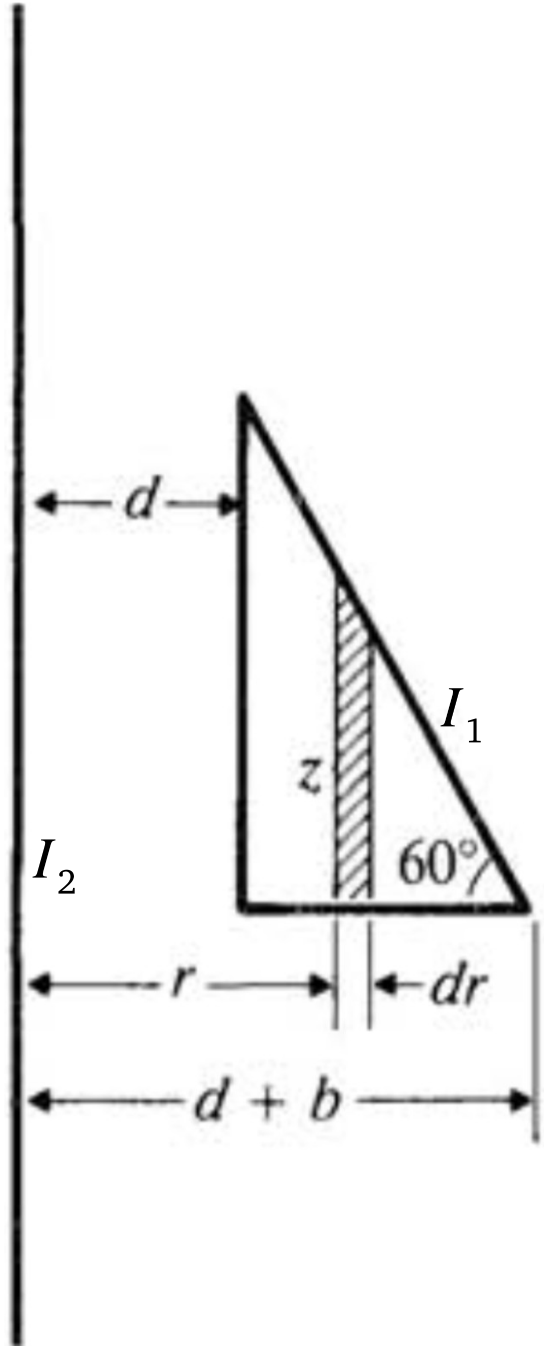
The equation of the sloped line of the triangle is

$$z = [(d+b) - r]\tan\frac{\pi}{3} = \sqrt{3}(d+b-r) \quad \Rightarrow \quad \mathrm{d}\boldsymbol{a}_1 = z\,\mathrm{d}r\,\hat{\boldsymbol{\phi}}$$

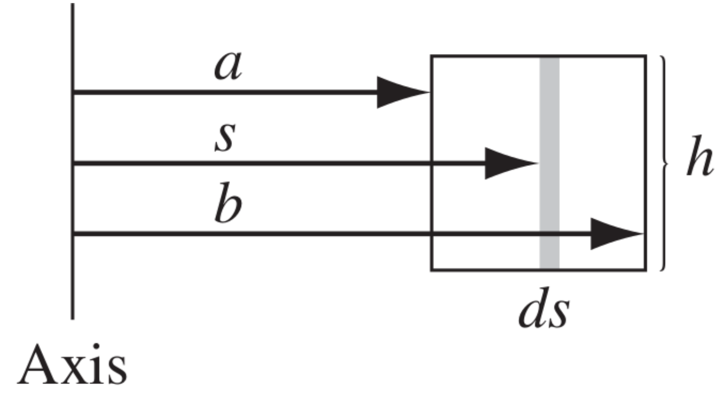
$$\Rightarrow \quad \Phi_1 = \int \frac{\mu_0 I_2}{2\pi r} z\,\mathrm{d}r = \sqrt{3}\frac{\mu_0 I_2}{2\pi}\int_d^{d+b}\frac{d+b-r}{r}\mathrm{d}r$$

$$= \frac{\sqrt{3}\mu_0 I_2}{2\pi}\left((d+b)\ln\frac{d+b}{d} - b\right) \Rightarrow M_{12} I_2$$

$$\Rightarrow \quad M = M_{12} = \frac{\sqrt{3}\mu_0}{2\pi}\left((d+b)\ln\frac{d+b}{d} - b\right)$$



<u>Example</u>: Find the self-inductance of a toroidal coil with rectangular cross section (inner radius a, outer radius b, height h), that carries a total of N turns.

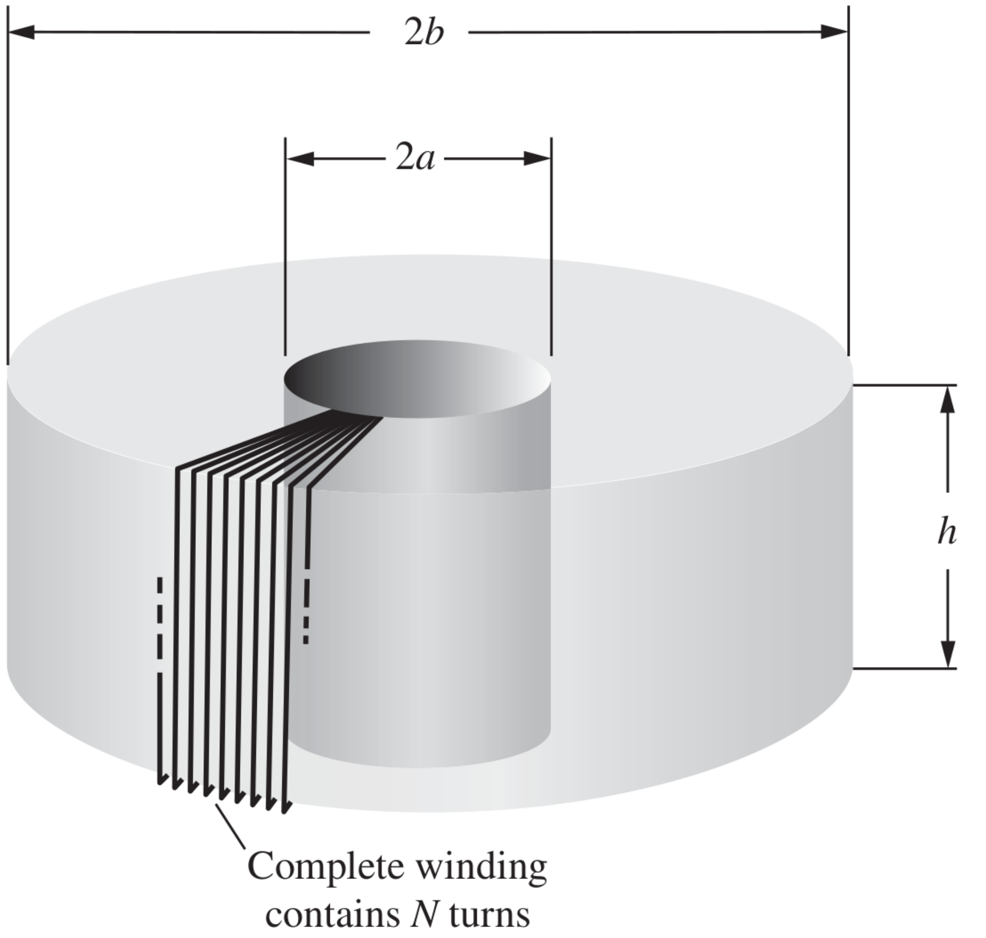


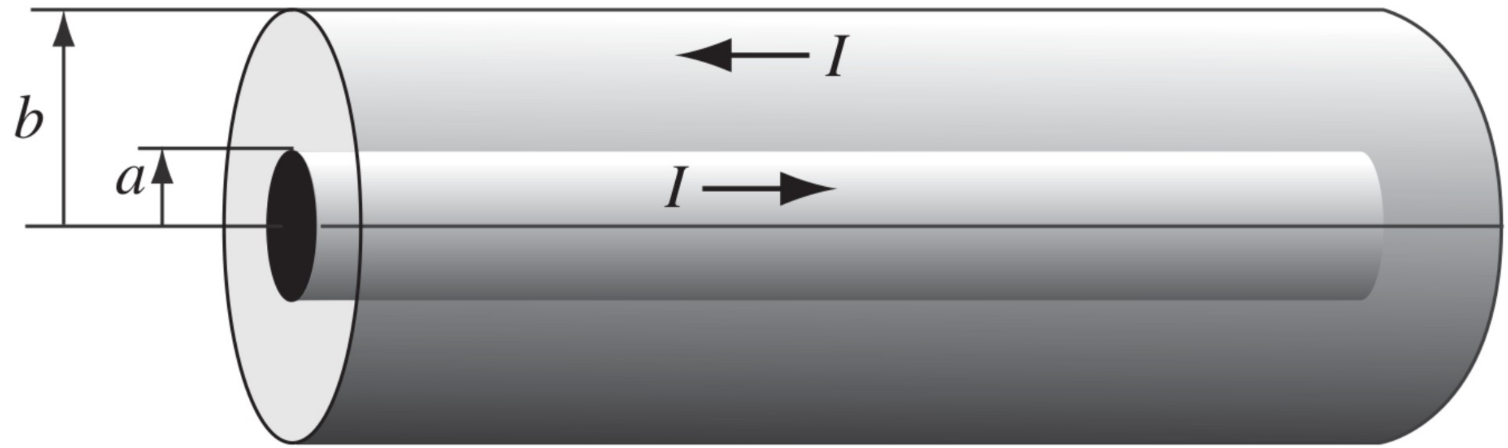
- The magnetic field inside the toroid is $B = \frac{\mu_0 N I}{2 \pi s}$

$$\Rightarrow \ \Phi_{\text{single}} = \int \mathbf{B} \cdot \mathrm{d}\boldsymbol{a} = \frac{\mu_0 N I}{2\pi} h \int_a^b \frac{\mathrm{d}s}{s} = \frac{\mu_0 N I h}{2\pi} \ln \frac{b}{a}$$

- The *total* flux is N times this,

so the self-inductance $L = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a}$





<u>Example</u>: Find the magnetic energy stored in a section of length ℓ.

- According to Ampère's law, only the field between the cylinders is nonzero,

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi} \quad \Rightarrow \quad \text{energy density } w = \frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi s}\right)^2 = \frac{\mu_0 I^2}{8\pi^2 s^2}$$

$$\Rightarrow \quad W = \int w \, d\tau = \int \frac{\mu_0 I^2}{8\pi^2 s^2} 2\pi \ell s \, d s = \frac{\mu_0 I^2 \ell}{4\pi}\int_a^b \frac{d s}{s} = \frac{\mu_0 I^2 \ell}{4\pi}\ln\frac{b}{a}$$

$$W = \frac{1}{2} L I^2 \quad \Rightarrow \quad L = \frac{\mu_0 \ell}{2\pi}\ln\frac{b}{a}$$ **external inductance** of a coaxial line

- This method of calculating self-inductance is especially useful when the current is not confined to a single path, but spreads over some surface or volume, so that different parts of the current enclose different amounts of flux. [Problem 5.27]

Quasi-Static Magnetic Fields in Conductors; Eddy Currents; Magnetic Diffusion

- **Quasi-static**: the finite speed of light can be neglected and fields are treated as if they propagated instantaneously, $c \to \infty \quad \Rightarrow \quad \frac{1}{c} \to 0$

- It is the regime where the system is small compared with the EM wavelength. It permits neglect of the contribution of the Maxwell displacement current to Ampere's law. And the magnetic fields dominate.

$$\Rightarrow \quad \nabla \times \mathbf{H} = \boldsymbol{J}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \boldsymbol{J} = \sigma \mathbf{E} \quad \text{Ohm's law}$$

$$\Rightarrow \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi \quad \Leftarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$$= -\frac{\partial \mathbf{A}}{\partial t} \quad \Leftarrow \quad \Phi = 0 \quad \Leftarrow \quad \rho \to 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{E} = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{A} = \text{const} \Rightarrow 0$$

$$\nabla \times \mathbf{B} = \mu \boldsymbol{J} = \mu \sigma \mathbf{E} \quad \Leftarrow \quad \mathbf{B} = \mu \mathbf{H} \quad \Rightarrow \nabla \times \nabla \times \mathbf{A} = \nabla (\cancel{\nabla \cdot \mathbf{A}}) - \nabla^2 \mathbf{A} = -\mu \sigma \frac{\partial \mathbf{A}}{\partial t}$$

$$\Rightarrow \quad \nabla^2 \mathbf{A} = \mu \sigma \frac{\partial \mathbf{A}}{\partial t} \quad \text{diffusion equation} \quad \Rightarrow \quad \nabla^2 \mathbf{E} = \mu \sigma \frac{\partial \mathbf{E}}{\partial t} \quad \text{for} \quad \frac{\partial \sigma}{\partial t} = 0$$

$$\Rightarrow \quad \nabla^2 \mathbf{B} = \mu \sigma \frac{\partial \mathbf{B}}{\partial t}, \quad \nabla^2 \boldsymbol{J} = \mu \sigma \frac{\partial \boldsymbol{J}}{\partial t} \quad \text{for} \quad \sigma = \text{const}$$

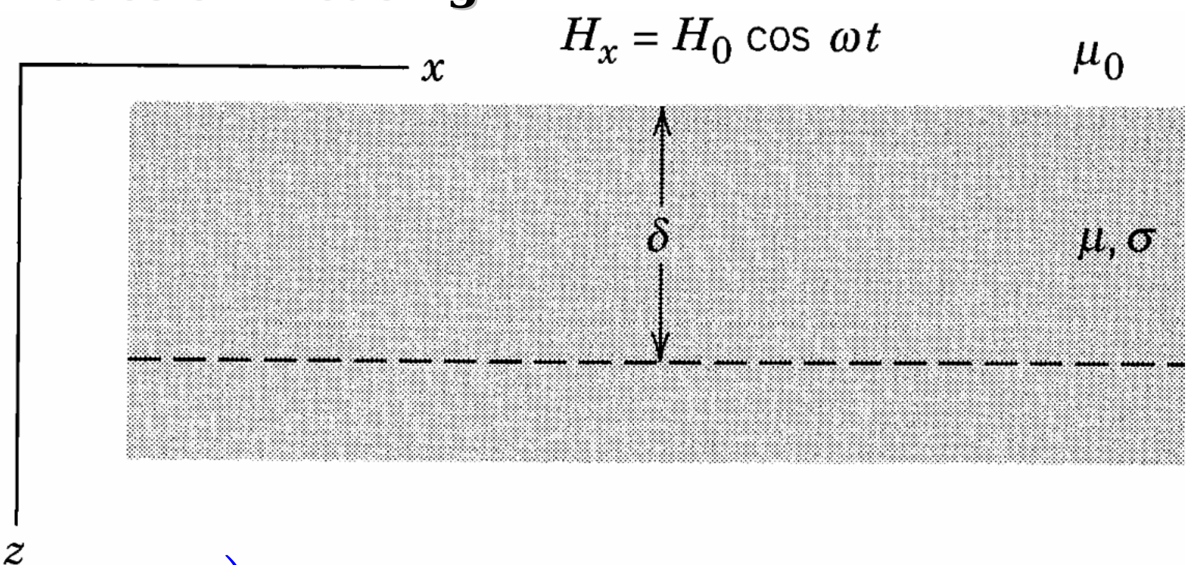
- The diffusion equation allows us to estimate the time for decay of an initial configuration of fields with typical spatial variation.

$$\nabla^2 \mathbf{A} = O\left(\frac{\mathbf{A}}{L^2}\right), \quad \frac{\partial \mathbf{A}}{\partial t} = O\left(\frac{\mathbf{A}}{\tau}\right)$$

$$\Rightarrow \quad \tau = O\left(\mu \sigma L^2\right) \quad \Rightarrow \quad L = O\left(\frac{1}{\sqrt{\mu \sigma \nu}}\right) \Leftarrow \nu = \frac{1}{\tau}$$

- For a copper sphere of radius 1cm, the decay time of some initial $\mathbf{B}$ field inside ~5-10 ms; for the molten iron core of the earth ~ 10^5 years.

A. Skin Depth, Eddy Currents, Induction Heating



- Boundary conditions

$$\mathbf{H}^{\parallel}\big|_{z=0^+} = \mathbf{H}^{\parallel}\big|_{z=0^-} \quad B^{\perp}\big|_{z=0^+} = B^{\perp}\big|_{z=0^-} \quad + \quad \mathbf{H}\big|_{z=0^-} = H_x\,\hat{\mathbf{x}} = H_0 \cos \omega t\,\hat{\mathbf{x}}$$

- The linearity of the diffusion equation implies that there is only an x-component throughout the half-space, $H_x(z>0,t)$.

$$H_x(z,t) = h(z)\, e^{-i\omega t} \Rightarrow \left(\frac{d^2}{dz^2} + i\mu\sigma\omega\right) h(z) = 0 \Rightarrow h(z) = e^{ikz} \quad \text{trial solution}$$

$$\Rightarrow k^2 = i\mu\sigma\omega \Rightarrow k = \pm\frac{1+i}{\delta} \Leftarrow \delta \equiv \sqrt{\frac{2}{\mu\sigma\omega}} \quad \text{skin depth}$$

- For copper at room temperature $\frac{1}{\sigma} = 1.68\times 10^{-8}\ \Omega\cdot\text{m} \Rightarrow \delta = 6.52\times 10^{-2}/\sqrt{\nu}\ \text{m}$

- For seawater $\delta = 240/\sqrt{\nu}\ \text{m} \Leftarrow \nu = \frac{\omega}{2\pi}$

- $$H_x(z,t) = \alpha\, e^{-\frac{z}{\delta}} e^{i\left(\frac{z}{\delta}-\omega t\right)} + \beta\, e^{\frac{z}{\delta}} e^{-i\left(\frac{z}{\delta}+\omega t\right)} \Leftarrow \begin{array}{l} \beta = 0 \Leftarrow H_x(z\to\infty,t) = \text{finite} \\ \alpha = H_0 \Leftarrow H_x(0^+,t) = H_0\, e^{-i\omega t} \end{array}$$

$$\Rightarrow H_x(z>0,t) = H_0\, e^{-\frac{z}{\delta}} \cos\left(\frac{z}{\delta} - \omega t\right) \Leftarrow \text{only the real part counts}$$

- The magnetic field falls off exponentially in z, with a spatial oscillation of the same scale, being confined mainly to a depth less than the skin depth.

- Only a y-component of $\mathbf{E}$: $E_y = \frac{1}{\sigma}\frac{\mathrm{d}H_x}{\mathrm{d}z} = \frac{i-1}{\sigma\delta} H_0 e^{-\frac{z}{\delta}} e^{i\left(\frac{z}{\delta}-\omega t\right)} \Leftarrow \nabla\times\mathbf{H} = \boldsymbol{J} = \sigma\mathbf{E}$

$$\Rightarrow \quad E_y = \frac{\mu\omega\delta}{\sqrt{2}} H_0 e^{-\frac{z}{\delta}} \cos\left(\frac{z}{\delta} - \omega t + \frac{3\pi}{4}\right) \quad \Leftarrow \text{ taking the real part } \Leftarrow \quad \frac{1}{\sigma\delta} = \frac{\mu\omega\delta}{2}$$

$$\Rightarrow \quad \frac{E_y}{c B_x} = \frac{E_y}{c\mu H_x} = O\left(\frac{\omega\delta}{c}\right) \ll 1 \quad \Leftarrow \text{ quasi-static assumption} \quad \Rightarrow \quad \mathbf{B} \text{ dominates}$$

$$\Rightarrow \quad J_y(z>0) = \sigma E_y = \frac{\sqrt{2}}{\delta} H_0 e^{-\frac{z}{\delta}} \cos\left(\frac{z}{\delta} - \omega t + \frac{3\pi}{4}\right)$$

$$\Rightarrow \quad K_y(t) \equiv \int_0^\infty J_y(z,t)\,\mathrm{d}z = -H_0 \cos\omega t \quad \Leftarrow \quad \text{effective surface (Eddy) current}$$

- For very small skin depth, the volume current flow in the region within $O(\delta)$ of the surface acts as a surface current to reduce the magnetic field to 0 for $z \gg \delta$.

- The time-averaged power input per unit volume $P_{\text{resistive}} = \langle \boldsymbol{J}\cdot\mathbf{E}\rangle = \frac{\mu}{2}\omega H_0^2 e^{-2\frac{z}{\delta}}$

- The heating of the conducting medium to a depth of the order of the skin depth is the basis of induction furnaces and of microwave cookers.

B. Diffusion of Magnetic Fields in Conducting Media

- Consider 2 infinite uniform current sheets, parallel to each other and located a distance $2a$ apart, at $z=-a$ and $z=+a$. For $t<0$

$$\mathbf{H}=\begin{cases} H_0\,\hat{\mathbf{x}} & \text{for } |z|<a \\ 0 & \text{otherwize}\end{cases} \Leftarrow \quad \boldsymbol{J}=J_y\,\hat{\mathbf{y}} \quad \Leftarrow \quad J_y=H_0\left[\delta(z+a)-\delta(z-a)\right]$$

- For $\boldsymbol{J}(t\geq 0)=0$, $\mathbf{A}$ & $\mathbf{H}$ decay according to the diffusion equation: $\nabla^2\mathbf{H}=\mu\,\sigma\,\dfrac{\partial\,\mathbf{H}}{\partial\,t}$

- Use a Laplace transform to separate the space & time dependences

$$H_x(z,t)=\int_0^\infty e^{-p\,t}\,\bar{h}(z,p)\,\mathrm{d}\,p \quad\Rightarrow\quad \left(\frac{\mathrm{d}^2}{\mathrm{d}\,z^2}+k^2\right)\bar{h}(z,p)=0 \quad\Leftarrow\quad k^2=\mu\,\sigma\,p$$

$$\text{symmetric about } z=0 \quad\Rightarrow\quad \bar{h}\propto\cos k\,z \quad\Rightarrow\quad H_x(z,t)=\int_0^\infty e^{-\frac{k^2\,t}{\mu\,\sigma}}\,h(k)\cos k\,z\,\mathrm{d}\,k$$

$$H_x(z,0^+)=\int_0^\infty h(k)\cos k\,z\,\mathrm{d}\,k=H_0\left[\Theta(z+a)-\Theta(z-a)\right] \quad\Leftarrow\quad \Theta:\text{step function}$$

$$\Rightarrow\quad \frac{1}{2}\int_{-\infty}^{+\infty}h(k)\,e^{i\,k\,z}\,\mathrm{d}\,k=H_0\left[\Theta(z+a)-\Theta(z-a)\right] \quad\Leftarrow\quad h(-k)=h(k) \quad \text{for symmetry}$$

$$\Rightarrow\quad h(k)=\frac{H_0}{\pi}\int_{-a}^{a}e^{-i\,k\,z}\,\mathrm{d}\,z=\frac{2\,H_0}{\pi\,k}\sin k\,a \quad\Leftarrow\quad \text{Fourier integral} \quad\Rightarrow\quad \kappa\equiv k\,a$$

$$\Rightarrow \quad H_x(z, t>0) = \frac{2H_0}{\pi}\int_0^\infty e^{-\kappa^2 \nu t}\frac{\sin\kappa}{\kappa}\cos\frac{\kappa z}{a}\,d\kappa \quad \Leftarrow \quad \nu \equiv \frac{1}{\mu\sigma a^2}$$

characteristic decay rate

Error function $\Xi(\xi) = -\Xi(-\xi) \equiv \frac{2}{\sqrt{\pi}}\int_0^\xi e^{-x^2}\,dx = \operatorname{sgn}(\xi)\frac{2}{\pi}\int_0^\infty e^{-\frac{x^2}{4\xi^2}}\frac{\sin x}{x}\,dx$

$$\Rightarrow \quad \Xi(\xi\to\infty) \to 1 - \frac{e^{-\xi^2}}{\sqrt{\pi}\,\xi}\left(1 - \frac{1}{2\xi^2} + \cdots\right), \quad \Xi(|\xi| \ll 1) \approx 2\frac{\xi}{\sqrt{\pi}}\left(1 - \frac{\xi^2}{3} + \cdots\right)$$

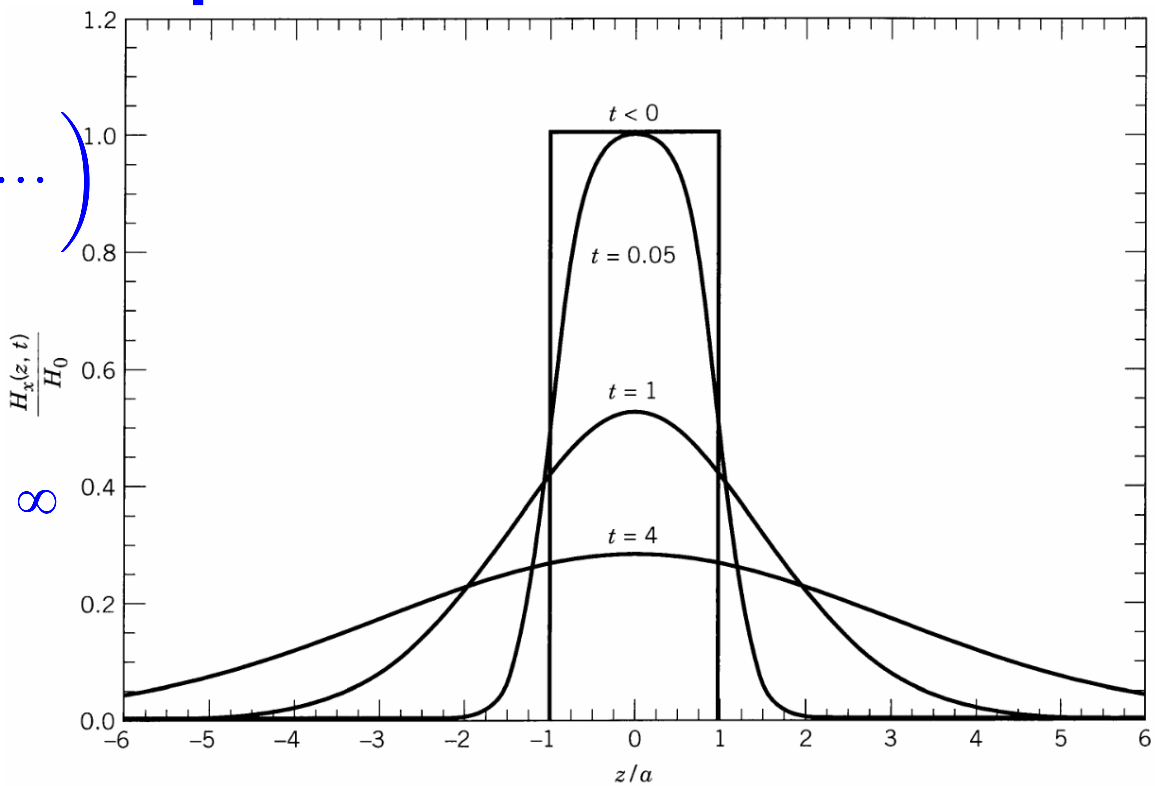
$$\Rightarrow \quad H_x = \frac{H_0}{2}\left[\Xi\left(\frac{a+z}{2a\sqrt{\nu t}}\right) + \Xi\left(\frac{a-z}{2a\sqrt{\nu t}}\right)\right] \to H_0[\Theta(z+a) - \Theta(z-a)] \quad \text{for } \nu t \to 0$$

$$\approx \frac{H_0}{\sqrt{\pi\nu t}}e^{-\frac{z^2}{4\nu a^2 t}}\left(1 + \frac{\frac{z^2}{2\nu a^2 t} - 1}{12\nu t} + \cdots\right)$$

$H_x \to 0$ as $\nu t \to 0, \quad |z| > a$

$H_x \approx \frac{H_0}{\sqrt{\pi\nu t}}$ for $\nu t \gg \frac{|z|}{2a} \quad \Leftarrow \quad t \to \infty$

$H_x = H_{x,\max}$ at $\nu t \approx \frac{z^2}{2a^2}$



$$\sin\kappa\cos\frac{\kappa z}{a}=\frac{1}{2}\left(\sin\frac{\kappa(a+z)}{a}+\sin\frac{\kappa(a-z)}{a}\right)$$

$$\Rightarrow\quad H_x=\frac{H_0}{\pi}\int_0^\infty\frac{e^{-\kappa^2\nu t}}{\kappa}\left(\sin\frac{\kappa(a+z)}{a}+\sin\frac{\kappa(a-z)}{a}\right)d\kappa$$

$$=\frac{H_0}{\pi}\int_0^\infty e^{-\kappa^2\nu t}\left(\frac{\sin k_+}{k_+}d k_+ +\frac{\sin k_-}{k_-}d k_-\right)\quad\Leftarrow\quad k_\pm\equiv\kappa\left(1\pm\frac{z}{a}\right)$$

$$=\frac{H_0}{\pi}\int_0^\infty\left(e^{-\frac{k_+^2}{4\xi_+^2}}\frac{\sin k_+}{k_+}d k_+ +e^{-\frac{k_-^2}{4\xi_-^2}}\frac{\sin k_-}{k_-}d k_-\right)\quad\Leftarrow\quad\xi_\pm=\frac{a\pm z}{2a\sqrt{\nu t}}$$

$$=H_0\frac{\Xi(\xi_+)+\Xi(\xi_-)}{2}=\frac{H_0}{2}\left[\Xi\left(\frac{a+z}{2a\sqrt{\nu t}}\right)+\Xi\left(\frac{a-z}{2a\sqrt{\nu t}}\right)\right]\quad\nwarrow\quad\kappa^2\nu t=\frac{k_\pm^2}{4\xi_\pm^2}$$

$$\text{Define}\ \ I(a)=\int_0^\infty e^{-x^2}\cos a x\,d x\quad\Rightarrow\quad I(0)=\int_0^\infty e^{-x^2}d x=\frac{\sqrt{\pi}}{2}\quad\Leftarrow\quad\text{Gauss integral}$$

$$\Rightarrow\quad\frac{d I}{d a}=-\int_0^\infty x\,e^{-x^2}\sin a x\,d x=\frac{1}{2}\int_0^\infty\sin a x\,d e^{-x^2}$$

$$=\frac{1}{2}\left(e^{-x^2}\sin a x\Big|_0^\infty-\int_0^\infty e^{-x^2}d\sin a x\right)=-\frac{a}{2}\int_0^\infty e^{-x^2}\cos a x\,d x=-\frac{a}{2}I$$

$$\Rightarrow\quad\frac{d I}{I}=-\frac{a}{2}d a\quad\Rightarrow\quad I(a)=\int_0^\infty e^{-x^2}\cos a x\,d x=\frac{\sqrt{\pi}}{2}e^{-\frac{a^2}{4}}\quad\Leftarrow\quad I(0)=\frac{\sqrt{\pi}}{2}$$

Feynman Technique

$$\Xi(\xi>0)=\frac{2}{\pi}\int_0^\infty e^{-p^2}\frac{\sin x}{x}\,\mathrm{d}\,x \quad\Leftarrow\quad p\equiv\frac{x}{2\,\xi} \quad\Rightarrow\quad \frac{\mathrm{d}\,e^{-p^2}}{\mathrm{d}\,\xi}=2\,e^{-p^2}\frac{p^2}{\xi}\,,\quad \Xi(0)=0$$

$$\Rightarrow\quad \frac{\mathrm{d}\,\Xi}{\mathrm{d}\,\xi}=\frac{4}{\pi\,\xi}\int_0^\infty p^2\,e^{-p^2}\frac{\sin x}{x}\,\mathrm{d}\,x=\frac{4}{\pi\,\xi}\int_0^\infty p\,e^{-p^2}\sin(2\,\xi\,p)\,\mathrm{d}\,p$$

$$=-\frac{2}{\pi\,\xi}\int_0^\infty \sin(2\,\xi\,p)\,\mathrm{d}\,e^{-p^2} \quad\Leftarrow\quad \mathrm{d}\,e^{-p^2}=-2\,p\,e^{-p^2}\,\mathrm{d}\,p$$

$$=-\frac{2}{\pi\,\xi}\left(e^{-p^2}\sin(2\,\xi\,p)\Big|_0^\infty-\int_0^\infty e^{-p^2}\,\mathrm{d}\sin(2\,\xi\,p)\right)$$

$$=\frac{4}{\pi}\int_0^\infty e^{-p^2}\cos(2\,\xi\,p)\,\mathrm{d}\,p=\frac{2}{\sqrt{\pi}}\,e^{-\xi^2} \quad\Rightarrow\quad \mathrm{d}\,\Xi=\frac{2}{\sqrt{\pi}}\,e^{-\xi'^2}\,\mathrm{d}\,\xi'$$

$$\Rightarrow\quad \Xi(\xi)=\Xi(\xi)-\Xi(0)=\int_0^\xi \mathrm{d}\,\Xi=\frac{2}{\sqrt{\pi}}\int_0^\xi e^{-\xi'^2}\,\mathrm{d}\,\xi'=\frac{2}{\sqrt{\pi}}\int_0^\xi e^{-x^2}\,\mathrm{d}\,x$$