Chapter 3 Boundary-Value Problems in Electrostatics II

- Solutions of the Laplace equation are represented by expansions in series of the appropriate orthonormal functions in various geometries.

- The construction of Green functions in terms of orthonormal functions arises in the attempt to solve the Poisson equation in the various geometries.

Laplace Equation in Spherical Coordinates

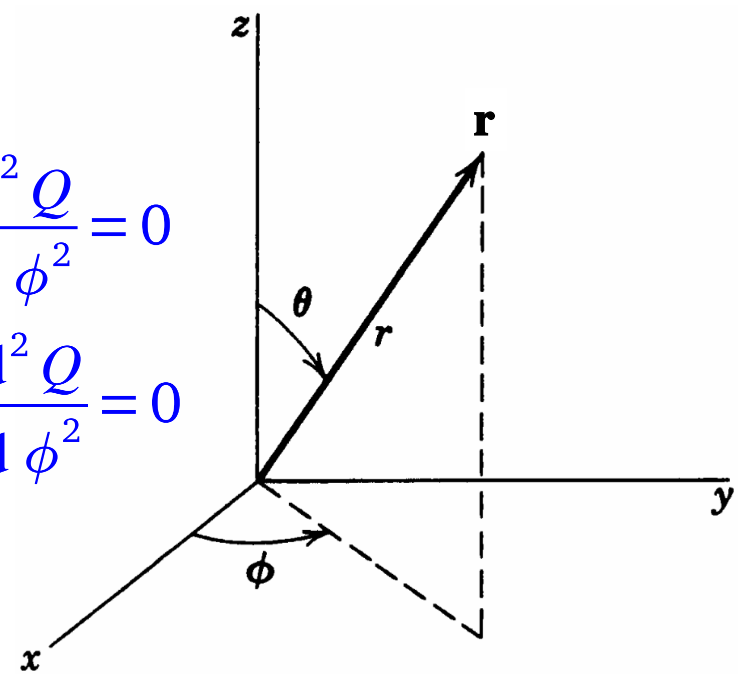
- In spherical coordinates, the Laplace equation is as follows:

$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

$$\Phi = \frac{U(r)}{r}P(\theta)Q(\phi) \quad \Leftarrow \quad \text{assuming a product form}$$

$$\Rightarrow \; PQ\frac{d^2U}{dr^2} + \frac{UQ}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{UP}{r^2\sin^2\theta}\frac{d^2Q}{d\phi^2} = 0$$

$$\Rightarrow \; \sin^2\theta\left[\frac{r^2}{U}\frac{d^2U}{dr^2} + \frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right] + \frac{1}{Q}\frac{d^2Q}{d\phi^2} = 0$$



- The ϕ dependence of the equation has now been isolated in the last term:

$$\frac{1}{Q}\frac{\mathrm{d}^2 Q}{\mathrm{d}\,\phi^2} = -m^2 \quad \Leftarrow \text{constant} \quad \Rightarrow \quad \boxed{Q = e^{\pm i\, m\, \phi}} \quad \Leftarrow \quad m : \begin{array}{l}\text{an integer for } Q \\ \text{being single-valued}\end{array}$$

- By similar considerations

$$\frac{\mathrm{d}^2 U}{\mathrm{d}\, r^2} - \frac{\ell\,(\ell+1)}{r^2}\, U = 0 \quad \Leftarrow \quad \ell : \text{real constant} \quad \Rightarrow \quad \boxed{U = A\, r^{\ell+1} + \frac{B}{r^{\ell}}}$$

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\,\theta}\left(\sin\theta\,\frac{\mathrm{d}\,P}{\mathrm{d}\,\theta}\right) + \left(\ell\,(\ell+1) - \frac{m^2}{\sin^2\theta}\right) P = 0$$

Legendre Equation and Legendre Polynomials

- $x = \cos\theta$, $\sin^2\theta = 1 - x^2$ $\Rightarrow$ $$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx} \right) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2} \right) P = 0$$

This equation is called the generalized Legendre equation, and its solutions are the associated Legendre functions. (Also $dx = -\sin\theta \, d\theta$)

- Consider the ordinary Legendre differential equation with $m=0$

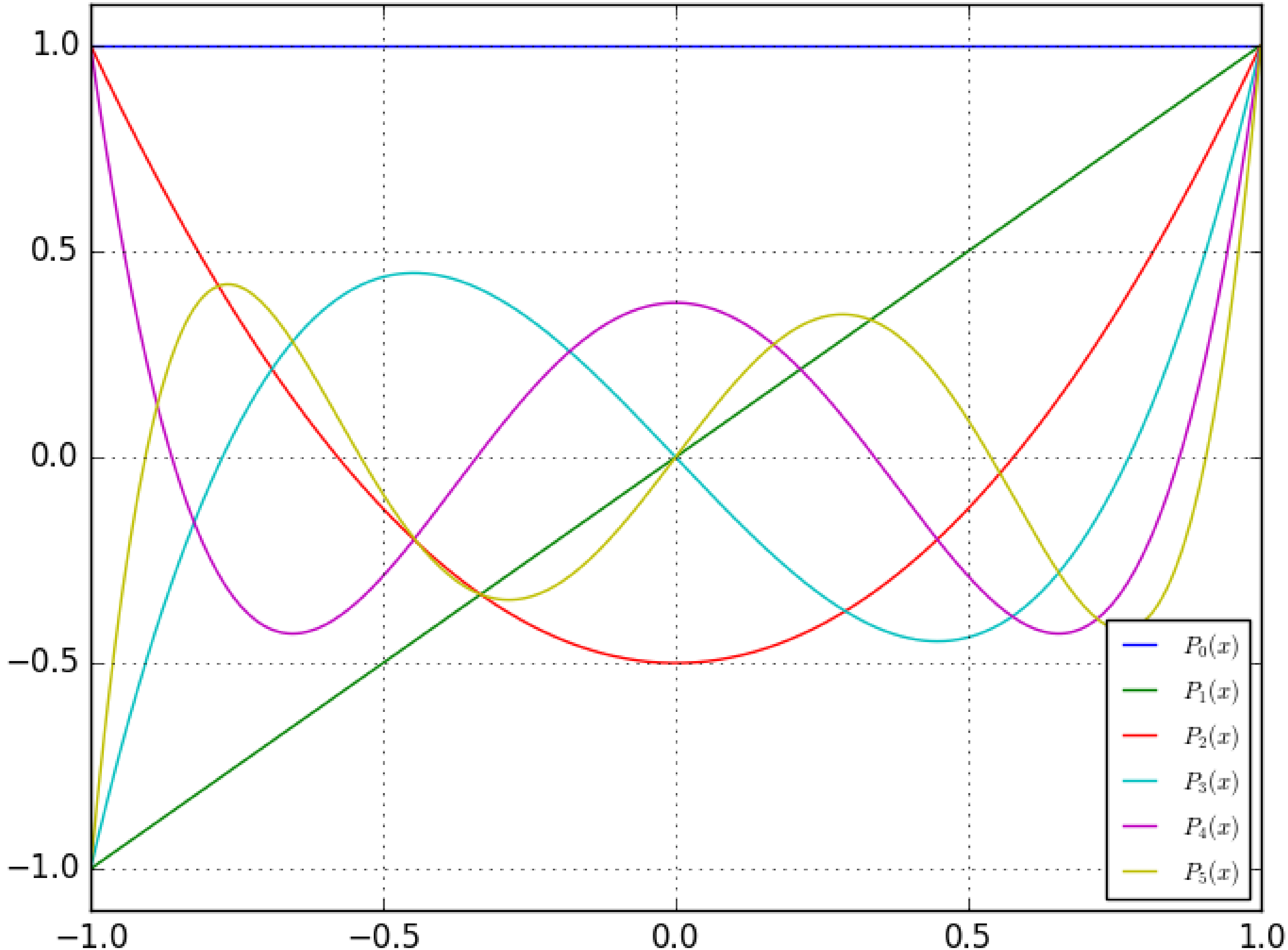
$$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx} \right) + \ell(\ell+1) P = 0 \text{ where } -1 \le x \le 1 \quad \Rightarrow \quad P(x) = x^{\alpha} \sum_{j=0}^{\infty} a_j x^j$$

$$\Rightarrow \sum_{j=0}^{\infty} \left((\alpha+j)(\alpha+j-1) a_j x^{\alpha+j-2} - [(\alpha+j)(\alpha+j+1) - \ell(\ell+1)] a_j x^{\alpha+j} \right) = 0$$

$$\Rightarrow \quad a_{j+2} = a_j \frac{(\alpha+j)(\alpha+j+1) - \ell(\ell+1)}{(\alpha+j+1)(\alpha+j+2)} \text{ for } j \ge 0 \quad \Leftarrow \quad \begin{matrix} \alpha(\alpha-1) = 0 \text{ for } a_0 \ne 0 \\ \alpha(\alpha+1) = 0 \text{ for } a_1 \ne 0 \end{matrix}$$

$$\text{For } a_0 \ne 0 \quad \Rightarrow \quad \alpha = \begin{bmatrix} 0 & \text{even power series of } x \\ 1 & \text{odd power series of } x \end{bmatrix}$$

- The 2 relations of α are equivalent, so choose one of a_0 and a_1 being nonzero.
- The series converges for $x^2 < 1$, regardless of the value of ℓ.
- The series diverges at $x = \pm 1$, unless it terminates.
- Since we want a solution that is finite, we demand that the series terminate.


1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0
$P_0(x)$
$P_1(x)$
$P_2(x)$
$P_3(x)$
$P_4(x)$
$P_5(x)$

- Since α and j are positive integers or 0, the recurrence relation will terminate only if ℓ *is 0 or a positive integer*.

- If ℓ is even (odd), then only the $\alpha=0$ ($\alpha=1$) series terminates.

- The polynomials in each case have x^ℓ as their highest power of x, the next highest being $x^{\ell-2}$, down to $x^0(x)$ for ℓ even (odd).

- These polynomials are normalized to be unity at $x=\pm 1$ and are called the *Legendre polynomials* of order ℓ,

$$P_0(x)=1, \quad P_1(x)=x, \quad P_2(x)=\frac{3x^2-1}{2}, \quad \cdots \quad \Rightarrow \quad P_\ell(x)=\frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

$$P_3(x)=\frac{5x^3-3x}{2}, \quad p_4(x)=\frac{35x^4-30x^2+3}{8}$$

Rodrigues' formula

- The Legendre polynomials form a complete orthogonal set of functions on the interval $-1\le x\le 1$:

$$\int_{-1}^{+1} P_{\ell'}\left[\frac{d}{dx}\left((1-x^2)\frac{dP_\ell}{dx}\right)+\ell(\ell+1)P_\ell\right]dx=0$$

$$\Rightarrow \int_{-1}^{+1}\left[(x^2-1)\frac{dP_\ell}{dx}\frac{dP_{\ell'}}{dx}+\ell(\ell+1)P_\ell P_{\ell'}\right]dx=0$$

$$\ell\leftrightarrow\ell' \Rightarrow [\ell(\ell+1)-\ell'(\ell'+1)]\int_{-1}^{+1}P_{\ell'}P_\ell\, dx=0 \Rightarrow \boxed{\int_{-1}^{+1}P_{\ell'}P_\ell\, dx=0 \quad \text{for } \ell\neq\ell'}$$

- Use Rodrigues' formula to determine the value for $\ell=\ell'$

$$N_\ell \equiv \int_{-1}^{+1} P_\ell^2 \, \mathrm{d}\, x = \frac{1}{4^\ell (\ell!)^2} \int_{-1}^{+1} \left(\frac{\mathrm{d}^\ell}{\mathrm{d}\, x^\ell} (x^2-1)^\ell \right) \left(\frac{\mathrm{d}^\ell}{\mathrm{d}\, x^\ell} (x^2-1)^\ell \right) \mathrm{d}\, x$$

$$= \frac{(-1)^\ell}{4^\ell (\ell!)^2} \int_{-1}^{+1} (x^2-1)^\ell \frac{\mathrm{d}^{2\ell}}{\mathrm{d}\, x^{2\ell}} (x^2-1)^\ell \, \mathrm{d}\, x = \frac{(2\ell)!}{4^\ell (\ell!)^2} \int_{-1}^{+1} (1-x^2)^\ell \, \mathrm{d}\, x$$

$$(1-x^2)^\ell = (1-x^2)(1-x^2)^{\ell-1} = (1-x^2)^{\ell-1} + \frac{x}{2\ell} \frac{\mathrm{d}}{\mathrm{d}\, x} (1-x^2)^\ell$$

$$\Rightarrow \quad N_\ell = \frac{2\ell-1}{2\ell} N_{\ell-1} + \frac{(2\ell-1)!}{4^\ell (\ell!)^2} \int_{-1}^{+1} x \, \mathrm{d}\, (1-x^2)^\ell = \frac{2\ell-1}{2\ell} N_{\ell-1} - \frac{1}{2\ell} N_\ell$$

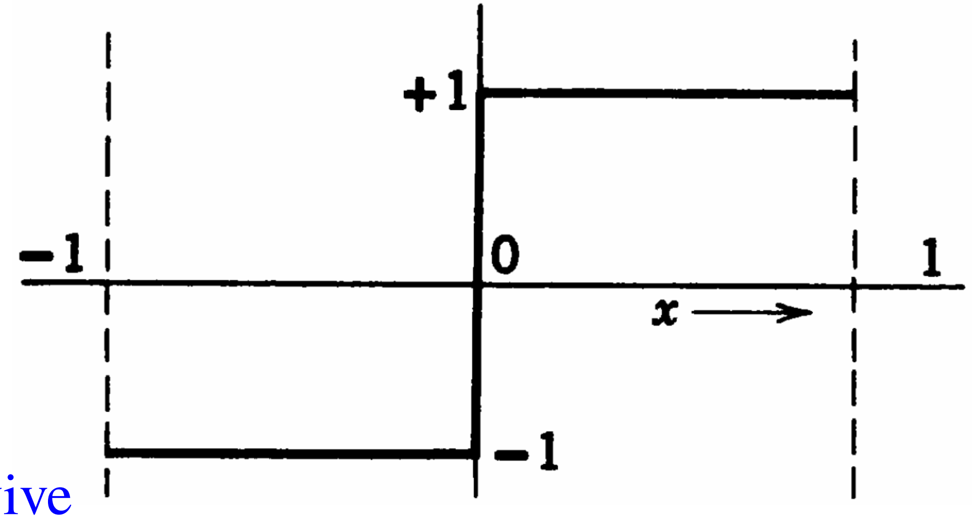
$$\Rightarrow \quad (2\ell+1) N_\ell = (2\ell-1) N_{\ell-1} = \cdots = (2\cdot 0+1) N_0 \quad \Rightarrow \quad \text{independent of } \ell$$

$$\Rightarrow \quad N_\ell = \frac{2}{2\ell+1} \quad \Leftarrow \quad N_0 = 2 \quad \Leftarrow \quad P_0 = 1$$

$$\Rightarrow \quad \boxed{\int_{-1}^{+1} P_\ell(x) P_{\ell'}(x) \, \mathrm{d}\, x = \frac{2}{2\ell+1} \delta_{\ell\ell'}} \quad \text{orthogonality condition} \quad \Rightarrow \quad U_\ell = \sqrt{\frac{2\ell+1}{2}} P_\ell$$

- For any function on the interval $-1 \le x \le 1$: $f(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x) \quad \Leftarrow \quad A_\ell = \frac{2\ell+1}{2} \int_{-1}^{+1} f(x) P_\ell(x) \, \mathrm{d}\, x$

Ex: consider $f(x) = \begin{cases} +1 & \text{for } x>0 \\ -1 & \text{for } x<0 \end{cases}$



$$A_\ell = \frac{2\ell+1}{2}\left(\int_0^1 P_\ell \, \mathrm{d}x - \int_{-1}^0 P_\ell \, \mathrm{d}x \right)$$

$$= (2\ell+1) \int_0^1 P_\ell \, \mathrm{d}x$$ only odd ℓ coefficients survive

$$= (-1)^{\frac{\ell-1}{2}} (2\ell+1) \frac{(\ell-2)!!}{(\ell+1)!!} \quad \Leftarrow \quad \begin{aligned} (2n+1)!! &= (2n+1)(2n-1)\cdots\times 3\times 1 \\ 2n!! &= \quad 2n \quad\; (2n-2)\cdots\times 4\times 2 \end{aligned}$$

$$\Rightarrow \quad f(x) = \frac{3}{2}P_1 - \frac{7}{8}P_3 + \frac{11}{16}P_5 - \cdots$$

- Some useful recurrence relations

$$\frac{\mathrm{d}P_{\ell+1}}{\mathrm{d}x} - \frac{\mathrm{d}P_{\ell-1}}{\mathrm{d}x} - (2\ell+1)P_\ell = 0, \quad (\ell+1)P_{\ell+1} - (2\ell+1)\,x\,P_\ell + \ell P_{\ell-1} = 0 \quad (\ast)$$

$$\frac{\mathrm{d}P_{\ell+1}}{\mathrm{d}x} - x\frac{\mathrm{d}P_\ell}{\mathrm{d}x} - (\ell+1)P_\ell = 0, \qquad (x^2-1)\frac{\mathrm{d}P_\ell}{\mathrm{d}x} - \ell\,x\,P_\ell + \ell P_{\ell-1} = 0$$

- $P_\ell(x) = \sum_{n=0}^{[\ell/2]} (-1)^n \dfrac{(2\ell-2n)!}{2^\ell\, n!\,(\ell-n)!\,(\ell-2n)!} x^{\ell-2n} \quad \Leftarrow$ *Mathematical methods for Physicists*, Arfken

$$\frac{1}{\sqrt{1-2\,x\,t+t^{2}}}=\sum_{n=0} P_{n}(x)\,t^{n} \quad \Leftarrow \quad \text{generating function}$$

$$\Rightarrow \sum_{n=0} P_{n}(0)\,t^{n}=\frac{1}{\sqrt{1+t^{2}}}=\sum_{k=0}(-1)^{k}\,C_{k}^{2k}\,\frac{t^{2k}}{2^{2k}} \quad \Rightarrow \quad P_{n}(0)=\begin{cases}(-1)^{n/2}\,\dfrac{C_{n/2}^{n}}{2^{n}}\,, & n \text{ even}\\ 0\,, & n \text{ odd}\end{cases}$$

$$\frac{\mathrm{d}\,P_{\ell+1}}{\mathrm{d}\,x}-\frac{\mathrm{d}\,P_{\ell-1}}{\mathrm{d}\,x}-(2\,\ell+1)\,P_{\ell}=0$$

$$\Rightarrow (2\,\ell+1)\int_{0}^{1} P_{\ell}\,\mathrm{d}\,x=P_{\ell+1}(1)-P_{\ell+1}(0)-P_{\ell-1}(1)+P_{\ell-1}(0)$$

$$=P_{\ell-1}(0)-P_{\ell+1}(0)=\frac{(-1)^{(\ell-1)/2}}{2^{\ell-1}}\,C_{(\ell-1)/2}^{\ell-1}-\frac{(-1)^{(\ell+1)/2}}{2^{\ell+1}}\,C_{(\ell+1)/2}^{\ell+1}$$

$$=(-1)^{\frac{\ell-1}{2}}\,\frac{(\ell-1)!}{[(\ell-1)!!]^{2}}-(-1)^{\frac{\ell+1}{2}}\,\frac{(\ell+1)!}{[(\ell+1)!!]^{2}}=(-1)^{\frac{\ell-1}{2}}\left(\frac{(\ell-2)!!}{(\ell-1)!!}+\frac{\ell!!}{(\ell+1)!!}\right)$$

$$=(-1)^{\frac{\ell-1}{2}}\,(2\,\ell+1)\,\frac{(\ell-2)!!}{(\ell+1)!!}\,, \qquad \ell \text{ odd}$$

Ex: consider

$$\int_{-1}^{+1} x\, P_\ell\, P_{\ell'}\, \mathrm{d}x = \frac{1}{2\ell+1}\int_{-1}^{+1} P_{\ell'}\left[(\ell+1)\,P_{\ell+1} + \ell\, P_{\ell-1}\right]\mathrm{d}x \quad \Leftarrow \quad (\ast)$$

$$= \begin{cases} \dfrac{2(\ell+1)}{(2\ell+1)(2\ell+3)}, & \ell' = \ell+1 \\[2ex] \dfrac{2\ell}{(2\ell-1)(2\ell+1)}, & \ell' = \ell-1 \end{cases} \quad \Leftarrow \quad \text{orthogonality condition}$$

Similarly,

$$\int_{-1}^{+1} x^2\, P_\ell\, P_{\ell'}\, \mathrm{d}x = \begin{cases} \dfrac{2(\ell+1)(\ell+2)}{(2\ell+1)(2\ell+3)(2\ell+5)}, & \ell' = \ell+2 \\[2ex] \dfrac{2(2\ell^2+2\ell-1)}{(2\ell-1)(2\ell+1)(2\ell+3)}, & \ell' = \ell \end{cases} \quad \Leftarrow \quad \text{assuming} \quad \ell' \geq \ell$$

Note: $P_\ell(1) = P_\ell(\cos 0) = 1\,, \quad P_\ell(-1) = P_\ell(\cos\pi) = (-1)^\ell \quad \Rightarrow \quad P_\ell(\pm 1) = (\pm 1)^\ell$

Boundary-Value Problems with Azimuthal Symmetry

- The general solution for a problem possessing azimuthal symmetry $m=0$

$$\Phi(r,\theta)=\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta) \Leftarrow \text{determining } A_\ell, B_\ell, \text{ from the boundary condition}$$

On the symmetric axis z $\Rightarrow$

$$\Phi(\theta=0)=\sum_{\ell=0}\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)$$

$$\Phi(\theta=\pi)=\sum_{\ell=0}\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)(-1)^\ell$$

- This expansion provides a means of obtaining the solution of potential problems from a knowledge of the potential in a limited domain, ie, on the symmetry axis.

Ex: let $\Phi_0(\theta)$ be the potential on the surface of a sphere of radius a, find the potential inside the sphere.

no charge at the origin $\Rightarrow$ $B_\ell=0$ $\Rightarrow$ $\Phi_0(\theta)=\sum_{\ell=0}^{\infty}A_\ell a^\ell P_\ell(\cos\theta)$

$$\Rightarrow A_\ell=\frac{2\ell+1}{2a^\ell}\int_0^\pi\Phi_0(\theta)P_\ell(\cos\theta)\sin\theta\,d\theta \Rightarrow \text{let } \Phi_0(\theta)=\begin{cases}+V, & 0\le\theta<\pi/2\\ -V, & \pi/2<\theta\le\pi\end{cases}$$

$$\Rightarrow \Phi(r,\theta)=V\left(\frac{3}{2}\frac{r}{a}P_1(\cos\theta)-\frac{7}{8}\frac{r^3}{a^3}P_3(\cos\theta)+\frac{11}{16}\frac{r^5}{a^5}P_5(\cos\theta)+\cdots\right) \text{ for } r<a$$

$$\Rightarrow \Phi(r,\theta)=V\left(\frac{3}{2}\frac{a^2}{r^2}P_1(\cos\theta)-\frac{7}{8}\frac{a^4}{r^4}P_3(\cos\theta)+\frac{11}{16}\frac{a^6}{r^6}P_5(\cos\theta)+\cdots\right) \text{ for } r>a$$

- For the problem of the hemispheres at equal and opposite potentials. We have

$$\Phi(z=r)=V\left(1-\frac{r^2-a^2}{r\sqrt{r^2+a^2}}\right)=\frac{V}{\sqrt{\pi}}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j!}\,\frac{4j-1}{2}\,\Gamma\left(\frac{2j-1}{2}\right)\frac{a^{2j}}{r^{2j}}$$

$$\Rightarrow \Phi(r,\theta)=\frac{V}{\sqrt{\pi}}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j!}\,\frac{4j-1}{2}\,\Gamma\left(\frac{2j-1}{2}\right)\frac{a^{2j}}{r^{2j}}P_{2j-1}(\cos\theta)\begin{matrix}=(2.27)\\=(3.36)\end{matrix}$$

- The potential at $\mathbf{r}$ due to a unit point charge at $\mathbf{r}'$:

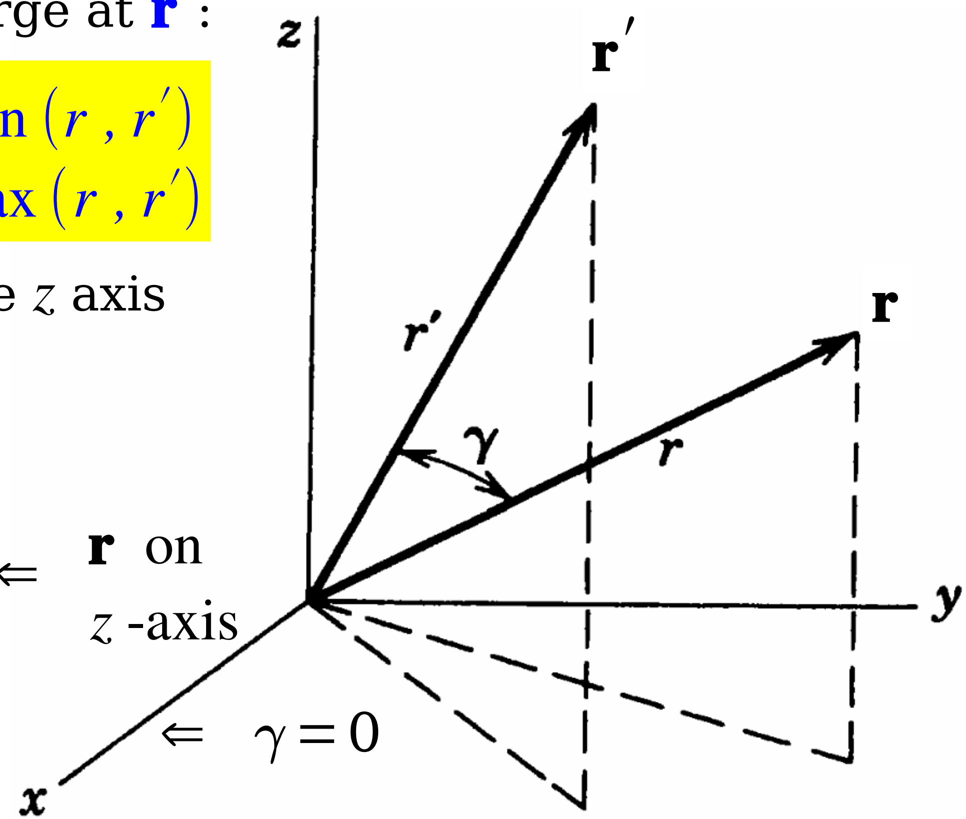
$$\boxed{\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}}P_\ell(\cos\gamma)\quad\Leftarrow\quad\begin{matrix}r_<=\min(r,r')\\ r_>=\max(r,r')\end{matrix}}$$

Proof: rotating axes so that $\mathbf{r}'$ lies along the z axis

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\gamma)$$

$$\rightarrow\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)\quad\Leftarrow\quad\gamma=0\quad\Leftarrow$$

$\mathbf{r}$ on z-axis



$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}}\rightarrow\frac{1}{|r-r'|}$$

$\Leftarrow\ \gamma=0$

$$\frac{1}{|r-r'|}=\frac{1}{r_>}\sum_{\ell=0}^{\infty}\left(\frac{r_<}{r_>}\right)^{\ell},$$

then multiply each term by $P_\ell(\cos\gamma)$ for points off the axis

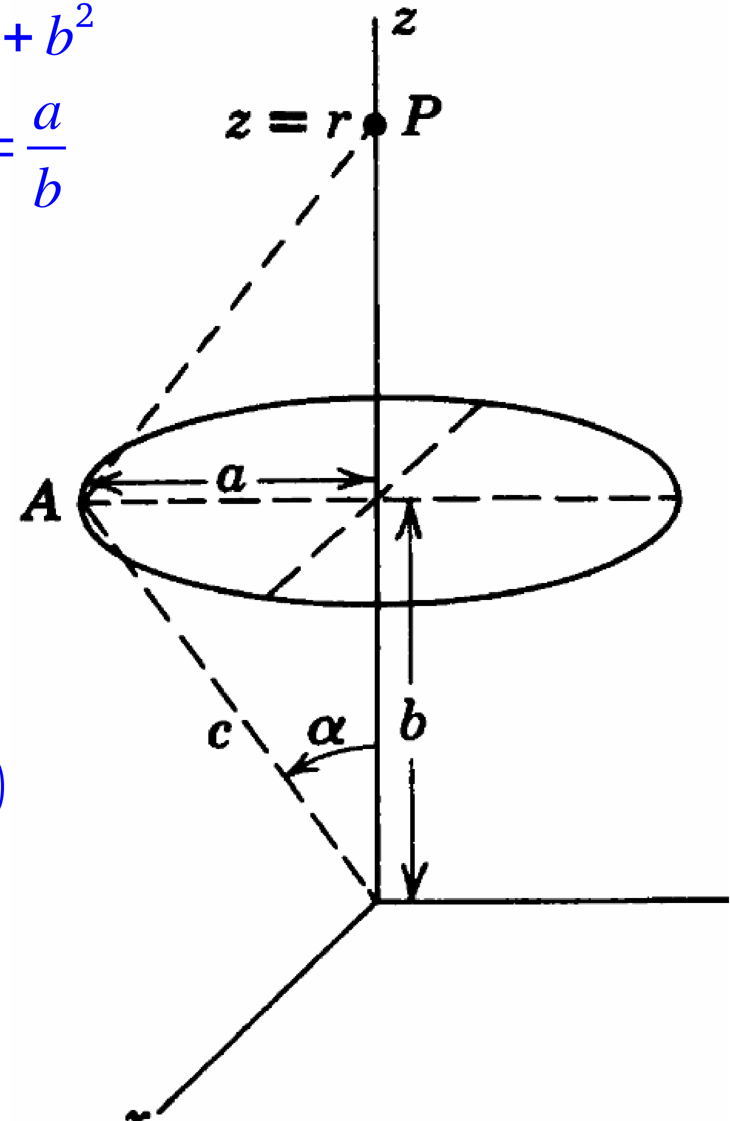
- The potential due to a total charge uniformly distributed around a circular ring

$$\Phi(z=r)=\frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{r^2+c^2-2cr\cos\alpha}} \Leftarrow \quad c^2=a^2+b^2, \quad \cos\alpha=\frac{a}{b}$$

$$=\begin{cases}\dfrac{q}{4\pi\epsilon_0}\displaystyle\sum_{\ell=0}^{\infty}\frac{c^\ell}{r^{\ell+1}}P_\ell(\cos\alpha) & r>c\\[2ex] \dfrac{q}{4\pi\epsilon_0}\displaystyle\sum_{\ell=0}^{\infty}\frac{r^\ell}{c^{\ell+1}}P_\ell(\cos\alpha) & r<c\end{cases}$$

$$\Rightarrow \quad \Phi(r,\theta)=\frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\alpha)P_\ell(\cos\theta)$$

where $\quad r_<=\min(r,c), \quad r_>=\max(r,c)$

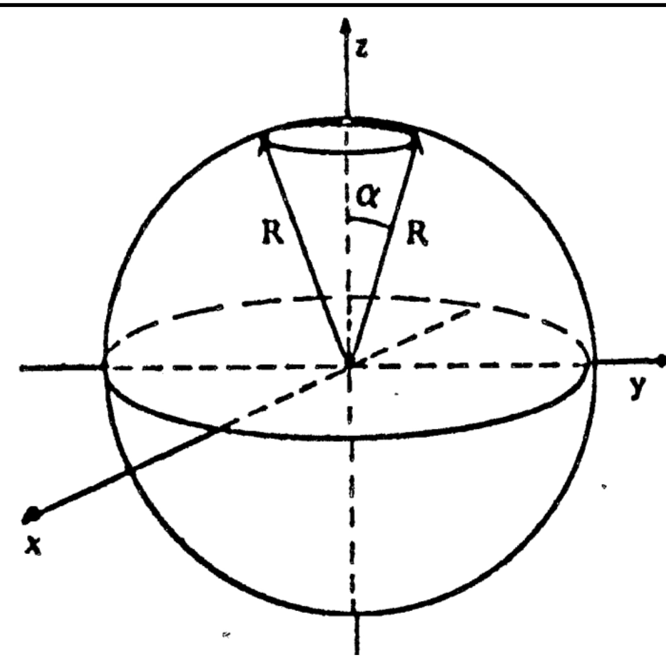


A spherical surface of radius R has charge uniformly distributed over its surface with a density $Q/4\pi R^2$, except for a spherical cap at the north pole, defined by the cone $\theta = \alpha$.

(a) Show that the potential inside the spherical surface can be expressed as

$$\Phi = \frac{Q}{8\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} \left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right] \frac{r^\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

where, for $\ell = 0$, $P_{\ell-1}(\cos\alpha) = -1$. What is the potential outside? [Problem 3.2a]



With the known surface charge density $\sigma = \begin{cases} 0, & 0 \le \theta < \alpha \\ Q/4\pi R^2, & \alpha \le \theta \le \pi \end{cases}$, we can express the potential on the z-axis inside the spherical surface as

$$\Phi(r < R, \theta = 0) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma(\mathbf{x}')R^2 \mathrm{d}\Omega'}{|\mathbf{x} - \mathbf{x}'|}$$

$$
\begin{aligned}
&= \frac{Q}{8\pi\epsilon_0}\int_\alpha^\pi \sum_{\ell=0}^{\infty}\frac{r^\ell}{R^{\ell+1}}P_\ell(\cos\theta')\sin\theta'\mathrm{d}\theta' = \frac{Q}{8\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{r^\ell}{R^{\ell+1}}\int_{-1}^{\cos\alpha}P_\ell(y)\mathrm{d}y \\
&= \frac{Q}{8\pi\epsilon_0}\left[\frac{\cos\alpha+1}{R}+\sum_{\ell=1}^{\infty}\frac{r^\ell}{R^{\ell+1}}\left.\frac{P_{\ell+1}(y)-P_{\ell-1}(y)}{2\ell+1}\right|_{-1}^{\cos\alpha}\right] \\
&= \frac{Q}{8\pi\epsilon_0}\left[\frac{P_1(\cos\alpha)-P_{-1}(\cos\alpha)}{R}+\sum_{\ell=1}^{\infty}\frac{r^\ell}{R^{\ell+1}}\frac{P_{\ell+1}(\cos\alpha)-P_{\ell-1}(\cos\alpha)}{2\ell+1}\right] \\
&= \frac{Q}{8\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{r^\ell}{R^{\ell+1}}\frac{P_{\ell+1}(\cos\alpha)-P_{\ell-1}(\cos\alpha)}{2\ell+1},
\end{aligned}
$$

where $y \equiv \cos\theta'$, $P_{-1}(\cos\alpha) \equiv -1$, and since $|\mathbf{x}-\mathbf{x}'|^{-1} = \sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\gamma)$ where $r_<$ ($r_>$) is the smaller (larger) of $|\mathbf{x}|$ and $|\mathbf{x}'|$ and $\cos\gamma = \hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'$, and the recurrence formula $\frac{\mathrm{d}P_{\ell+1}}{\mathrm{d}x} - \frac{\mathrm{d}P_{\ell-1}}{\mathrm{d}x} = (2\ell+1)P_\ell$. The potential at any point inside the sphere is now obtained by multiplying each member of these series by $P_\ell(\cos\theta)$

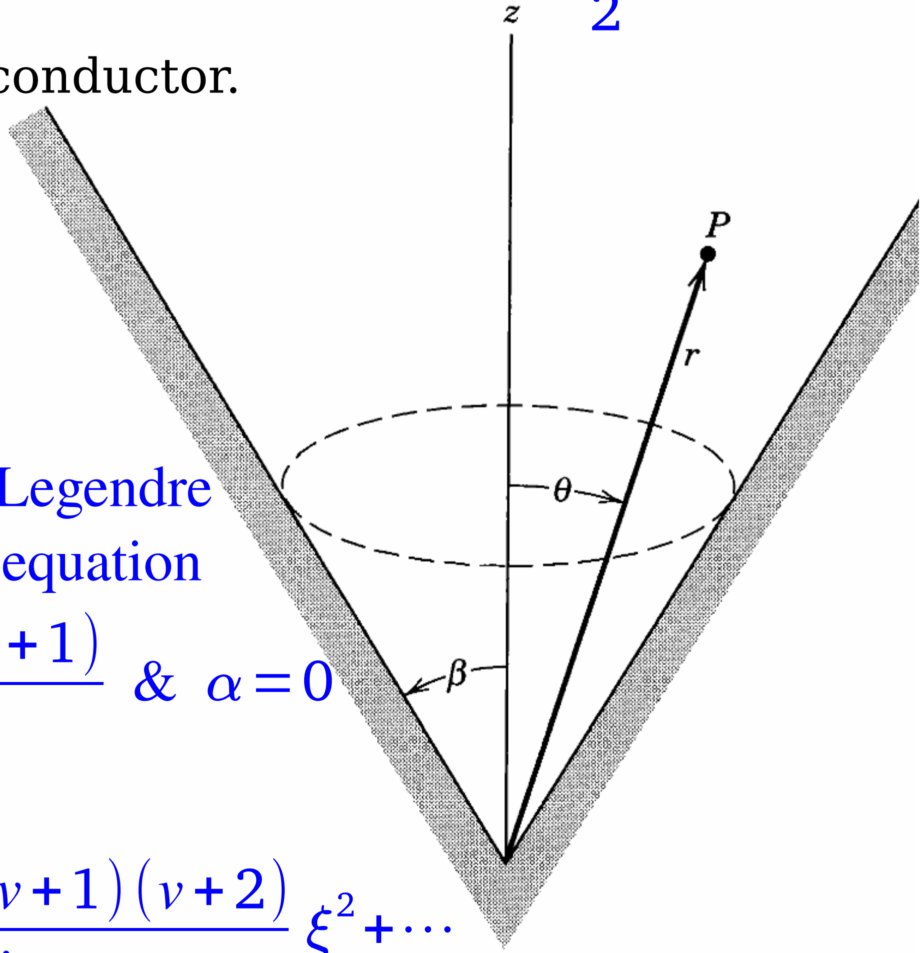
$$
\Phi_{\mathrm{in}}(r,\theta) = \frac{Q}{8\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{r^\ell}{R^{\ell+1}}\frac{P_{\ell+1}(\cos\alpha)-P_{\ell-1}(\cos\alpha)}{2\ell+1}P_\ell(\cos\theta).
$$

Outside the sphere, i.e., $r > R$, the potential can be obtained by swapping the position of r and R in Φ_{in},

$$
\Phi_{\mathrm{out}}(r,\theta) = \frac{Q}{8\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{R^\ell}{r^{\ell+1}}\frac{P_{\ell+1}(\cos\alpha)-P_{\ell-1}(\cos\alpha)}{2\ell+1}P_\ell(\cos\theta).
$$

Behavior of Fields in a Conical Hole or Near a Sharp Point

- For $\beta < \dfrac{\pi}{2}$, the region is a deep conical hole in a conductor. For $\beta > \dfrac{\pi}{2}$, the region is that surrounding a pointed conical conductor.



- Seek solutions finite and single-valued on the range of $x = \cos\theta$

$$\xi \equiv \frac{1-x}{2} \quad \Leftarrow \quad x \equiv \cos\theta \quad \Rightarrow \quad \cos\beta < x < 1$$

$$\Rightarrow \frac{d}{d\xi}\left(\xi(1-\xi)\frac{dP}{d\xi}\right) + \nu(\nu+1)P = 0 \quad \Leftarrow \quad \text{Legendre equation}$$

$$\Rightarrow \; P(\xi) = \xi^{\alpha}\sum_{j=0}^{\infty} a_j \xi^j \;\Rightarrow\; \frac{a_{j+1}}{a_j} = \frac{(j-\nu)(j+\nu+1)}{(j+1)^2} \;\;\&\;\; \alpha = 0$$

$$\text{set } a_0 = 1 \quad \Rightarrow \quad P(\xi = 0) = 1$$

$$\Rightarrow P_\nu(\xi) = 1 + \frac{(-\nu)(\nu+1)}{1!\,1!}\xi + \frac{(-\nu)(-\nu+1)(\nu+1)(\nu+2)}{2!\,2!}\xi^2 + \cdots$$

- If ν is 0 or a positive integer the series is exactly the Legendre *polynomials*.

- For ν not being an integer, the series is called a *Legendre function of the 1*[st] *kind and order* ν.

- Hypergeometric function $\;{}_2F_1(a,b;c;z)=1+\dfrac{ab}{c}\dfrac{z}{1!}+\dfrac{a(a+1)b(b+1)}{c(c+1)}\dfrac{z^2}{2!}+\cdots$

$$\Rightarrow \quad P_\nu(x)={}_2F_1\left(-\nu,\nu+1;1;\frac{1-x}{2}\right)$$

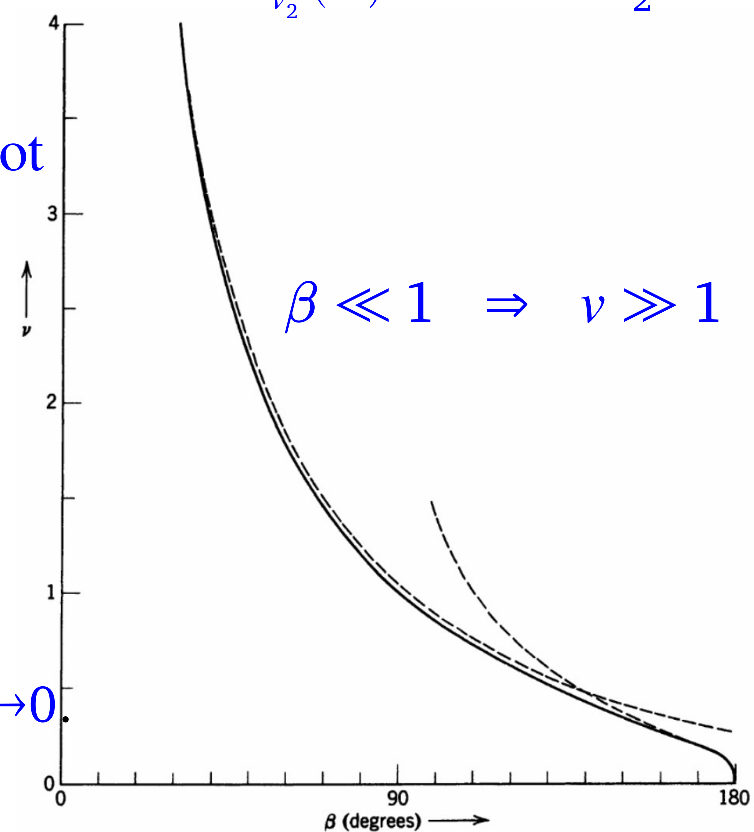
- The basic solution to the problem: $A\,r^\nu P_\nu(\cos\theta) \;\Leftarrow\; \nu>0$ for finite at the origin

- The potential vanishes at $\theta=\beta$: $P_\nu(\cos\beta)=0$ an eigenvalue condition for ν.

$$\Phi(r,\theta)=\sum_{k=1}^{\infty}A_k\,r^{\nu_k}P_{\nu_k}(\cos\theta) \;\Leftarrow\; \begin{cases}\cos\beta \text{ is the 1st zero for } P_{\nu_1}(x) & \text{for } \nu=\nu_1\\ \cos\beta \text{ is the 2nd zero for } P_{\nu_2}(x) & \text{for } \nu=\nu_2\\ \cdots \end{cases}$$

$$r\to 0 \quad\Rightarrow\quad \Phi\simeq A\,r^\nu P_\nu(\cos\theta) \quad\Leftarrow\quad \nu:\text{the smallest root}$$

$$\Rightarrow \begin{cases} E_r=-\dfrac{\partial\Phi}{\partial r}\simeq-\nu A\,r^{\nu-1}P_\nu(\cos\theta)\\[2ex] E_\theta=-\dfrac{1}{r}\dfrac{\partial\Phi}{\partial\theta}\simeq A\,r^{\nu-1}P'_\nu(\cos\theta)\sin\theta\\[2ex] \sigma(r)=-\epsilon_0 E_{\theta|\theta=\beta}\simeq-\epsilon_0 A\,r^{\nu-1}P'_\nu(\cos\beta)\sin\beta \end{cases}$$



- The fields and charge density all vary as $r^{\nu-1}$ as $r\to 0$.

- An approximate expression for ν can be from the Bessel function

$$P_\nu(\cos\theta) \simeq J_0\left((2\nu+1)\sin\frac{\theta}{2} \right) \quad \Leftarrow \quad \text{for large } \nu \text{ and } \theta < 1$$

- The 1st zero of $J_0(x)$ is at $x=2.405$ $\quad \Rightarrow \quad \nu \simeq \dfrac{2.405}{\beta} - \dfrac{1}{2}$

- $\beta \to 0 \quad \Rightarrow \quad |\mathbf{E}|,\ \sigma \propto r^{\nu-1} \to 0$ small fields & little charge deep in a conical hole

- $\beta = \dfrac{\pi}{2} \quad \Rightarrow \quad \nu = 1 \quad \Rightarrow \quad \sigma \propto 1\ (\text{const})$ (plane)

- $\beta > \dfrac{\pi}{2} \quad \Rightarrow \quad \nu < 1 \quad \Rightarrow \quad \text{singular at } r = 0$

- $\beta \to \pi \quad \Rightarrow \quad \nu \to 0 \quad \Rightarrow \quad \nu \simeq \left(2\ln\dfrac{2}{\pi-\beta} \right)^{-1} \quad \Rightarrow \quad \left[\begin{array}{l} \nu \simeq 0.2 \quad \Leftarrow \quad \pi - \beta \simeq 10^\circ \\ \nu \simeq 0.1 \quad \Leftarrow \quad \pi - \beta \simeq 1^\circ \end{array} \right.$

- The fields near a narrow conical point vary $r^{-1+\epsilon}$, $\epsilon \ll 1$, and very high fields exist around the point.

Associated Legendre Functions and the Spherical Harmonics

- The general potential problem can, however, have azimuthal variations.

$$\frac{d}{d\,x}\left(\left(1-x^2\right)\frac{d\,P}{d\,x}\right)+\left(\ell\left(\ell+1\right)-\frac{m^2}{1-x^2}\right)P=0 \quad \Leftarrow \quad x=\cos\theta$$

- Associated Legendre function: generalization of Legendre polynomial

$$P_\ell^m\left(x\right)=\left(-1\right)^m\left(1-x^2\right)^{\frac{m}{2}}\frac{d^m}{d\,x^m}P_\ell\left(x\right) \quad \Leftarrow \quad \ell\in\mathbb{N}\cup 0\,, \quad m=0\,,\cdots,\,\ell$$

$$=\frac{\left(-1\right)^m}{2^\ell\,\ell!}\sqrt{\left(1-x^2\right)^m}\,\frac{d^{m+\ell}}{d\,x^{m+\ell}}\left(x^2-1\right)^\ell \quad \Leftarrow \quad \begin{matrix}\text{Rodrigues' formula}\\ m=-\ell\,,\cdots,\,\ell\end{matrix}$$

$$\Rightarrow \quad \boxed{P_\ell^{-m}\left(x\right)=\left(-1\right)^m\frac{\left(\ell-m\right)!}{\left(\ell+m\right)!}P_\ell^m\left(x\right)}$$

$$\Rightarrow \quad \int_{-1}^{+1}P_{\ell'}^m\left(x\right)P_\ell^m\left(x\right)d\,x=\frac{2}{2\,\ell+1}\frac{\left(\ell+m\right)!}{\left(\ell-m\right)!}\delta_{\ell'\,\ell}$$

- *Spherical harmonics* (tesseral harmonics)

$$\boxed{Y_\ell^m\left(\theta\,,\phi\right)=\sqrt{\frac{2\,\ell+1}{4\,\pi}\frac{\left(\ell-m\right)!}{\left(\ell+m\right)!}}\,P_\ell^m\left(\cos\theta\right)e^{i\,m\,\phi} \quad \Rightarrow \quad Y_\ell^{-m}\left(\theta\,,\phi\right)=\left(-1\right)^m Y_\ell^{\star\,m}\left(\theta\,,\phi\right)}$$

$$\Rightarrow \int_0^{2\pi}\int_0^{\pi} Y_{\ell'}^{*m'}(\theta,\phi)\,Y_\ell^m(\theta,\phi)\sin\theta\,d\theta\,d\phi = \delta_{\ell'\ell}\,\delta_{m'm}$$

$\Leftarrow$ normalization orthogonality

$$\Rightarrow \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_\ell^{*m}(\theta',\phi')\,Y_\ell^m(\theta,\phi) = \delta(\phi-\phi')\,\delta(\cos\theta-\cos\theta')$$

$\Leftarrow$ completeness

- $\ell=0$ $\quad Y_0^0 = \dfrac{1}{\sqrt{4\pi}}$

$\ell=1$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\,e^{i\phi}$$

$$Y_1^0 = +\sqrt{\frac{3}{4\pi}}\cos\theta$$

$\ell=2$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\,e^{2i\phi}$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\,e^{i\phi}$$

$$Y_2^0 = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\left(3\cos^2\theta-1\right)$$

$\ell=3$

$$Y_3^3 = -\frac{1}{4}\sqrt{\frac{35}{4\pi}}\sin^3\theta\,e^{3i\phi}$$

$$Y_3^2 = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\theta\cos\theta\,e^{2i\phi}$$

$$Y_3^1 = -\frac{1}{4}\sqrt{\frac{21}{4\pi}}\sin\theta\left(5\cos^2\theta-1\right)e^{i\phi}$$

$$Y_3^0 = \frac{1}{2}\sqrt{\frac{7}{4\pi}}\left(5\cos^3\theta-3\cos\theta\right)$$

- For $m=0$, $\quad Y_\ell^0(\theta,\phi) = \sqrt{\dfrac{2\ell+1}{4\pi}}\,P_\ell(\cos\theta)$

- For an arbitrary function

$$g(\theta,\phi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_\ell^m Y_\ell^m(\theta,\phi) \;\Leftarrow\; A_\ell^m=\int g(\theta,\phi)\, Y_\ell^{\star m}(\theta,\phi)\, \mathrm{d}\Omega$$

- The expansion for $\theta=0$

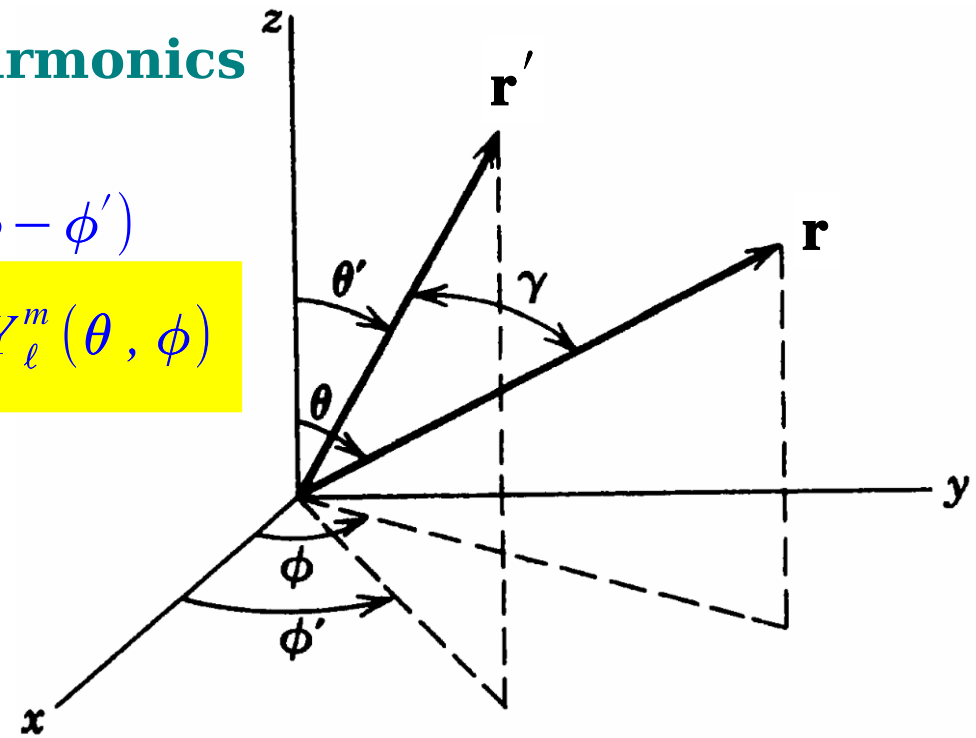
For $m\neq 0 \;\Rightarrow\; Y_\ell^m(\theta,\phi)\propto f(\theta,\phi)\sin\theta \;\Rightarrow\; Y_\ell^m(0,\phi)=0$

$$g(\theta=0,\phi)=\sum_{\ell=0}^{\infty}\sqrt{\frac{2\ell+1}{4\pi}}\,A_\ell^0 \;\Leftarrow\; A_\ell^0=\sqrt{\frac{2\ell+1}{4\pi}}\int g(\theta,\phi)\,P_\ell(\cos\theta)\,\mathrm{d}\Omega \qquad (\star)$$

All terms in the series with $m\neq 0$ vanish at $\theta=0$

- The general solution $\boxed{\Phi(r,\theta,\phi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(A_\ell^m r^\ell+\frac{B_\ell^m}{r^{\ell+1}}\right)Y_\ell^m(\theta,\phi)}$

Addition Theorem for Spherical Harmonics



- For $\mathbf{r}=(r\,,\theta\,,\phi)\,,\quad \mathbf{r}'=(r'\,,\theta'\,,\phi')$

$$\Rightarrow\quad \cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')$$

$$\Rightarrow\quad \boxed{P_\ell(\cos\gamma)=\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell}Y_\ell^{*m}(\theta'\,,\phi')\,Y_\ell^m(\theta\,,\phi)}$$

Proof: if let $\mathbf{r}'$ be on the z-axis

$$\Rightarrow\quad \nabla'^2 P_\ell(\cos\gamma)+\frac{\ell(\ell+1)}{r^2}P_\ell(\cos\gamma)=0$$

If rotate the axis to a new place

$$\nabla^2\text{:scalar operator}\quad\Rightarrow\quad\text{invariant under rotation}\quad\Rightarrow\quad\nabla'^2=\nabla^2$$

$$\Rightarrow\quad \nabla^2 P_\ell(\cos\gamma)+\frac{\ell(\ell+1)}{r^2}P_\ell(\cos\gamma)=0\quad\Rightarrow\quad P_\ell\ \text{is a spherical harmonic of}\ \ell$$

$$\Rightarrow\quad P_\ell(\cos\gamma)=\sqrt{\frac{4\pi}{2\ell+1}}\,Y_\ell^0(\gamma\,,\psi)=\sum_{m=-\ell}^{\ell}A_m(\theta'\,,\phi')\,Y_\ell^m(\theta\,,\phi)$$

$$\Rightarrow\quad A_m=\int Y_\ell^{*m}(\theta\,,\phi)\,P_\ell(\cos\gamma)\,\mathrm{d}\,\Omega=\sqrt{\frac{4\pi}{2\ell+1}}\int Y_\ell^{*m}(\theta\,,\phi)\,Y_\ell^0(\gamma\,,\psi)\,\mathrm{d}\,\Omega$$

Let $g(\gamma\,,\psi)=\sqrt{\dfrac{4\pi}{2\ell+1}}\,Y_\ell^m(\theta(\gamma\,,\psi)\,,\phi(\gamma\,,\psi))=\sum_k B_k\,Y_\ell^k(\gamma\,,\psi)$

$$\Rightarrow \quad B_0 = \sqrt{\frac{4\pi}{2\ell+1}} \int Y_\ell^m(\theta,\phi)\, Y_\ell^{\star\,0}(\gamma,\psi)\, \mathrm{d}\,\Omega = A_m^\star \quad \Rightarrow \quad A_m = B_0^\star$$

$$\text{Choose } \theta=\theta' \quad \Rightarrow \quad g(\gamma=0,\psi) = \left[\begin{array}{ll} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m(\theta,\phi)_{\gamma=0} & = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m(\theta',\phi') \\ \sum_k B_k\, Y_\ell^k(0,\psi) & = \sqrt{\frac{2\ell+1}{4\pi}}\, B_0 \end{array} \right.$$

$$\Rightarrow \quad B_0 = \frac{4\pi}{2\ell+1} Y_\ell^m(\theta',\phi') \quad \Rightarrow \quad A_m(\theta',\phi') = \frac{4\pi}{2\ell+1} Y_\ell^{\star\,m}(\theta',\phi') \quad \text{q.e.d.}$$

- Another theorem $P_\ell(\cos\gamma) = P_\ell(\cos\theta)\, P_\ell(\cos\theta')$

$$+2\sum_{m=1}^{\ell} \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos\theta)\, P_\ell^m(\cos\theta') \cos m(\phi-\phi')$$

- $\gamma = 0 \quad \Rightarrow \quad \sum_{m=-\ell}^{\ell} |Y_\ell^m(\theta,\phi)|^2 = \dfrac{2\ell+1}{4\pi}$

- A useful application of the addition theorem

$$\boxed{\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_\ell^{\star\,m}(\theta',\phi')\, Y_\ell^m(\theta,\phi)}$$

Two point charges q and $-q$ are located on the z axis at $z = +a$ and $z = -a$, respectively.

(a) Find the electrostatic potential as an expansion in spherical harmonics and powers of r for both $r > a$ and $r < a$. [Problem 3.6a]

The electric potential for the two point charges at $a\hat{\mathbf{z}}$ and $-a\hat{\mathbf{z}}$ is

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{r} - a\hat{\mathbf{z}}|} - \frac{1}{|\mathbf{r} + a\hat{\mathbf{z}}|}\right).$$

Since $\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*(\theta', \phi')Y_{\ell m}(\theta, \phi)$, and also $a\hat{\mathbf{z}} = (a, \theta' = 0, \phi')$, $-a\hat{\mathbf{z}} = (a, \theta' = \pi, \phi')$ in the spherical coordinate where ϕ' is arbitrary, and there thus exists azimuthal symmetry, then the the potential expanded in spherical harmonics is

$$\begin{aligned}
\Phi(r,\theta,\phi) &= \frac{q}{\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}\left[Y_{\ell m}^*(0,\phi') - Y_{\ell m}^*(\pi,\phi')\right]Y_{\ell m}(\theta,\phi) \\
&= \frac{q}{\epsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}[Y_{\ell 0}^*(0,\phi') - Y_{\ell 0}^*(\pi,\phi')]Y_{\ell 0}(\theta,\phi) \\
&= \frac{q}{\sqrt{4\pi}\epsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{\sqrt{2\ell+1}}\frac{r_<^\ell}{r_>^{\ell+1}}[P_\ell(1) - P_\ell(-1)]Y_{\ell 0}(\theta,\phi)
\end{aligned}$$

$$= \frac{q}{\sqrt{4\pi}\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{\sqrt{2\ell+1}} \frac{r_<^\ell}{r_>^{\ell+1}} [1 - (-1)^\ell] Y_{\ell 0}(\theta, \phi)$$

$$= \frac{q}{\sqrt{\pi}\epsilon_0} \sum_{k=0}^{\infty} \frac{1}{\sqrt{4k+3}} \frac{r_<^{2k+1}}{r_>^{2k+2}} Y_{2k+1,0}(\theta, \phi),$$

where where $r_<$ ($r_>$) is the smaller (larger) of r and a, and we use $Y_{\ell 0}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)$ and $P_\ell(\pm 1) = (\pm 1)^\ell$.

Laplace Equation in Cylindrical Coordinates; Bessel Functions

The Laplace equations: $\nabla^2\Phi = \boxed{\dfrac{\partial^2\Phi}{\partial\rho^2} + \dfrac{1}{\rho}\dfrac{\partial\Phi}{\partial\rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2\Phi}{\partial\phi^2} + \dfrac{\partial^2\Phi}{\partial z^2} = 0} \;\Rightarrow\; \Phi = R(\rho)\,Q(\phi)\,Z(z)$

$$\Rightarrow\; \frac{d^2 Z}{d z^2} - k^2 Z = 0\,,\quad \frac{d^2 Q}{d\phi^2} + \nu^2 Q = 0\,,\quad \frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R = 0$$

$\Rightarrow\; Z(z) = e^{\pm k z}\,,\quad Q(\phi) = e^{\pm i\nu\phi} \;\Leftarrow\; \nu\in\mathbb{Z}\,,\quad k\in\mathbb{R}(+)\,,$ and let $x \equiv k\rho$

$\Rightarrow\; \boxed{\dfrac{d^2 R}{d x^2} + \dfrac{1}{x}\dfrac{dR}{dx} + \left(1 - \dfrac{\nu^2}{x^2}\right)R = 0} \;\Leftarrow$ Bessel equation $\;\Rightarrow\; R(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j$

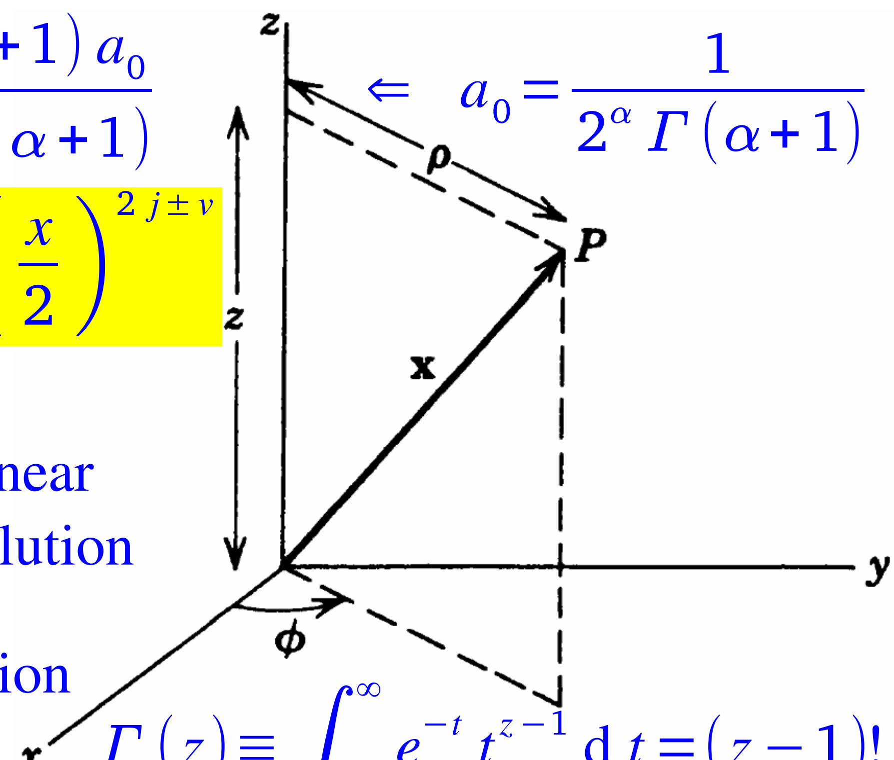
$$\Rightarrow\; \begin{matrix}\alpha = \pm\nu\,, \\ a_{\text{odd}} = 0\,,\end{matrix}\quad a_{2j} = -\frac{a_{2j-2}}{4j(j+\alpha)} = \frac{(-1)^j\,\Gamma(\alpha+1)\,a_0}{2^{2j}\,j!\,\Gamma(j+\alpha+1)} \;\Leftarrow\; a_0 = \frac{1}{2^\alpha\,\Gamma(\alpha+1)}$$

$\Rightarrow\;$ 2 solutions: $\boxed{J_{\pm\nu}(x) = \sum_{j=0}^{\infty}\dfrac{(-1)^j}{j!\,\Gamma(j\pm\nu+1)}\left(\dfrac{x}{2}\right)^{2j\pm\nu}}$

$\nu\notin\mathbb{Z} \;\Rightarrow\; J_{\pm\nu}$ are linearly independent

$\nu\in\mathbb{Z} \;\Rightarrow\; J_{-\nu} = (-1)^\nu J_\nu \;\Leftarrow$ need another linear independent solution

$\Rightarrow\; \boxed{N_\nu = \dfrac{J_\nu\cos\nu\pi - J_{-\nu}}{\sin\nu\pi}} \;\Leftarrow$ Neumann function



$$\Gamma(z) \equiv \int_0^\infty e^{-t}\,t^{z-1}\,dt = (z-1)!$$

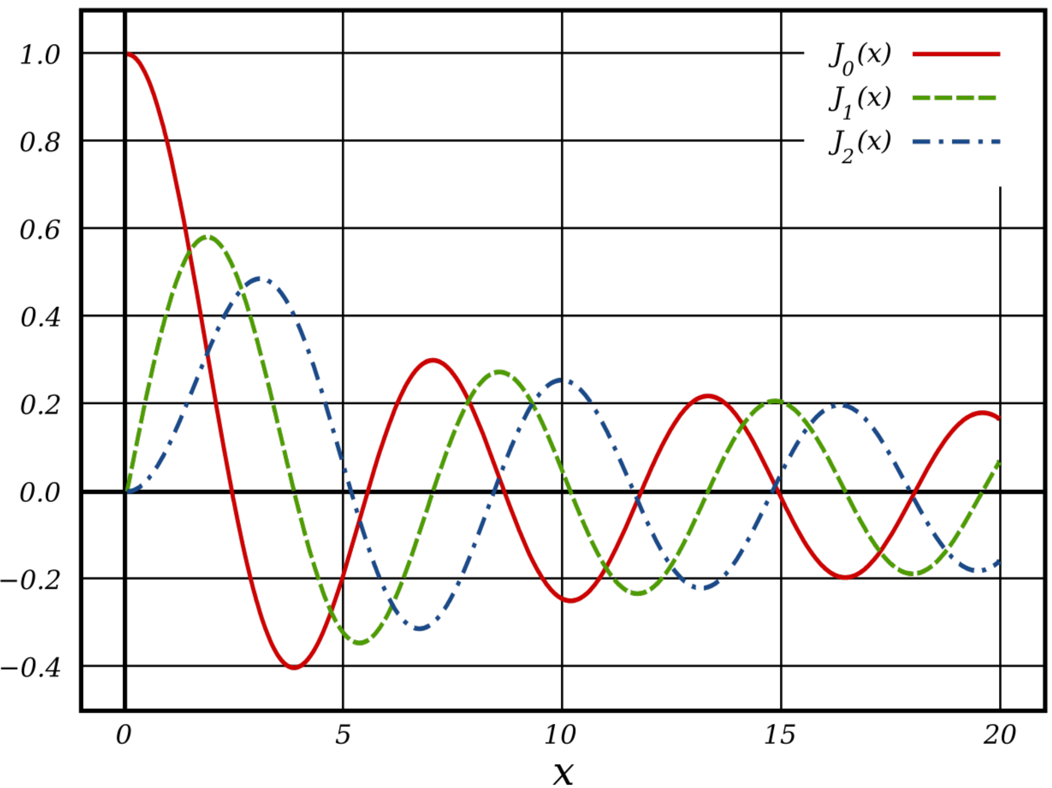
- Replace $J_{\pm\nu}$ with J_ν and N_ν no matter if ν is an integer or not.
- Hankel functions: $H_\nu^{(1)} \equiv J_\nu + i\,N_\nu\,,\quad H_\nu^{(2)} \equiv J_\nu - i\,N_\nu$ usually used in radiation.
- $\Omega_\nu \in \left\{ J_\nu\,,\, N_\nu\,,\, H_\nu^{(1)}\,,\, H_\nu^{(2)} \right\} \quad \Rightarrow \quad \Omega_{\nu-1} + \Omega_{\nu+1} = \dfrac{2\,\nu}{x}\,\Omega_\nu\,,\quad \Omega_{\nu-1} - \Omega_{\nu+1} = 2\,\dfrac{d\,\Omega_\nu}{d\,x} \qquad (\#)$
- $x \ll 1 \quad \Rightarrow \quad J_\nu \to \dfrac{1}{\Gamma(\nu+1)}\left(\dfrac{x}{2}\right)^\nu,\quad N_\nu \to \begin{cases} \dfrac{2}{\pi}\left(\ln\dfrac{x}{2} + 0.5772\cdots\right), & \nu = 0 \\ -\dfrac{\Gamma(\nu)}{\pi}\left(\dfrac{2}{x}\right)^\nu, & \nu > 0 \end{cases}$

  $x \gg 1\,,\ \nu \quad \Rightarrow \quad J_\nu \to \sqrt{\dfrac{2}{\pi\,x}}\cos\left(x - \dfrac{2\,\nu+1}{4}\pi\right),\quad N_\nu \to \sqrt{\dfrac{2}{\pi\,x}}\sin\left(x - \dfrac{2\,\nu+1}{4}\pi\right)$
- $J_\nu(x_{\nu n}) = 0 \ \Leftarrow\ x_{\nu n}$ is the n^{th} root $\Rightarrow$

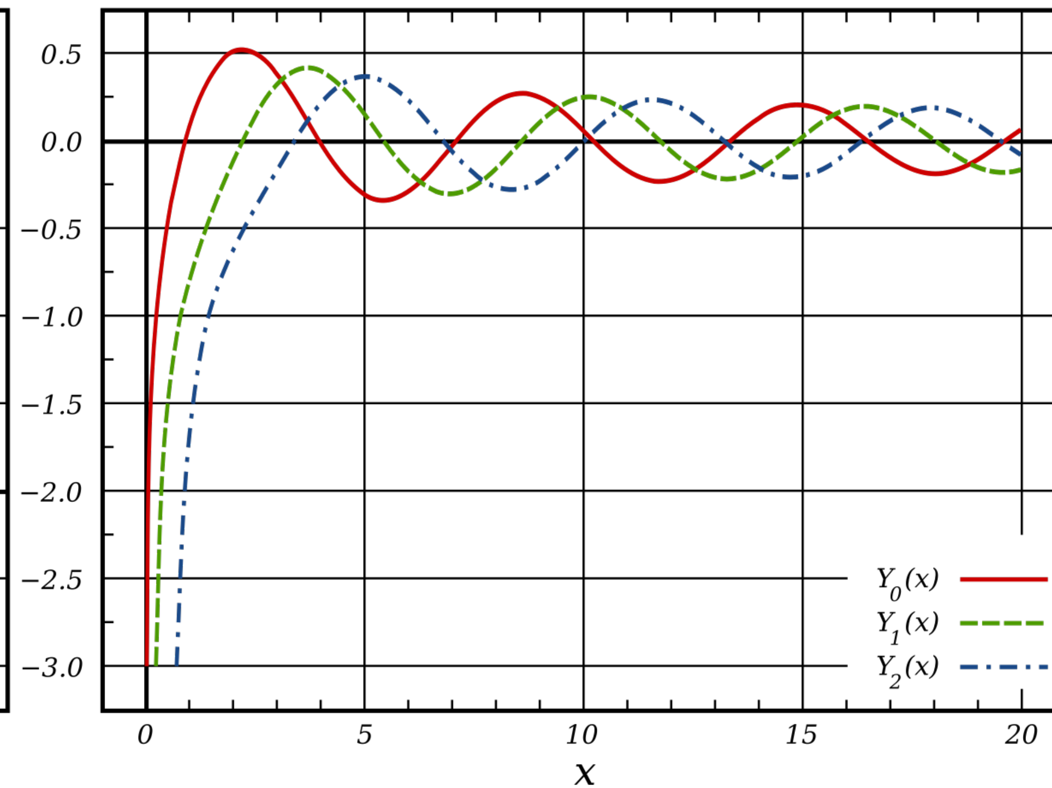
$$\begin{array}{llll} \nu = 0\,, & x_{0n} = 2.405\,, & 5.520\,, & 8.654\,,\ \cdots \\ \nu = 1\,, & x_{1n} = 3.832\,, & 7.016\,, & 10.173\,,\ \cdots \\ \nu = 2\,, & x_{2n} = 5.136\,, & 8.417\,, & 11.620\,,\ \cdots \end{array}$$

$x_{\nu n} \simeq \pi\left(n + \dfrac{\nu}{2} - \dfrac{1}{4}\right) \quad \Leftarrow$ asymptotic formula

$J_\nu(x)$



$N_\nu(x)$



$$J_m(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\varphi - m\varphi)\,\mathrm{d}\varphi = \frac{1}{2\pi i^m}\int_0^{2\pi} e^{i(x\cos\varphi - m\varphi)}\,\mathrm{d}\varphi$$

⇧ Problem 3.16d

• $\sqrt{\rho}\, J_\nu\left(x_{\nu n} \frac{\rho}{a} \right)$ for fixed $\nu \geq 0\,,\ \ n = 1, 2\,, \cdots$ form an orthogonal set on $0 \leq \rho \leq a$

Proof: $\frac{1}{\rho} \frac{\mathrm{d}}{\mathrm{d}\,\rho}\left[\rho \frac{\mathrm{d}}{\mathrm{d}\,\rho} J_\nu\left(x_{\nu n} \frac{\rho}{a} \right)\right] + \left(\frac{x_{\nu n}^2}{a^2} - \frac{\nu^2}{\rho^2} \right) J_\nu\left(x_{\nu n} \frac{\rho}{a} \right) = 0$

$$\Rightarrow \int_0^a J_\nu\left(x_{\nu n'} \frac{\rho}{a} \right) \left[\frac{\mathrm{d}}{\mathrm{d}\,\rho}\left[\rho \frac{\mathrm{d}}{\mathrm{d}\,\rho} J_\nu\left(x_{\nu n} \frac{\rho}{a} \right)\right] + \left(\frac{x_{\nu n}^2}{a^2} - \frac{\nu^2}{\rho^2} \right) \rho\, J_\nu\left(x_{\nu n} \frac{\rho}{a} \right) \right] \mathrm{d}\,\rho = 0 \quad \Leftarrow \quad \times \rho\, J_\nu\left(x_{\nu n'} \frac{\rho}{a} \right)$$

$$\Rightarrow \quad \rho\, J_\nu\left(x_{\nu n'} \frac{\rho}{a} \right) \frac{\mathrm{d}}{\mathrm{d}\,\rho} J_\nu\left(x_{\nu n} \frac{\rho}{a} \right) \Bigg|_0^a \qquad [= x_{\nu n}\, J_\nu\,(x_{\nu n'})\, J'_\nu\,(x_{\nu n})]$$

$$- \int_0^a \rho \frac{\mathrm{d}}{\mathrm{d}\,\rho} J_\nu\left(x_{\nu n'} \frac{\rho}{a} \right) \frac{\mathrm{d}}{\mathrm{d}\,\rho} J_\nu\left(x_{\nu n} \frac{\rho}{a} \right) \mathrm{d}\,\rho$$

$$+ \int_0^a \left(\frac{x_{\nu n}^2}{a^2} - \frac{\nu^2}{\rho^2} \right) \rho\, J_\nu\left(x_{\nu n'} \frac{\rho}{a} \right) J_\nu\left(x_{\nu n} \frac{\rho}{a} \right) \mathrm{d}\,\rho = 0 \quad (\star')$$

write down the same equation, with n and n' interchanged, and sbutract

$$\Rightarrow \quad (x_{\nu n}^2 - x_{\nu n'}^2) \int_0^a \rho\, J_\nu\left(x_{\nu n'} \frac{\rho}{a} \right) J_\nu\left(x_{\nu n} \frac{\rho}{a} \right) \mathrm{d}\,\rho = 0 \quad \Leftarrow \quad \text{orthogonality condition}$$

- The normalization integral:

$$\int_0^a \rho\, J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right) J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) \mathrm{d}\rho = \frac{a^2}{2} J^2_{\nu\pm1}(x_{\nu n})\,\delta_{n n'}$$

Proof:

$$(\#) \quad\Rightarrow\quad J_{\nu\pm1} = \frac{\nu}{x} J_\nu \mp J'_\nu \quad\Rightarrow\quad J_{\nu\pm1}(x_{\nu n}) = \mp J'_\nu(x_{\nu n}) \quad\Rightarrow\quad J'^{\,2}_\nu(x_{\nu n}) = J^2_{\nu\pm1}(x_{\nu n})$$

$$(\star') - (n \leftrightarrow n') \quad\Rightarrow\quad x_{\nu n} J_\nu(x_{\nu n'}) J'_\nu(x_{\nu n}) - x_{\nu n'} J_\nu(x_{\nu n}) J'_\nu(x_{\nu n'})$$

$$= \frac{x^2_{\nu n'} - x^2_{\nu n}}{a^2} \int_0^a \rho\, J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right) J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) \mathrm{d}\rho$$

$$x_{\nu n'} = x_{\nu n} + \epsilon \quad\Rightarrow\quad J_\nu(x_{\nu n'}) \approx J_\nu(x_{\nu n}) + \epsilon\, J'_\nu(x_{\nu n})$$

$$\Rightarrow\quad x_{\nu n} J_\nu(x_{\nu n'}) J'_\nu(x_{\nu n}) - x_{\nu n'} J_\nu(x_{\nu n}) J'_\nu(x_{\nu n'}) \approx \epsilon\, x_{\nu n} J'^{\,2}_\nu(x_{\nu n}) \quad\Leftarrow\quad J_\nu(x_{\nu n}) = 0$$

$$\Rightarrow\quad \frac{x^2_{\nu n'} - x^2_{\nu n}}{a^2} \int_0^a \rho\, J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right) J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) \mathrm{d}\rho \approx \frac{2\, x_{\nu n}\,\epsilon}{a^2} \int_0^a \rho\, J^2_\nu\left(x_{\nu n}\frac{\rho}{a}\right) \mathrm{d}\rho$$

$$\Rightarrow \int_0^a \rho\, J^2_\nu\left(x_{\nu n}\frac{\rho}{a}\right) \mathrm{d}\rho = \frac{a^2}{2} J'^{\,2}_\nu(x_{\nu n}) = \frac{a^2}{2} J^2_{\nu\pm1}(x_{\nu n}) \qquad \text{QED}$$

- Assuming the set of Bessel functions is complete, the Fourier-Bessel series of f

$$f(\rho) = \sum_{n=1}^{\infty} A_{\nu n} J_\nu\left(x_{\nu n} \frac{\rho}{a} \right), \quad 0 \le \rho < a, \quad \nu \ge -1$$

$$\text{where } A_{\nu n} = \frac{2}{a^2 J_{\nu+1}^2(x_{\nu n})} \int_0^a \rho f(\rho) J_\nu\left(x_{\nu n} \frac{\rho}{a} \right) \mathrm{d}\rho$$

- The Fourier-Bessel series is appropriate to functions vanishing at $\rho = a$, ie homogeneous Dirichlet boundary conditions on a cylinder.

- An alternative expansion is in a series of functions $\sqrt{\rho}\, J_\nu\left(y_{\nu n} \frac{\rho}{a} \right)$ where $y_{\nu n}$ is the n^{th} root of $J'_\nu(y) = 0$, and is useful for functions with vanishing slope at $\rho = a$.

- The reason is that, in proving the orthogonality of the functions, it demands

$$\rho J_\nu(k\rho) \frac{\mathrm{d}}{\mathrm{d}\rho} J_\nu(k'\rho) - \rho J_\nu(k'\rho) \frac{\mathrm{d}}{\mathrm{d}\rho} J_\nu(k\rho) = 0 \quad \text{at} \quad \begin{matrix} \rho = 0 \\ \rho = a \end{matrix}$$

$$\Rightarrow \quad k a = \begin{matrix} x_{\nu n} \\ y_{\nu n} \end{matrix} \quad \Rightarrow \quad \begin{matrix} J_\nu(x_{\nu n}) = 0 \\ J'_\nu(y_{\nu n}) = 0 \end{matrix} \quad \Rightarrow \quad \rho \frac{\mathrm{d}}{\mathrm{d}\rho} J_\nu(k\rho) + \lambda J_\nu(k\rho) = 0 \quad \begin{matrix} \text{at the endpoints} \\ \text{in general} \end{matrix}$$

- Some of the other possibilities

Neumann series: $\sum_{n=0}^{\infty} a_n J_{\nu+n}(z)$, Schlomilch series: $\sum_{n=0}^{\infty} a_n J_{\nu}(n x)$

Kapteyn series: $\sum_{n=0}^{\infty} a_n J_{\nu+n}((\nu+n) z) \Leftarrow$ Kepler motion of planets and of radiation by moving charges

- If $\frac{d^2 Z}{d z^2} + k^2 Z = 0 \Rightarrow Z(z) = \sin k z, \ \cos k z$ $\Downarrow$ $x = k\rho$

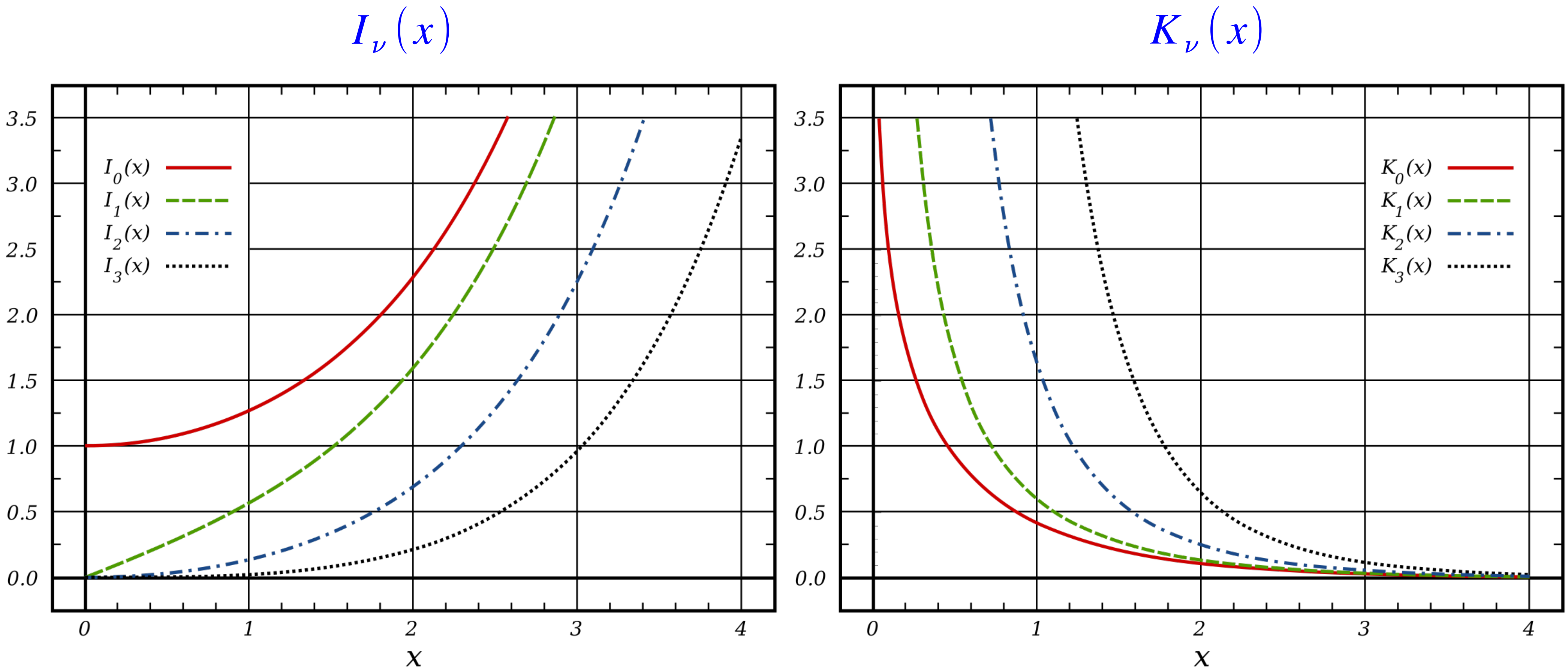
$$\Rightarrow \frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \left(k^2 + \frac{\nu^2}{\rho^2}\right) R = 0 \Rightarrow \frac{d^2 R}{d x^2} + \frac{1}{x}\frac{dR}{dx} - \left(1 + \frac{\nu^2}{x^2}\right) R = 0$$

$$\Rightarrow \boxed{I_\nu(x) = i^{-\nu} J_\nu(i x), \quad K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(i x)} \Leftarrow \text{modified Bessel functions}$$

For $\nu \geq 0$

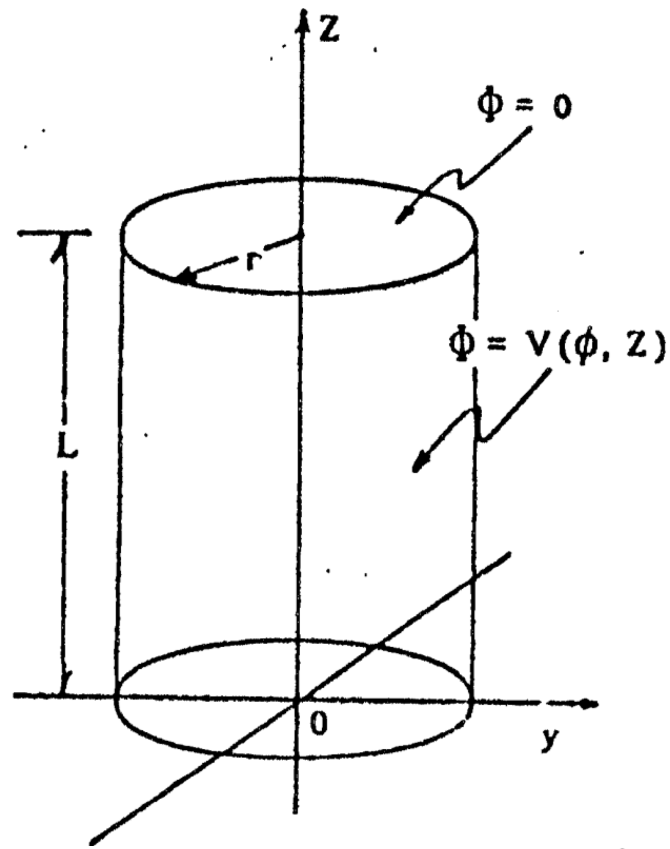
$$x \ll 1 \Rightarrow I_\nu(x) \to \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu, \quad K_\nu(x) \to \begin{cases} -\left(\ln\frac{x}{2} + 0.5772\cdots\right), & \nu = 0 \\ \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu, & \nu \neq 0 \end{cases}$$

$$x \gg 1, \nu \Rightarrow I_\nu(x) \to \frac{e^x}{\sqrt{2\pi x}}\left[1 + O\left(\frac{1}{x}\right)\right], \quad K_\nu(x) \to \sqrt{\frac{\pi}{2x}}\, e^{-x}\left[1 + O\left(\frac{1}{x}\right)\right]$$



$I_\nu(x)$
$I_0(x)$
$I_1(x)$
$I_2(x)$
$I_3(x)$
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
0
1
2
3
4
x
$K_\nu(x)$
$K_0(x)$
$K_1(x)$
$K_2(x)$
$K_3(x)$
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
0
1
2
3
4
x

A hollow right circular cylinder of radius b has its axis coincident with the z axis and its ends at $z = 0$ and $z = L$. The potential on the end faces is zero, while the potential on the cylindrical surface is given as $V(\phi, z)$. Using the appropriate separation of variables in cylindrical coordinates, find a series solution for the potential anywhere inside the cylinder. [Problem 3.9]



This problem can be described with a Laplace equation $\nabla^2\Phi(\mathbf{x}) = 0$ since there is no charge in the domain of interest. Because of the cylindrical geometry of the problem, the Laplace equation is written in cylindrical coordinates:

$$\frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0.$$

With the separation of variables $\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$, the Laplace equation turns to the three ordinary differential equations:

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} + k^2 Z = 0, \quad \frac{\mathrm{d}^2 Q}{\mathrm{d}\phi^2} + m^2 Q = 0, \quad \frac{\mathrm{d}^2 R}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}R}{\mathrm{d}\rho} - \left(k^2 + \frac{v^2}{\rho^2}\right) R = 0.$$

Their elementary solutions are $Z = c \sin kz + d \cos kz$, $Q = A \cos m\phi + B \sin m\phi$, and $R = \alpha I_m(k\rho) + \beta K_m(k\rho)$ where I_m and K_m are modified Bessel functions. Now we bring the boundary conditions into the consideration. Since $Z(0) = Z(L) = 0$, it leads to $d = 0$ and $k = n\pi/L$ where $n \in \mathbb{N}$. For the potential to be single-valued when the full azimuth is allowed, therefore $m \in \mathbb{Z}$. We expect $R(\rho = 0)$ is finite in the hollow cylinder, then $\beta = 0$. Combining the result, the general solution of Φ is

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} I_m(\frac{n\pi}{L}\rho)(A_{mn} \cos m\phi + B_m n \sin m\phi) \sin\frac{n\pi z}{L},$$

where the coefficients A_{mn}'s and B_{mn}'s can be obtained by applying the boundary condition on the cylinder surface. The boundary condition is

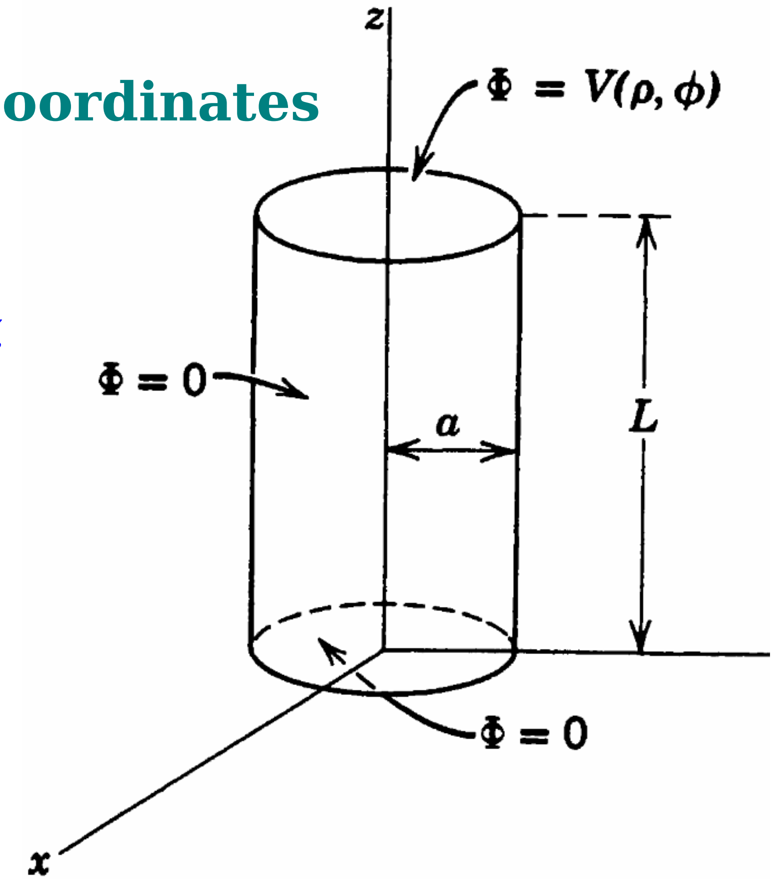
$$\Phi(b, \phi, z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} I_m(\frac{n\pi}{L}b)(A_{mn} \cos m\phi + B_m n \sin m\phi) \sin\frac{n\pi z}{L} = V(\phi, z),$$

therefore,

$$\begin{bmatrix} A_{mn} \\ B_{mn} \end{bmatrix} = \frac{2}{\pi L I_m(n\pi b/L)} \int_0^{2\pi}\int_0^L V(\phi', z') \begin{bmatrix} \cos m\phi' \\ \sin m\phi' \end{bmatrix} \sin\frac{n\pi z'}{L}\mathrm{d}z'\mathrm{d}\phi', \quad m \neq 0,$$

$$A_{0n} = \frac{1}{\pi L I_0(n\pi b/L)} \int_0^{2\pi}\int_0^L V(\phi', z') \sin\frac{n\pi z'}{L}\mathrm{d}z'\mathrm{d}\phi'.$$

Boundary-Value Problems in Cylindrical Coordinates



- The potential on the side and the bottom is 0, the top is $\Phi = V(\rho, \phi)$

$$\Rightarrow \quad Q(\phi) = A \sin m\phi + B \cos m\phi, \quad Z(z) = \sinh kz$$

$$\Rightarrow \quad R(\rho) = C\, J_m(k\rho) + D\, N_m(k\rho) \quad \Leftarrow \quad m \in \mathbb{Z}$$

$$R(0) = \text{finite} \quad \Rightarrow \quad D = 0 \quad \Leftarrow \quad N_m(\rho \to 0) \to \infty$$

$$R(ka) = 0 \quad \Rightarrow \quad k_{mn} = \frac{x_{mn}}{a}, \quad n \in \mathbb{N} \quad \Leftarrow J_m(x_{mn}) = 0$$

$$\Rightarrow \quad \Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho) \sinh k_{mn} z \times \left(A_{mn} \sin m\phi + B_{mn} \cos m\phi \right)$$

$$\Rightarrow \quad V = \sum_{m,n} J_m(k_{mn}\rho) \sinh k_{mn} L \left(A_{mn} \sin m\phi + B_{mn} \cos m\phi \right)$$

$$A_{mn} = \frac{2 \operatorname{csch} k_{mn} L}{\pi a^2 J_{m+1}^2(k_{mn} a)} \int_0^{2\pi} \sin m\phi \, d\phi \int_0^a \rho V J_m(k_{mn}\rho) \, d\rho$$

$$\Rightarrow \quad B_{mn} = \frac{2 \operatorname{csch} k_{mn} L}{\pi a^2 J_{m+1}^2(k_{mn} a)} \int_0^{2\pi} \cos m\phi \, d\phi \int_0^a \rho V J_m(k_{mn}\rho) \, d\rho$$

$$\text{using } \frac{B_{0n}}{2} \text{ for } m = 0$$

- Push $a \to \infty$, $0 \le z \to \infty \;\Rightarrow\; \Phi(z \to \infty) = 0$

$$\Rightarrow \quad \Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \int_0^{\infty} e^{-kz} J_m(k\rho) [A_m(k) \sin m\phi + B_m(k) \cos m\phi] \, \mathrm{d}k \qquad (\$)$$

If $\Phi(z=0) = V$

$$\Rightarrow \quad V(\rho, \phi) = \sum_{m=0}^{\infty} \int_0^{\infty} J_m(k\rho) [A_m(k) \sin m\phi + B_m(k) \cos m\phi] \, \mathrm{d}k$$

$$\Rightarrow \int_0^{\infty} J_m(k'\rho) \begin{bmatrix} A_m(k') \\ B_m(k') \end{bmatrix} \mathrm{d}k' = \frac{1}{\pi} \int_0^{2\pi} V(\rho, \phi) \begin{bmatrix} \sin m\phi \\ \cos m\phi \end{bmatrix} \mathrm{d}\phi$$

- These radial integral equations of the 1st kind can be solved since they are

Hankel transforms: $F_\nu(k) = \int_0^{\infty} f(\rho) J_\nu(k\rho) \rho \, \mathrm{d}\rho$

- Using the integral relation $\boxed{\int_0^{\infty} x J_m(kx) J_m(k'x) \, \mathrm{d}x = \frac{\delta(k-k')}{k}}$ [Prob. 3.16a]

$$\Rightarrow \quad \begin{bmatrix} A_m(k) \\ B_m(k) \end{bmatrix} = \frac{k}{\pi} \int_0^{\infty} \rho \, \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi \, V(\rho, \phi) J_m(k\rho) \begin{bmatrix} \sin m\phi \\ \cos m\phi \end{bmatrix} \quad \& \quad \text{using } \frac{1}{2} B_0(k) \text{ for } m = 0$$

$$\int_0^a \rho\, J_v(k_{v n'} \rho)\, J_v(k_{v n} \rho)\, \mathrm{d}\rho = \frac{a^2}{2} J_{v\pm 1}^2(x_{v n})\, \delta_{n n'} \quad \Leftarrow \quad k_{v n} \equiv \frac{x_{v n}}{a}, \quad k_{v n'} \equiv \frac{x_{v n'}}{a}$$

$$\Rightarrow \sum_{n'} k_{v n'} (k_{v,\, n'+1} - k_{v n'}) \int_0^a \rho\, J_v(k_{v n'} \rho)\, J_v(k_{v n} \rho)\, \mathrm{d}\rho = \mathrm{LHS}$$

$$= \frac{a^2}{2} k_{v n} (k_{v,\, n+1} - k_{v n})\, J_{v+1}^2(x_{v n}) = x_{v n} \frac{x_{v,\, n+1} - x_{v n}}{2} J_{v+1}^2(x_{v n}) = \mathrm{RHS}$$

$$a \to \infty \quad \Rightarrow \quad \begin{array}{l} \text{discrete } \ k_{v n} \\ \to \text{continuous } \ \mathrm{d}k \end{array} \quad \Rightarrow \quad \mathrm{LHS} \to \int_0^\infty k\, \mathrm{d}k \int_0^\infty \rho\, J_v(k' \rho)\, J_v(k \rho)\, \mathrm{d}\rho$$

$$x_{v n} \to n\pi + \frac{2v-1}{4}\pi \gg 1 \quad \text{for} \quad a \to \infty$$

$$J_{v+1}^2(x_{v n}) \to \frac{2}{\pi x_{v n}} \cos^2\left(x_{v n} - \frac{2v+3}{4}\pi \right) \to \frac{2}{\pi x_{v n}} \cos^2(n-1)\pi = \frac{2}{\pi x_{v n}}$$

$$\Rightarrow \quad \mathrm{RHS} = x_{v n} \frac{x_{v,\, n+1} - x_{v n}}{2} J_{v+1}^2(x_{v n}) \to 1 = \int_0^\infty \delta(k - k')\, \mathrm{d}k$$

$$\Rightarrow \quad \int_0^\infty \rho\, J_m(k\rho)\, J_m(k'\rho)\, \mathrm{d}\rho = \frac{\delta(k-k')}{k}$$

- $J_\nu(kx)$'s for fixed ν, $\mathrm{Re}(\nu)>-1$, form a complete, orthogonal (in k) set on the interval $0<x<\infty$. For m fixed

$$A(x)=\int_0^\infty \widetilde{A}(k)\,J_\nu(k\,x)\,\mathrm{d}\,k \quad \Leftarrow \quad \widetilde{A}(k)=k\int_0^\infty x\,A(x)\,J_\nu(k\,x)\,\mathrm{d}\,x \qquad \text{Hankel transform}$$

- Spherical Bessel function $j_\ell(z)\equiv\sqrt{\dfrac{\pi}{2\,z}}\,J_{\ell+1/2}(z),\quad \ell\in\mathbb{N}\cup 0$ [See Chapter 9]

$$\Rightarrow \quad \int_0^\infty r^2\,j_\ell(k\,r)\,j_\ell(k'\,r)\,\mathrm{d}\,r=\frac{\pi}{2\,k^2}\,\delta(k-k') \quad \Leftarrow \quad \text{orthogonality relation}$$

$$\int_0^\infty k^2\,j_\ell(k\,r)\,j_\ell(k\,r')\,\mathrm{d}\,k=\frac{\pi}{2\,r^2}\,\delta(r-r') \quad \Leftarrow \quad \text{completeness relation}$$

- The Fourier-spherical Bessel expansion for a given ℓ

$$A(r)=\int_0^\infty \widetilde{A}(k)\,j_\ell(k\,r)\,\mathrm{d}\,k \quad \Leftarrow \quad \widetilde{A}(k)=\frac{2\,k^2}{\pi}\int_0^\infty r^2\,A(r)\,j_\ell(k\,r)\,\mathrm{d}\,r$$

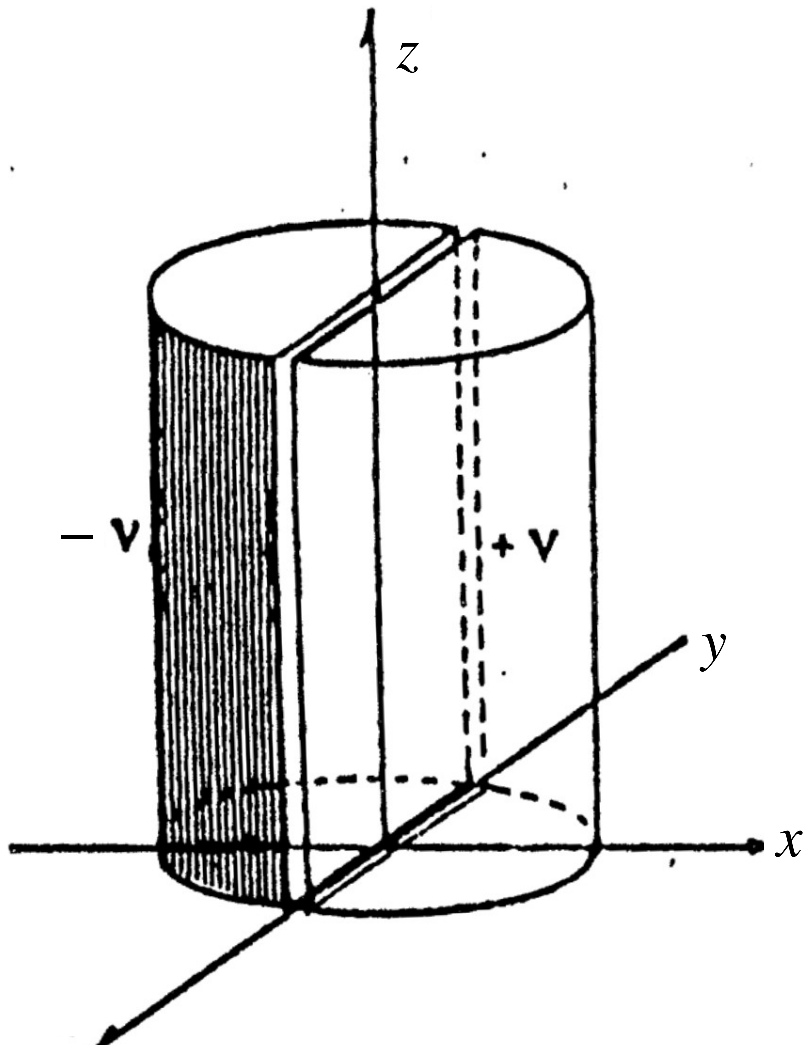
- Useful for current decay in conducting media or time-dependent magnetic diffusion with angular symmetry. [Problems in Chapter 5]

- The treatment of the Laplace equation in rectangular coordinates has been shown in Sec. 2.9.

For the cylinder in Problem 3.9 the cylindrical surface is made of two equal half-cylinders, one at potential V and the other at potential $-V$, so that

$$V(\phi, z) = \begin{cases} V & \text{for } -\pi/2 < \phi < \pi/2 \\ -V & \text{for } \quad \pi/2 < \phi < 3\pi/2 \end{cases}$$

(a) Find the potential inside the cylinder. [Problem 3.10]



Since the boundary condition on the cylindrical surface, $V(\phi, z)$, is an even function of ϕ, it implies that all the coefficients B_{mn} vanish. The left coefficients are

$$\begin{aligned} A_{mn} &= \frac{2}{\pi L I_m(n\pi b/L)} \int_0^{2\pi} \int_0^L V(\phi', z') \cos m\phi' \sin \frac{n\pi z'}{L} \mathrm{d}z' \mathrm{d}\phi' \\ &= \frac{2V[1-(-1)^n]}{n\pi^2 I_m(n\pi b/L)} \left[\int_{-\pi/2}^{\pi/2} \cos m\phi' \mathrm{d}\phi' - \int_{\pi/2}^{3\pi/2} \cos m\phi' \mathrm{d}\phi' \right] \\ &= \frac{4V[1-(-1)^n][1-(-1)^m]}{mn\pi^2 I_m(n\pi b/L)} \sin \frac{m\pi}{2}. \end{aligned}$$

The result indicates that A_{mn} is nonvanishing only when m and n are both odd. Then the nonvanishing coefficients are $A_{2k+1,2\ell+1} = \dfrac{16V(-1)^k(2k+1)^{-1}(2\ell+1)^{-1}}{\pi^2 I_{2k+1}((2\ell+1)\pi b/L)}$ where $k, \ell \in \{0\} \cup \mathbb{N}$. Therefore, the potential can be expressed as

$$\Phi(\rho, \phi, z) = \frac{16V}{\pi^2} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} (-1)^k \frac{I_{2k+1}((2\ell+1)\pi\rho/L)}{I_{2k+1}((2\ell+1)\pi b/L)} \frac{\cos[(2k+1)\phi]}{2k+1} \frac{\sin[(2\ell+1)\pi z/L]}{2\ell+1}.$$

Expansion of Green Functions in Spherical Coordinates

- To handle problems involving distributions of charge and boundary values for the solutions of the Poisson equation, it needs to determine the Green function that satisfies the appropriate boundary conditions.

- For the case of no boundary surfaces, the expansion of the Green function

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\frac{r_<^{\ell}}{r_>^{\ell+1}}Y_\ell^{*m}(\theta',\phi')\,Y_\ell^m(\theta,\phi)$$

- To obtain a similar expansion for the Green function for the "exterior" problem with a spherical boundary at $r=a$, which is found from the image form method

$$G(\mathbf{r},\mathbf{r}')=\sum_{\ell,m}\frac{4\pi}{2\ell+1}\left[\frac{r_<^{\ell}}{r_>^{\ell+1}}-\frac{1}{a}\left(\frac{a^2}{r\,r'}\right)^{\ell+1}\right]Y_\ell^{*m}(\theta',\phi')\,Y_\ell^m(\theta,\phi)$$

$$\text{where }\ \frac{r_<^{\ell}}{r_>^{\ell+1}}-\frac{1}{a}\left(\frac{a^2}{r\,r'}\right)^{\ell+1}=\begin{cases}\dfrac{1}{r'^{\,\ell+1}}\left(r^{\ell}-\dfrac{a^{2\ell+1}}{r^{\ell+1}}\right), & r<r' \quad\Rightarrow\quad \begin{array}{c}\text{vanishes for}\\ r=a \text{ or } r'\to\infty\end{array}\\[2ex] \dfrac{1}{r^{\ell+1}}\left(r'^{\,\ell}-\dfrac{a^{2\ell+1}}{r'^{\,\ell+1}}\right), & r>r' \quad\Rightarrow\quad \begin{array}{c}\text{vanishes for}\\ r'=a \text{ or } r\to\infty\end{array}\end{cases}$$

- Symmetric in r and r'.

- The reason for different linear combinations for $r<r'$ and for $r>r'$ is connected with the fact that the Green function is a solution of the Poisson equation with a delta function inhomogeneity.

- From first principles, a Green function for a Dirichlet potential problem

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -4\pi\,\delta(\mathbf{r}-\mathbf{r}') \quad \Leftarrow \quad G(\mathbf{r},\mathbf{r}') = 0 \text{ for } \mathbf{r} \text{ or } \mathbf{r}' \text{ on the boundary}$$

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{r^2}\,\delta(r-r')\,\delta(\phi-\phi')\,\delta(\cos\theta-\cos\theta')$$

$$= \frac{1}{r^2}\,\delta(r-r')\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_\ell^{*m}(\theta',\phi')\,Y_\ell^m(\theta,\phi)$$

$$\Rightarrow \quad G(\mathbf{r},\mathbf{r}') = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_\ell^m(r\,|\,r',\theta',\phi')\,Y_\ell^m(\theta,\phi)$$

$$\Rightarrow \begin{bmatrix} A_\ell^m(r\,|\,r',\theta',\phi') = g_\ell(r,r')\,Y_\ell^{*m}(\theta',\phi') \\ \dfrac{1}{r}\dfrac{\mathrm{d}^2}{\mathrm{d}\,r^2}\left(r\,g_\ell(r,r')\right) - \dfrac{\ell(\ell+1)}{r^2}\,g_\ell(r,r') = -\dfrac{4\pi}{r^2}\,\delta(r-r') \end{bmatrix}$$

$$\Rightarrow \quad g_\ell(r,r') = \begin{bmatrix} A\;\; r^\ell + \dfrac{B}{r^{\ell+1}} & \text{for} \quad r<r' \\ A'\, r^\ell + \dfrac{B'}{r^{\ell+1}} & \text{for} \quad r>r' \end{bmatrix}$$

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\left(\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right)\right]Y_\ell^m=0$$

$$\Rightarrow\quad\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_\ell^m=-\ell(\ell+1)\,Y_\ell^m$$

$$\mathbb{L}^2\equiv-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\quad\Rightarrow\quad\mathbb{L}^2\,Y_\ell^m=\ell(\ell+1)\,Y_\ell^m$$

$$\Rightarrow\quad\nabla^2\Phi=\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\,\Phi)-\frac{\mathbb{L}^2}{r^2}\Phi=0\quad\Rightarrow\quad\frac{d^2U}{dr^2}=\frac{\ell(\ell+1)}{r^2}U\quad\Leftarrow\quad\Phi=\frac{U}{r}Y_\ell^m$$

$\Rightarrow$ Y_ℓ^m is also kind of eigenfunciton of ∇^2 up to a factor of $\frac{1}{r^2}$

$$\mathbb{L}_z\equiv-i\left(x\,\partial_y-y\,\partial_x\right)=-i\,\partial_\phi\quad\Rightarrow\quad\mathbb{L}_z\,Y_\ell^m=m\,Y_\ell^m\quad\Leftarrow\quad\partial_x\equiv\frac{\partial}{\partial x},\cdots$$

$$\mathbb{L}_x\equiv-i\left(y\,\partial_z-z\,\partial_y\right)=i\left(+\sin\phi\,\partial_\theta+\cot\theta\cos\phi\,\partial_\phi\right)$$

$$\mathbb{L}_y\equiv-i\left(z\,\partial_x-x\,\partial_z\right)=i\left(-\cos\phi\,\partial_\theta+\cot\theta\sin\phi\,\partial_\phi\right)$$

$$\Rightarrow\quad\mathbb{L}_\pm=\mathbb{L}_x\pm i\,\mathbb{L}_y=e^{\pm i\phi}\left(\pm\partial_\theta+i\cot\theta\,\partial_\phi\right)$$

$$\Rightarrow\quad\mathbb{L}^2=\mathbb{L}_x^2+\mathbb{L}_y^2+\mathbb{L}_z^2=\frac{1}{2}\left(\mathbb{L}_+\mathbb{L}_-+\mathbb{L}_-\mathbb{L}_+\right)+\mathbb{L}_z^2$$

- A, B, A', B' are functions of r' to be determined by the boundary conditions, the requirement implied by $\delta(r-r')$, and the symmetry of $g_\ell(r, r')$ in r and r'.

- Suppose that the boundary surfaces are concentric spheres at $r=a$ and $r=b$

$$\Rightarrow \quad g_\ell(a, r') = g_\ell(b, r') = 0 \quad \Rightarrow \quad g_\ell(r, r') = \begin{cases} A\, r^\ell \left(1 - \dfrac{a^{2\ell+1}}{r^{2\ell+1}}\right), & \text{for } r < r' \\ \dfrac{B'}{r^{\ell+1}} \left(1 - \dfrac{r^{2\ell+1}}{b^{2\ell+1}}\right), & \text{for } r > r' \end{cases}$$

$$\Rightarrow \quad g_\ell(r, r') = C\, \frac{r_<^\ell}{r_>^{\ell+1}} \left(1 - \frac{a^{2\ell+1}}{r_<^{2\ell+1}}\right) \left(1 - \frac{r_>^{2\ell+1}}{b^{2\ell+1}}\right) \quad \Leftarrow \quad \text{symmetry in } r \text{ and } r'$$

$$\int_{r'_-}^{r'_+} \left(\frac{d^2}{dr^2}(r\, g_\ell) - \frac{\ell(\ell+1)}{r} g_\ell \right) dr = -\int_{r'_-}^{r'_+} \frac{4\pi}{r} \delta(r-r')\, dr \quad \Leftarrow \quad \begin{array}{l} r'_+ = r' + \epsilon \\ r'_- = r' - \epsilon \end{array}$$

$$\Rightarrow \quad \frac{d}{dr}(r\, g_\ell)_{r'_+} - \frac{d}{dr}(r\, g_\ell)_{r'_-} = -\frac{4\pi}{r'}$$

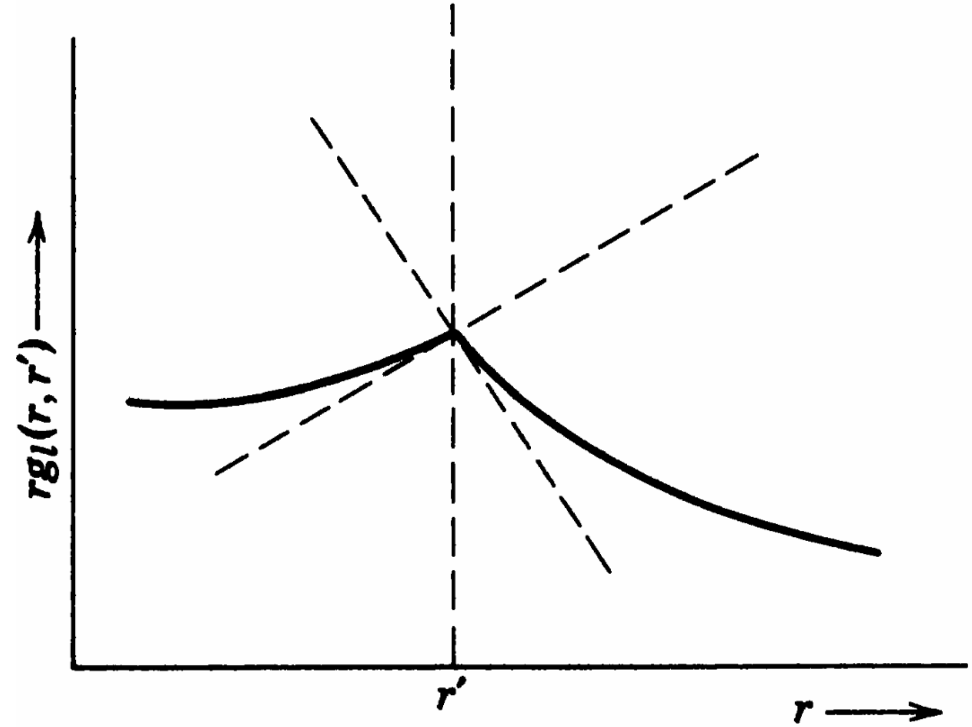
$$\frac{d}{dr}(r\, g_\ell)_{r'_+} = C \left(r'^{\ell} - \frac{a^{2\ell+1}}{r'^{\ell+1}} \right) \frac{d}{dr} \left(\frac{1}{r^\ell} - \frac{r^{\ell+1}}{b^{2\ell+1}} \right)_{r=r'}$$

$$= -\frac{C}{r'} \left[1 - \left(\frac{a}{r'}\right)^{2\ell+1} \right] \left[\ell + (\ell+1) \left(\frac{r'}{b}\right)^{2\ell+1} \right]$$

$$\frac{\mathrm{d}}{\mathrm{d}\,r}\left(r\,g_\ell\right)_{r'_-}$$

$$= \frac{C}{r'}\left[1 - \left(\frac{r'}{b}\right)^{2\ell+1}\right]\left[\ell + 1 + \ell\left(\frac{a}{r'}\right)^{2\ell+1}\right]$$

$$\Rightarrow \quad C = \frac{4\pi}{2\ell+1}\,\frac{1}{1-\left(\frac{a}{b}\right)^{2\ell+1}}$$



$$\Rightarrow \quad G(\mathbf{r},\mathbf{r}') = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi\,Y_\ell^{*m}(\theta',\phi')\,Y_\ell^m(\theta,\phi)}{(2\ell+1)\left[1-\left(\frac{a}{b}\right)^{2\ell+1}\right]}\,\frac{r_<^\ell}{r_>^{\ell+1}}\left(1-\frac{a^{2\ell+1}}{r_<^{2\ell+1}}\right)\left(1-\frac{r_>^{2\ell+1}}{b^{2\ell+1}}\right)$$

$$\rightarrow \sum_{\ell,m}\frac{4\pi}{2\ell+1}\,\frac{r_<^\ell}{r_>^{\ell+1}}\left(1-\frac{a^{2\ell+1}}{r_<^{2\ell+1}}\right)Y_\ell^{*m}(\theta',\phi')\,Y_\ell^m(\theta,\phi) \qquad \text{for } b\rightarrow\infty$$

$$\rightarrow \sum_{\ell,m}\frac{4\pi}{2\ell+1}\,\frac{r_<^\ell}{r_>^{\ell+1}}\,Y_\ell^{*m}(\theta',\phi')\,Y_\ell^m(\theta,\phi) \qquad \text{for } \begin{array}{l} a\rightarrow 0 \\ b\rightarrow\infty \end{array}$$

$$\rightarrow \sum_{\ell,m}\frac{4\pi}{2\ell+1}\,\frac{r_<^\ell}{r_>^{\ell+1}}\left(1-\frac{r_>^{2\ell+1}}{b^{2\ell+1}}\right)Y_\ell^{*m}(\theta',\phi')\,Y_\ell^m(\theta,\phi) \qquad \text{for } a\rightarrow 0$$

Solution of Potential Problems with the Spherical Green Function Expansion

- The general solution to the Poisson equation with specified values of the potential on the boundary surface

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \rho(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}')\, \mathrm{d}^3x' - \frac{1}{4\pi}\oint_{\mathcal{S}} \Phi(\mathbf{r}')\, \frac{\partial G}{\partial n'}\, \mathrm{d}a'$$

- Consider the potential *inside* a sphere of radius b and set a=0

$$\frac{\partial G}{\partial n'} = \frac{\partial G}{\partial r'}_{|r'=b} = -\frac{4\pi}{b^2}\sum_{\ell,m}\left(\frac{r}{b}\right)^{\ell} Y_\ell^{*m}(\theta',\phi')\, Y_\ell^m(\theta,\phi) \quad \text{and} \quad \begin{array}{l} \Phi(\mathbf{r}') = V(\theta,\phi) \\ \rho(\mathbf{r}') = 0 \end{array}$$

$$\Rightarrow \quad \Phi(\mathbf{r}) = \sum_{\ell,m}\left(\oint V(\theta',\phi')\, Y_\ell^{*m}(\theta',\phi')\, \mathrm{d}\Omega'\right)\left(\frac{r}{b}\right)^{\ell} Y_\ell^m(\theta,\phi)$$

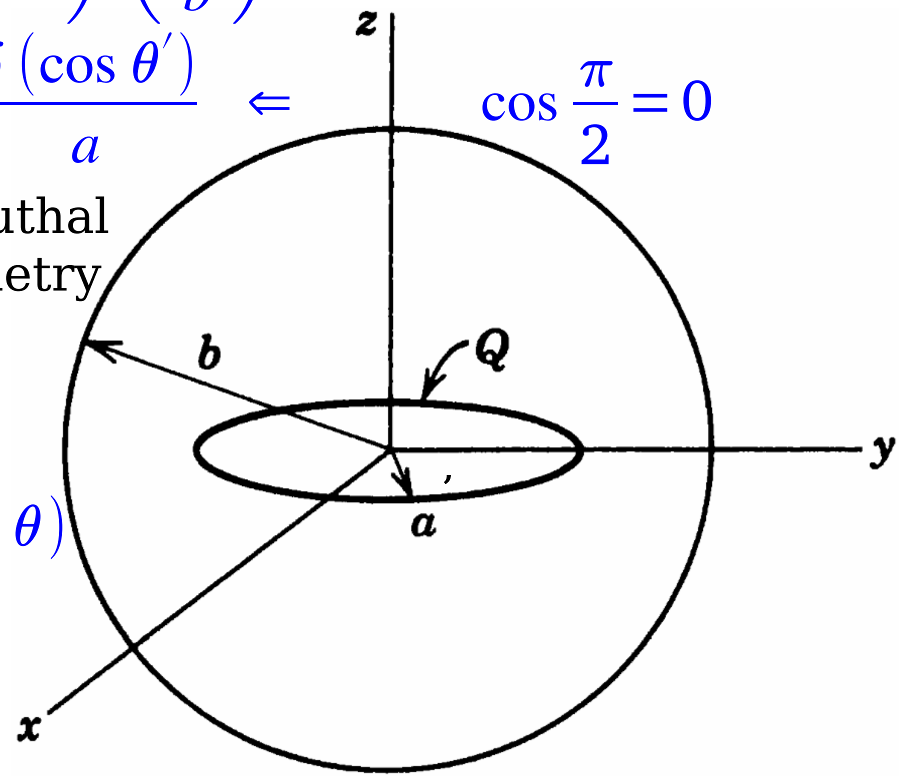
- Consider $V = 0 \;\; \& \;\; \rho(\mathbf{r}') = \dfrac{Q}{2\pi a}\,\delta(r'-a)\,\dfrac{\delta(\cos\theta')}{a} \quad \Leftarrow \quad \cos\dfrac{\pi}{2} = 0$

only terms with m=0 survive because of azimuthal symmetry

$$\Rightarrow \Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}')\, \mathrm{d}^3x'$$

$$= \frac{Q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty} P_\ell(0)\, \frac{r_<^{\ell}}{r_>^{\ell+1}}\left(1 - \frac{r_>^{2\ell+1}}{b^{2\ell+1}}\right) P_\ell(\cos\theta)$$

where $\quad r_< = \min(r, a), \quad r_> = \max(r, a)$



$$P_{2n+1}(0)=0\,,\quad P_{2n}(0)=\frac{(-1)^n(2n-1)!!}{2^n\,n!}\quad\Leftarrow\quad(2n-1)!!\equiv\frac{(2n)!}{(2n)!!}=\frac{(2n)!}{2^n\,n!}$$

$$\Rightarrow\quad\Phi(\mathbf{r})=\frac{Q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n\,n!}\frac{r_<^{2n}}{r_>^{2n+1}}\left(1-\frac{r_>^{4n+1}}{b^{4n+1}}\right)P_{2n}(\cos\theta)$$

$$\to\frac{Q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n\,n!}\frac{r_<^{2n}}{r_>^{2n+1}}P_{2n}(\cos\theta)\quad\Leftarrow\quad b\to\infty\quad\Rightarrow\quad\text{Sec. III}$$

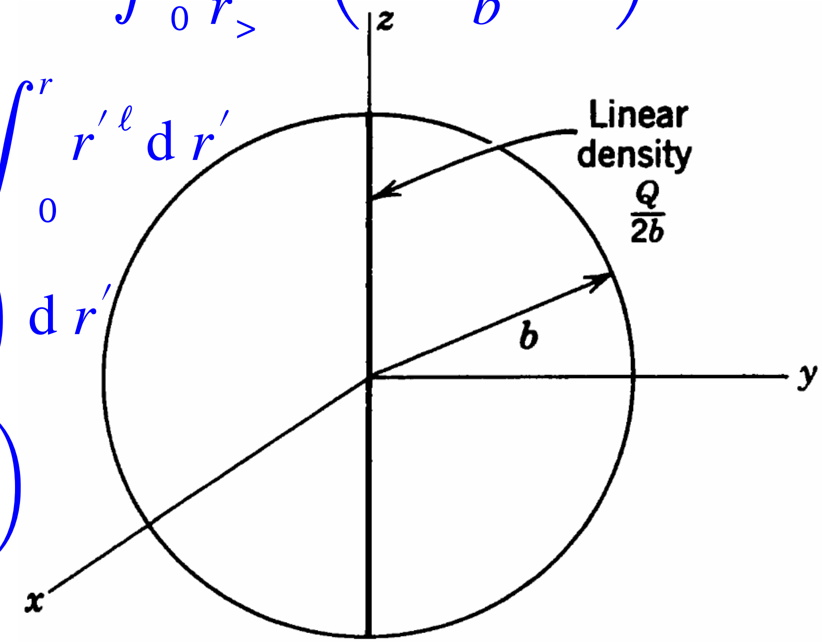
- 2nd example: $V=0\ \ \&\ \ \rho(\mathbf{r}')=\dfrac{Q}{2b}\dfrac{\delta(\cos\theta'-1)+\delta(\cos\theta'+1)}{2\pi r'^2}$

$$\Rightarrow\quad\Phi(\mathbf{r})=\frac{Q}{8\pi\epsilon_0 b}\sum_{\ell=0}^{\infty}\left[P_\ell(1)+P_\ell(-1)\right]P_\ell(\cos\theta)\int_0^b\frac{r_<^\ell}{r_>^{\ell+1}}\left(1-\frac{r_>^{2\ell+1}}{b^{2\ell+1}}\right)\mathrm{d}r'$$

$$\int_0^b\frac{r_<^\ell}{r_>^{\ell+1}}\left(1-\frac{r_>^{2\ell+1}}{b^{2\ell+1}}\right)\mathrm{d}r'=\left(\frac{1}{r^{\ell+1}}-\frac{r^\ell}{b^{2\ell+1}}\right)\int_0^r r'^{\ell}\,\mathrm{d}r'$$

$$+\,r^\ell\int_r^b\left(\frac{1}{r'^{\ell+1}}-\frac{r'^{\ell}}{b^{2\ell+1}}\right)\mathrm{d}r'$$

$$=\frac{2\ell+1}{\ell(\ell+1)}\left(1-\frac{r^\ell}{b^\ell}\right)$$



$$\text{For } \ell=0: \int_0^b \left(\frac{1}{r_>}-\frac{1}{b}\right) \mathrm{d}\, r' = \left(\frac{1}{r}-\frac{1}{b}\right)\int_0^r \mathrm{d}\, r' + \int_r^b \left(\frac{1}{r'}-\frac{1}{b}\right)\mathrm{d}\, r' = \ln\frac{b}{r}$$

$$\Rightarrow \quad \Phi(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0 b}\left[\ln\frac{b}{r} + \sum_{\ell=1}^{\infty}\frac{4\ell+1}{2\ell(2\ell+1)}\left(1-\frac{r^{2\ell}}{b^{2\ell}}\right)P_{2\ell}(\cos\theta)\right]$$

$$\text{where } P_\ell(-1) = (-1)^\ell$$

- The potential diverges along the z axis for $\cos\theta=\pm1$, except at $r=b$ exactly.

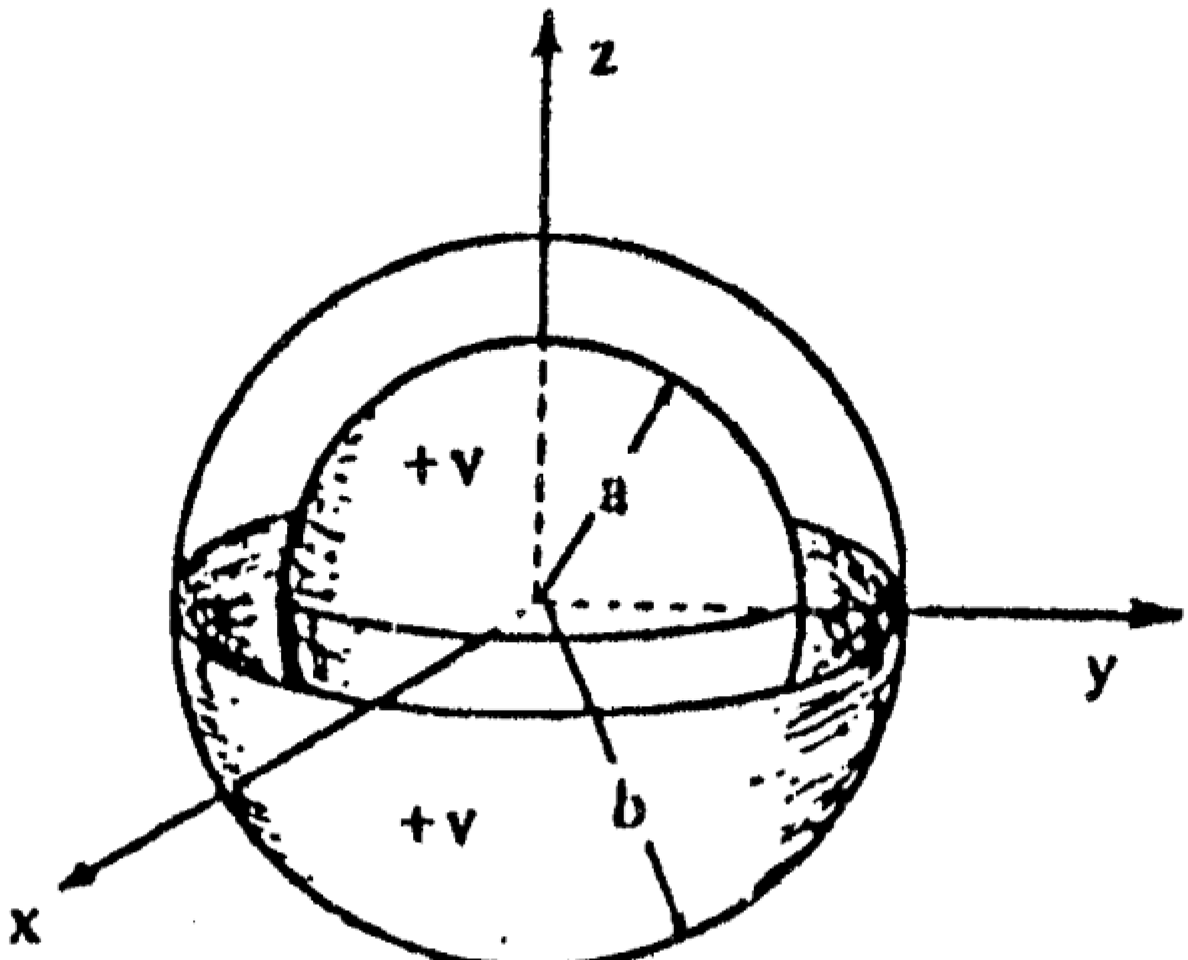
- The surface-charge density on the grounded sphere

$$\sigma(\theta) = \epsilon_0 \frac{\partial\Phi}{\partial r}_{|r=b} = -\frac{Q}{4\pi b^2}\left(1+\sum_{\ell=1}^{\infty}\frac{4\ell+1}{2\ell+1}P_{2\ell}(\cos\theta)\right)$$

- The leading term shows that the total charge induced on the sphere is $-Q$, the other terms integrating to 0 over the surface of the sphere.

Solve for the potential in Problem 3.1, using the appropriate Green function obtained in the text, and verify that the answer obtained in this way agrees with the direct solution from the differential equation. [Problem 3.13]

3.1 Two concentric spheres have radii a, b ($b > a$) and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential V. The other hemispheres are at zero potential



The Green function with the Dirichlet boundary condition for two concentric spheres has been derived as Eq. (3.125) in Jackson's *Classical Electrodynamics*:

$$G(\mathbf{x},\mathbf{x}') = 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{Y^*_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi)}{(2\ell+1)[1-(a/b)^{2\ell+1}]}\left(r_<^\ell - \frac{a^{2\ell+1}}{r_<^{\ell+1}}\right)\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right),$$

where $r_<$ ($r_>$) is the smaller (larger) of $|\mathbf{x}|$ and $|\mathbf{x}'|$. Since ther is azimuthal symmetry in this problem, only the $m=0$ terms survives. And the Green function becomes

$$G(\mathbf{x},\mathbf{x}') = \sum_{\ell=0}^{\infty} \frac{P_\ell(\cos\theta')P_\ell(\cos\theta)}{1-(a/b)^{2\ell+1}}\left(r_<^\ell - \frac{a^{2\ell+1}}{r_<^{\ell+1}}\right)\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right).$$

There is no charge distribution between the two concentric spheres, therefore, according to the Green theorem, the electric potential within is

$$\begin{aligned}
\Phi(\mathbf{x}) &= -\frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial G}{\partial n'}(\mathbf{x},\mathbf{x}')\mathrm{d}a' = -\frac{1}{4\pi}\left[\int_{S_a}\Phi(\mathbf{x}')\frac{\partial G}{\partial n'}\mathrm{d}a' + \int_{S_b}\Phi(\mathbf{x}')\frac{\partial G}{\partial n'}\mathrm{d}a'\right] \\
&= \frac{1}{4\pi}\left[a^2\oint \Phi(r',\theta')\left.\frac{\partial G}{\partial r'}\right|_{r'=a}\mathrm{d}\Omega' - b^2\oint \Phi(r',\theta')\left.\frac{\partial G}{\partial n'}\right|_{r'=b}\mathrm{d}\Omega'\right] \\
&= \frac{V}{2}\sum_{\ell=0}\frac{(2\ell+1)P_\ell(\cos\theta)}{1-(a/b)^{2\ell+1}}\left[\left(\frac{a^{\ell+1}}{r^{\ell+1}} - \frac{a^{\ell+1}r^\ell}{b^{2\ell+1}}\right)\int_0^{\pi/2}P_\ell(\cos\theta')\sin\theta'\mathrm{d}\theta'\right. \\
&\qquad \left. + \left(\frac{r^\ell}{b^\ell} - \frac{a^{2\ell+1}}{b^\ell r^{\ell+1}}\right)\int_{\pi/2}^{\pi}P_\ell(\cos\theta')\sin\theta'\mathrm{d}\theta'\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{V}{2}\sum_{\ell=0} \frac{(2\ell+1)P_\ell(\cos\theta)}{1-(a/b)^{2\ell+1}} \left[\frac{a^{\ell+1}}{r^{\ell+1}} - \frac{a^{\ell+1}r^\ell}{b^{2\ell+1}} + (-1)^\ell \left(\frac{r^\ell}{b^\ell} - \frac{a^{2\ell+1}}{b^\ell r^{\ell+1}}\right)\right] q_\ell \\
&= \frac{V}{2} - \frac{V}{2}\sum_{n=0}^{\infty} \frac{(4n+3)(2n-1)!!}{(-2)^{n+1}(n+1)!} \left[\frac{b^{2n+1}+a^{2n+1}}{b^{4n+3}-a^{4n+3}} \left(\frac{a^2b^2}{r^2}\right)^{n+1}\right. \\
&\qquad\qquad \left. - \frac{a^{2n+2}+b^{2n+2}}{b^{4n+3}-a^{4n+3}} r^{2n+1}\right] P_{2n+1}(\cos\theta),
\end{aligned}
$$

where $q_\ell \equiv \int_{\pi/2}^{0} P_\ell(\cos\theta)\mathrm{d}\cos\theta = \int_0^1 P_\ell(x)\mathrm{d}x$ as in Problem 3.1, and the Dirichlet boundary conditions on the two concentric spheres are used. Thus, the result is in agreement with the one in Problem 3.1.

Expansion of Green Functions in Cylindrical Coordinates

- The Green function: $\nabla_x^2 G(\mathbf{r},\mathbf{r}') = -4\pi\,\delta(\rho-\rho')\,\dfrac{\delta(\phi-\phi')}{\rho}\,\delta(z-z')$

- The delta functions can be written in terms of orthonormal functions:

$$\delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}$$

$$\delta(z-z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(z-z')}\,\mathrm{d}k = \frac{1}{\pi}\int_0^{\infty}\cos k(z-z')\,\mathrm{d}k$$

$$\Rightarrow \quad G(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} e^{im(\phi-\phi')}\cos k(z-z')\,g_m(k,\rho,\rho')\,\mathrm{d}k$$

$$\Rightarrow \quad \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}g_m}{\mathrm{d}\rho}\right) - \left(k^2+\frac{m^2}{\rho^2}\right)g_m = -\frac{4\pi}{\rho}\,\delta(\rho-\rho') \quad \Rightarrow \quad \begin{matrix} I_m(k\rho) \\ K_m(k\rho) \end{matrix} \quad \text{for } \rho\neq\rho'$$

$$\text{Let } \begin{cases} \psi_1(k\rho) = A\,I_m(k\rho) + B\,K_m(k\rho) & \text{satifies the boundary condition for } \rho<\rho' \\ \psi_2(k\rho) = C\,I_m(k\rho) + D\,K_m(k\rho) & \text{satifies the boundary condition for } \rho>\rho' \end{cases}$$

$$\Rightarrow \quad g_m(k,\rho,\rho') = \psi_1(k\rho_<)\,\psi_2(k\rho_>) \quad \Leftarrow \quad \text{the symmetry of the Green function}$$

$$\Rightarrow \quad \frac{\mathrm{d}g_m}{\mathrm{d}\rho}_{|\rho'+\epsilon} - \frac{\mathrm{d}g_m}{\mathrm{d}\rho}_{|\rho'-\epsilon} = k\begin{vmatrix}\psi_1 & \psi_2 \\ \psi_1' & \psi_2'\end{vmatrix}\left(=k\,\underbrace{W[\psi_1,\psi_2]}_{\text{Wronskian}}\right) = -\frac{4\pi}{\rho'}$$

$e^{im\phi}$'s form an orthogonal set. $\Rightarrow$ $f(\phi) = \sum_{\ell=-\infty}^{\infty} A_\ell e^{i\ell\phi}$

$$\Rightarrow \int_0^{2\pi} f(\phi') \left(\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \right) d\phi' = \frac{1}{2\pi} \int_0^{2\pi} \sum_{\ell=-\infty}^{\infty} A_\ell e^{i\ell\phi'} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} d\phi'$$

$$= \sum_{\ell,m=-\infty}^{\infty} A_\ell e^{im\phi} \left(\frac{1}{2\pi} \int_0^{2\pi} e^{i(\ell-m)\phi'} d\phi' \right) \Leftarrow \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)\phi} d\phi = \delta_{mn}$$

$$= \sum_{\ell,m=-\infty}^{\infty} A_\ell e^{im\phi} \delta_{m\ell} = \sum_{\ell=-\infty}^{\infty} A_\ell e^{i\ell\phi} = f(\phi)$$

$$\Rightarrow \delta(\phi-\phi') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \Leftarrow \int_0^{2\pi} f(\phi') \delta(\phi-\phi') d\phi' = f(\phi)$$

$F(k) = \int_{-\infty}^{\infty} f(z') e^{-ikz'} dz'$ + $f(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ikz} dk$ $\Leftarrow$ Fourier Transform

$$\Rightarrow f(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ikz} dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(z') e^{-ikz'} dz' \right) e^{ikz} dk$$

$$= \int_{-\infty}^{\infty} f(z') \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(z-z')} dk \right) dz'$$

$$\Rightarrow \delta(z-z') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(z-z')} dk = \frac{1}{\pi} \int_0^{\infty} \cos k(z-z') dk$$

- The Sturm-Liouville type equation: $\frac{d}{d\,x}\left(p(x)\frac{d\,\psi}{d\,x}\right)+q(x)\,\psi=0$
- The Wronskian of 2 linearly independent solutions of a Sturm-Liouville type equation is proportional to $\frac{1}{p(x)}$ $\Rightarrow$ $W[\psi_1(x),\psi_2(x)]=-\frac{4\pi}{x}$

Proof: $\frac{d}{d\,x}\left(p(x)\frac{d\,\psi}{d\,x}\right)+q(x)\,\psi=0 \quad\Rightarrow\quad \psi''+(\ln p)'\,\psi'+q\,p^{-1}\,\psi=0$

$$\Rightarrow \quad \begin{matrix} \psi_1''+(\ln p)'\,\psi_1'+q\,p^{-1}\,\psi_1=0 \\ \psi_2''+(\ln p)'\,\psi_2'+q\,p^{-1}\,\psi_2=0 \end{matrix} \quad\Rightarrow\quad \begin{matrix} \psi_1''\,\psi_2+(\ln p)'\,\psi_1'\,\psi_2+q\,p^{-1}\,\psi_1\,\psi_2=0 \\ \psi_1\,\psi_2''+(\ln p)'\,\psi_1\,\psi_2'+q\,p^{-1}\,\psi_1\,\psi_2=0 \end{matrix}$$

$$\Rightarrow \quad (\psi_1''\,\psi_2-\psi_1\,\psi_2'')+(\ln p)'\,(\psi_1'\,\psi_2-\psi_1\,\psi_2')=0$$

$$\Rightarrow \quad \frac{d}{d\,x}(\psi_1'\,\psi_2-\psi_1\,\psi_2')+(\ln p)'\,(\psi_1'\,\psi_2-\psi_1\,\psi_2')=0$$

$$\Rightarrow \quad \frac{d}{d\,x}W[\psi_1,\psi_2]+(\ln p)'\,W[\psi_1,\psi_2]=0 \quad\Rightarrow\quad d\ln W=-d\ln p$$

$$\Rightarrow \quad W=\frac{c}{p} \quad\Rightarrow\quad W \propto \frac{1}{p}$$

- No boundary surfaces $\Rightarrow$ $\begin{array}{l} g_m(k, 0, \rho') \text{ finite} \\ g_m(k, \infty, \rho') = 0 \end{array}$ $\Rightarrow$ $\begin{array}{l} \psi_1(k\rho) = A\, I_m(k\rho) \\ \psi_2(k\rho) = \quad K_m(k\rho) \end{array}$

$\Rightarrow \quad A = 4\pi \quad \Leftarrow \quad W[I_m(x), K_m(x)] = -\dfrac{1}{x} \quad \Leftarrow$ Use the limit froms of I_m and K_m to calculate.

$$\Rightarrow \quad \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty I_m(k\rho_<)\, K_m(k\rho_>) \cos k(z - z')\, e^{i m(\phi - \phi')}\, \mathrm{d}k$$

$$= \frac{4}{\pi} \int_0^\infty \cos k(z - z')\, \mathrm{d}k \left(\frac{I_0(k\rho_<)\, K_0(k\rho_>)}{2} + \sum_{m=1}^{\infty} I_m(k\rho_<)\, K_m(k\rho_>) \cos m(\phi - \phi') \right)$$

- $\mathbf{r}' \to 0$ $\Rightarrow$ only the $m = 0$ term survives $\Rightarrow$ $\dfrac{1}{\sqrt{\rho^2 + z^2}} = \dfrac{2}{\pi} \displaystyle\int_0^\infty K_0(k\rho) \cos kz\, \mathrm{d}k$

$$\Rightarrow \quad K_0\left(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')}\right) \quad \Leftarrow \quad \rho^2 \to \rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')$$

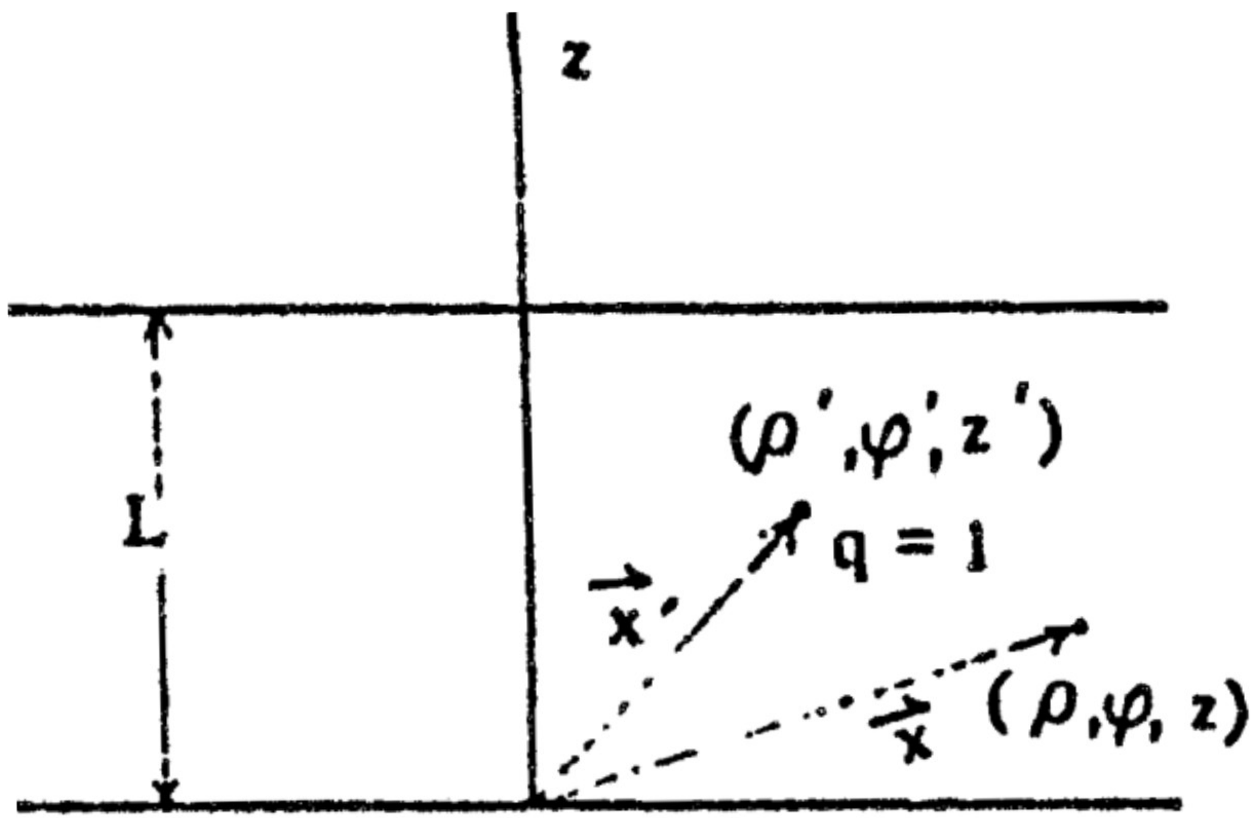
$$= I_0(k\rho_<)\, K_0(k\rho_>) + 2 \sum_{m=1}^{\infty} I_m(k\rho_<)\, K_m(k\rho_>) \cos m(\phi - \phi')$$

$$\Rightarrow \ln \frac{1}{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')} = 2 \ln \frac{1}{\rho_>} + 2 \sum_{m=1}^{\infty} \left(\frac{\rho_<}{\rho_>} \right)^m \frac{\cos m(\phi - \phi')}{m} \quad \Leftarrow \quad k \to 0$$

The Dirichlet Green function for the unbounded space between the planes at $z = 0$ and $z = L$ allows discussion of a point charge or a distribution of charge between parallel conducting planes held at zero potential. [Problem 3.17]

(a) Using cylindrical coordinates show that one form of the Green function is

$$G(\mathbf{x}, \mathbf{x}') = \frac{4}{L} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi}{L}\rho_<\right) K_m\left(\frac{n\pi}{L}\rho_>\right)$$



The Green function $G(\mathbf{x}, \mathbf{x}')$ should satisfy the equation, $\nabla^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta^3(\mathbf{x}-\mathbf{x}')$. In cylindrical coordinates it reads,

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right] G(\mathbf{x}, \mathbf{x}') = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z'). \tag{0.4}$$

And G should vanish at $z, z' = 0$ and $z, z' = L$. The ϕ and z delta functions can be written in terms of orthonormal functions:

$$\delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}, \qquad \delta(z-z') = \frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi z}{L}\sin\frac{n\pi z'}{L}.$$

We thus expand $G(\mathbf{x}, \mathbf{x}')$ in the similar fashion:

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{\pi L}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} g_{mn}(\rho, \rho')e^{im(\phi-\phi')}\sin\frac{n\pi z}{L}\sin\frac{n\pi z'}{L}.$$

Then the substitution into Eq.(0.4) leads to

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} - (\frac{n^2\pi^2}{L^2} + \frac{m^2}{\rho^2})\right] g_{mn} = -\frac{4\pi}{\rho}\delta(\rho-\rho'). \tag{0.5}$$

For $\rho \neq \rho'$, $g_{mn}(\rho, \rho')$'s are modified Bessel functions $I_m(k\rho)$ and $K_m(k\rho)$ where $k \equiv n\pi/L$. Since g_{mn} must be well behaved at $\rho \to 0$ and $\rho \to \infty$, thus

$$g_{mn}(\rho, \rho') \propto \begin{cases} I_m(k\rho), & \rho < \rho', \\ K_m(k\rho), & \rho > \rho'. \end{cases}$$

The symmetry in ρ and ρ' requires that $g_{mn}(\rho,\rho') = AI_m(k\rho_<)K_m(k\rho_>)$, where $\rho_<$ ($\rho_>$) is the smaller (larger) of ρ and ρ'. The normalization A is determined by the discontinuity in slope implied by the delta function in Eq.(0.5):

$$\frac{\mathrm{d}}{\mathrm{d}\rho}g_{mn}\bigg|_+ - \frac{\mathrm{d}}{\mathrm{d}\rho}g_{mn}\bigg|_- = AkW[I_m,K_m] = -\frac{A}{\rho} = -\frac{4\pi}{\rho},$$

where $|_\pm$ means evaluated at $\rho = \rho' \pm \epsilon$ and ϵ is infinitesimal. And $W[I_m,K_m]$ is the Wronskian of I_m and K_m. Thus $A = 4\pi$, and $g_{mn} = 4\pi I_m(k\rho_<)K_m(k\rho_>)$. Therefore

$$G(\mathbf{x},\mathbf{x}') = \frac{4}{L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sin\frac{n\pi z}{L}\sin\frac{n\pi z'}{L}I_m(\frac{n\pi}{L}\rho_<)K_m(\frac{n\pi}{L}\rho_>).$$

(b) Show that an alternative form of the Green function is

$$G(\mathbf{x},\mathbf{x}') = 2\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')\,\frac{\sinh(kz_<)\sinh[k(L-z_>)]}{\sinh(kL)}$$

For the alternative form, the ρ and ϕ delta functions are written in terms of orthonormal functions:

$$\frac{1}{\rho}\delta(\rho-\rho') = \int_0^\infty kJ_m(k\rho)J_m(k\rho')\mathrm{d}k, \qquad \delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}.$$

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty e^{im(\phi-\phi')}kJ_m(k\rho)J_m(k\rho')g_m(k,z,z')\mathrm{d}k.$$

Then the substitution into Eq.(0.4) leads to

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} g_m - k^2 g_m = -4\pi\delta(z - z'). \tag{0.6}$$

Since $g_m(k, z, z')$ must vanish at $z = 0$ and $z = L$, thus

$$g_m(k, z, z') \propto \begin{cases} \sinh kz, & z < z', \\ \sinh k(L - z), & z > z'. \end{cases}$$

The symmetry in z and z' requires that $g_m(k, z, z') = A \sinh kz_< \sinh k(L - z_>)$, where $z_<$ ($z_>$) is the smaller (larger) of z and z'. The normalization A is determined by the discontinuity in slope implied by the delta function in Eq.(0.6):

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}z} g_m\right|_+ - \left.\frac{\mathrm{d}}{\mathrm{d}z} g_m\right|_- &= -Ak \sinh kz \cosh k(L - z) - Ak \sinh k(L - z) \cosh kz \\ &= -Ak \sinh kL = -4\pi. \end{aligned}$$

where $|_\pm$ means evaluated at $z = z' \pm \epsilon$ and ϵ is infinitesimal. Thus $A = \dfrac{4\pi}{k \sinh kL}$, and $g_m = \dfrac{4\pi}{k \sinh kL} \sinh kz_< \sinh k(L - z_>)$. Therefore

$$G(\mathbf{x}, \mathbf{x}') = 2 \sum_{m=-\infty}^{\infty} \int_0^\infty e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \frac{\sinh kz_< \sinh k(L - z_>)}{\sinh kL} \mathrm{d}k.$$

Eigenfunction Expansions for Green Functions

- Elliptic differential equation $\nabla^2 \psi(\mathbf{r}) + [f(\mathbf{r}) + \lambda]\,\psi(\mathbf{r}) = 0$

- If the solutions are required to satisfy homogeneous boundary conditions on the surface S of the volume of V, then the equation will not in general have well-behaved solutions, except for certain values of λ.

- The values of λ, denoted by λ_n, are called *eigenvalues* (or *characteristic values*) and the solutions $\psi_n(\mathbf{r})$ are called *eigenfunctions* : $\nabla^2 \psi_n(\mathbf{r}) + [f(\mathbf{r}) + \lambda_n]\,\psi_n(\mathbf{r}) = 0$

- The eigenfunctions are orthogonal: $\int_{\mathcal{V}} \psi_m^*(\mathbf{r})\,\psi_n(\mathbf{r})\,\mathrm{d}^3x = \delta_{mn}$ & assumed completeness.

- The spectrum of eigenvalues λ_n may be a discrete set, or a continuum, or both.

- To find the Green function: $\nabla^2 G(\mathbf{r},\mathbf{r}') + [f(\mathbf{r}) + \lambda]\,G(\mathbf{r},\mathbf{r}') = -4\pi\,\delta(\mathbf{r}-\mathbf{r}')$

$$\Rightarrow \quad G(\mathbf{r},\mathbf{r}') = \sum_n a_n(\mathbf{r}')\,\psi_n(\mathbf{r}) \quad \Rightarrow \quad \sum_m a_m(\mathbf{r}')\,(\lambda - \lambda_m)\,\psi_m(\mathbf{r}) = -4\pi\,\delta(\mathbf{r}-\mathbf{r}')$$

$$\Rightarrow \int_{\mathcal{V}} \sum_m a_m(\mathbf{r}')\,(\lambda - \lambda_m)\,\psi_m(\mathbf{r})\,\psi_n^*(\mathbf{r})\,\mathrm{d}^3x = -4\pi \int_{\mathcal{V}} \delta(\mathbf{r}-\mathbf{r}')\,\psi_n^*(\mathbf{r})\,\mathrm{d}^3x$$

$$\Rightarrow \quad a_n(\mathbf{r}') = \frac{4\pi}{\lambda_n - \lambda}\,\psi_n^*(\mathbf{r}') \quad \Rightarrow \quad \boxed{G(\mathbf{r},\mathbf{r}') = 4\pi \sum_n \frac{\psi_n^*(\mathbf{r}')\,\psi_n(\mathbf{r})}{\lambda_n - \lambda}}$$

- For a continuous spectrum $G(\mathbf{r}, \mathbf{r}') = 4\pi \int \frac{\psi^*(\mathbf{r}', \lambda')\,\psi(\mathbf{r}, \lambda')}{\lambda' - \lambda}\,\mathrm{d}\lambda'$

- Consider $f(\mathbf{r}) = \lambda = 0 \quad \Rightarrow \quad \nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\,\delta(\mathbf{r} - \mathbf{r}')$

- The eigenfunctions for the wave equation: $\left(-\partial_t^2 + c^2\,\nabla^2\right)\Psi = 0 \quad \Rightarrow \quad \Psi = \psi\, e^{-i\omega t}$, $\omega = c\,k$

$$\left(\nabla^2 + k^2\right)\psi_{\boldsymbol{k}}(\mathbf{r}) = 0 \quad \Rightarrow \quad \psi_{\boldsymbol{k}}(\mathbf{r}) = \frac{e^{i\,\boldsymbol{k}\cdot\mathbf{r}}}{(2\pi)^{3/2}} \quad \Rightarrow \quad \int \psi_{\boldsymbol{k}'}^*(\mathbf{r})\,\psi_{\boldsymbol{k}}(\mathbf{r})\,\mathrm{d}^3 x = \delta(\boldsymbol{k} - \boldsymbol{k}')$$

$$\Rightarrow \quad G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{2\pi^2} \int \frac{e^{i\,\boldsymbol{k}\cdot(\mathbf{r} - \mathbf{r}')}}{k^2}\,\mathrm{d}^3 k \quad \Rightarrow \text{3d Fourier integration of } \frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

- 2nd ex: a box with $x = y = z = 0$, $x = a$, $y = b$, $z = c$: For a Dirichlet problem,

$$\left(\nabla^2 + k_{\ell m n}^2\right)\psi_{\ell m n}(x, y, z) = 0$$

$$\Rightarrow \quad \begin{aligned} \psi_{\ell m n}(x, y, z) &= \sqrt{\frac{8}{a\,b\,c}}\,\sin\frac{\ell\pi x}{a}\,\sin\frac{m\pi y}{b}\,\sin\frac{n\pi z}{c} \\ k_{\ell m n}^2 &= \pi^2\left(\frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) \end{aligned}$$

$$\Rightarrow \quad G(\mathbf{r},\mathbf{r}') = \frac{32\pi}{abc} \sum_{\ell,m,n=1}^{\infty} \frac{\sin\frac{\ell\pi x}{a}\sin\frac{\ell\pi x'}{a}\sin\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}\sin\frac{n\pi z}{c}\sin\frac{n\pi z'}{c}}{k_{\ell m n}^2}$$

$$= \frac{16\pi}{ab} \sum_{\ell,m=1}^{\infty} \sin\frac{\ell\pi x}{a}\sin\frac{\ell\pi x'}{a}\sin\frac{m\pi y}{b}\sin\frac{m\pi y'}{b} \frac{\sinh(K_{\ell m} z_<)\sinh[K_{\ell m}(c-z_>)]}{K_{\ell m}\sinh(K_{\ell m} c)}$$

(the z coordinate is singled out for special treatment like in Sec 3.9 & 3.11)

$$\Rightarrow \quad \frac{\sinh(K_{\ell m} z_<)\sinh[K_{\ell m}(c-z_>)]}{K_{\ell m}\sinh(K_{\ell m} c)} = \frac{2}{c}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi z'}{c}\sin\frac{n\pi z}{c}}{k_{\ell m n}^2} \quad \Leftarrow \quad \frac{K_{\ell m}^2}{\pi^2} = \frac{\ell^2}{a^2}+\frac{m^2}{b^2}$$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z-z')^2}} \quad \Leftarrow \quad \begin{array}{l}\mathbf{r} = (\rho,\ \phi,\ z)\\ \mathbf{r}' = (\rho',\ \phi',\ z')\end{array}$$

$$= \sum_{m=-\infty}^{\infty}\int_0^{\infty} J_m(k\rho)\,J_m(k\rho')\,e^{im(\phi-\phi')}\,e^{-k(z_>-z_<)}\,\mathrm{d}k \quad \Leftarrow \quad \text{Problem 3.16b}$$

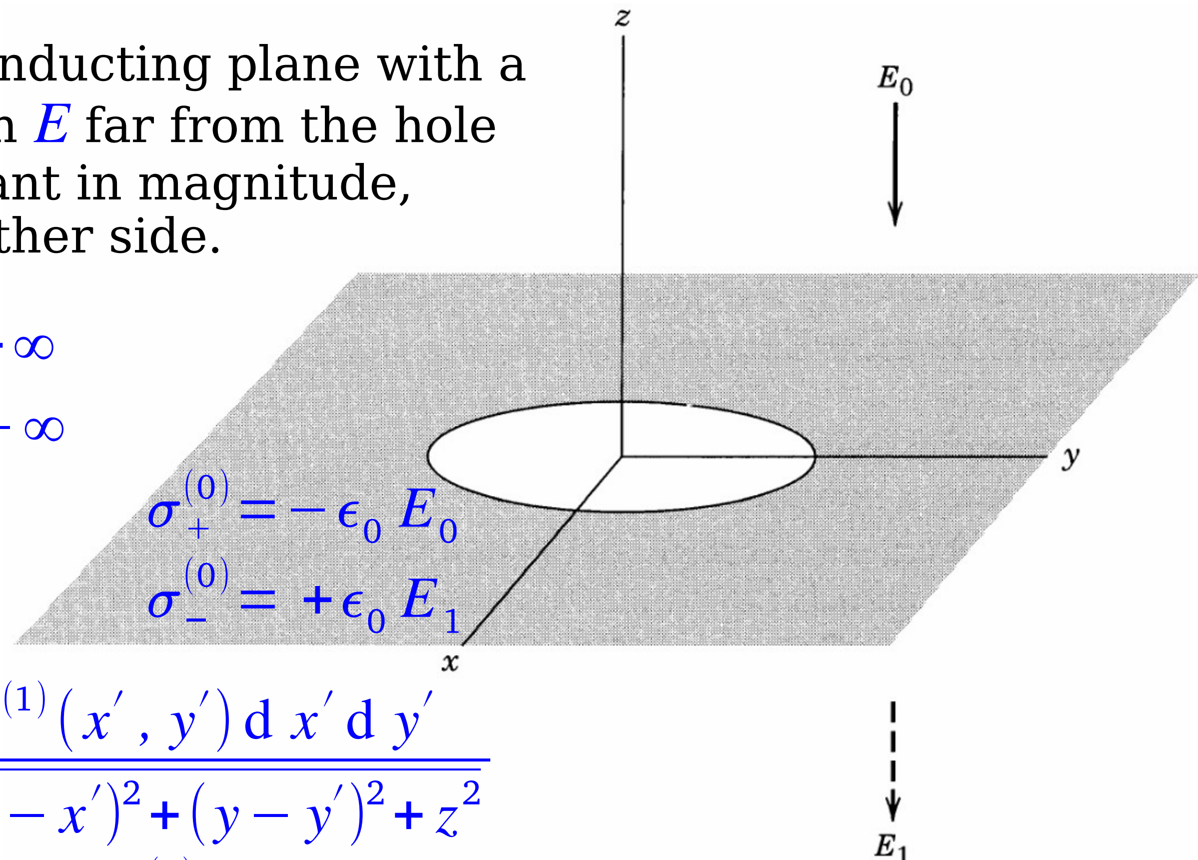
$$\Rightarrow \quad \frac{1}{\sqrt{\rho^2+z^2}} = \sum_{m=-\infty}^{\infty}\int_0^{\infty} J_m(k\rho)\,J_m(0)\,e^{im\phi}\,e^{-k|z|}\,\mathrm{d}k \quad \Leftarrow \quad \text{set } \mathbf{r}'=0$$

$$= \int_0^{\infty} J_0(k\rho)\,e^{-k|z|}\,\mathrm{d}k \quad \Leftarrow \quad J_m(0) = \delta_{m0}$$

Mixed Boundary Conditions; Conducting Plane with a Circular Hole

- Mixed boundary conditions are more difficult to handle than the normal one.

- An infinitely thin, grounded, conducting plane with a circular hole of radius a, and with E far from the hole being normal to the plane, constant in magnitude, and having different values on either side.



$$\sigma_+^{(0)} = -\epsilon_0 E_0 \qquad \sigma_-^{(0)} = +\epsilon_0 E_1$$

$$\mathbf{E} = \begin{cases} \mathbf{E}_0 = -E_0 \hat{\mathbf{z}} & \text{for } z \to +\infty \\ \mathbf{E}_1 = -E_1 \hat{\mathbf{z}} & \text{for } z \to -\infty \end{cases}$$

$$\Rightarrow \quad \Phi = \begin{cases} E_0 z + \Phi^{(1)} & \text{for } z > 0 \\ E_1 z + \Phi^{(1)} & \text{for } z < 0 \end{cases}$$

$$\Rightarrow \quad \Phi^{(1)}(x, y, z) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma^{(1)}(x', y')\, d x'\, d y'}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}}$$

$$\Rightarrow \quad \begin{matrix} \Phi^{(1)}(z) = \Phi^{(1)}(-z) \\ \mathbf{E}^{(1)} = -\nabla \Phi^{(1)} \end{matrix} \quad \Rightarrow \quad \begin{matrix} E_x^{(1)}(z) = E_x^{(1)}(-z) \\ E_y^{(1)}(z) = E_y^{(1)}(-z) \end{matrix}, \quad E_z^{(1)}(z) = -E_z^{(1)}(-z)$$

- The *total z-component* of E field must be continuous across $z=0$ in the hole

$$-E_0 + E_z^{(1)}\big|_{z=0^+} = -E_1 + E_z^{(1)}\big|_{z=0^-} \text{ for } \rho < a \quad \Rightarrow \quad E_z^{(1)}\big|_{z=0^+} = -E_z^{(1)}\big|_{z=0^-} = \frac{E_0 - E_1}{2}$$

- The potential is 0 on the surface by hypothesis: $\Phi(a \le \rho < \infty, z=0) = 0 \quad \Rightarrow \quad \Phi^{(1)}(a \le \rho < \infty, z=0) = 0$

- An electrostatic boundary-value problem with the *mixed boundary conditions*:

$$\left.\frac{\partial \Phi^{(1)}}{\partial z}\right|_{z=0^+} = \frac{E_1 - E_0}{2} \quad \text{for } 0 \le \rho < a \qquad (\&) \Rightarrow$$

$$\Phi^{(1)}_{|z=0} = 0 \quad \text{for } a \le \rho < \infty$$

Use ($) and azimuthal symmetry

$$\Phi^{(1)}(\rho, z) = \int_0^\infty A(k)\, J_0(k\rho)\, e^{-k|z|}\, \mathrm{d}k$$

- For large ρ or $|z|$ the rapid oscillations of $J_0(k\rho)$ or the decrease of $e^{-k|z|}$ imply that the integral receives its important contributions from the region around $k=0$

$$A(k) = \sum_{\ell=0}^\infty \frac{k^\ell}{\ell!} \frac{\mathrm{d}^\ell A}{\mathrm{d}k^\ell}(0) \quad \Rightarrow \quad \Phi^{(1)} = \sum_{\ell=0}^\infty \frac{\mathrm{d}^\ell A}{\mathrm{d}k^\ell}(0)\, B_\ell$$

$$\Rightarrow \quad B_\ell = \frac{1}{\ell!} \int_0^\infty k^\ell J_0(k\rho)\, e^{-k|z|}\, \mathrm{d}k = \frac{(-1)^\ell}{\ell!} \frac{\mathrm{d}^\ell}{\mathrm{d}|z|^\ell} \int_0^\infty J_0(k\rho)\, e^{-k|z|}\, \mathrm{d}k$$

$$= \frac{(-1)^\ell}{\ell!} \frac{\mathrm{d}^\ell}{\mathrm{d}|z|^\ell} \frac{1}{\sqrt{\rho^2 + z^2}} = \frac{P_\ell(|\cos\theta|)}{r^{\ell+1}} \quad \Leftarrow$$

Problem 3.16c

$r^2 = \rho^2 + z^2, \quad z = r\cos\theta$

$$\Rightarrow \quad \Phi^{(1)} = \sum_{\ell=0}^\infty \frac{P_\ell(|\cos\theta|)}{r^{\ell+1}} \frac{\mathrm{d}^\ell A}{\mathrm{d}k^\ell}(0) \quad \Leftarrow$$

multipole expansion

$A(0)$ total charge

$\dfrac{\mathrm{d}A}{\mathrm{d}k}(0)$ dipole moment

$\cdots$

- For the mixed boundary value problem

$$(\&) \;\Rightarrow\; \begin{aligned} &\int_0^\infty k\, A(k)\, J_0(k\rho)\, \mathrm{d}k = \frac{E_0 - E_1}{2} \quad \text{for } 0 \le \rho < a \\ &\int_0^\infty A(k)\, J_0(k\rho)\, \mathrm{d}k = 0 \quad \text{for } a \le \rho < \infty \end{aligned} \;\Leftarrow\; \text{dual integral equations}$$

- Consider the dual integral equations

$$\begin{aligned} &\int_0^\infty y\, g(y)\, J_n(y x)\, \mathrm{d}y = x^n \quad \text{for } 0 \le x < 1 \\ &\int_0^\infty g(y)\, J_n(y x)\, \mathrm{d}y = 0 \quad \text{for } 1 \le x < \infty \end{aligned} \;\Rightarrow\; \begin{aligned} g(y) &= \frac{\Gamma(n+1)\, j_{n+1}(y)}{\Gamma(n+3/2)\sqrt{\pi}} \\ &= \frac{\Gamma(n+1)\, J_{n+3/2}(y)}{\Gamma(n+3/2)\sqrt{2y}} \end{aligned}$$

$$\begin{bmatrix} n = 0 \\ x = \dfrac{\rho}{a} \\ y = k a \end{bmatrix} \Rightarrow \begin{aligned} A(k) &= \frac{a^2}{\pi}(E_0 - E_1)\, j_1(ka) = \frac{E_0 - E_1}{\pi}\left(\frac{\sin ka}{k^2} - \frac{a \cos ka}{k}\right) \\ &= \frac{a^2 (E_0 - E_1)}{3\pi}\left(ka - \frac{(ka)^3}{10} + \cdots\right) \Rightarrow \begin{aligned} &A(0) = 0 \\ &\frac{\mathrm{d}A}{\mathrm{d}k}(0) \neq 0 \end{aligned} \end{aligned}$$

- The total charge with $\Phi^{(1)}$ is 0 and the leading term is the $\ell = 1$ contribution:

$$\Phi^{(1)} \to \frac{a^3 (E_0 - E_1)}{3\pi} \frac{|z|}{r^3} = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} \;\Rightarrow\; \mathbf{p} = \mp \frac{4}{3} \epsilon_0 a^3 (\mathbf{E}_0 - \mathbf{E}_1) \quad \text{for } z \gtrless 0$$

effective electric dipole moment

- The reversal of the effective dipole moment depending on whether the observer is above or below the plane is because that a true dipole potential is odd in z, whereas it is even here.

- The idea that a small hole in a plane conducting sheet is equivalent far from the opening to a dipole normal to the surface is important in discussing the consequences of such openings in the walls of waveguides and cavities.

- The added potential in the neighborhood of the opening

$$\Phi^{(1)} = a^2 \frac{E_0 - E_1}{\pi} \int_0^\infty j_1(k\,a)\, J_0(k\,\rho)\, e^{-k|z|}\, \mathrm{d}\,k$$

different form from Jackson's , but the same result

$$= \frac{E_0 - E_1}{\pi} \left(\sqrt{a^2 - R_-^2} - |z| \sin^{-1} \frac{a}{R_+} \right) \quad \Leftarrow \quad R_\pm = \frac{\sqrt{(\rho + a)^2 + z^2} \pm \sqrt{(a - \rho)^2 + z^2}}{2}$$

$$\Rightarrow \quad \Phi^{(1)}(\rho = 0\,,\, z) = \frac{E_0 - E_1}{\pi} \left(a - |z| \sin^{-1} \frac{a}{\sqrt{a^2 + z^2}} \right)$$

$$\Phi^{(1)}(0\,,\,|z| \gg a) \to \frac{E_0 - E_1}{\pi} \frac{a^3}{3\,z^2}$$

$$\Rightarrow \quad \Phi^{(1)}(0\,,\,|z| \to 0) \to \frac{E_0 - E_1}{\pi}\, a$$

$$\Phi^{(1)}(0 \le \rho < a\,,\, 0) = \frac{E_0 - E_1}{\pi} \sqrt{a^2 - \rho^2}$$

$$\int_0^\infty j_1(k\,a)\,J_0(k\,\rho)\,e^{-k|z|}\,\mathrm{d}\,k = -\frac{1}{a}\int_0^\infty J_0\,e^{-k|z|}\,\mathrm{d}\,j_0 \quad \Leftarrow \quad \frac{\mathrm{d}\,j_0}{\mathrm{d}\,x} = -\,j_1\,,\quad j_0(x) = \frac{\sin x}{x}$$

$$= \frac{1}{a} - \frac{|z|}{a}\int_0^\infty j_0\,J_0\,e^{-k|z|}\,\mathrm{d}\,k - \frac{\rho}{a}\int_0^\infty j_0\,J_1\,e^{-k|z|}\,\mathrm{d}\,k \quad \Leftarrow \quad \frac{\mathrm{d}\,J_0}{\mathrm{d}\,x} = -\,J_1$$

$$\int_0^\infty J_0(k\,\rho)\,e^{-k|z|}\,\frac{\sin k\,a}{k}\,\mathrm{d}\,k = \quad \sin^{-1}\frac{a}{R_+}$$

$$\Leftarrow \quad R_\pm = \frac{\sqrt{(\rho+a)^2+z^2} \pm \sqrt{(a-\rho)^2+z^2}}{2}$$

$$\int_0^\infty J_1(k\,\rho)\,e^{-k|z|}\,\frac{\sin k\,a}{k}\,\mathrm{d}\,k = \frac{a-\sqrt{a^2-R_-^2}}{\rho}$$

6.752 of Table of Integrals, Series, and Products, Gradshteyn & Ryzhik (2007)

$$\Rightarrow \quad \int_0^\infty j_1(k\,a)\,J_0(k\,\rho)\,e^{-k|z|}\,\mathrm{d}\,k = \frac{1}{a} - \frac{|z|}{a^2}\sin^{-1}\frac{a}{R_+} - \frac{a-\sqrt{a^2-R_-^2}}{a^2}$$

$$= \frac{1}{a^2}\left(\sqrt{a^2-R_-^2} - |z|\sin^{-1}\frac{a}{R_+}\right)$$

$$\Rightarrow \quad \Phi^{(1)} = a^2\,\frac{E_0-E_1}{\pi}\int_0^\infty j_1(k\,a)\,J_0(k\,\rho)\,e^{-k|z|}\,\mathrm{d}\,k$$

$$= \frac{E_0-E_1}{\pi}\left(\sqrt{a^2-R_-^2} - |z|\sin^{-1}\frac{a}{R_+}\right)$$

- The tangential (a radial field) and the normal electric field in the opening

$$\mathbf{E}_{\parallel}(0 \le \rho < a, 0) = \frac{E_0 - E_1}{\pi} \frac{\boldsymbol{\rho}}{\sqrt{a^2 - \rho^2}}$$

$$E_z(0 \le \rho < a, 0) = -\frac{E_0 + E_1}{2}$$

- The magnitude of $\mathbf{E}$ has a square root singularity at the edge of the opening.

Selected problems: 3, 7, 12, 19, 22, 25