Chapter 2 Boundary-Value Problems in Electrostatics I

- The correct Green function is not necessarily easy to be found.

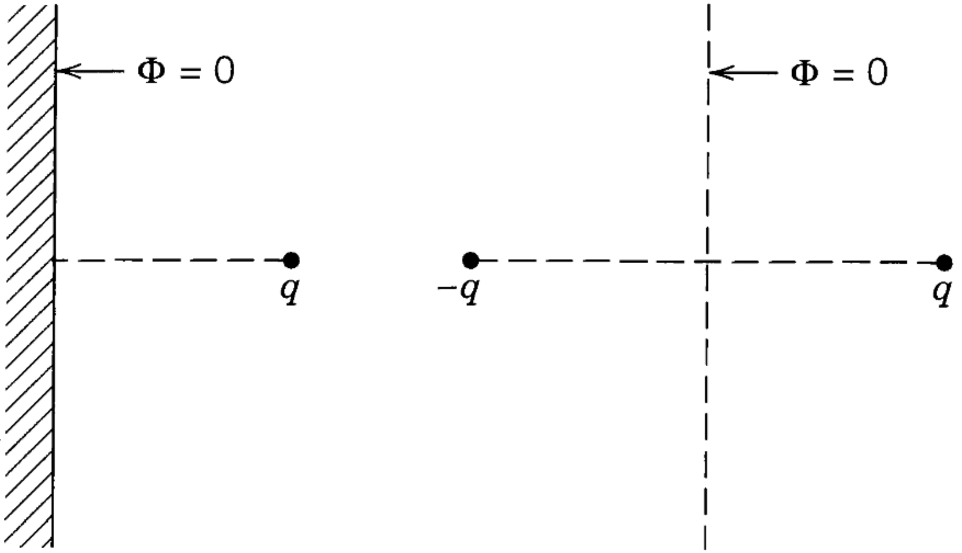
- 3 techniques to electrostatic boundary value problems:

(1) the method of images

(2) expansion in orthogonal functions

~~(3) finite element analysis (numerical method)~~

Method of Images

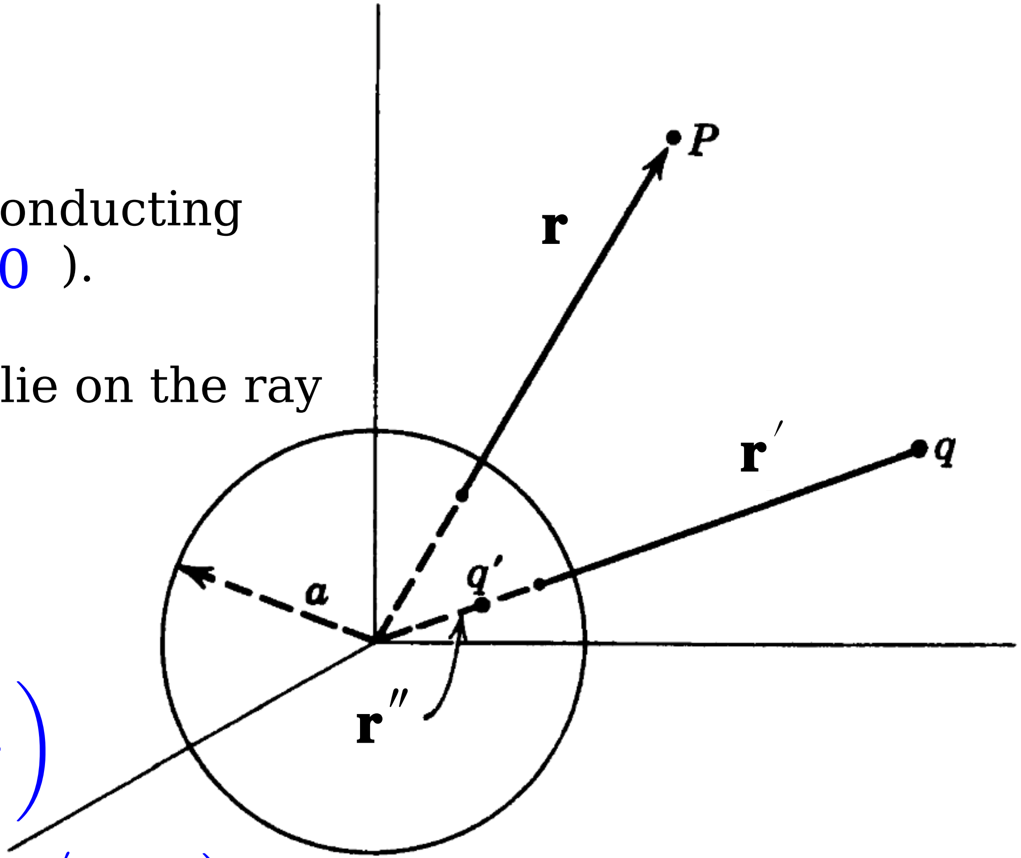


- The method deals with the problem of point charges in the presence of boundary surfaces, eg, conductors either grounded or held at fixed potentials.

- Infer from the geometry of the situation that some suitably placed charges, *external to the region of interest*, can simulate the boundary conditions. The charges are called *image charges*.

- The method replaces the actual problem with boundaries by an enlarged region with image charges but not boundaries.

Point Charge in the Presence of a Grounded Conducting Sphere



- A point charge is outside a grounded conducting sphere. Find the potential ($\Phi(|\mathbf{r}|=a)=0$).

- By symmetry the image charge q' will lie on the ray from the origin to the charge q, then

$$\Phi(\mathbf{r})=\frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\mathbf{r}-\mathbf{r}'|}+\frac{q'}{|\mathbf{r}-\mathbf{r}''|}\right)$$

$$=\frac{1}{4\pi\epsilon_0}\left(\frac{q}{|r\,\hat{\mathbf{r}}-r'\,\hat{\mathbf{r}}'|}+\frac{q'}{|r\,\hat{\mathbf{r}}-r''\,\hat{\mathbf{r}}'|}\right)$$

$$\Rightarrow\quad \Phi(r=a)=\frac{1}{4\pi\epsilon_0}\left(\frac{q}{|a\,\hat{\mathbf{r}}-r'\,\hat{\mathbf{r}}'|}+\frac{q'}{|a\,\hat{\mathbf{r}}-r''\,\hat{\mathbf{r}}'|}\right)=0 \quad\Rightarrow\quad \begin{array}{l}\Phi(\mathbf{r}=+a\,\hat{\mathbf{r}}')=0\\ \Phi(\mathbf{r}=-a\,\hat{\mathbf{r}}')=0\end{array}$$

$$\Rightarrow\quad \frac{q}{r'-a}=-\frac{q'}{a-r''},\quad \frac{q}{r'+a}=\frac{q'}{a+r''}\quad\Rightarrow\quad r''=\frac{a^2}{r'},\quad q'=-\frac{a}{r'}q$$

- As q is closer to the sphere, the q' grows and moves out from the center of the sphere.

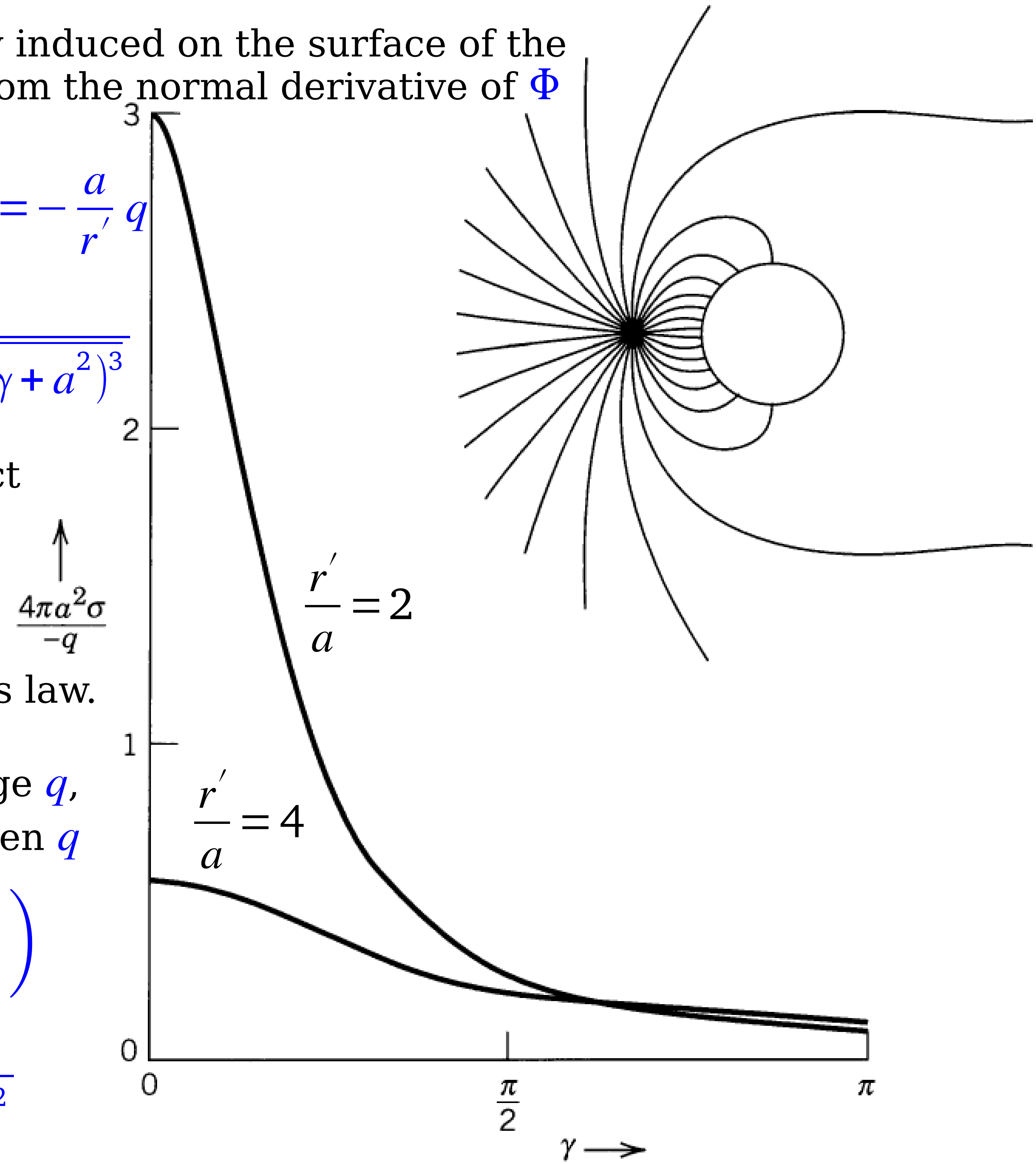
- When q is just outside the surface of the sphere, q' is equal and opposite in and lies just beneath the surface.

- The actual charge density induced on the surface of the sphere can be calculated from the normal derivative of Φ at the surface:

$$\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial r}\Big|_{r=a} \quad \Rightarrow \int \sigma \, d a = -\frac{a}{r'} q$$

$$= \frac{q}{4 \pi a} \frac{a^2 - r'^2}{\sqrt{(r'^2 - 2 a r' \cos\gamma + a^2)^3}}$$

- It is easy to show by direct integration that the total induced charge on the sphere is equal to the magnitude of the image charge according to Gauss's law.



- For the force on the charge q, write down the force between q and q'

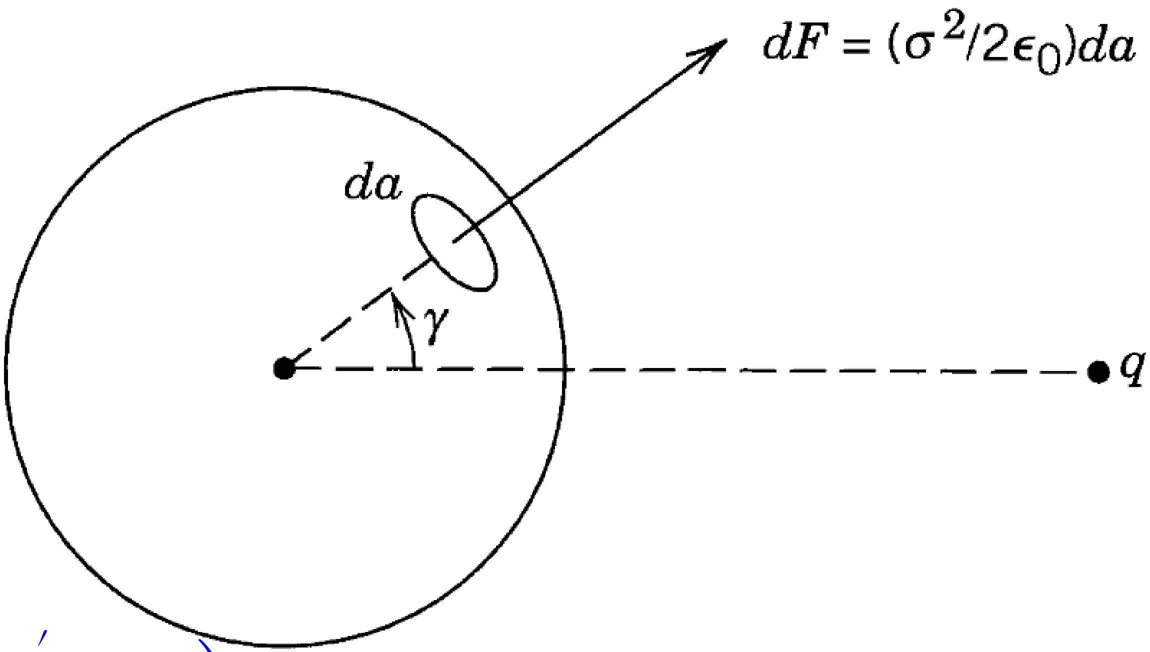
$$r' - r'' = r' \left(1 - \frac{a^2}{r'^2}\right)$$

$$\Rightarrow \quad |\mathbf{F}| = \frac{q^2}{4 \pi \epsilon_0} \frac{a r'}{(r'^2 - a^2)^2}$$

- The alternative method for the force is to calculate the total force acting on the surface of the sphere

$$|\mathbf{F}| = \int \mathrm{d}\, F_x = \int \frac{\sigma^2}{2\,\epsilon_0} \cos\gamma \,\mathrm{d}\, a = \frac{q^2 \left(r'^2 - a^2\right)^2}{32\,\pi^2\,\epsilon_0} \int \frac{\cos\gamma}{\left(r'^2 + a^2 - 2\,a\,r' \cos\gamma\right)^3} \,\mathrm{d}\,\Omega$$

$$= \frac{q^2}{4\,\pi\,\epsilon_0} \frac{a\,r'}{\left(r'^2 - a^2\right)^2}$$



- The whole discussion has been based on the understanding that q is outside the sphere. Actually, the results apply equally for q inside the sphere.

- $$\Phi(\mathbf{r}) = \frac{1}{4\,\pi\,\epsilon_0} \left(\frac{q}{|\mathbf{r} - \mathbf{r}'|} - \frac{a\,r'\,q}{\left|r'^2\,\mathbf{r} - a^2\,\mathbf{r}'\right|} \right)$$

Point Charge in the Presence of a Charged, Insulated, Conducting Sphere

- Consider an insulated conducting sphere with total charge Q in the presence of a point charge q.

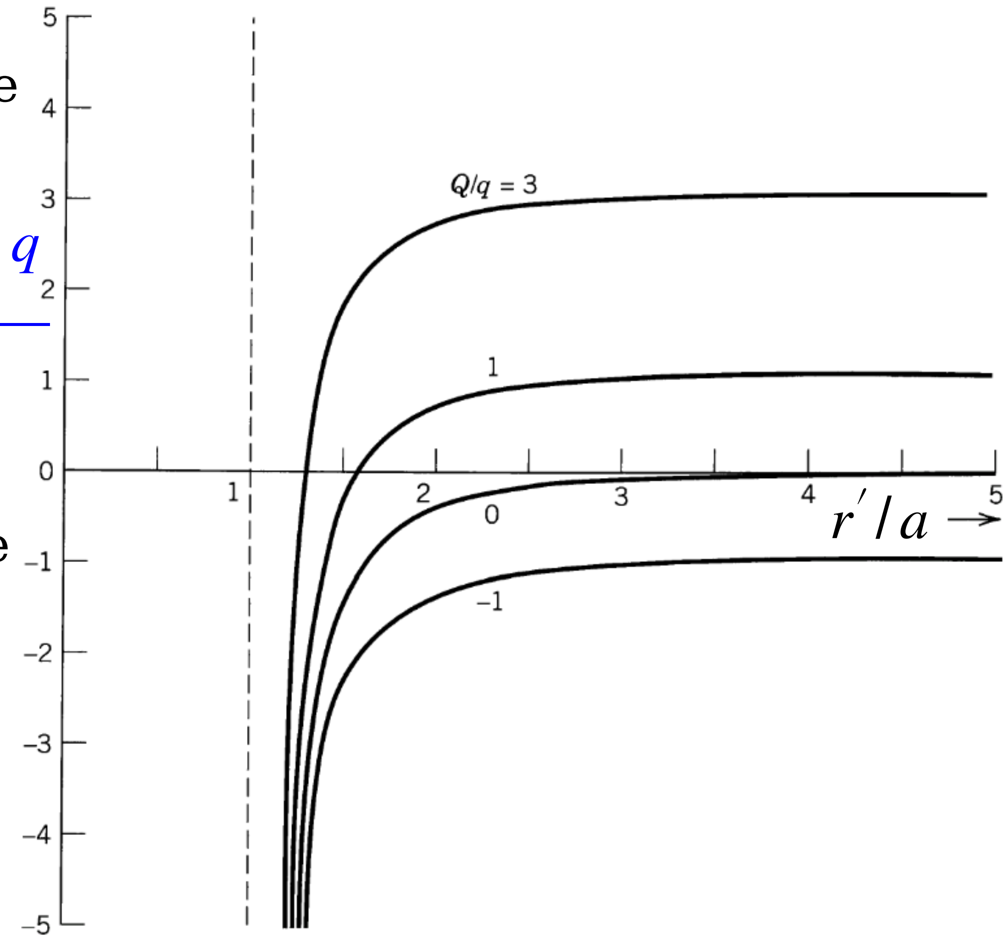
- Start with the grounded conducting sphere (with its charge q' distributed over its surface). Then disconnect the ground wire and add to the sphere an amount of charge $(Q - q')$. This brings the total charge on the sphere up to Q.

- The added charge $(Q-q')$ will distribute itself uniformly over the surface. Then

$$4\pi\epsilon_0 \Phi(\mathbf{r}) = \frac{q}{|\mathbf{r}-\mathbf{r}'|} - \frac{\frac{a}{r'}q}{\left|\mathbf{r} - \frac{a^2}{r'^2}\mathbf{r}'\right|} + \frac{Q + \frac{a}{r'}q}{r}$$

- The force acting on the charge q can be written down from Coulomb's law

$$\mathbf{F} = \frac{q}{4\pi\epsilon_0 r'^2}\left(Q - \frac{q a^3 (2r'^2 - a^2)}{r'(r'^2 - a^2)^2}\right)\hat{\mathbf{r}}'$$



- In the limit of $r' \gg a$, the force reduces to the usual Coulomb's law for 2 small charged bodies. But close to the sphere the force is modified because of the induced charge distribution on the surface of the sphere.

- If the sphere is charged oppositely to q, or is uncharged, the force is attractive at all distances.

- Even if the charge Q is the same sign as q, the force becomes attractive at very close distances.

- In the limit of $Q \gg q$, the point of zero force (unstable equilibrium point) is very close to the sphere, at $r' \simeq a\left(1+\frac{1}{2}\sqrt{\frac{q}{Q}}\right)$

$$\mathbf{F}=0 \quad \Rightarrow \quad Q-\frac{q\,a^3\left(2\,r'^2-a^2\right)}{r'\left(r'^2-a^2\right)^2}=0 \quad \Rightarrow \quad r'\left(r'^2-a^2\right)^2=\frac{q}{Q}\,a^3\left(2\,r'^2-a^2\right)$$

$$\text{Using }\begin{array}{c}\text{perturbation}\\ \text{method}\end{array} \quad \Rightarrow \quad \frac{q}{Q}=0 \text{ get } r'^{(0)}=a \quad \Rightarrow \quad r'^{(1)}=r'^{(0)}+\delta \text{ get } \delta=\frac{a}{2}\sqrt{\frac{q}{Q}}$$

- This example explains why an excess of charge on the surface does not leave the surface because of mutual repulsion of the individual charges.

- As soon as an element of charge is removed from the surface, the image force tends to attract it back.

Point Charge Near a Conducting Sphere at Fixed Potential

- The potential is the same as for the charged sphere, except that the charge $(Q - q')$ at the center is replaced by a charge $Q' = 4\pi\epsilon_0 a V$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\mathbf{r}-\mathbf{r}'|} - \frac{a r' q}{\left|r'^2 \mathbf{r} - a^2 \mathbf{r}'\right|}\right) + \frac{a}{r} V$$

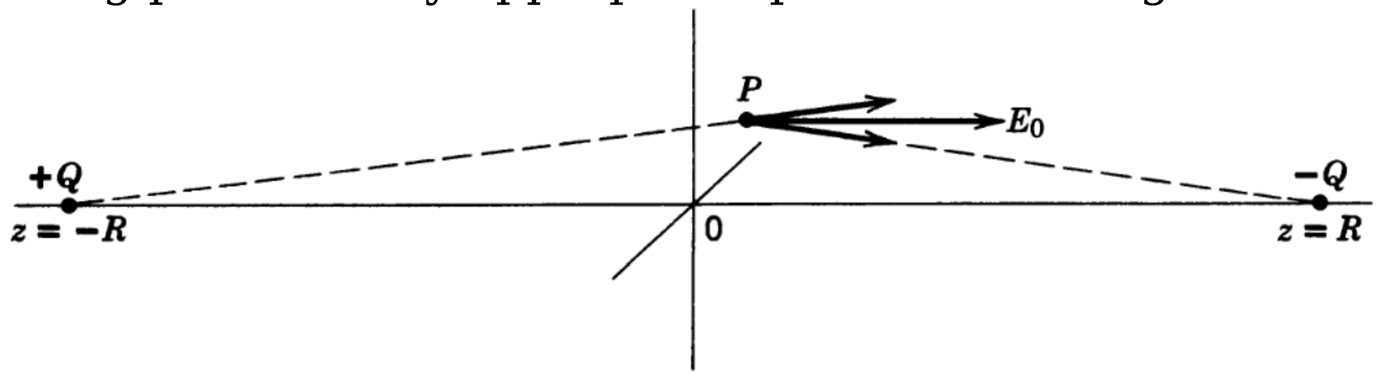
$$\Rightarrow \quad \mathbf{F} = \frac{q}{r'^2}\left(a V - \frac{1}{4\pi\epsilon_0}\frac{q a r'^3}{\left(r'^2 - a^2\right)^2}\right)\hat{\mathbf{r}}'$$

- For $4\pi\epsilon_0 a V \gg q$, the unstable equilibrium point: $r' \simeq a\left(1 + \frac{1}{2}\sqrt{\frac{q}{4\pi\epsilon_0 a V}}\right)$

Conducting Sphere in a Uniform Electric Field by Method of Images

- Consider a conducting sphere of radius a in a uniform electric field. A uniform field can be thought of as being produced by appropriate positive and negative charges at infinity.

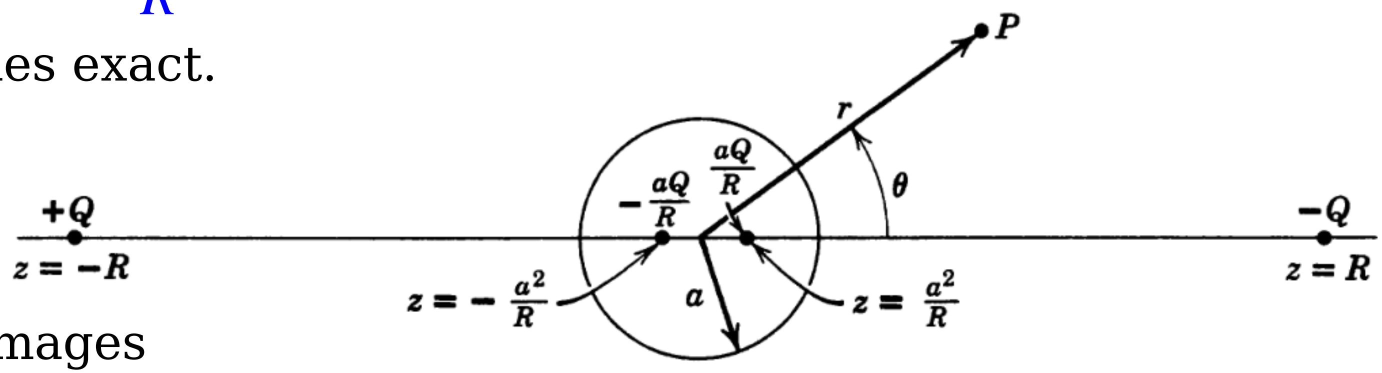
- The electric field near the origin: $E_0 \simeq \dfrac{1}{2\pi\epsilon_0}\dfrac{Q}{R^2}$



(a)

- In the limit as $R, Q \to \infty$ with $\dfrac{Q}{R^2}$ constant, this approximation becomes exact.

- A conducting sphere is placed at the origin, the potential will be that due to the charges and their images



(b)

$$4\pi\epsilon_0\Phi = \frac{Q}{\sqrt{r^2+R^2+2rR\cos\theta}} - \frac{aQ}{\sqrt{r^2R^2+a^4+2a^2rR\cos\theta}} - \frac{Q}{\sqrt{r^2+R^2-2rR\cos\theta}} + \frac{aQ}{\sqrt{r^2R^2+a^4-2a^2rR\cos\theta}}$$

- For $R \gg r$: $\Phi = -\frac{2Q}{4\pi\epsilon_0 R^2}\left(r - \frac{a^3}{r^2}\right)\cos\theta + \cdots$

- To the limit $E_0 \equiv \frac{Q}{2\pi\epsilon_0 R^2}$

becomes the applied uniform field: $\Phi = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta$

- The 1st term $-E_0 z$ is the potential of a uniform field. The 2nd is the potential due to the induced surface-charge density or, equivalently, the image charges.

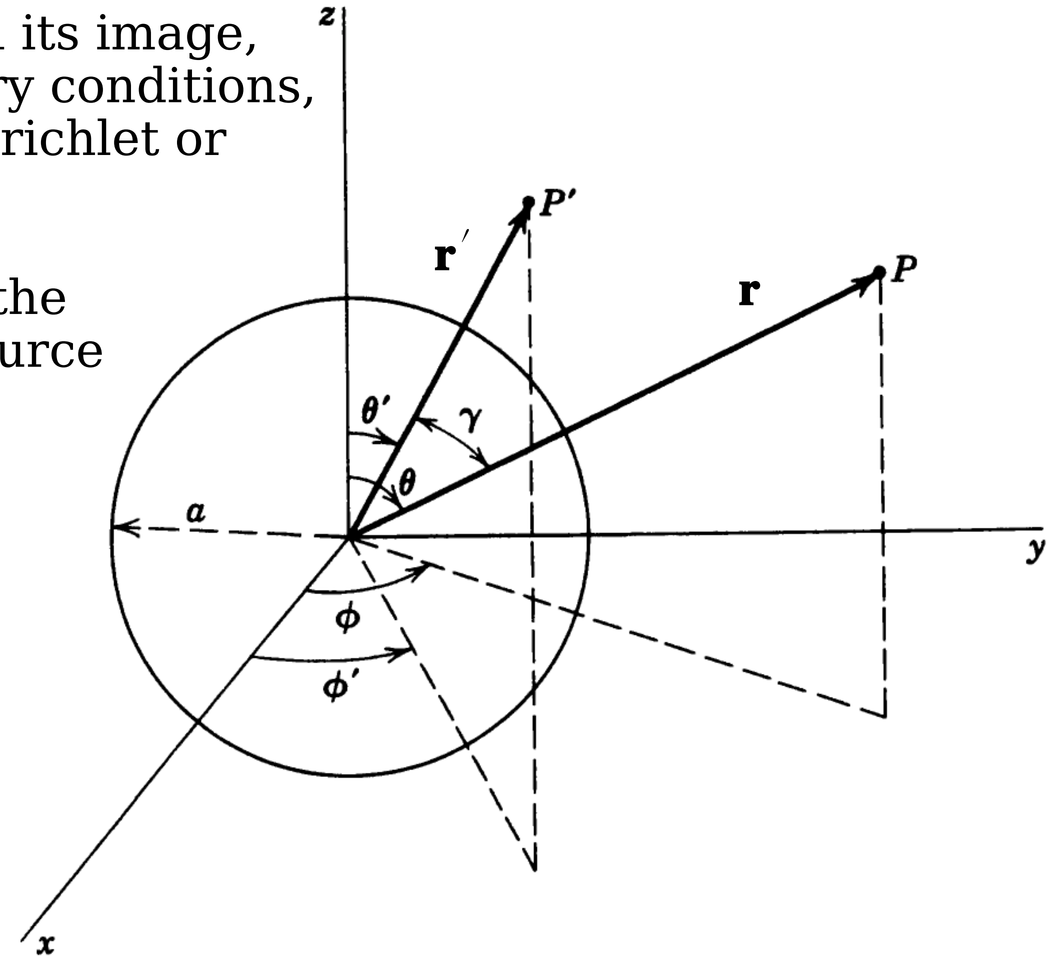
- The image charges form a dipole of strength $D = \frac{a}{R} Q \times 2\frac{a^2}{R} = 4\pi\epsilon_0 E_0 a^3$

- The induced surface-charge density $\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial r}\Big|_{r=a} = 3\epsilon_0 E_0 \cos\theta$

- The surface integral of this charge density vanishes, (but the surface charge density doesn't vanish,) so that there is no difference between a grounded and an insulated sphere.

Green Function for the Sphere; General Solution for the Potential

- The potential due to a unit source and its image, chosen to satisfy homogeneous boundary conditions, is the Green function appropriate for Dirichlet or Neumann boundary conditions.



- For Dirichlet boundary conditions on the sphere, the Green function for a unit source and its image is

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{a\,r'}{|r'^2\mathbf{r} - a^2\mathbf{r}'|}$$

$$= \frac{1}{\sqrt{r^2 + r'^2 - 2\,r\,r'\cos\gamma}} - \frac{a}{\sqrt{r^2 r'^2 + a^4 - 2\,r\,r'\,a^2\cos\gamma}}$$

- The symmetry in the variables is obvious, as is the condition that $G = 0$ if either $\mathbf{r}$ or $\mathbf{r}'$ is on the surface of the sphere.

- For the boundary problem in Sec. 2.2, we now can derive the solution as

$$4\pi\,\Phi(\mathbf{r}) = \int_{\mathcal{V}} \frac{q}{\epsilon_0}\,\delta(\mathbf{r}'-\mathbf{y})\,G(\mathbf{r},\mathbf{y})\,\mathrm{d}^3 y + \oint_{\mathcal{S}} \left(G\frac{\partial \Phi}{\partial n'} - \Phi\frac{\partial G}{\partial n'} \right) \mathrm{d}a' \quad \Leftarrow \quad \Phi(a) = 0$$

- For solutions of a Poisson equation we need not only G, but also $\frac{\partial G}{\partial n'}$

$$\left.\frac{\partial G}{\partial n'}\right|_{r'=a} = \left(\hat{\mathbf{n}}'\cdot\nabla' G\right)_{r'=a} = \left[\left(-\hat{\mathbf{r}}'\right)\cdot\nabla' G\right]_{r'=a} = \frac{a^2-r^2}{a\sqrt{\left(r^2+a^2-2\,a\,r\cos\gamma\right)^3}}$$

here r' means the source space, not only the coordinate of single charge.

- The condition is essentially the induced surface-charge density.

- The solution of the Laplace equation *outside* a sphere with the potential specified on its surface is

$$\nabla^2\Phi = 0 \quad\Rightarrow\quad \Phi(\mathbf{r}) = \frac{1}{4\pi}\int \Phi(a,\theta',\phi')\frac{a\left(r^2-a^2\right)}{\sqrt{\left(r^2+a^2-2\,a\,r\cos\gamma\right)^3}}\,\mathrm{d}\Omega'$$

$$\text{where}\quad \cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\left(\phi-\phi'\right)$$

- For the *interior* problem, $\left.\frac{\partial G}{\partial n'}\right|_{r'=a} = \left(\hat{\mathbf{r}}'\cdot\nabla' G\right)_{r'=a} = \frac{r^2-a^2}{a\sqrt{\left(r^2+a^2-2\,a\,r\cos\gamma\right)^3}}$

$$\Rightarrow\quad \Phi(\mathbf{r}) = \frac{1}{4\pi}\int \Phi(a,\theta',\phi')\frac{a\left(a^2-r^2\right)}{\sqrt{\left(r^2+a^2-2\,a\,r\cos\gamma\right)^3}}\,\mathrm{d}\Omega'$$

- For a problem with a charge distribution, we must add to the potential integral the appropriate charge density with the Green function.

Conducting Sphere with Hemispheres at Different Potentials

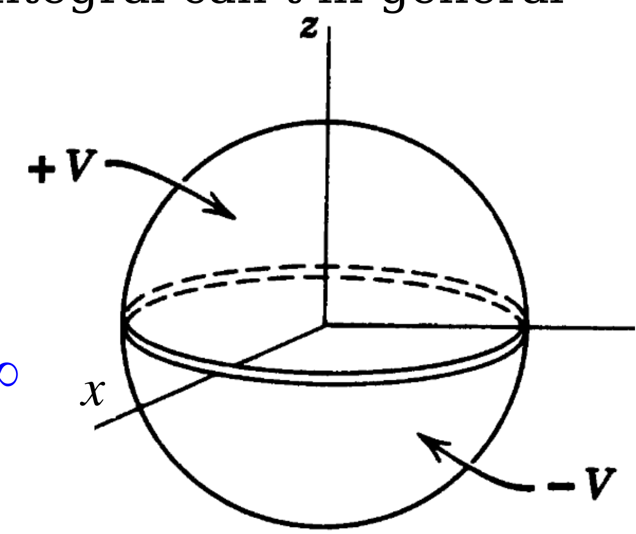
- $\Phi(r,\theta,\phi)$ the solution for the potential $+\quad \xi^2 = r^2 + a^2 \quad ⇙ \quad \begin{matrix}\theta' \to \pi - \theta' \\ \phi' \to \phi' + \pi\end{matrix}$

$$= \frac{V}{4\pi}\int_0^{2\pi} d\phi' \left(\int_0^1 \frac{a(r^2-a^2)\, d\cos\theta'}{\sqrt{(r^2+a^2-2ar\cos\gamma)^3}} - \int_{-1}^0 \frac{a(r^2-a^2)\, d\cos\theta'}{\sqrt{(r^2+a^2-2ar\cos\gamma)^3}} \right)$$

$$= \frac{Va(r^2-a^2)}{4\pi}\int_0^{2\pi} d\phi' \int_0^1 \left(\frac{1}{\sqrt{(\xi^2-2ar\cos\gamma)^3}} - \frac{1}{\sqrt{(\xi^2+2ar\cos\gamma)^3}} \right) d\cos\theta'$$

- Because of the complicated dependence of angle, the integral can't in general be integrated in closed form.

- Consider the potential on the $+z$ axis



$$\cos\gamma = \cos\theta' \quad \Leftarrow \quad \theta = 0$$

$$\Rightarrow \quad \Phi(z) = V\left(1 - \frac{z^2-a^2}{z\sqrt{z^2+a^2}}\right) \quad \Rightarrow \quad \begin{matrix}\Phi \simeq \dfrac{3Va^2}{2z^2}, & z\to\infty \\ \Phi = V, & z = a\end{matrix}$$

- In the absence of a closed expression for the integral, we can expand the denominator in power series and integrate term by term, defining $\alpha = \dfrac{ar}{r^2+a^2}$

$$\Phi = \frac{Va(r^2-a^2)}{4\pi\sqrt{(r^2+a^2)^3}}\int_0^{2\pi} d\phi' \int_0^1 \left(\frac{1}{\sqrt{(1-2\alpha\cos\gamma)^3}} - \frac{1}{\sqrt{(1+2\alpha\cos\gamma)^3}} \right) d\cos\theta'$$

- In the expansion of the radicals only odd powers of $\alpha\cos\gamma$ will appear

$$\frac{1}{\sqrt{(1-2\alpha\cos\gamma)^3}}-\frac{1}{\sqrt{(1+2\alpha\cos\gamma)^3}}=6\alpha\cos\gamma+35\alpha^3\cos^3\gamma+\cdots$$

$$\Rightarrow\int_0^{2\pi}d\phi'\int_0^1\begin{bmatrix}\cos\gamma\\ \cos^3\gamma\end{bmatrix}d\cos\theta'=\begin{bmatrix}\pi\cos\theta\\ \frac{\pi}{4}\cos\theta\,(3-\cos^2\theta)\end{bmatrix}$$

$$\Rightarrow\quad\Phi(r,\theta,\phi)=\frac{3a(r^2-a^2)V}{2\sqrt{(r^2+a^2)^3}}\alpha\cos\theta\left(1+\frac{35}{24}\alpha^2(3-\cos^2\theta)+\cdots\right)\quad\Leftarrow\quad\text{in }\alpha$$

$$=\frac{3}{2}\frac{a^2}{r^2}V\left[\cos\theta-\frac{7}{12}\frac{a^2}{r^2}\left(\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta\right)+\cdots\right]\quad\Leftarrow\quad\text{in }\frac{a^2}{r^2}$$

- Only odd powers of $\cos\theta$ appear, as required by the symmetry of the problem.

- For large values of $\frac{r}{a}$ this expansion converges rapidly and so is a useful representation for the potential.

- It is easily verified (by the Taylor expansion) that, for $\cos\theta=1$, this expression agrees with the expansion of the expression for the potential on the axis.

- The special choice in the 2nd expression is related to the Legendre polynomials.

A closed volume is bounded by conducting surfaces that are the n sides of a regular polyhedron ($n = 4, 6, 8, 12, 20$). The n surfaces are at different potentials V_i, $i = 1, 2, \ldots, n$. Prove in the simplest way you can that the potential at the center of the polyhedron is the average of the potential on the n sides. <= 2.28

Assume $G_D(\mathbf{x}, \mathbf{x}')$ is the Green's function that satisfies Dirichlet boundary conditions inside a polyhedron. Let $\mathbf{x}_c$ to be the center of a polyhedron. For no charge inside the polyhedron, the potential at $\mathbf{x}_c$ is

$$\Phi(\mathbf{x}_c) = -\frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial G_D}{\partial n'}(\mathbf{x}_c, \mathbf{x}')\mathrm{d}a' = -\frac{1}{4\pi}\sum_{i=1}^{n} V_i \int_{S_i} \frac{\partial G_D}{\partial n'}(\mathbf{x}_c, \mathbf{x}')\mathrm{d}a',$$

where $S = \sum S_i$ and since $\Phi_{|S_i} = V_i$. Since $\mathbf{x}_c$ is the center of the polyhedron, therefore, it is the center of symmetry. There will be no difference for each side S_i from the viewpoint of $\mathbf{x}_c$. Therefore, the surface integrations over differenct side S_i, i.e., $\int_{S_i} \frac{\partial G_D}{\partial n'}(\mathbf{x}_c, \mathbf{x}')\mathrm{d}a'$ are the same. Then we can rewrite the expression of the potential at $\mathbf{x}_c$ as

$$\begin{aligned}\Phi(\mathbf{x}_c) &= -\frac{1}{4\pi}\sum_{i=1}^{n} V_i \int_{S_i} \frac{\partial G_D}{\partial n'}(\mathbf{x}_c, \mathbf{x}')\mathrm{d}a' = -\frac{1}{4\pi}\sum_{i=1}^{n} V_i \times \frac{1}{n}\oint_S \frac{\partial G_D}{\partial n'}(\mathbf{x}_c, \mathbf{x}')\mathrm{d}a' \\ &= -\frac{1}{4n\pi}\sum_{i=1}^{n} V_i \int_V \nabla'^2 G_D(\mathbf{x}_c, \mathbf{x}')\mathrm{d}^3x' = \frac{1}{n}\sum_{i=1}^{n} V_i,\end{aligned}$$

where the Gauss theorem is applied and since $\nabla'^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}')$. Therefore, the potential at the center of a polyhedron is the average of the potential on its sides.

Orthogonal Functions and Expansions

- The orthogonal set chosen depends on the symmetries or near symmetries involved.

- Consider an interval (a, b) in a variable ξ with a set of real or complex *orthonormal* functions $U_n(\xi)$. The orthogonality condition:

$$\int_a^b U_n^*(\xi)\, U_m(\xi)\, \mathrm{d}\xi = \delta_{mn}$$

- An arbitrary function can be expanded in a series of the orthonormal functions

$$f(\xi) = \sum_{n=1}^{\infty} a_n U_n(\xi) \quad \Leftarrow \quad \text{completeness of the function set}$$

$$= \sum_{n=1}^{\infty} \left(\int_a^b U_n^*(\xi')\, f(\xi')\, \mathrm{d}\xi' \right) U_n(\xi) \quad \Leftarrow \quad a_n = \int_a^b U_n^*(\xi)\, f(\xi)\, \mathrm{d}\xi$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} U_n^*(\xi')\, U_n(\xi) = \delta(\xi' - \xi) \quad \Leftarrow \quad \text{closure relation or completeness}$$

- The most famous orthogonal functions are the sines and cosines, an expansion in terms of them being a *Fourier series*.

- The orthonormal functions $\sqrt{\frac{2}{a}} \sin\frac{2\pi m x}{a}, \quad \sqrt{\frac{2}{a}} \cos\frac{2\pi m x}{a} \quad \text{for } x \in \left[-\frac{a}{2}, \frac{a}{2}\right]$

- The constant function is $\frac{1}{\sqrt{a}}$ for $m = 0$

- A function is customarily written in the form:

$$f(x)=\frac{1}{2}A_0+\sum_{m=1}^{\infty}\left(A_m\cos\frac{2\pi m x}{a}+B_m\sin\frac{2\pi m x}{a}\right)$$

where $A_m=\frac{2}{a}\int_{-a/2}^{+a/2} f(x)\cos\frac{2\pi m x}{a}\,\mathrm{d}x,\quad B_m=\frac{2}{a}\int_{-a/2}^{+a/2} f(x)\sin\frac{2\pi m x}{a}\,\mathrm{d}x$

- Suppose that the space is 2d, and the variable ξ ranges over the interval (a, b) while the variable η has the interval (c, d). The orthonormal functions in each dimension are $U_n(\xi)$ and $V_m(\eta)$. Then the expansion of an arbitrary function is

$$f(\xi,\eta)=\sum_n\sum_m a_{nm}U_n(\xi)V_m(\eta)\quad\Leftarrow\quad a_{nm}=\int_a^b \mathrm{d}\xi\int_c^d U_n^{\star}(\xi)V_m^{\star}(\eta)f(\xi,\eta)\,\mathrm{d}\eta$$

- $(a,b)\to(-\infty,+\infty)\quad\Rightarrow\quad U_m(x)\to U(m,x)$

$$\Rightarrow\int_a^b U_n^*(x)U_m(x)\,\mathrm{d}x\ \to\ \int_{-\infty}^{+\infty}U^*(n,x)U(m,x)\,\mathrm{d}x=\delta(m-n)\ \leftarrow\ \delta_{mn}$$

- For *Fourier integral*, start with: $U_m(x)=\frac{1}{\sqrt{a}}e^{i\frac{2m\pi x}{a}}\ \Leftarrow\ \left[\begin{array}{l} m=0,\pm1,\pm2,\cdots \\ x\in\left(-\frac{a}{2},\frac{a}{2}\right)\end{array}\right.$

$$\Rightarrow\quad f(x)=\frac{1}{\sqrt{a}}\sum_{m=-\infty}^{\infty}A_m e^{i\frac{2m\pi x}{a}}\quad\Leftarrow\quad A_m=\frac{1}{\sqrt{a}}\int_{-a/2}^{+a/2}e^{-i\frac{2m\pi x'}{a}}f(x')\,\mathrm{d}x'$$

- $a \to \infty \quad \Rightarrow \quad \left| \begin{array}{l} \dfrac{2\pi m}{a} \to k \\ \displaystyle\sum_m \to \int_{-\infty}^{+\infty} \mathrm{d}\, m \to \frac{a}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\, k \\ A_m \to \sqrt{\dfrac{2\pi}{a}}\, A(k) \end{array} \right.$

$$\Rightarrow \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A(k)\, e^{i k x}\, \mathrm{d}\, k \quad \Leftarrow \quad A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i k x} f(x)\, \mathrm{d}\, x$$

$$\Rightarrow \quad \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(k-k')x}\, \mathrm{d}\, x = \delta(k-k'), \qquad \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i k (x-x')}\, \mathrm{d}\, k = \delta(x-x')$$

orthogonality condition   completeness relation

- The 2 continuous variables x and k are complete equivalent.

Separation of Variables; Laplace Equation in Rectangular Coordinates

- Equations involving the 3-dim Laplacian operator are known to be separable in 11 different coordinate systems.

- Discuss only 3 of these — rectangular, spherical, and cylindrical.

- The Laplace equation in rectangular coordinates: $\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$

- Assume $\Phi(x, y, z) = X(x)Y(y)Z(z) \Rightarrow \frac{1}{X}\frac{d^2 X}{d x^2} + \frac{1}{Y}\frac{d^2 Y}{d y^2} + \frac{1}{Z}\frac{d^2 Z}{d z^2} = 0$

- Hold for arbitrary values of the independent coordinates, each of the 3 terms must be separately constant:

$$\frac{1}{X}\frac{d^2 X}{d x^2} = -\alpha^2, \quad \frac{1}{Y}\frac{d^2 Y}{d y^2} = -\beta^2, \quad \frac{1}{Z}\frac{d^2 Z}{d z^2} = \alpha^2 + \beta^2 \quad \Rightarrow \quad \Phi = e^{\pm i \alpha x} e^{\pm i \beta y} e^{\pm \sqrt{\alpha^2+\beta^2} z}$$

- By linear superposition, the solution can construct a very large class of solutions to the Laplace equation.

- To determine α and β it is necessary to impose specific boundary conditions on the potential.

- To find the potential everywhere inside the box,

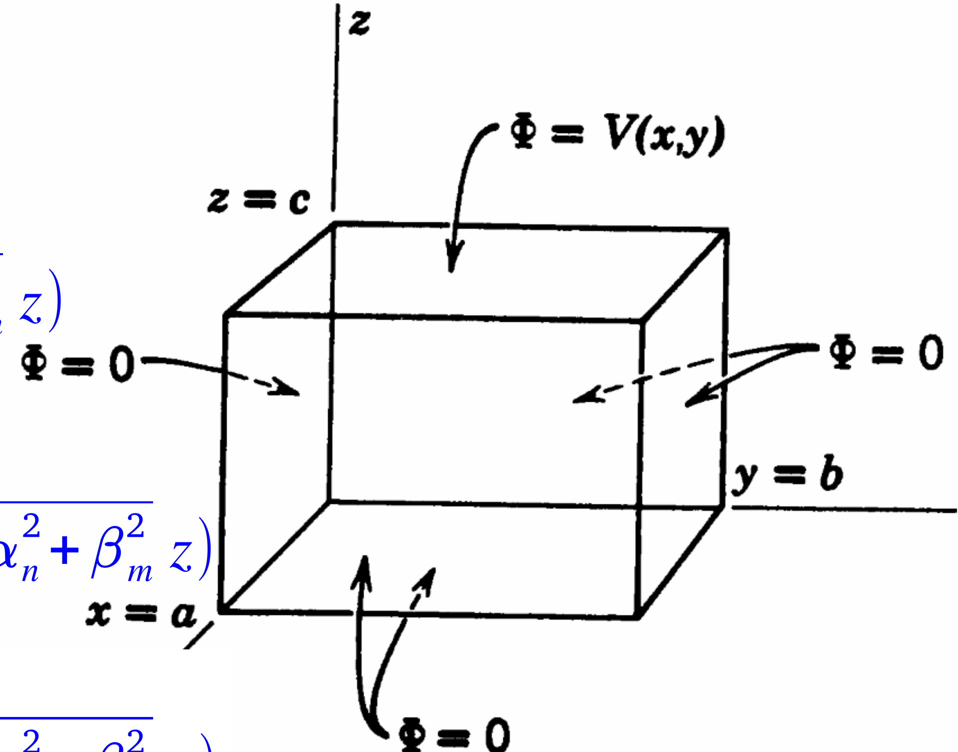
$$\Phi(0, y, z) = 0, \quad \Phi(x, 0, z) = 0, \quad \Phi(x, y, 0) = 0 \quad \Rightarrow \quad \begin{array}{l} X = \sin \alpha x, \quad Y = \sin \beta y \\ Z = \sinh\left(\sqrt{\alpha^2 + \beta^2}\, z\right) \end{array}$$

$$\Phi(a, y, z)=0 \quad \Phi(x, b, z)=0 \quad \Rightarrow \quad \alpha_n = \frac{n\pi}{a}, \quad \beta_m = \frac{m\pi}{b}$$

$$\Rightarrow \quad \Phi_{nm} = \sin(\alpha_n x)\sin(\beta_m y)\sinh\left(\sqrt{\alpha_n^2+\beta_m^2}\, z\right)$$

$$\Rightarrow \quad \Phi(x, y, z) = \sum_{n,m=1}^{\infty} A_{nm}\Phi_{nm}$$

$$= \sum_{n,m=1}^{\infty} A_{nm}\sin(\alpha_n x)\sin(\beta_m y)\sinh\left(\sqrt{\alpha_n^2+\beta_m^2}\, z\right)$$



$$\Phi(x, y, c) = V(x, y)$$

$$= \sum_{n,m=1}^{\infty} A_{nm}\sin(\alpha_n x)\sin(\beta_m y)\sinh\left(\sqrt{\alpha_n^2+\beta_m^2}\, c\right)$$

$$\Rightarrow \quad A_{nm} = \frac{4}{a\, b \sinh\left(c\sqrt{\alpha_n^2+\beta_m^2}\right)} \int_0^a \mathrm{d}x \int_0^b \mathrm{d}y\, V(x, y)\sin(\alpha_n x)\sin(\beta_m y)$$

- If the rectangular box has potentials different from 0 on all 6 sides, the required solution for the potential inside the box can be obtained by a linear superposition of 6 solutions, one for each side.

- The problem of the solution of the Poisson equation, ie, the potential inside the box with a charge distribution inside, as well as prescribed boundary conditions on the surface, requires the construction of the appropriate Green function.

A 2d Potential Problem; Summation of a Fourier Series

- Consider the solution by separation of variables of the 2d Laplace equation in Cartesian coordinates, in which the potential is independent of z,

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

$$\Phi \sim e^{\pm i \alpha x} e^{\pm \alpha y} \Leftarrow \alpha \in \mathbb{R} \text{ or } \mathbb{C}$$

- Boundary conditions:

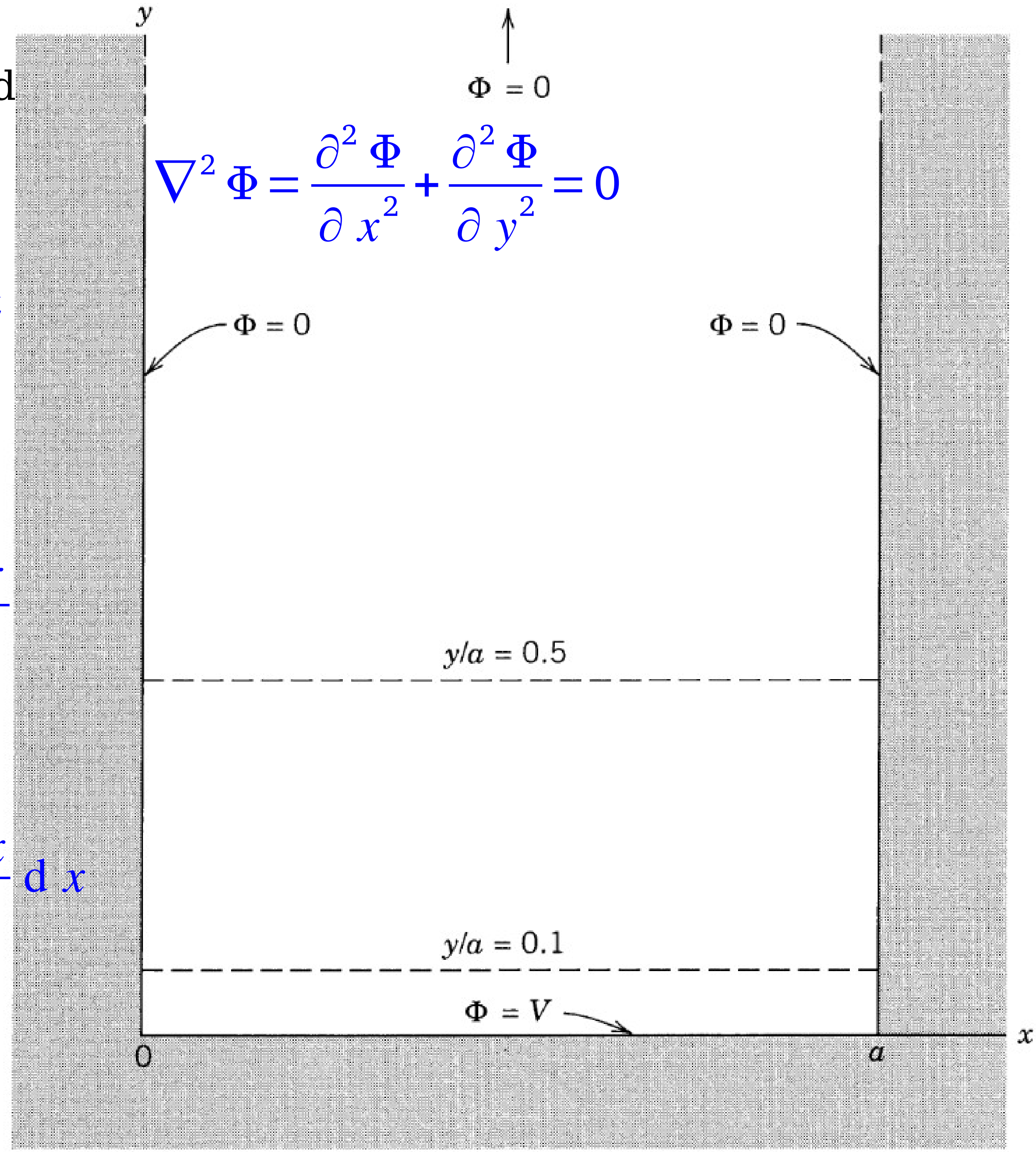
$$\Phi(0, y) = \Phi(a, y) = 0$$
$$\Phi(x, \infty) = 0, \quad \Phi(x, 0) = V$$

$$\Rightarrow \Phi \sim e^{-\alpha y} \sin \alpha x \Leftarrow \alpha = \frac{n \pi}{a}$$

$$\Rightarrow \Phi = \sum_{n=1}^{\infty} A_n e^{-\frac{n \pi y}{a}} \sin \frac{n \pi x}{a}$$

$$\Leftarrow A_n = \frac{2}{a} \int_0^a \Phi(x, 0) \sin \frac{n \pi x}{a} \, \mathrm{d}x$$

$$= \begin{cases} \dfrac{4V}{n \pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$



$$\Rightarrow \quad \Phi(x,y) = \sum_{n\ \mathrm{odd}} \frac{4V}{n\pi} e^{-\frac{n\pi y}{a}} \sin\frac{n\pi x}{a} \quad \Rightarrow \quad \Phi \to \frac{4V}{\pi} e^{-\frac{\pi y}{a}} \sin\frac{\pi x}{a} \quad \text{for } y \geq \frac{a}{\pi}$$

- The smooth behavior in x of the asymptotic solution sets in for $y \geq a$, regardless of the complexities of $\Phi(x,0)$.

- $\sin\theta = \Im\left[e^{i\theta}\right]$

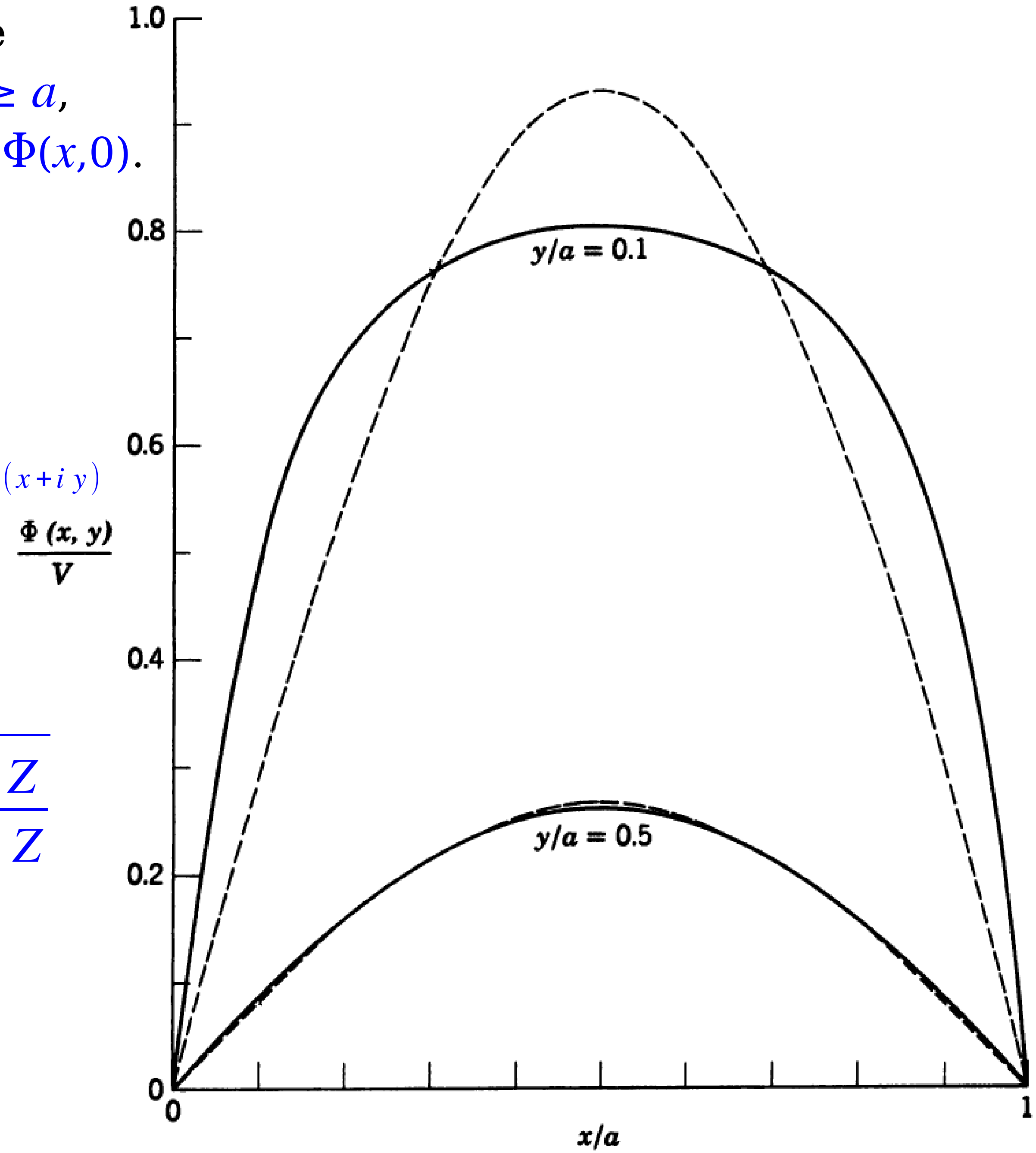
$$\Rightarrow \quad \Phi = \Im\left[\sum_{n\ \mathrm{odd}} \frac{4V}{n\pi} e^{\frac{i n\pi}{a}(x+iy)}\right]$$

$$= \frac{4V}{\pi} \Im\left[\sum_{n\ \mathrm{odd}} \frac{Z^n}{n}\right] \Leftarrow \quad Z \equiv e^{\frac{i\pi}{a}(x+iy)}$$

$$\ln(1+Z) = Z - \frac{Z^2}{2} + \frac{Z^3}{3} - \frac{Z^4}{4} + \cdots$$

$$\Rightarrow \quad \sum_{n\ \mathrm{odd}} \frac{Z^n}{n} = \frac{1}{2} \ln\frac{1+Z}{1-Z} = \ln\sqrt{\frac{1+Z}{1-Z}}$$

$$\Rightarrow \quad \Phi(x,y) = \frac{2V}{\pi} \Im\left[\ln\frac{1+Z}{1-Z}\right]$$



- The imaginary part of a logarithm is equal to the phase of its argument,

$$F = A + i\,B = S\,(\cos\phi + i\,\sin\phi) = S\,e^{i\,\phi} \quad\Rightarrow\quad \ln F = \ln\left(S\,e^{i\,\phi}\right) = \ln S + i\,\phi$$

- $$\frac{1+Z}{1-Z} = \frac{1-|Z|^2 + 2\,i\,\Im[Z]}{|1-Z|^2} \quad\Rightarrow\quad \cos\Theta \;\propto\; 1-|Z|^2, \quad \sin\Theta \;\propto\; 2\,\Im[Z]$$

$$\Rightarrow \quad \text{the phase of the argument of } \ln\frac{1+Z}{1-Z} = \tan^{-1}\frac{2\,\Im[Z]}{1-|Z|^2} \quad\Leftarrow\quad \tan\Theta = \frac{\sin\Theta}{\cos\Theta}$$

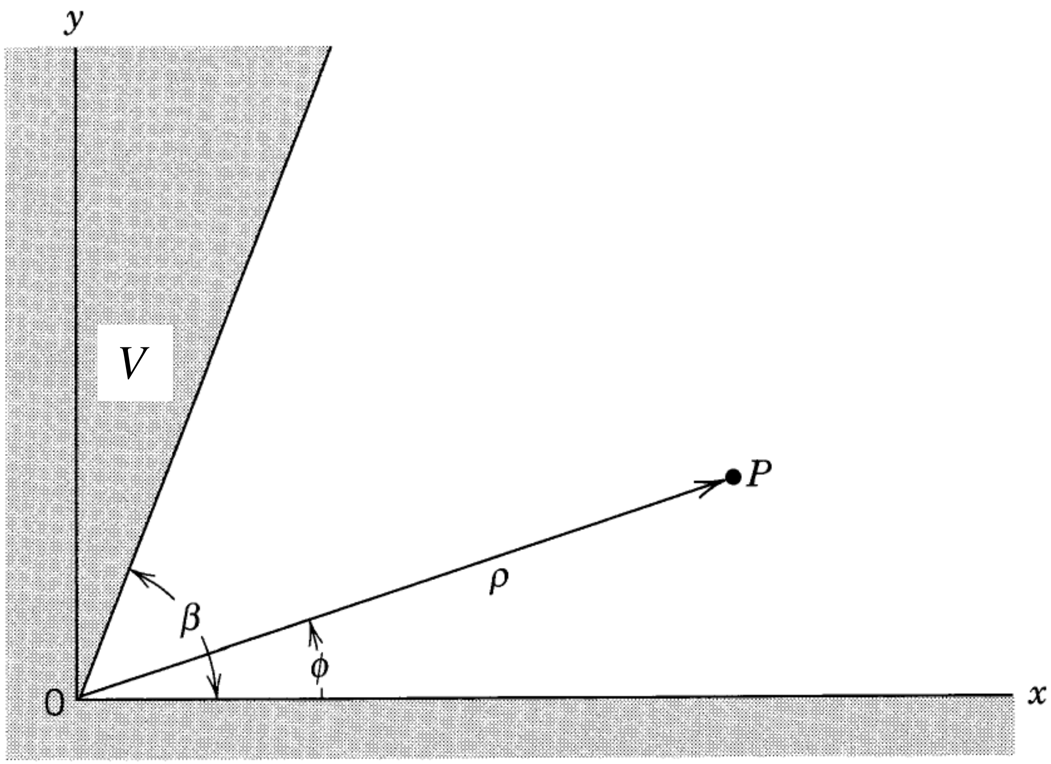
$$Z = e^{\frac{i\,\pi}{a}(x+i\,y)} \quad\Rightarrow\quad \Im[Z] = e^{-\frac{\pi\,y}{a}}\sin\frac{\pi\,x}{a}, \quad 1-|Z|^2 = 1-e^{-2\frac{\pi\,y}{a}}$$

$$\Rightarrow \quad \Phi(x\,,\,y) = \frac{2\,V}{\pi}\tan^{-1}\frac{\sin\frac{\pi\,x}{a}}{\sinh\frac{\pi\,y}{a}} \quad\Leftarrow\quad 0 \le \tan^{-1}\frac{\sin\frac{\pi\,x}{a}}{\sinh\frac{\pi\,y}{a}} \le \frac{\pi}{2} \quad\Leftarrow\quad \sinh\beta = \frac{e^{\beta}-e^{-\beta}}{2}$$

- The infinite series has now been transformed into the explicit closed form.

- The real or the imaginary part of an analytic function satisfies the Laplace equation in two dimensions as a result of the Cauchy-Riemann equations.

Fields and Charge Densities in 2d Corners and Along Edges



- The geometry suggests use of polar rather than Cartesian coordinates.

- The Laplace equation in 2d:

$$\nabla^2 \Phi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

- Separation of variables:

$$\Phi(\rho, \phi) = R(\rho)\Psi(\phi)$$

$$\Rightarrow \quad \frac{\rho}{R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{1}{\Psi}\frac{d^2\Psi}{d\phi^2} = 0$$

$$\Rightarrow \quad \frac{\rho}{R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = \nu^2, \quad \frac{1}{\Psi}\frac{d^2\Psi}{d\phi^2} = -\nu^2$$

$$\Rightarrow \quad \begin{cases} R(\rho) = a\rho^{\nu} + b\rho^{-\nu} \\ \Psi(\phi) = A\cos(\nu\phi) + B\sin(\nu\phi) \end{cases} \quad \text{but for } \nu = 0 \quad \Rightarrow \quad \begin{cases} R(\rho) = a_0 + b_0 \ln \rho \\ \Psi(\phi) = A_0 + B_0 \phi \end{cases}$$

- If there is no restriction on ϕ, it is necessary that ν be a positive or negative integer or 0 to ensure that the potential is single-valued $\Rightarrow \; B_0 = 0$.

- The general solution:

$$\Phi(\rho, \phi) = a_0 + b_0 \ln \rho + \sum_{n=1}^{\infty}\left(a_n \rho^n \sin(n\phi + \alpha_n) + \frac{b_n}{\rho^n}\sin(n\phi + \beta_n)\right)$$

- If there is no charge at the origin, all the b_n are 0. If the origin is excluded, the b_n can be different from 0.

- The logarithmic term is equivalent to a line charge on the axis with charge density per unit length $\lambda = -2\pi\epsilon_0 b_0$.

- $\Phi(\rho, 0) = \Phi(\rho, \beta) = V$

$$\Rightarrow \quad b_0 = b_n = \alpha_n = 0 \,,\, a_0 = V \,, \quad \sin(\nu\beta) = 0 \quad \Leftarrow \quad \nu = \frac{m\pi}{\beta} \,, \quad m = 1,2\,,\cdots$$

$$\Rightarrow \quad \Phi(\rho, \phi) = V + \sum_{m=1}^{\infty} a_m \rho^{\frac{m\pi}{\beta}} \sin\frac{m\pi\phi}{\beta}$$

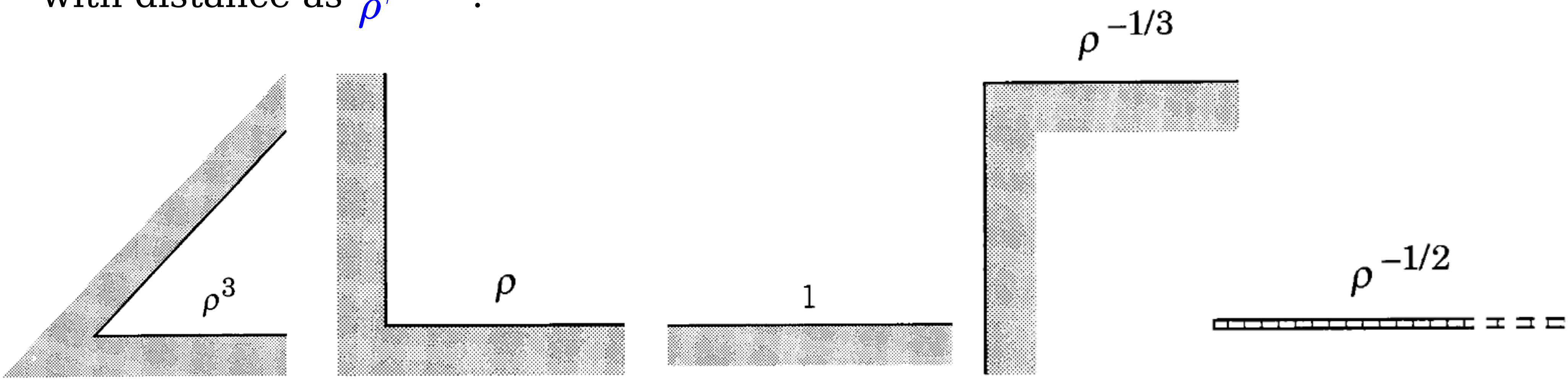
- The undetermined coefficients a_m depend on the potential remote from $\rho=0$.

- For small enough ρ only the 1st term in the series will be important.

$$\Phi(\rho, \phi) \simeq V + a_1 \rho^{\frac{\pi}{\beta}} \sin\frac{\pi\phi}{\beta} \quad \Rightarrow \quad \left[\begin{array}{l} E_\rho = \quad -\dfrac{\partial\Phi}{\partial\rho} \simeq -\dfrac{a_1\pi}{\beta}\rho^{\frac{\pi}{\beta}-1} \sin\dfrac{\pi\phi}{\beta} \\ E_\phi = -\dfrac{1}{\rho}\dfrac{\partial\Phi}{\partial\phi} \simeq -\dfrac{a_1\pi}{\beta}\rho^{\frac{\pi}{\beta}-1} \cos\dfrac{\pi\phi}{\beta} \end{array} \right.$$

$$\Rightarrow \quad \sigma(\rho, 0) = \sigma(\rho, \beta) = \epsilon_0 E_\phi(\rho, 0) \simeq -\frac{\epsilon_0 a_1 \pi}{\beta}\rho^{\frac{\pi}{\beta}-1} \quad \Leftarrow \text{ surface charge density}$$

- The components of the field and the surface-charge density near $\rho=0$ all vary with distance as $\rho^{\frac{\pi}{\beta}-1}$.



- For a very deep corner (small β) the power of ρ becomes very large, and no charge accumulates in such a corner.

- When $\beta>\pi$, the 2d corner becomes an edge and the field and the surface-charge density become singular as $\rho\to 0$.

- The 2d electrostatic considerations apply to many 3d situations, even with time-varying fields.

- The singular behavior of the fields near sharp edges is the reason for the effectiveness of lightning rods.

Selected problems: 3, 7, 13, 15, 23, 26

Starting with the series solution (2.71) for the two-dimensional potential problem with the potential specified on the surface of a cylinder of radius b, evaluate the coefficients formally, substitute them into the series, and sum it to obtain the potential *inside* the cylinder in the form of Poisson's integral:

$$\Phi(\rho, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \Phi(b, \phi') \frac{b^2 - \rho^2}{b^2 + \rho^2 - 2b\rho \cos(\phi' - \phi)} d\phi'$$

What modification is necessary if the potential is desired in the region of space bounded by the cylinder and infinity? [Problem 2.12]

Equation (2.71) reads as

$$\Phi(\rho, \phi) = a_0 + c_0 \ln \rho + \sum_{n=1}^{\infty} [a_n \rho^n \sin(n\phi + \alpha_n) + c_n \rho^{-n} \sin(n\phi + \beta_n)].$$

For the potential inside the cylinder, it requires the potential Φ to be regular for $0 \leq \rho \leq b$ and it indicates $c_0 = c_n = 0$ for all c_n's. Then the potential becomes

$$\Phi_{\text{in}}(\rho, \phi) = a_0 + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \rho^n (A_n \sin n\phi + B_n \cos n\phi),$$

where

$$A_0 = 2a_0 = \frac{1}{\pi} \int_0^{2\pi} \Phi(b, \phi') \mathrm{d}\phi', \quad A_n = a_n \cos \alpha_n = \frac{1}{\pi b^n} \int_0^{2\pi} \Phi(b, \phi') \sin n\phi' \mathrm{d}\phi',$$

$$B_n = a_n \sin \alpha_n = \frac{1}{\pi b^n} \int_0^{2\pi} \Phi(b, \phi') \cos n\phi' \mathrm{d}\phi'.$$

Rewrite the potential by replacing these coefficients with the above expressions

$$
\begin{aligned}
\Phi_{\text{in}}(\rho, \phi) &= \frac{1}{2\pi}\int_0^{2\pi} \Phi(b, \phi')\mathrm{d}\phi' + \sum_{n=1}^{\infty} \rho^n \left[\left(\frac{1}{\pi b^n}\int_0^{2\pi} \Phi(b, \phi') \sin n\phi' \mathrm{d}\phi'\right) \sin n\phi \right. \\
&\qquad \left. + \left(\frac{1}{\pi b^n}\int_0^{2\pi} \Phi(b, \phi') \cos n\phi' \mathrm{d}\phi'\right) \cos n\phi \right] \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Phi(b, \phi') \left[1 + 2\sum_{n=1}^{\infty} \frac{\rho^n}{b^n} \cos n(\phi - \phi')\right] \mathrm{d}\phi' \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Phi(b, \phi') \left[\sum_{n=0}^{\infty} \frac{\rho^n}{b^n} [e^{in(\phi-\phi')} + e^{-in(\phi-\phi')}] - 1\right] \mathrm{d}\phi' \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Phi(b, \phi') \left[\frac{1}{1 - \rho e^{i(\phi-\phi')}/b} + \frac{1}{1 - \rho e^{-i(\phi-\phi')}/b} - 1\right] \mathrm{d}\phi' \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Phi(b, \phi') \frac{b^2 - \rho^2}{b^2 + \rho^2 - 2b\rho\cos(\phi - \phi')} \mathrm{d}\phi'.
\end{aligned}
$$

For the potential outside the cylinder, it requires the potential Φ to be finite for $b \leq \rho < \infty$ and it indicates $c_0 = a_n = 0$ for all a_n's. Then the potential becomes

$$
\Phi_{\text{out}}(\rho, \phi) = a_0 + \sum_{n=1}^{\infty} c_n \rho^{-n} \sin(n\phi + \beta_n) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \rho^{-n}(C_n \sin n\phi + D_n \cos n\phi),
$$

where

$$C_n = c_n \cos\beta_n = \frac{b^n}{\pi}\int_0^{2\pi}\Phi(b,\phi')\sin n\phi'\mathrm{d}\phi', \quad D_n = c_n \sin\beta_n = \frac{b^n}{\pi}\int_0^{2\pi}\Phi(b,\phi')\cos n\phi'\mathrm{d}\phi'.$$

Following the same reasoning, it is easy to find

$$\Phi_{\text{out}}(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{\rho^2 - b^2}{b^2+\rho^2-2b\rho\cos(\phi-\phi')}\mathrm{d}\phi'.$$

Therefore, the modification needed from Φ_{in} to Φ_{out} is to swap ρ and b in their expressions.