Chapter 1 Introduction to Electrostatics

- *Electrostatics* — phenomena involving time-independent distributions of charge and fields.

- Electrostatics developed as a science of *macroscopic* phenomena. Some idealizations like point charge may fail to have meaning microscopically.

Coulomb's Law

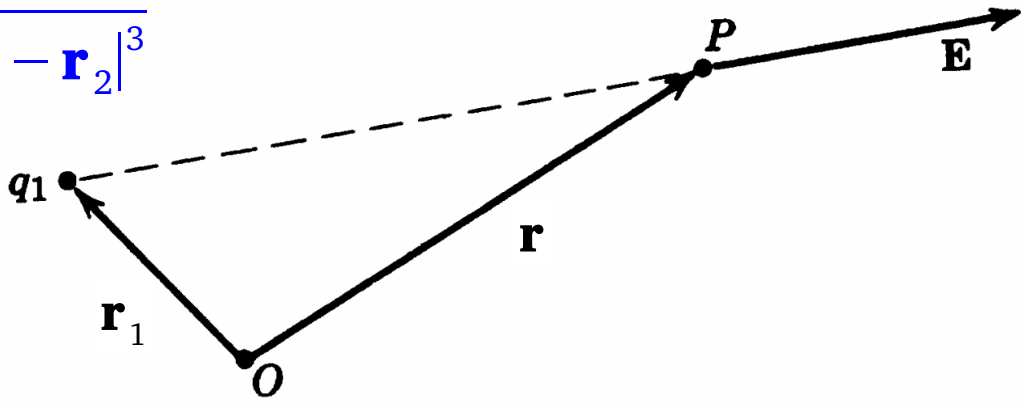
- The force between 2 small charged bodies separated in a distance
  - varies directly as the magnitude of each charge,
  - varies inversely as the square of the distance,
  - is directed along the line joining the charges,
  - attractive if oppositely charged and repulsive if the same type of charge,
  - the vector summation rule applies.

Electric Field

- Electric field: force/(unit charge) at a given point in a limiting process $\mathbf{F} = q\,\mathbf{E}$

- Coulomb's Law: $\mathbf{F}(\mathbf{r}_1, \mathbf{r}_2) = k\,q_1\,q_2\,\dfrac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}$

- the electric field: $\mathbf{E}(\mathbf{r}) = k\,q_1\,\dfrac{\mathbf{r} - \mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|^3}$



- In the SI system: $k = \dfrac{1}{4\pi\epsilon_0} = 10^{-7} c^2$ $\Leftarrow$ free space permittivity $\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{F/M}$

- The linear superposition law:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} q_i \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3} \Rightarrow \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \mathrm{d}^3 x'$$

where $\Delta q = \rho(\mathbf{r}') \Delta x' \Delta y' \Delta z'$, $\mathrm{d}^3 x' = \mathrm{d}x' \mathrm{d}y' \mathrm{d}z'$

- **Dirac delta function**: a mathematically improper function with the properties

1. $\delta(x - a) = 0$ for $x \neq a$ in 1d

2. $\displaystyle\int \delta(x - a)\, \mathrm{d}x = \begin{cases} 1 & \text{if the region of integration includes } x = a \\ 0 & \text{otherwise} \end{cases}$

3. $\displaystyle\int f(x)\, \delta(x - a)\, \mathrm{d}x = f(a)$

4. $\displaystyle\int f(x) \frac{\mathrm{d}\delta}{\mathrm{d}x}(x - a)\, \mathrm{d}x = -\frac{\mathrm{d}f}{\mathrm{d}x}(a)$ $\Leftarrow$ using integration by parts

5. $\displaystyle\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{\left|\dfrac{\mathrm{d}f}{\mathrm{d}x}\right|_{x = x_i}}$ where x_i's are roots of $f(x)$.

Definition of the Dirac delta function: $\int_{-\infty}^{+\infty} f(x)\,\delta(x-y)\,\mathrm{d}\,x = f(y)$

$$\int_{-\infty}^{+\infty} g(x)\,\delta(f(x))\,\mathrm{d}\,x = \int g\,\delta(f)\,\frac{\mathrm{d}\,x}{\mathrm{d}\,f}\,\mathrm{d}\,f \quad \Leftarrow \quad \begin{array}{l}\text{if } u = f(x),\\ \text{then } x = f^{-1}(u) = f^{-1}(f)\end{array}$$

$$= \int_{-}^{+} g\,\delta(f)\left|\frac{\mathrm{d}\,x}{\mathrm{d}\,f}\right|\mathrm{d}\,f = \int_{-}^{+} \frac{g\,\delta(f)}{\left|\frac{\mathrm{d}\,f}{\mathrm{d}\,x}\right|}\,\mathrm{d}\,f$$

$$f(x) = 0 \text{ if } x = z_i \text{ the root} \quad \Rightarrow \int_{-}^{+} \frac{g\,\delta(f)}{\left|\frac{\mathrm{d}\,f}{\mathrm{d}\,x}\right|}\,\mathrm{d}\,f = \int_{z_i-\epsilon}^{z_i+\epsilon} \frac{g(x)\,\delta(x-z_i)}{\left|\frac{\mathrm{d}\,f}{\mathrm{d}\,x}\right|_{x=z_i}}\,\mathrm{d}\,x$$

$$\Rightarrow \quad \int_{-\infty}^{+\infty} g(x)\,\delta(f(x))\,\mathrm{d}\,x = \int_{-\infty}^{+\infty} g(x)\sum_i \frac{\delta(x-z_i)}{\left|\frac{\mathrm{d}\,f}{\mathrm{d}\,x}\right|_{x=z_i}}\,\mathrm{d}\,x \quad \Leftarrow \quad \text{if more than 1 root}$$

$$\Rightarrow \quad \delta(f(x)) = \sum_i \frac{\delta(x-z_i)}{\left|\frac{\mathrm{d}\,f}{\mathrm{d}\,x}\right|_{x=z_i}} \quad \text{where } z_i\text{'s are roots.}$$

6. $\delta(\mathbf{r}-\mathbf{R}) = \delta(x_1 - X_1)\,\delta(x_2 - X_2)\,\delta(x_3 - X_3)$ with Cartesian coordinates in 3d

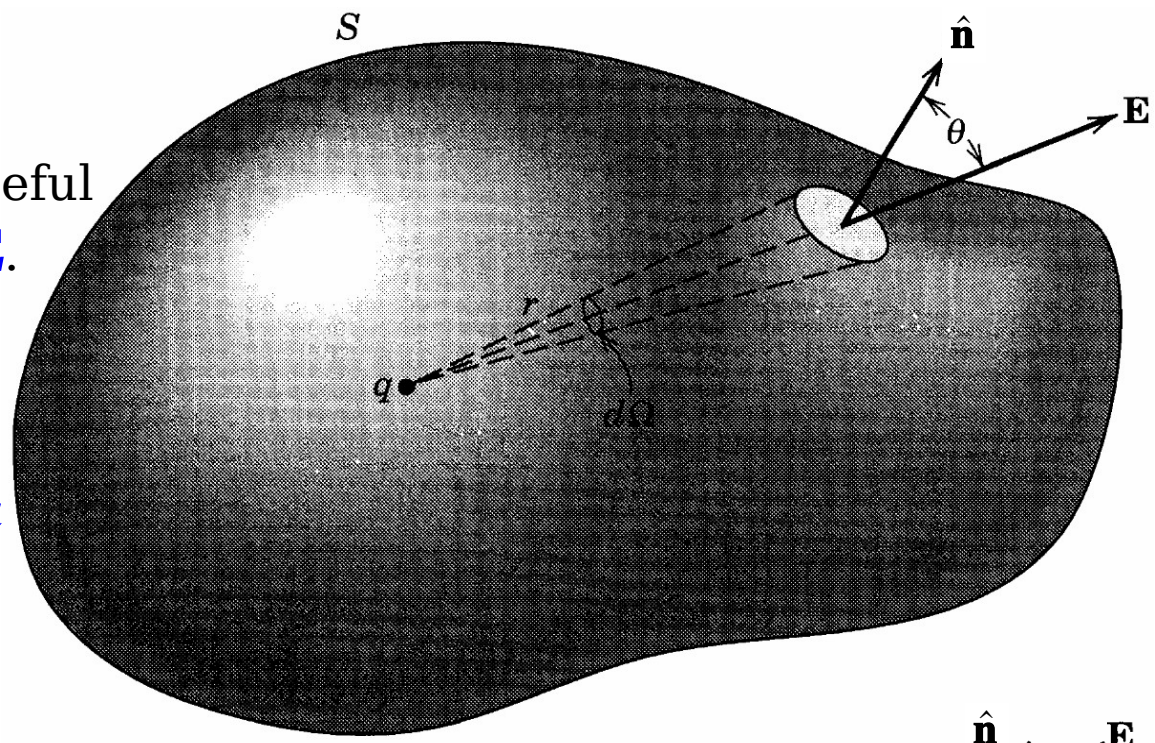
7. $$\int_{\Delta\nu} \delta(\mathbf{r}-\mathbf{R})\,\mathrm{d}^3 x = \begin{cases} 1 & \text{if } \Delta V \text{ constains } \mathbf{r}=\mathbf{R} \\ 0 & \text{if } \Delta V \text{ does not constain } \mathbf{r}=\mathbf{R} \end{cases} \Rightarrow \quad [\delta(\mathbf{r}-\mathbf{R})] = \frac{1}{V}$$

- A discrete set of point charges can be described with delta functions

$$\rho(\mathbf{r}) = \sum_{i=1}^{n} q_i\,\delta(\mathbf{r}-\mathbf{r}_i)$$

Gauss's Law

- *Gauss's law* is sometimes more useful and leads to a differential eqn for **E**.



$$\mathbf{E}\cdot \mathrm{d}\,\boldsymbol{a} = \frac{q}{4\,\pi\,\epsilon_0}\,\frac{\cos\theta}{r^2}\,\mathrm{d}\,a$$

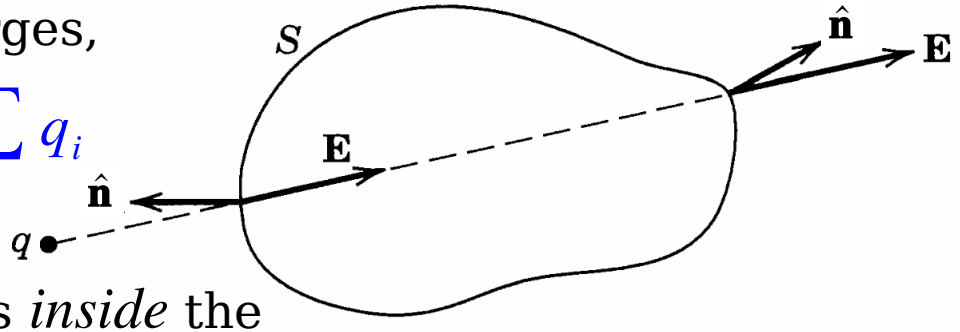
$$= \frac{q}{4\,\pi\,\epsilon_0}\,\mathrm{d}\,\Omega \quad \Leftarrow \quad r^2\,\mathrm{d}\,\Omega = \cos\theta\,\mathrm{d}\,a$$

$$\Rightarrow \epsilon_0 \oint_S \mathbf{E}\cdot \mathrm{d}\,\boldsymbol{a} = \begin{cases} q & \text{if } q \textit{ inside } S \\ 0 & \text{if } q \textit{ outside } S \end{cases}$$

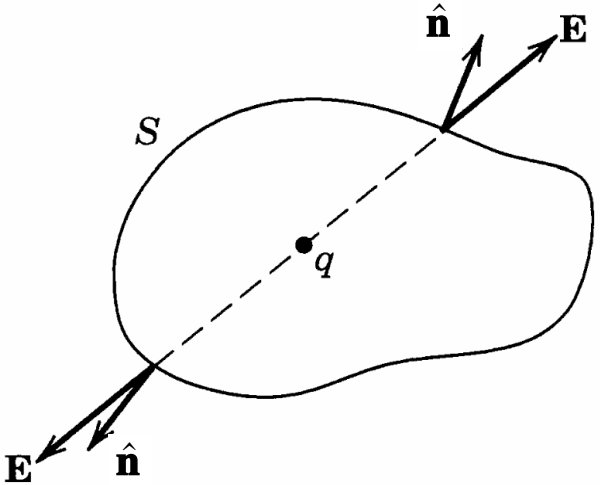
- For a set of charges,

$$\oint_S \mathbf{E}\cdot \mathrm{d}\,\boldsymbol{a} = \frac{1}{\epsilon_0}\sum_i q_i$$

the sum is over only those charges *inside* the surface S, $\mathrm{d}\,\boldsymbol{a} \equiv \hat{\mathbf{n}}\,\mathrm{d}\,a$



q outside S



q inside S

- For a continuous charge density:

$$\oint_S \mathbf{E}\cdot \mathrm{d}\,\boldsymbol{a} = \frac{1}{\epsilon_0}\int_V \rho\,(\mathbf{r})\,\mathrm{d}^3\,x$$

- The equation is one of the basic equations of electrostatics. It depends upon (a) the inverse square law for the force between charges; (b) the central nature of the force; (c) the linear superposition of the effects of different charges.

Differential Form of Gauss's Law

- The *divergence theorem*: for any well-behaved vector field defined within a volume surrounded by the closed surface $\oint_S \mathbf{A}\cdot \mathrm{d}\,\boldsymbol{a} = \int_V \nabla\cdot\mathbf{A}\,\mathrm{d}^3 x$
- Apply the divergence theorem $\oint_S \mathbf{E}\cdot \mathrm{d}\,\boldsymbol{a} = \int_V \nabla\cdot\mathbf{E}\,\mathrm{d}^3 x = \frac{1}{\epsilon_0}\int_V \rho(\mathbf{r})\,\mathrm{d}^3 x$

$$\Rightarrow \int_V \left(\nabla\cdot\mathbf{E} - \frac{\rho}{\epsilon_0}\right)\mathrm{d}^3 x = 0 \text{ for an arbitrary volume } \mathcal{V} \Rightarrow \nabla\cdot\mathbf{E} - \frac{\rho}{\epsilon_0} = 0$$

the differential form of Gauss's law of electrostatics.

Another Equation of Electrostatics and the Scalar Potential

- A vector field can be specified almost completely if its divergence and curl are given everywhere in space.

- $\nabla \times \nabla \psi = 0 \;\text{ for all }\; \psi$ $\quad\Downarrow\quad \dfrac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\nabla \dfrac{1}{|\mathbf{r}-\mathbf{r}'|}$

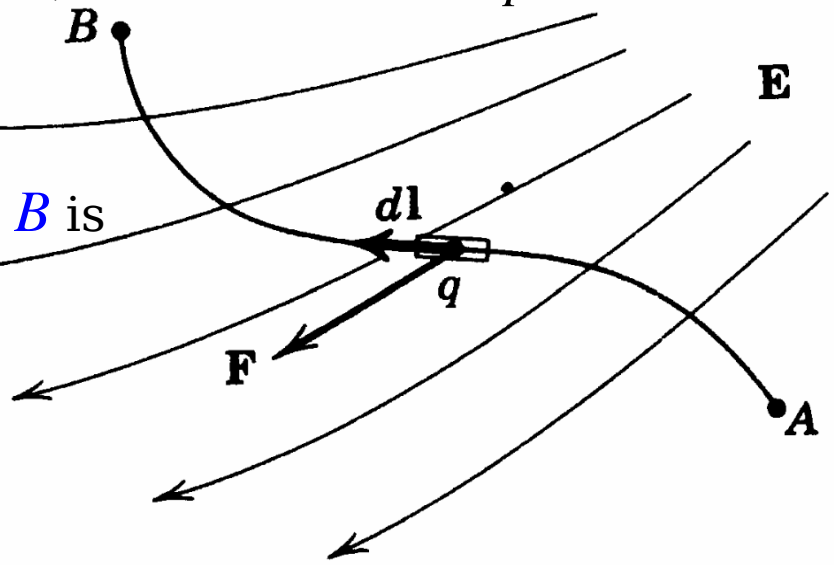
- Look for an equation specifying curl $\mathbf{E}$ as a function of position,

$$\nabla \times \mathbf{E} = 0 \quad \Leftarrow \quad \mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\nabla \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3 x' = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}^3 x'$$

- Note that $\nabla \times \mathbf{E} = 0$ depends on the central nature of the force, and on the fact that the force is a function of relative distances only, but does not depend on the inverse square nature.

- Since a scalar is easier to deal with than a vector, define the *scalar potential*

$$\mathbf{E} = -\nabla \Phi \quad \Rightarrow \quad \Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3 x'$$



- The work done in moving the charge from A to B is

$$W = -\int_A^B \mathbf{F}\cdot \mathrm{d}\,\boldsymbol{\ell} = -q\int_A^B \mathbf{E}\cdot \mathrm{d}\,\boldsymbol{\ell} \quad \Leftarrow \quad \mathbf{F} = q\,\mathbf{E}$$

$$= q\int_A^B \nabla\Phi\cdot \mathrm{d}\,\boldsymbol{\ell} = q\int_A^B \mathrm{d}\,\Phi = q\left(\Phi_B - \Phi_A\right)$$

- $q\,\Phi$ can be interpreted as the potential energy of the test charge in the electrostatic field.

- The line integral of the electric field between 2 points is independent of the path and is the negative of the potential difference between the points:

$$\int_A^B \mathbf{E}\cdot \mathrm{d}\,\boldsymbol{\ell} = -\left(\Phi_B - \Phi_A\right) \;\Rightarrow\; \oint \mathbf{E}\cdot \mathrm{d}\,\boldsymbol{\ell} = 0 \quad (\star) \;\Rightarrow$$

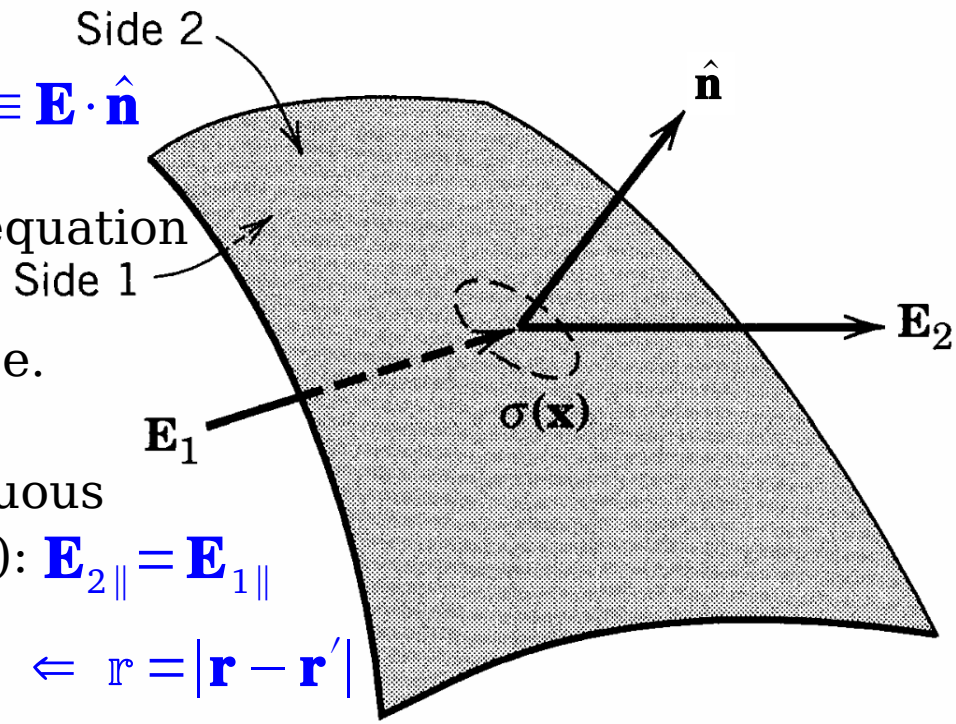
If the path is closed, the line integral is 0.

- *Stokes's theorem*: $\oint_C \mathbf{A}\cdot \mathrm{d}\,\boldsymbol{\ell} = \int_S \nabla\times\mathbf{A}\cdot \mathrm{d}\,\boldsymbol{a}$

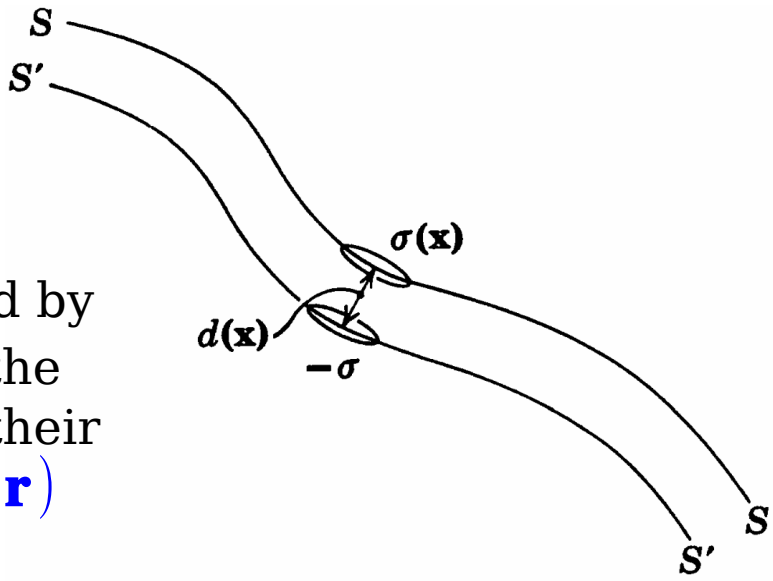
- With the line integral of the electric field being independent of the path and the application of the Stokes's theorem $\Rightarrow \quad \nabla\times\mathbf{E} = 0$

- A shorthand definition: $\vec{\mathbb{r}} \equiv \mathbf{r} - \mathbf{r}' \;\Rightarrow\; \mathbb{r} = \left|\mathbf{r} - \mathbf{r}'\right| \;\Rightarrow\; \hat{\mathbb{r}} = \dfrac{\vec{\mathbb{r}}}{\mathbb{r}} = \dfrac{\mathbf{r} - \mathbf{r}'}{\left|\mathbf{r} - \mathbf{r}'\right|}$

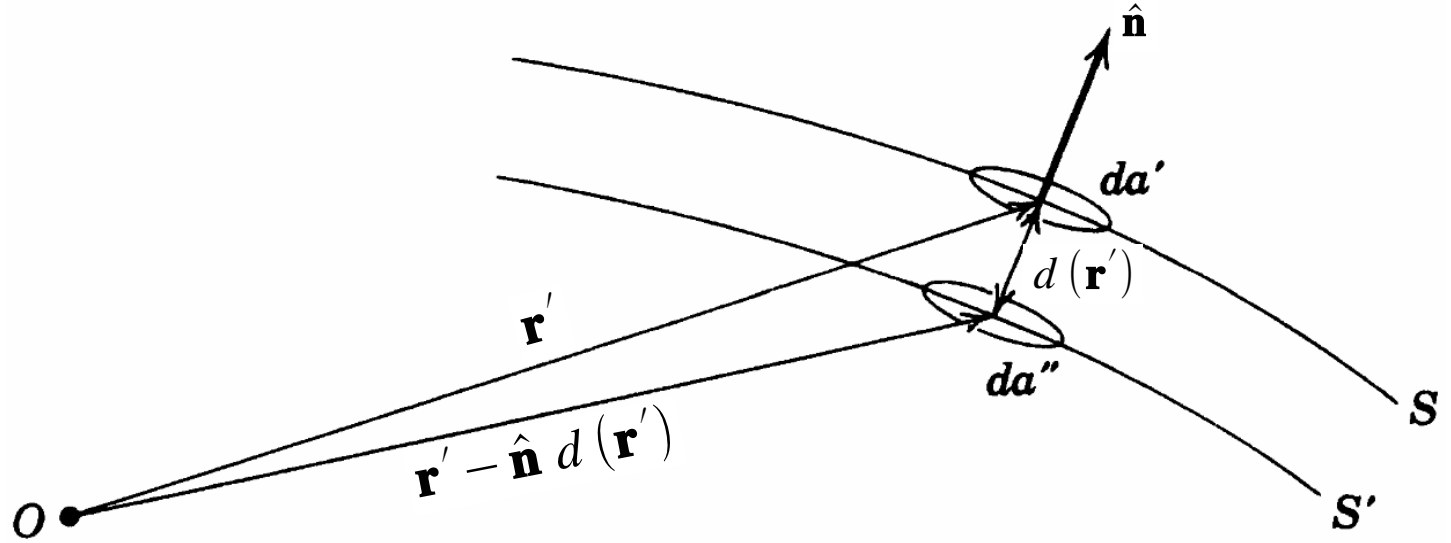
Surface Distributions of Charges & Dipoles and Discontinuities in the Electric Field & Potential



- Gauss's law tells $E_{2\perp} - E_{1\perp} = \frac{\sigma}{\epsilon_0} \Leftarrow E_\perp \equiv \mathbf{E}\cdot\hat{\mathbf{n}}$
- This does not determine $\mathbf{E}_1$ and $\mathbf{E}_2$. The equation means that there is a discontinuity in the normal component of $\mathbf{E}$ in crossing a surface.
- The tangential component of $\mathbf{E}$ is continuous across a boundary surface from equation (*): $\mathbf{E}_{2\parallel} = \mathbf{E}_{1\parallel}$
- In this case $\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\mathbf{r}')}{\mathbb{r}}\,\mathrm{d}a' \Leftarrow \mathbb{r} = |\mathbf{r}-\mathbf{r}'|$
- For volume or surface distributions of charge, the potential is continuous, $\Phi_2 = \Phi_1$, even within the charge distribution. $\mathbf{E}$ is bounded, even if discontinuous across a surface distribution of charge.



- With point or line charges, or dipole layers, the potential is no longer continuous.
- The dipole-layer distribution of strength is formed by letting S' approach infinitesimally close to S while the surface-charge density becomes infinite such that their product approaches the limit: $\lim_{d(\mathbf{r})\to 0} \sigma(\mathbf{r})\,d(\mathbf{r}) = D(\mathbf{r})$



- $\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_s \frac{\sigma(\mathbf{r}')}{\mathbb{r}}\,\mathrm{d}a' + \frac{1}{4\pi\epsilon_0}\int_{s'} \frac{-\sigma(\mathbf{r}')}{|\vec{\mathbb{r}} + \hat{\mathbf{n}}\,d|}\,\mathrm{d}a'' \Leftarrow \vec{\mathbb{r}} = \mathbf{r} - \mathbf{r}',\ \mathbb{r} = |\mathbf{r} - \mathbf{r}'|$

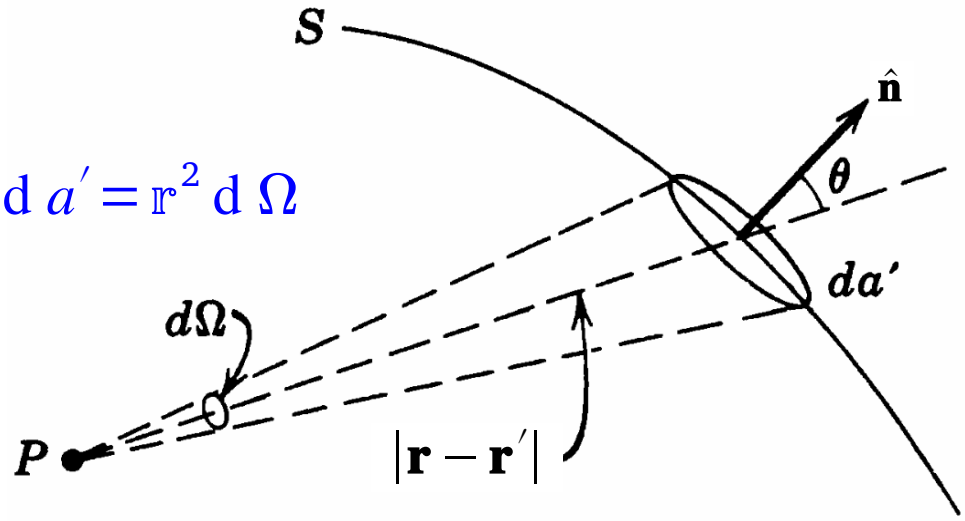
$$\approx -\frac{1}{4\pi\epsilon_0}\int_s \sigma\, d\,\hat{\mathbf{n}}\cdot\nabla\frac{1}{\mathbb{r}}\,\mathrm{d}a' \Leftarrow \frac{1}{|\vec{\mathbb{r}} + \mathbf{a}|} = \frac{1}{\mathbb{r}} + \mathbf{a}\cdot\nabla\frac{1}{\mathbb{r}} + \cdots,\ \nabla\frac{1}{\mathbb{r}} = -\frac{\hat{\mathbb{r}}}{\mathbb{r}^2}$$

$$= \frac{1}{4\pi\epsilon_0}\int_s D(\mathbf{r}')\frac{\hat{\mathbb{r}}}{\mathbb{r}^2}\cdot\hat{\mathbf{n}}\,\mathrm{d}a' = -\frac{1}{4\pi\epsilon_0}\int_s D(\mathbf{r}')\frac{\cos\theta}{\mathbb{r}^2}\,\mathrm{d}a'$$

The $-$ sign comes from that the angle is viewed by the observer.

$$= -\frac{1}{4\pi\epsilon_0}\int_s D(\mathbf{r}')\,\mathrm{d}\Omega \Leftarrow \cos\theta\,\mathrm{d}a' = \mathbb{r}^2\,\mathrm{d}\Omega$$

$$\rightarrow -\frac{\Delta\Omega\, D}{4\pi\epsilon_0} \quad \text{if } D = \text{const}$$



- $\mathrm{d}\,\Omega$ has a + sign if θ is an acute angle, ie, when the observation point views the "inner" side of the dipole layer.

- For a constant surface-dipole-moment density, the potential is the product of the moment divided by $4\,\pi\,\epsilon_0$ and the solid angle subtended at the observation point by the surface, regardless of its shape.

- In the integrand we notice that it is the sum of the potential of a point dipole with dipole moment $\mathrm{d}\,\mathbf{p} = D\,\mathrm{d}\,\boldsymbol{a}' \;\Rightarrow\; \Phi(\mathbf{r}) = \dfrac{1}{4\,\pi\,\epsilon_0}\displaystyle\int_S \frac{\hat{\mathbb{r}}}{\mathbb{r}^2}\cdot\mathrm{d}\,\mathbf{p}$

- There is a discontinuity in potential in crossing a double layer. The total potential jump in crossing the surface is: $\Phi_2 - \Phi_1 = \dfrac{D}{\epsilon_0} = -\dfrac{-2\,\pi\,D}{4\,\pi\,\epsilon_0} - \left(-\dfrac{2\,\pi\,D}{4\,\pi\,\epsilon_0}\right)$

- The potential has a discontinuity of $\dfrac{D}{\epsilon_0}$ in crossing from the inner to the outer side, being $-\dfrac{D}{2\,\epsilon_0}$ on the inner side and $+\dfrac{D}{2\,\epsilon_0}$ on the outer.

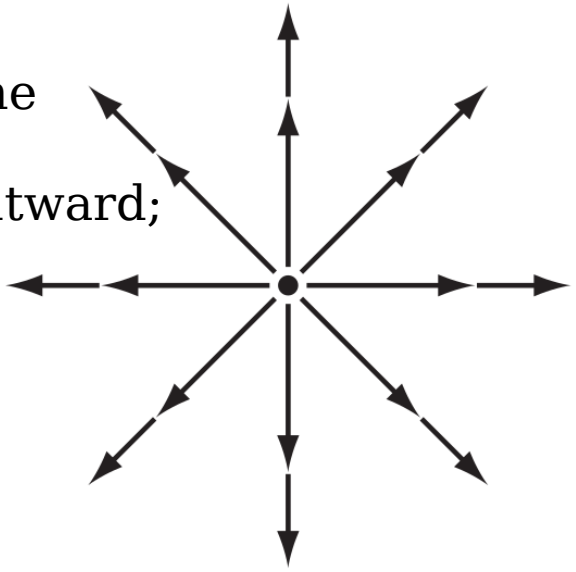
Poisson and Laplace Equations

- $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \Rightarrow \quad \nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \quad \Leftarrow$ Poisson equation

  $\nabla \times \mathbf{E} = 0 \quad \Rightarrow \quad \mathbf{E} = -\nabla \Phi \qquad \nabla^2 \Phi = 0 \quad \Leftarrow$ Laplace equation (for $\rho = 0$)

  $\Rightarrow \quad \Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3 x' \quad$ for $\quad \Phi(\infty) = 0$

- To verify the result directly and avoid being singular in the resulting integrand: $\mathbf{v} \equiv -\nabla \frac{1}{r} = \frac{\hat{\mathbf{r}}}{r^2}$ is directed radially outward; it is likely to have a large positive divergence from it. But



$$\nabla \cdot \mathbf{v} = -\nabla^2 \frac{1}{r} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{1}{r^2} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} (1) = 0 \quad ? \quad (?)$$

- If we integrate over a sphere of radius R, centered at the origin, the surface integral $\int \mathbf{v} \cdot d\boldsymbol{a} = \int \frac{\hat{\mathbf{r}}}{R^2} \cdot R^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}} = \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi = 4\pi \quad (\$)$

- But the *volume* integral, $-\int \nabla^2 \frac{1}{r} \, d^3 x = \int \nabla \cdot \mathbf{v} \, d^3 x = 0 \quad \Leftarrow \quad \nabla \cdot \mathbf{v} = 0 \quad \Leftarrow \quad (?)$

- Since $\int_v \nabla \cdot \mathbf{v}\, d^3x = \oint_s \mathbf{v} \cdot d\boldsymbol{a}$. Does this mean that the divergence theorem is false?

- The source of the problem is at $r=0$, where $\mathbf{v}$ blows up. It is true that $\nabla \cdot \mathbf{v}=0$ everywhere *except* the origin, but right *at* the origin the situation is complicated.

- The surface integral ($) is *independent of* R; if the divergence theorem is right (and it is), we should get $\int \nabla \cdot \mathbf{v}\, d^3x = 4\pi$ for *any* sphere centered at the origin, no matter how small. So the entire contribution must come from the point $r=0$.

- Thus, $\nabla \cdot \mathbf{v}$ has the property that it vanishes everywhere except at one point, and yet its integral (over any volume containing that point) is 4π.

- This is where the **Dirac delta function** comes in.

- The divergence of $\frac{\hat{\mathbf{r}}}{r^2}$ is 0 everywhere except at the origin, and yet its integral over any volume containing the origin is a constant (4π). These are precisely the defining conditions for the Dirac delta function;

$$\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\,\delta(\mathbf{r}-\mathbf{r}') \quad \text{or} \quad \nabla^2 \frac{1}{\mathbb{r}} = -4\pi\,\delta(\vec{\mathbb{r}})$$

1.5

The time-averaged potential of a neutral hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right)$$

where q is the magnitude of the electronic charge, and $\alpha^{-1} = a_0/2$, a_0 being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically.

The corresponding electric field are

$$\mathbf{E} = -\nabla\Phi = -\frac{\partial\Phi}{\partial r}\mathbf{e}_r = \frac{qe^{-\alpha r}}{8\pi\epsilon_0}\left[\alpha^2 + \frac{2\alpha}{r} + \frac{2}{r^2}\right]\mathbf{e}_r.$$

Since Φ is singular at $r = 0$, we divide the calculation of the charge density ρ into two parts, i.e., one for $r \neq 0$ and the other for $r \to 0$. And then these two parts are combined after the calculation to have ρ for the whole space.

For $r \neq 0$, the Poisson equation of Φ gives

$$\rho_1 = -\epsilon_0\nabla^2\Phi = -\epsilon_0\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) = -\frac{q\alpha^3}{8\pi}e^{-\alpha r}.$$

For $r \to 0$, the Gauss's law gives

$$\lim_{r\to 0}\int_0^r \rho_0 \mathrm{d}^3x = \lim_{r\to 0}\epsilon_0\oint \mathbf{E}\cdot\mathbf{n}\mathrm{d}a = \lim_{r\to 0}\frac{qe^{-\alpha r}}{4\pi}\frac{1+\alpha r+\alpha^2 r^2/2}{r^2}\times 4\pi r^2$$

$$= q\lim_{r\to 0}\left[1+\alpha r+\frac{\alpha^2 r^2}{2}\right]e^{-\alpha r} = q.$$

This leads $\rho_0 = q\delta(\mathbf{r})$. Therefore, combining the two parts gives

$$\rho = \rho_0 + \rho_1 = q\delta(\mathbf{r}) - q\frac{\alpha^3}{8\pi}e^{-\alpha r} = q\delta(\mathbf{r}) - q\frac{e^{-\alpha r}}{a_0^3\pi}.$$

Physically, we identify ρ_0 as the charge density of proton, the nucleus of a hydrogen, and ρ_1 as the charge density of the electron cloud of the hydrogen.

Remark: the charge at $r = 0$ can be obtained alternatively by

$$\lim_{r\to 0}\int_0^r \rho_0 \mathrm{d}^3x = \lim_{r\to\infty}\left[\epsilon_0 \oint \mathbf{E}\cdot\mathbf{n}\mathrm{d}a - \int \rho_1 \mathrm{d}^3x\right] = 0 - (-q) = q.$$

In the calculation it also indicates that the total charge is zero for a neutral hydrogen.

Green's Theorem

- To handle the boundary conditions it is necessary to develop some new mathematical tools. The divergence theorem:

$$\int_v \nabla\cdot\mathbf{A}\,\mathrm{d}^3x=\oint_s \mathbf{A}\cdot\mathrm{d}\boldsymbol{a}\ \text{ and }\ \mathbf{A}=\phi\nabla\psi\ \Rightarrow\ \nabla\cdot(\phi\nabla\psi)=\phi\nabla^2\psi+\nabla\phi\cdot\nabla\psi,\quad \phi\nabla\psi\cdot\hat{\mathbf{n}}=\phi\frac{\partial\psi}{\partial n},\quad \mathrm{d}\boldsymbol{a}=\hat{\mathbf{n}}\,\mathrm{d}a$$

$$\Rightarrow \int_v(\phi\nabla^2\psi+\nabla\phi\cdot\nabla\psi)\,\mathrm{d}^3x=\oint_s \phi\frac{\partial\psi}{\partial n}\,\mathrm{d}a \ \Leftarrow\ \text{Green's 1st identity}\ \Rightarrow\ \phi\leftrightarrow\psi$$

$$\Rightarrow \int_v(\phi\nabla^2\psi-\psi\nabla^2\phi)\,\mathrm{d}^3x=\oint_s\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)\mathrm{d}a \ \Leftarrow\ \text{Green's 2nd identiy}$$
$$\text{Green theorem}$$

- Convert the Poisson differential equation into an integral equation

$$\nabla^2\Phi=-\frac{\rho}{\epsilon_0},\quad \psi=\frac{1}{\mathbb{r}}=\frac{1}{|\mathbf{r}-\mathbf{r}'|}\ \Rightarrow\ \nabla^2\psi=\nabla^2\frac{1}{\mathbb{r}}=-4\pi\delta(\vec{\mathbb{r}})\ \Leftarrow\ \vec{\mathbb{r}}=\mathbf{r}-\mathbf{r}'$$

$$\Rightarrow \int_v\left(-4\pi\Phi(\mathbf{r}')\delta(\vec{\mathbb{r}})+\frac{\rho(\mathbf{r}')}{\epsilon_0}\frac{1}{\mathbb{r}}\right)\mathrm{d}^3x'=\oint_s\left(\Phi\frac{\partial}{\partial n'}\frac{1}{\mathbb{r}}-\frac{1}{\mathbb{r}}\frac{\partial\Phi}{\partial n'}\right)\mathrm{d}a'$$

$$\Rightarrow \begin{bmatrix}\Phi(\mathbf{r})\\0\end{bmatrix}=\frac{1}{4\pi\epsilon_0}\int_v\frac{\rho(\mathbf{r}')}{\mathbb{r}}\mathrm{d}^3x'+\frac{1}{4\pi}\oint_s\left(\frac{1}{\mathbb{r}}\frac{\partial\Phi}{\partial n'}-\Phi\frac{\partial}{\partial n'}\frac{1}{\mathbb{r}}\right)\mathrm{d}a'\ \Leftarrow\begin{bmatrix}\mathbf{r}\in V\\\mathbf{r}\notin V\end{bmatrix}$$

- For $\mathbf{r} \in V$, if $S \to \infty$, $E(S) \propto \frac{1}{\mathbb{r}^{1+\Delta}} \Rightarrow \Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathbb{r}} \mathrm{d}^3 x'$

- $\rho(\mathbf{r}') = 0 \Rightarrow \Phi(\mathbf{r}) = \frac{1}{4\pi} \oint_S \left(\frac{1}{\mathbb{r}} \frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'} \frac{1}{\mathbb{r}} \right) \mathrm{d}\, a'$

$$= \oint_S f\left(\Phi, \frac{\partial \Phi}{\partial n'} \right) \mathrm{d}\, a'$$

- This result is not a solution to a boundary-value problem, but only an integral statement, since the arbitrary specification of both Φ and $\partial\Phi$ (Cauchy boundary conditions) is an overspecification of the problem.

Uniqueness of the Solution with Dirichlet or Neumann Boundary Conditions

- *Dirichlet problem/Dirichlet boundary condition*: specification of the potential $\Phi_{|S}$ on a closed surface defines a unique potential problem.

- *Neumann boundary condition*: specification of the electric field $\mathbf{E}_{|S}$ (normal derivative of the potential) everywhere on the surface defines a unique problem.

- To show the uniqueness of the solution of the Poisson equation inside a volume subject to Dirichlet/Neumann boundary conditions on the closed bounding surface.

- Assume 2 solutions satisfy the same boundary conditions and $\Psi = \Phi_1 - \Phi_2$

$$\Rightarrow \quad \nabla^2 \Psi = 0 \quad \text{in} \quad \mathcal{V}, \quad \text{and} \quad \Psi_{|S} = 0 \quad \text{(Dirichlet)}, \quad \text{or} \quad \frac{\partial \Psi}{\partial n}_{|S} = 0 \quad \text{(Neumann)}$$

$$\Rightarrow \int_{\mathcal{V}} \left(\Psi \nabla^2 \Psi + \nabla \Psi \cdot \nabla \Psi \right) \mathrm{d}^3 x = \oint_S \Psi \frac{\partial \Psi}{\partial n} \mathrm{d}\, a \quad \Leftarrow \quad \phi = \psi = \Psi \quad \Rightarrow \int_{\mathcal{V}} |\nabla \Psi|^2 \mathrm{d}^3 x = 0$$

$$\Rightarrow \quad \nabla \Psi = 0 \ \text{inside} \ \mathcal{V} \quad \Rightarrow \quad \Psi = \text{const} \quad \Rightarrow \quad \begin{array}{lll} \Psi = 0 & \text{(Dirichlet)} & \Rightarrow \quad \Phi_1 = \Phi_2 \\ \Psi = \text{const} & \text{(Neumann)} & \Rightarrow \quad \Phi_1 = \Phi_2 + \text{const} \end{array}$$

- There is also a unique solution to a problem with mixed boundary conditions.

- A solution to the Poisson equation doesn't necessarily exist with arbitrary Φ and $\partial \Phi$ specified on a closed boundary.

Formal Solution of Electrostatic Boundary-Value Problem with Green Function

- $\nabla'^2 \frac{1}{\mathbb{r}} = -4\pi\delta(\vec{\mathbb{r}})$ $\Leftarrow$ the potential of a unit point source $\Leftarrow$ $\vec{\mathbb{r}} = \mathbf{r} - \mathbf{r}'$, $\mathbb{r} = |\mathbf{r} - \mathbf{r}'|$
- The function is only one of a class of functions depending on the variables $\mathbf{r}$ and $\mathbf{r}'$, and called *Green functions*, which satisfy $\nabla'^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\vec{\mathbb{r}})$

$$\Rightarrow \quad G(\mathbf{r}, \mathbf{r}') = \frac{1}{\mathbb{r}} + F(\mathbf{r}, \mathbf{r}') \quad \Rightarrow \quad \nabla'^2 F(\mathbf{r}, \mathbf{r}') = 0$$

- With the generalized concept of a Green function and its additional freedom $F(\mathbf{r}, \mathbf{r}')$, there arises the possibility that we can use Green's theorem and choose $F(\mathbf{r}, \mathbf{r}')$ to eliminate one or the other of the 2 surface integrals, obtaining a result that involves only Dirichlet or Neumann boundary conditions.

- $\phi = \Phi$, $\psi = G(\mathbf{r}, \mathbf{r}')$ $\Rightarrow$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\, G(\mathbf{r}, \mathbf{r}')\, \mathrm{d}^3 x' + \frac{1}{4\pi}\oint_S \left(G(\mathbf{r}, \mathbf{r}')\frac{\partial \Phi}{\partial n'} - \Phi(\mathbf{r}')\frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} \right) \mathrm{d}\, a'$$

- For *Dirichlet boundary conditions* we demand: $G_D(\mathbf{r}, \mathbf{r}') = 0$ for $\mathbf{r}'$ on $\mathcal{S}$

$$\Rightarrow \quad \Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \rho(\mathbf{r}')\, G_D(\mathbf{r}, \mathbf{r}')\, \mathrm{d}^3 x' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{r}')\frac{\partial G_D}{\partial n'}\, \mathrm{d}\, a'$$

- Gauss's theorem gives $\oint_S \frac{\partial G}{\partial n'} \mathrm{d}a' = \oint_S \nabla G \cdot \mathrm{d}\boldsymbol{a}' = \int_V \nabla^2 G \, \mathrm{d}^3 x' = -4\pi$

- For *Neumann boundary conditions*, the simplest allowable one is

$$\frac{\partial G_N}{\partial n'}(\mathbf{r}, \mathbf{r}') = -\frac{4\pi}{S} \quad \text{for } \mathbf{r}' \text{ on } S \quad \text{instead of} \quad \frac{\partial G_N}{\partial n'}(\mathbf{r}, \mathbf{r}') = 0 \quad \text{for } \mathbf{r}' \text{ on } S$$

where S is the total area of the boundary surface

$$\Rightarrow \quad \Phi(\mathbf{r}) = \langle \Phi \rangle_S + \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') G_N(\mathbf{r}, \mathbf{r}') \, \mathrm{d}^3 x' + \frac{1}{4\pi} \oint_S G_N \frac{\partial \Phi}{\partial n'} \mathrm{d}a'$$

- The customary Neumann problem is the so-called exterior problem in which the volume is bounded by 2 surfaces, one closed and finite, the other at infinity. Then the surface area is infinite; the average value vanishes.

- The mathematical symmetry property $G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$ merely represents the physical interchangeability of the source and the observation points.

- For the physical meaning of $\frac{F(\mathbf{r}, \mathbf{r}')}{4\pi\epsilon_0}$, it is a solution of the Laplace equation

inside V and represents the potential of charges *external to the volume* V.

1.10

Prove the *mean value theorem*: For charge-free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of *any* sphere centered on that point.

With Green's theorem

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\mathrm{d}^3x = \oint_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right]\mathrm{d}a,$$

let $\phi = \Phi$, the scalar electric potential, $\psi = G(\mathbf{x}, \mathbf{x}')$, a Green function, then

$$\int_V (\Phi\nabla^2 G - G\nabla^2\Phi)\mathrm{d}^3x = \oint_S \left[\Phi\frac{\partial G}{\partial n} - G\frac{\partial\Phi}{\partial n}\right]\mathrm{d}a,$$

where Φ and G satisfy the following Poisson equations

$$\nabla'^2\Phi(\mathbf{x}') = -\frac{\rho(\mathbf{x}')}{\epsilon_0}, \quad \nabla'^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}').$$

With the subsitution of the Poisson equations, it gives

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x}, \mathbf{x}')\mathrm{d}^3x' + \frac{1}{4\pi}\oint_S \left[G(\mathbf{x}, \mathbf{x}')\frac{\partial\Phi}{\partial n'}(\mathbf{x}') - \Phi(\mathbf{x}')\frac{\partial G}{\partial n'}(\mathbf{x}, \mathbf{x}')\right]\mathrm{d}a' \\ &= \frac{1}{4\pi}\oint_S \left[G(\mathbf{x}, \mathbf{x}')\frac{\partial\Phi}{\partial n'}(\mathbf{x}') - \Phi(\mathbf{x}')\frac{\partial G}{\partial n'}(\mathbf{x}, \mathbf{x}')\right]\mathrm{d}a',\end{aligned}$$

for $\rho = 0$ in a charge-free space. Let's choose the point $\mathbf{x}$ where the potential is to be evaluated to be the origin of the coordinate. Choose $G(\mathbf{x}, \mathbf{x}') = |\mathbf{x} - \mathbf{x}'|^{-1}$ where $\mathbf{x}'$ belongs to the surface S of a spherical boundary of radius R centered at the point $\mathbf{x} = 0$. With

$$G(0, \mathbf{x}') = \frac{1}{R}, \ \frac{\partial G}{\partial n'}(0, \mathbf{x}') = \nabla' G(0, \mathbf{x}') \cdot \mathbf{n}' = -\frac{1}{R^2}, \ \frac{\partial \Phi}{\partial n'}(\mathbf{x}') = \nabla' \Phi(\mathbf{x}') \cdot \mathbf{n}' = -\mathbf{E} \cdot \mathbf{n}',$$

we have

$$\begin{aligned} \Phi(0) &= -\frac{1}{4\pi} \oint_S \left[\frac{1}{R} \mathbf{E} \cdot \mathbf{n}' - \Phi(\mathbf{x}') \frac{1}{R^2} \right] \mathrm{d}a' \\ &= \frac{1}{4\pi R^2} \oint_S \Phi(\mathbf{x}') \mathrm{d}a' - \frac{1}{4\pi R} \int_V \nabla' \cdot \mathbf{E} \mathrm{d}^3 x' = \frac{1}{4\pi R^2} \oint_S \Phi(\mathbf{x}') \mathrm{d}a' = \langle \Phi(\mathbf{x}') \rangle_S, \end{aligned}$$

where the Gauss theorem is applied in the above equation and $\nabla \cdot \mathbf{E} = 0$ in a charge-free space. Therefore, for charge-free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of any sphere centered on that point.

Electrostatic Potential Energy & Energy Density; Capacitance

- If a point charge is brought from infinity to a point in a scalar potential (which vanishes at infinity), the work done on the charge is

$$W_i = q_i \, \Phi(\mathbf{r}_i) = \frac{q_i}{4\pi\epsilon_0} \sum_{j=1}^{n-1} \frac{q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \quad \Leftarrow \quad \Phi(\mathbf{r}_i) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{n-1} \frac{q_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

$$\Rightarrow \quad W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j<i} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} = \frac{1}{8\pi\epsilon_0} \sum_i \sum_j \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \quad \text{for } i \neq j \quad \text{no self-energy term}$$

$$\Rightarrow \quad W = \frac{1}{8\pi\epsilon_0} \iint \frac{\rho(\mathbf{r}) \, \rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3x \, d^3x' = \frac{1}{2} \int \rho(\mathbf{r}) \, \Phi(\mathbf{r}) \, d^3x \quad \text{for a continuous charge distribution}$$

With the integral form, the self-energy is usually included.

- Expresses the electrostatic potential energy in terms of the positions of the charges and emphasize the interactions between charges via Coulomb forces.

- An alternative approach is to emphasize the electric field and to interpret the energy as being stored in the electric field surrounding the charges

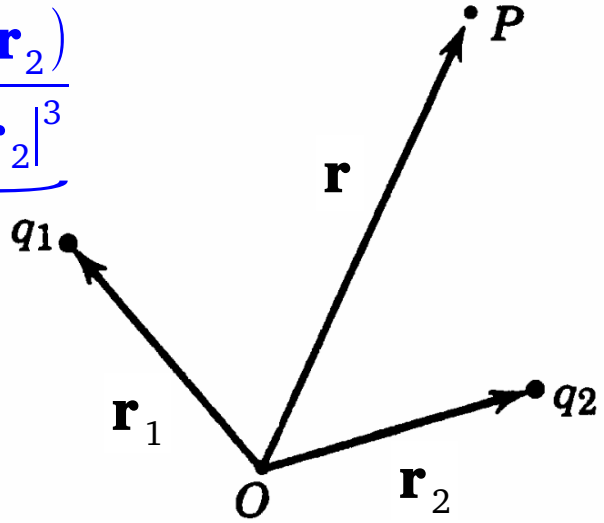
$$\Rightarrow \quad W = \frac{1}{2} \int \rho(\mathbf{r}) \, \Phi(\mathbf{r}) \, d^3x = -\frac{\epsilon_0}{2} \int \Phi \, \nabla^2 \Phi \, d^3x \quad \Leftarrow \quad \Phi \, \nabla^2 \Phi = \nabla \cdot (\Phi \, \nabla \Phi) - (\nabla \Phi)^2$$

$$= \frac{\epsilon_0}{2} \int |\nabla \Phi|^2 \, d^3x = \frac{\epsilon_0}{2} \int |\mathbf{E}|^2 \, d^3x \quad \Rightarrow \quad w = \frac{\epsilon_0}{2} |\mathbf{E}|^2 \geq 0 \quad \Leftarrow \quad \text{energy density}$$

- The energy density is positive definite. This contradicts our impression that the potential energy of 2 charges of opposite sign is negative. This apparent contradiction comes from "self-energy" contributions to the energy density. Ex:

$$\mathbf{E} = \frac{q_1}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}_1}{|\mathbf{r}-\mathbf{r}_1|^3} + \frac{q_2}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}_2}{|\mathbf{r}-\mathbf{r}_2|^3}$$

$$\Rightarrow \quad 32\pi^2\epsilon_0 w = \underbrace{\frac{q_1^2}{|\mathbf{r}-\mathbf{r}_1|^4} + \frac{q_2^2}{|\mathbf{r}-\mathbf{r}_2|^4}}_{\text{self energy}} + 2\underbrace{q_1 q_2 \frac{(\mathbf{r}-\mathbf{r}_1)\cdot(\mathbf{r}-\mathbf{r}_2)}{|\mathbf{r}-\mathbf{r}_1|^3|\mathbf{r}-\mathbf{r}_2|^3}}_{w_{\text{int}}}$$



Define $\boldsymbol{\rho} \equiv \dfrac{\mathbf{r}-\mathbf{r}_1}{|\mathbf{r}_1-\mathbf{r}_2|}$, $\hat{\mathbf{n}} \equiv \dfrac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|}$ $\Rightarrow$ $\boldsymbol{\rho}+\hat{\mathbf{n}} = \dfrac{\mathbf{r}-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|}$

$$\Rightarrow \quad 16\pi^2\epsilon_0 W_{\text{int}} = q_1 q_2 \int \frac{(\mathbf{r}-\mathbf{r}_1)\cdot(\mathbf{r}-\mathbf{r}_2)}{|\mathbf{r}-\mathbf{r}_1|^3|\mathbf{r}-\mathbf{r}_2|^3}\,\mathrm{d}^3x = \frac{q_1 q_2}{|\mathbf{r}_1-\mathbf{r}_2|}\int \frac{\boldsymbol{\rho}\cdot(\boldsymbol{\rho}+\hat{\mathbf{n}})}{\rho^3|\boldsymbol{\rho}+\hat{\mathbf{n}}|^3}\,\mathrm{d}^3\rho$$

$$= \frac{q_1 q_2}{|\mathbf{r}_1-\mathbf{r}_2|}\int \nabla_\rho\frac{1}{\rho}\cdot\nabla_\rho\frac{1}{|\boldsymbol{\rho}+\hat{\mathbf{n}}|}\,\mathrm{d}^3\rho = -\frac{q_1 q_2}{|\mathbf{r}_1-\mathbf{r}_2|}\int \frac{\mathrm{d}^3\rho}{|\boldsymbol{\rho}+\hat{\mathbf{n}}|}\nabla_\rho^2\frac{1}{\rho}$$

$$\Rightarrow \quad W_{\text{int}} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\mathbf{r}_1-\mathbf{r}_2|}\int \frac{\delta^3(\rho)}{|\boldsymbol{\rho}+\hat{\mathbf{n}}|}\,\mathrm{d}^3\rho = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\mathbf{r}_1-\mathbf{r}_2|} \quad \Leftarrow \quad \text{as expected}$$

- Forces acting between charged bodies can be obtained by calculating the change in the total electrostatic energy of the system under small virtual displacements.

- To calculate the force/area on the surface of a conductor with a (fixed) surface-charge density, imagine a small outward displacement of an elemental area of the surface $w = \frac{\epsilon_0}{2} |\mathbf{E}|^2 = \frac{\sigma^2}{2\,\epsilon_0} \quad \Rightarrow \quad \Delta W = -\frac{\sigma^2}{2\,\epsilon_0} \Delta a \, \Delta x$ an outward force/area (pressure) equal to w at the surface of the conductor.

- This result is normally derived by taking the product of the surface-charge density and the electric field, with care taken to eliminate the electric field due to the element of surface-charge density itself.

- For a system of n conductors, the electrostatic potential energy can be expressed in terms of the potentials alone and certain geometrical quantities called coefficients of capacity

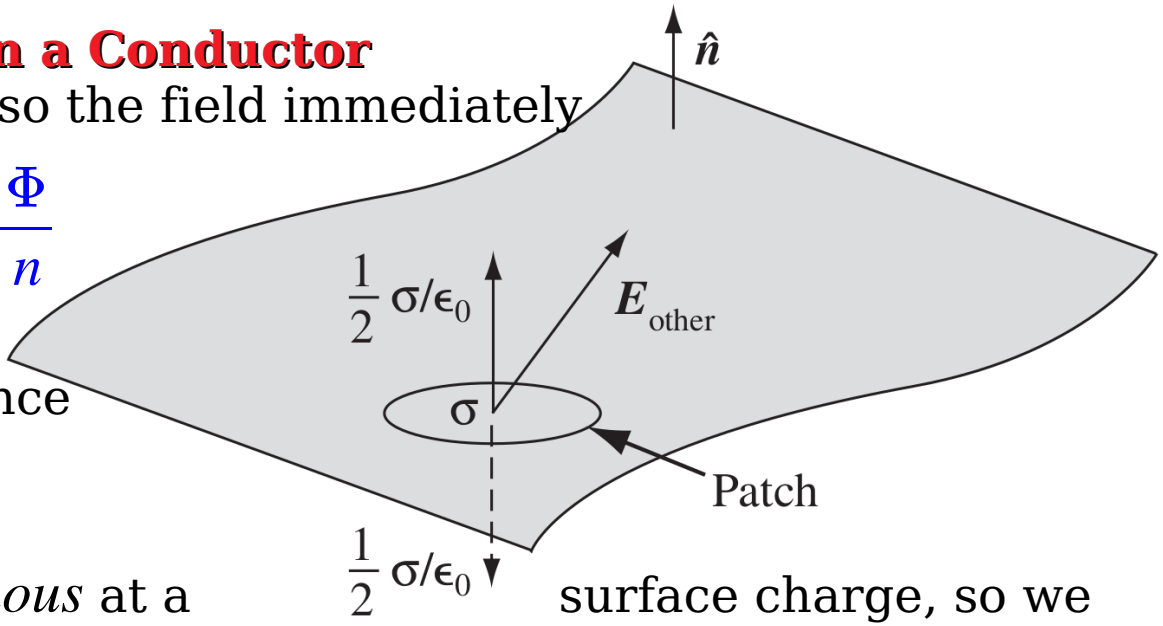
$$V_i = \sum_{j=1}^{n} p_{i\,j}\, Q_j\,, \quad i = 1, 2\,, \cdots, n \quad \Rightarrow \quad Q_i = \sum_{j=1}^{n} C_{i\,j}\, V_j\,, \quad i = 1, 2\,, \cdots, n$$

$$C_{i\,i}: \text{capacities or capacitances}\,, \quad C_{i\,j}\,(i \neq j): \text{coefficients of induction}$$

- *The capacitance of a conductor is the total charge on the conductor when it is maintained at unit potential, all other conductors being held at zero potential.*

Surface Charge and the Force on a Conductor

- The field inside a conductor is 0, so the field immediately *outside* is: $\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \quad \Rightarrow \quad \sigma = -\epsilon_0 \frac{\partial \Phi}{\partial n}$



- In the presence of an electric field, a surface charge will experience a force; the force/area, $\boldsymbol{f} = \sigma \mathbf{E}$.

- But the electric field is *discontinuous* at a surface charge, so we should use the *average* of the two: $\boldsymbol{f} = \sigma \mathbf{E}_{\text{average}} = \frac{1}{2}\sigma\left(\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}}\right)$

- The discontinuity is due entirely to the charge on the patch, which puts out a field ($\frac{\sigma}{2\epsilon_0}$) on either side, pointing away from the surface

$$\mathbf{E}_{\text{above}} = \mathbf{E}_{\text{other}} + \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$$

$$\mathbf{E}_{\text{below}} = \mathbf{E}_{\text{other}} - \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$$

$$\Rightarrow \quad \mathbf{E}_{\text{other}} = \frac{1}{2}\left(\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}}\right) = \mathbf{E}_{\text{average}}$$

- Averaging is a device for removing the contribution of the patch itself.

- The argument applies to *any* surface charge, especially, to a conductor,

$$\mathbf{E}_{\text{outside}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}\,, \quad \mathbf{E}_{\text{inside}} = 0 \;\Rightarrow\; \boldsymbol{f} = \sigma \mathbf{E}_{\text{average}} = \frac{\sigma}{2}\left(\mathbf{E}_{\text{outside}} + \mathbf{E}_{\text{inside}}\right) = \frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{n}}$$

- The capacitance of 2 conductors carrying equal and opposite charges in the presence of other grounded conductors is defined as the ratio of the charge on one conductor to the potential difference between them.

- The potential energy for the system of conductors: $$W = \frac{1}{2} \sum_{i=1}^{n} Q_i V_i = \frac{1}{2} \sum_{i,\, j=1}^{n} C_{ij} V_i V_j$$

- The expression of the energy in terms of the potentials and the C_{ij}, or in terms of the charges Q_i and the coefficients p_{ij} permits the application of variational methods to obtain approximate values of capacitances.

Selected problems: 3, 6, 9, 13, 17

1.15

Prove *Thomson's theorem*: If a number of surfaces are fixed in position and a given total charge is placed on each surface, then the electrostatic energy in the region bounded by the surfaces is an absolute minimum when the charges are placed so that every surface is an equipotential, as happens when they are conductors.

Let $\mathbf{E} = -\nabla\Phi$ be the electric field from the charge distribution that the total charge in the ith conductor with its surface S_i is q_i, where $i = 1, 2, \ldots$, and with which every surface S_i is an equipotential, i.e., $\Phi|_{S_i} = \Phi_i$, a constant potential on S_i. The total electrostatic energy $\mathcal{E}$ related to $\mathbf{E}$ in the region V bounded by S_i's is

$$\mathcal{E} = \frac{\epsilon_0}{2}\int_V |\mathbf{E}|^2 \mathrm{d}^3x.$$

Now let $\mathbf{E}'$ be the electric field from an arbitrary charge distribution that the total charge in the ith conductor is still q_i, but with which each surface S_i is *not* necessarily an equipotential surface. The total electrostatic energy $\mathcal{E}'$ related to $\mathbf{E}'$ in V bounded by S_i's is

$$\mathcal{E}' = \frac{\epsilon_0}{2}\int_V |\mathbf{E}'|^2 \mathrm{d}^3x.$$

The difference between these two total electrostatic energies is

$$\mathcal{E}' - \mathcal{E} = \frac{\epsilon_0}{2}\int_V (|\mathbf{E}'|^2 - |\mathbf{E}|^2)\mathrm{d}^3x = \frac{\epsilon_0}{2}\int_V (\mathbf{E}' - \mathbf{E})^2 \mathrm{d}^3x + \epsilon_0 \int_V (\mathbf{E}'\cdot\mathbf{E} - |\mathbf{E}|^2)\mathrm{d}^3x.$$

The second integral in the RHS of the above equation vanishes,

$$\int_V (\mathbf{E}' \cdot \mathbf{E} - |\mathbf{E}|^2) \mathrm{d}^3x = \int_V \mathbf{E} \cdot (\mathbf{E}' - \mathbf{E}) \mathrm{d}^3x = \int_V (-\nabla\Phi) \cdot (\mathbf{E}' - \mathbf{E}) \mathrm{d}^3x$$
$$= \int_V [\nabla \cdot (\Phi\mathbf{E} - \Phi\mathbf{E}') - \Phi(\nabla \cdot \mathbf{E} - \nabla \cdot \mathbf{E}')] \mathrm{d}^3x = \sum_i \oint_{S_i} \Phi(\mathbf{E} - \mathbf{E}') \cdot \mathbf{n} \mathrm{d}a$$
$$= \sum_i \Phi_i \left(\oint_{S_i} \mathbf{E} \cdot \mathbf{n} \mathrm{d}a - \oint_{S_i} \mathbf{E}' \cdot \mathbf{n} \mathrm{d}a \right) = \sum_i \Phi_i (q_i - q_i) = 0,$$

where $\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{E}' = 0$ is used since there is not any charge density in V. Then

$$\mathcal{E}' - \mathcal{E} = \frac{\epsilon_0}{2} \int_V (\mathbf{E}' - \mathbf{E})^2 \mathrm{d}^3x \geq 0,$$

and thus $\mathcal{E}' \geq \mathcal{E}$. Therefore, the electrostatic energy in the region bounded by the surfaces is an absolute minimum when the charges are placed so that every surface is an equipotential, as happens when they are conductors.