Chapter 13 Collisions, Energy Loss, and Scattering of Charged Particles; Cherenkov and Transition Radiation

- Consider collisions between swiftly moving, charged particles, with special emphasis on the exchange of energy between collision partners and on the accompanying deflections from the incident direction as well as Cherenkov radiation and transition radiation.

- If the fast incident charged particle is heavier than an electron, the collisions with electrons and with nuclei have different consequences.

- The electrons can take up appreciable amounts of energy from the incident particle without causing significant deflections, whereas the massive nuclei absorb very little energy but because of their greater charge cause scattering of the incident particle.

- Loss of energy by the incident particle occurs almost entirely in collisions with electrons. The deflection of the particle from its incident direction results from essentially elastic collisions with the atomic nuclei.

- A full quantum-mechanical treatment is needed to obtain exact results, even though all the essential features are classical or semi-classical in origin.

Rutherford Scattering

- The incident beam is characterized by specifying its *flux intensity* I, which gives the number of particles/normal area/time.

- The (*differential scattering*) *cross section for scattering in a given direction* is

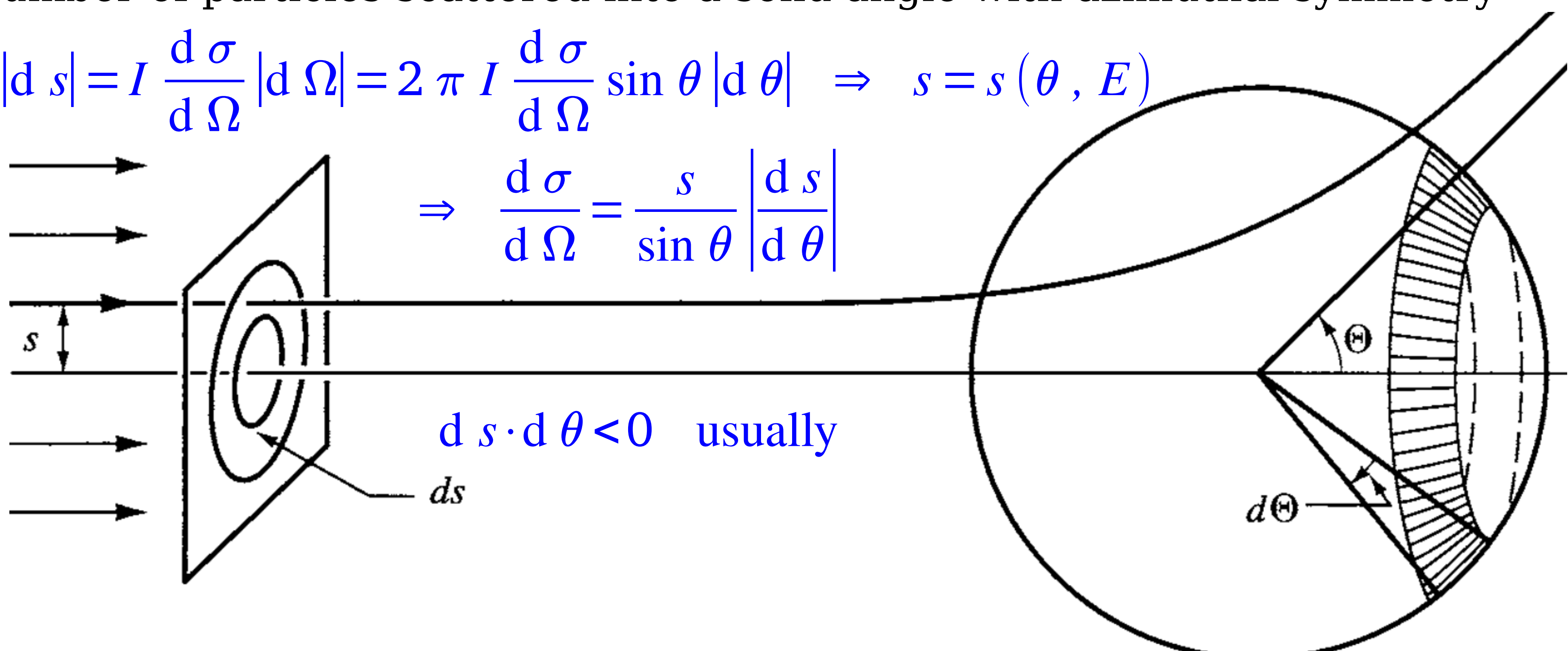
$$\frac{d\sigma}{d\Omega}(\theta,\phi)\, d\Omega = \frac{\text{number of particles scattered into solid angle } d\Omega \text{ per unit time}}{\text{incident intensity}}$$

- If v_0 is the incident speed of the particle $\Rightarrow \quad \ell = m v_0 s = s\sqrt{2mE}$

- The number of particles scattered into a solid angle with azimuthal symmetry

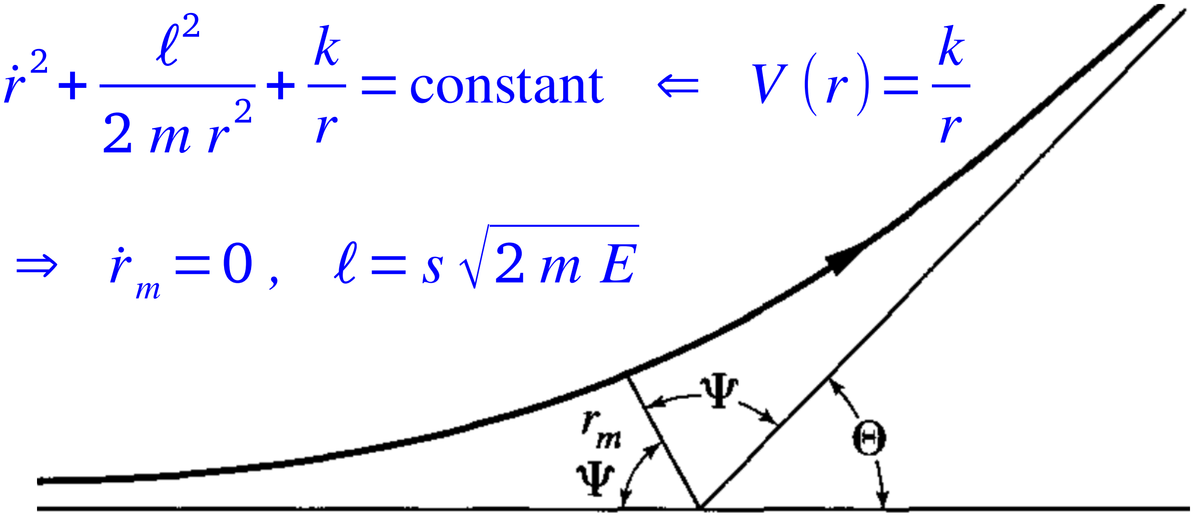
$$2\pi I s|ds| = I \frac{d\sigma}{d\Omega}|d\Omega| = 2\pi I \frac{d\sigma}{d\Omega} \sin\theta\, |d\theta| \quad \Rightarrow \quad s = s(\theta, E)$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{s}{\sin\theta}\left|\frac{ds}{d\theta}\right|$$



$$ds \cdot d\theta < 0 \quad \text{usually}$$

$$\mathcal{E} = \frac{1}{2}\, m \left(\dot{r}^2 + r^2\, \dot{\theta}^2 \right) + V(r) = \frac{1}{2}\, m\, \dot{r}^2 + \frac{\ell^2}{2\, m\, r^2} + \frac{k}{r} = \text{constant} \quad \Leftarrow \quad V(r) = \frac{k}{r}$$



$$\Rightarrow \quad \dot{r} = \sqrt{\frac{2}{m} \left(E - \frac{k}{r} - \frac{\ell^2}{2\, m\, r^2} \right)} \quad \Rightarrow \quad \dot{r}_m = 0\,, \quad \ell = s \sqrt{2\, m\, E}$$

$$\Rightarrow \quad r_m = \frac{2\, s^2\, E}{\sqrt{4\, s^2\, E^2 + k^2} - k}$$

$$\Rightarrow \quad \mathrm{d}\, t = \frac{m\, r\, \mathrm{d}\, r}{\sqrt{2\, m\, r \left(r\, E - k \right) - \ell^2}} \quad \Rightarrow \quad \mathrm{d}\, \theta = \frac{\ell\, \mathrm{d}\, r}{r \sqrt{2\, m\, r \left(r\, E - k \right) - \ell^2}} \quad \Rightarrow \quad \ell = m\, r^2 \frac{\mathrm{d}\, \theta}{\mathrm{d}\, t}$$

$$\Rightarrow \quad \psi = \int_{r_m}^{\infty} \frac{s \sqrt{E}\, \mathrm{d}\, r}{r \sqrt{r \left(r\, E - k \right) - E\, s^2}} = \int_0^{u_m} \frac{s \sqrt{E}\, \mathrm{d}\, u}{\sqrt{E - k\, u - E\, s^2\, u^2}} \quad \Leftarrow \quad u \equiv \frac{1}{r}\,, \quad u_m = \frac{1}{r_m}$$

$$= \cos^{-1} \frac{2\, s^2\, E\, u + k}{\sqrt{k^2 + 4\, s^2\, E^2}} \Bigg|_u^0 \quad \Leftarrow \quad \int \frac{\mathrm{d}\, x}{\sqrt{\alpha + \beta\, x - x^2}} = - \cos^{-1} \frac{2\, x - \beta}{\sqrt{\beta^2 + 4\, \alpha}} + C$$

$$= \cos^{-1} \frac{k}{\sqrt{k^2 + 4\, s^2\, E^2}} - \cos^{-1} 1 = \cos^{-1} \frac{k}{\sqrt{k^2 + 4\, s^2\, E^2}} = \frac{\pi - \theta}{2}$$

$$\Rightarrow \quad \sin \frac{\theta}{2} = \cos \frac{\pi - \theta}{2} = \frac{k}{\sqrt{k^2 + 4\, s^2\, E^2}} \quad \Rightarrow \quad \cot \frac{\theta}{2} = \frac{2\, s\, E}{k} \quad \Rightarrow \quad s = \frac{k}{2\, E} \cot \frac{\theta}{2}$$

$$\Rightarrow \quad \frac{\mathrm{d}\, \sigma}{\mathrm{d}\, \Omega} = \frac{s}{\sin \theta} \left| \frac{\mathrm{d}\, s}{\mathrm{d}\, \theta} \right| = \left(\frac{k}{2\, E} \csc^2 \frac{\theta}{2} \right)^2 \quad \Rightarrow \quad \text{Replace } k = z\, e^2\,, \quad E \sim p\, v$$

Energy Transfer in a Coulomb Collision Between Heavy Incident Particle & Stationary Free Electron; Energy Loss in Hard Collisions

- A swift particle of charge $z e$ and mass M (energy $E = \gamma M c^2$, momentum $P = \gamma \beta M c$) collides with an atomic electron of charge $-e$ and mass m.

- For energetic collisions the binding of the electron in the atom is neglected; the electron can be considered free and initially at rest in the laboratory.

- For $M \gg m$, the collision is best viewed as elastic Coulomb scattering in the rest frame of the incident particle.

- The Rutherford scattering formula $\dfrac{d\sigma}{d\Omega} = \left(\dfrac{\mathbb{z} e^2}{2 p v} \csc^2 \dfrac{\theta}{2} \right)^2$ where $p = \gamma \beta m c$

and $v = \beta c$ are the momentum and speed of the electron in the rest frame of the heavy particle (exact in the limit $M/m \to \infty$).

- The cross section can be given a Lorentz-invariant form by relating the scattering angle to the 4-momentum transfer squared, $Q^2 = -(\vec{p} - \vec{p}\,')^2$;

For elastic scattering

$$Q^2 = 4 p^2 \sin^2 \frac{\theta}{2} \quad \Rightarrow \quad \frac{d\sigma}{d Q^2} = 4\pi \left(\frac{\mathbb{z} e^2}{\beta c Q^2} \right)^2 \quad \Leftarrow \quad \beta^2 = 1 - \left(\frac{M m c^2}{\vec{p} \cdot \vec{P}} \right)^2$$

- The cross section for a given energy loss T by the incident particle is proportional to the above equation.

- In the electron's rest frame, the invariant $Q^2 = 2\,m\,T \quad \Rightarrow \quad \frac{d\,\sigma}{d\,T} = \frac{2\,\pi\,\mathbb{Z}^2\,e^4}{m\,c^2\,\beta^2\,T^2}$

the cross section/energy for energy loss T by the massive incident particle in a Coulomb collision with a free stationary electron.

- $T_{\min} < T < T_{\max} \quad \Leftarrow \quad T_{\min} \geq \hbar \langle \omega \rangle$: mean effective atomic binding energy

- Find $T_{\max}$ by recognizing that the most energetic collision in the rest frame of the incident particle occurs when the electron reverses its direction.

- In the LAB frame, after the collision the electron has

$$\begin{aligned} \mathcal{E}' &= \gamma\,m\,c^2 \\ p' &= \gamma\,m\,c\,\beta \end{aligned} \quad \Leftarrow \quad T_{\max} = \mathcal{E} - m\,c^2 = \gamma\left(\mathcal{E}' + \beta\,c\,p'\right) - m\,c^2 = 2\,\gamma^2\,\beta^2\,m\,c^2$$

- The exact answer for $T_{\max}$ has a factor in the denominator $D = 1 + \frac{2\,m\,\mathcal{E}}{M^2\,c^2} + \frac{m^2}{M^2}$

- For equal masses, $T_{max} = (\gamma - 1)\,M\,c^2$.

- When the spin of the electron is taken into account, there is a quantum-mechanical correction to the energy loss cross section,

$$\frac{d\,\sigma}{d\,T}_{|qm} = \frac{2\,\pi\,\mathbb{Z}^2\,e^4}{m\,c^2\,\beta^2\,T^2}\left(1 - \beta^2 \sin^2\frac{\theta}{2}\right) = \frac{2\,\pi\,\mathbb{Z}^2\,e^4}{m\,c^2\,\beta^2\,T^2}\left(1 - \beta^2\frac{T}{T_{\max}}\right)$$

- The energy loss/distance in collisions with energy transfer > $\varepsilon\ (\ll T_{\max})$ for a heavy particle passing through matter with N atoms/volume, with Z electrons,

$$\frac{\mathrm{d}\,\mathcal{E}}{\mathrm{d}\,x}\left(T>\varepsilon\right)=N\,Z\int_{\varepsilon}^{T_{\max}}T\,\frac{\mathrm{d}\,\sigma}{\mathrm{d}\,T}\,\mathrm{d}\,T=2\,\pi\,N\,Z\,\frac{\mathbb{Z}^2\,e^4}{m\,c^2\,\beta^2}\left(\ln\frac{2\,\gamma^2\,\beta^2\,m\,c^2}{\varepsilon}-\beta^2\right)\quad(\star)$$

- The term with $-\beta^2$ is the relativistic spin contribution.

- The equation represents the energy loss in close collisions, valid provided $\varepsilon\gg\hbar\langle\omega\rangle$ because binding has been ignored.

- Classically, in the rest frame of the heavy particle the incident electron approaches at impact parameter b. There is a one-to-one correspondence between b and the scattering angle θ (see Problem 13.1).

- The energy transfer $T\left(b\right)=\dfrac{2\,\mathbb{Z}^2\,e^4}{m\,v^2\left(b^2+b^{c\,2}_{\min}\right)}\quad\Leftarrow\quad b^c_{\min}=\dfrac{\mathbb{Z}\,e^2}{p\,v}\quad\Rightarrow\quad T\propto\dfrac{1}{b^2}$ for $b\gg b^c_{\min}$

- If the energy transfer is greater than ε, the impact parameter must be less than the maximum: $b_{\max}\left(\varepsilon\right)=\sqrt{\dfrac{2}{m\,\varepsilon}}\,\dfrac{\mathbb{Z}\,e^2}{v}$

- When the heavy particle passes through matter it sees electrons at all possible impact parameters, with weighting according to the area of an annulus, $2\,\pi\,b\,\mathrm{d}\,b$.

- The classical energy loss/distance for collisions with transfer greater than ε is

$$\frac{d\,\mathcal{E}}{d\,x}(T>\varepsilon)=2\,\pi\,N\,Z\int_0^{b_{\max}} T(b)\,b\,d\,b=4\,\pi\,N\,Z\,\frac{\mathbb{z}^2\,e^4}{m\,c^2\,\beta^2}\ln\frac{b_{\max}(\varepsilon)}{b^c_{\min}}\quad\Rightarrow\quad\begin{array}{l}(\star)\\ \text{no spin}\end{array}$$

- Same result (for a spinless particle) quantum mechanically and classically is a consequence of the validity of the Rutherford cross section in both regimes.

- To find a classical result for the total energy loss/distance, we must address the influence of atomic binding.

- The incident heavy particle produces appreciable time-varying electromagnetic fields at the atom for a time $\Delta\,t\approx\dfrac{b}{\gamma\,v}\quad\Leftarrow\quad\text{eqn}\,(11.153)$

- If the characteristic time $\Delta\,t$ is long compared to the atomic period $\dfrac{2\,\pi}{\langle\omega\rangle}$, the atom responds adiabatically—it stretches slowly during the encounter and returns to normal, without appreciable energy being transferred.

- If $\Delta\,t$ is very short compared to the characteristic period, the electron can be treated as almost free.

- The dividing line is $\langle\omega\rangle\,\Delta\,t\sim 1\quad\Rightarrow\quad\bar{b}_{\max}\approx\dfrac{\gamma\,v}{\langle\omega\rangle}$, beyond which no significant energy transfer is possible.

$$\Rightarrow \quad \frac{\mathrm{d}\,\mathcal{E}}{\mathrm{d}\,x}_{|\text{classical}} = 4\,\pi\,N\,Z\,\frac{\mathbb{Z}^2\,e^4}{m\,c^2\,\beta^2}\ln\,B_c \quad (@) \quad \Leftarrow \quad B_c = \lambda\,\frac{\gamma^2\,\beta^3\,m\,c^2}{\mathbb{Z}\,e^2\,\langle\omega\rangle} = \lambda\,\frac{\gamma^2\,\beta^3\,m\,c^2}{\eta\,\hbar\,\langle\omega\rangle}$$

where $\eta = \dfrac{\mathbb{Z}\,e^2}{\hbar\,v}$: characteristic of quantum-mechanical Coulomb scattering $\Rightarrow$ $\eta \ll 1$: quantum; $\eta \gg 1$: classical

- With many different electronic frequencies, $\langle\omega\rangle$ is the geometric mean of all the frequencies ω_i, weighted with the oscillator strength f_i: $Z\ln\langle\omega\rangle = \sum f_j \ln\omega_j$

Energy Loss from Soft Collisions; Total Energy Loss

- The energy loss in collisions with energy transfers less than ε really can be treated properly only by quantum mechanics. We can "explain" the result in semiclassical language afterwards. Bethe expression

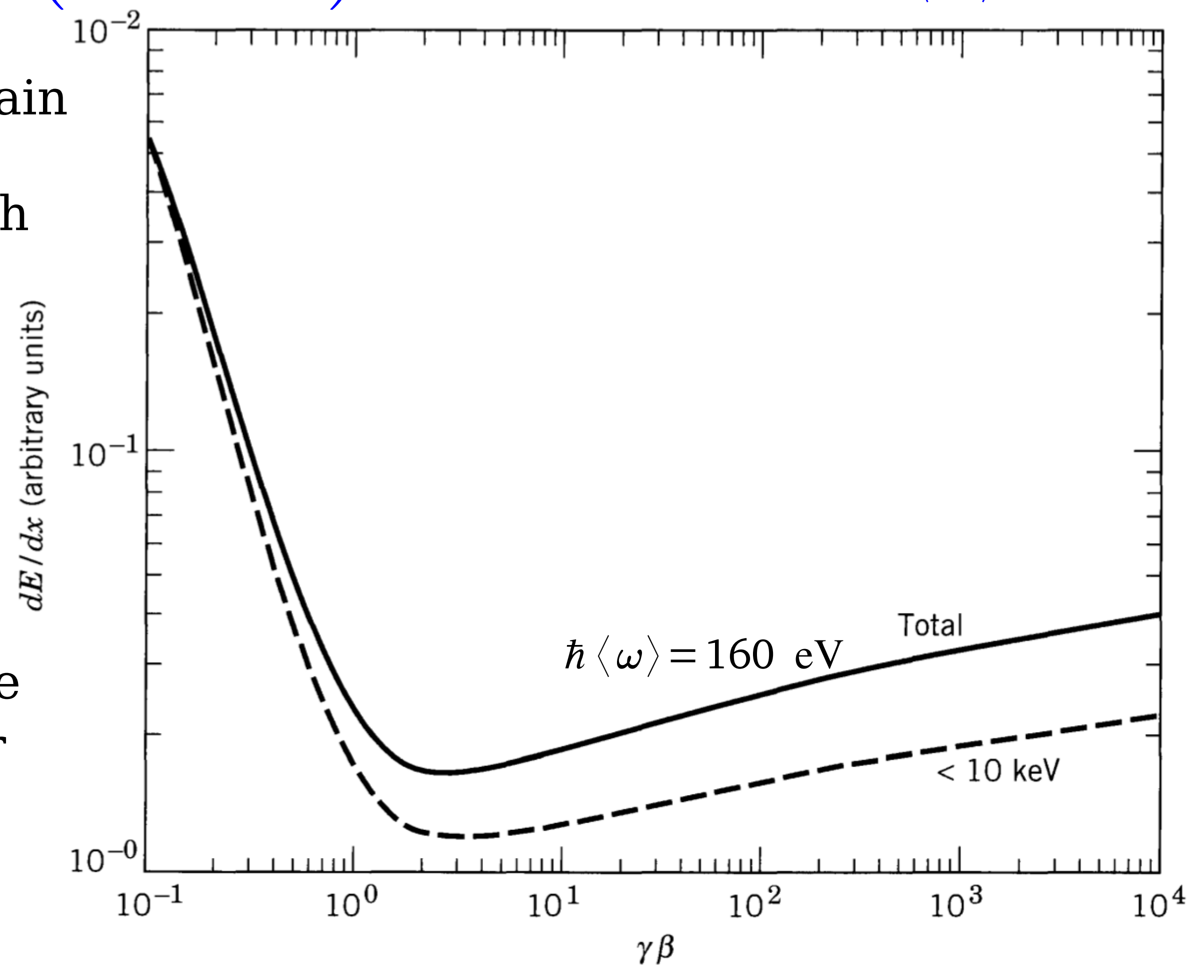
- $$\frac{d\,\mathcal{E}}{d\,x}(T<\varepsilon)=2\,\pi\,N\,Z\,\frac{\mathbb{z}^2 e^4}{m\,c^2\,\beta^2}\left(\ln B_q^2(\varepsilon)-\beta^2\right)\quad(\$)\qquad\Leftarrow\qquad B_q(\varepsilon)=\frac{\gamma\,v\sqrt{2\,m\,\varepsilon}}{\hbar\langle\omega\rangle}$$

$$\Rightarrow\quad \text{Total}\ \frac{d\,\mathcal{E}}{d\,x}=4\,\pi\,N\,Z\,\frac{\mathbb{z}^2 e^4}{m\,c^2\,\beta^2}\left(\ln B_q-\beta^2\right)\quad(\#)\qquad\Leftarrow\qquad B_q=\frac{2\,\gamma^2\beta^2\,m\,c^2}{\hbar\langle\omega\rangle}$$

- For low energies ($\gamma\beta<1$) the main dependence is as $\dfrac{1}{\beta^2}$, while at high energies $\propto\ln\gamma$.

- $\dfrac{d\,\mathcal{E}}{d\,x}$ has its minimum at $\gamma\beta\approx 3$.

- Explain semiclassically why $\dfrac{d\,\mathcal{E}}{d\,x}$ in the soft collision increases more slowly than the total as $\ln\gamma$ rather than $\ln\gamma^2$.



- $B_c = \dfrac{b_{\max}}{b^c_{\min}} \Leftarrow b^c_{\min} = \dfrac{\mathbb{Z}\, e^2}{\gamma\, m\, v^2} \Rightarrow b_{\min}$ different for $\eta < 1 \Rightarrow b^q_{\min}$ quantum level

electron momentum $p = \gamma\, m\, v \Rightarrow \Delta p \to 0 \Rightarrow \Delta p \ll p \Rightarrow$ uncertainty in impact parameter $\Delta b \gg \dfrac{\hbar}{p}$

$$\Rightarrow b^q_{\min} = \frac{\hbar}{\gamma\, m\, v} \Rightarrow \frac{b^c_{\min}}{b^q_{\min}} = \eta = \frac{\mathbb{Z}\, e^2}{\hbar\, v}$$

- In calculating energy loss, the larger of the 2 minimum impact parameters should be used. When $\eta > 1$, the classical lower limit applies; for $\eta < 1$, the quantum low limit applies for B^q the correct expression for B.

- The soft collisions contributing to ($) come semiclassically from the more distant collisions. The momentum transfer to the struck electron is $\delta p = \sqrt{2\, m\, T}$

- To be certain that the collision produces an energy transfer $< \varepsilon$,

$$\Delta p < \delta p_{\max} = \sqrt{2\, m\, \varepsilon} \Rightarrow \Delta b > \frac{\hbar}{\sqrt{2\, m\, \varepsilon}} \Rightarrow b^q_{\min}(\varepsilon) \approx \frac{\hbar}{\sqrt{2\, m\, \varepsilon}}$$

$$\Rightarrow B_q(\varepsilon) = \frac{b_{\max}}{b^q_{\min}(\varepsilon)} \approx \frac{\gamma\, v \sqrt{2\, m\, \varepsilon}}{\hbar \langle \omega \rangle} \Leftarrow b_{\max} = \frac{\gamma\, v}{\langle \omega \rangle}$$

- At high energies the dominant energy dependence is through

$$\frac{d\,\mathcal{E}}{d\,x} \propto \ln B \approx \ln \frac{b_{\max}}{b_{\min}}$$

- For the total energy loss, the maximum impact parameter is proportional to γ, while the quantum-mechanical minimum impact parameter $b^q_{\min}$ is inversely proportional to γ. The ratio varies as γ^2.

- For energy loss restricted to energy transfers $< \varepsilon$, the minimum impact parameter $b^q_{\min}(\varepsilon)$ is independent of $\gamma \Rightarrow B_q(\varepsilon) \propto \gamma$.

- The semiclassical description in terms of impact parameters contains a conceptual difficulty. The energy transfer T in each collision is related directly to the impact parameter b. When $b \gg b^c_{\min} \Rightarrow T(b) \approx \dfrac{2\,\mathbb{Z}^2 e^4}{m\,v^2\,b^2}$ [Problem 13.1].

- With increasing b the energy transfer decreases rapidly until at

$$b = b_{\max} \approx \frac{\gamma\,v}{\langle\omega\rangle} \Rightarrow T(b_{\max}) \approx \frac{\mathbb{Z}^2}{\gamma^2}\frac{v_0^4}{v^4}\frac{\hbar\langle\omega\rangle}{I_{\mathrm{H}}}\hbar\langle\omega\rangle \Leftarrow v_0 = \frac{c}{137} \text{ electron's orbital speed}$$

$$I_{\mathrm{H}} = 13.6\ \mathrm{eV}\text{: ionization potential} \Rightarrow \text{for } v \gg v_0 \Rightarrow T(b_{\max}) \ll I_{\mathrm{H}} \Leftarrow \hbar\langle\omega\rangle \le Z\,I_{\mathrm{H}}$$

- Since energy must be transferred to the atom in discrete quantum jumps. A tiny amount of energy such as $T(b_{\max})$ simply cannot be absorbed by the atom.

- So the classical expression for $T(b)$ should be used only if it is large enough. But this requirement would set a different upper limit on the impact parameters from $b_{\max} \approx \dfrac{\gamma\,v}{\langle\omega\rangle}$ and lead to wrong results.

- The classical concept of the transfer of a small amount of energy in every collision is incorrect quantum-mechanically. Instead, while on the average over many collisions, a small energy is transferred, the small average results from appreciable amounts of energy transferred in a very small fraction of those collisions.

- A meaningful semiclassical description requires

(a) the statistical interpretation;

(b) use the uncertainty principle to set appropriate minimum impact parameters.

- For electrons ($M = m$) instead of a heavy particle of mass ($M \gg m$), kinematic modifications occur in the energy loss in hard collisions.

$$T_{\max} : 2\gamma^2 \beta^2 m c^2 \rightarrow (\gamma - 1) m c^2 \quad \Rightarrow \quad (\ast) : \frac{2\gamma^2 \beta^2 m c^2}{\varepsilon} \rightarrow \frac{(\gamma-1) m c^2}{\varepsilon}$$

$$B_q = \frac{2\gamma^2 \beta^2 m c^2}{\hbar \langle \omega \rangle} \quad \rightarrow \quad B_q(\text{electrons}) = \frac{\sqrt{2}\,\gamma\,\beta\,\sqrt{\gamma - 1}\, m c^2}{\hbar \langle \omega \rangle} \approx \frac{\sqrt{2\gamma^3}\, m c^2}{\hbar \langle \omega \rangle}$$

- The expressions for $\frac{\mathrm{d}E}{\mathrm{d}x}$ represent the *average* collisional energy loss per unit distance by a particle traversing matter. With Poisson statistics for the number of collisions producing a given energy transfer T, it can be shown that the mean square deviation in energy from the mean is $\Omega^2 = 2\pi N Z \mathbb{z}^2 e^4 (\gamma^2 + 1) t$

- This result holds provided $\Omega \ll \bar{E}\,, \quad \Omega \ll (E_0 - \bar{E})\,, \quad \Omega \gg T_{\max} \approx 2\gamma^2 \beta^2 m c^2$

Density Effect in Collisional Energy Loss

- For particles that are not relativistic, the observed energy loss is given by (#) for particles of all kinds in media of all types. For ultra-relativistic particles, the observed energy loss < the predicted by (#), especially for dense substances.

- From the figure in page 9, the observed energy loss increases beyond the minimum with a slope of roughly ½ that of the theoretical curve, corresponding to only one power γ of in (#) instead of 2.

- This reduction in energy loss, known as the density effect, by Fermi (1940).

- Assumed that it is ok to calculate the effect of the incident particle's fields on one electron in one atom at a time, and then sum up incoherently the energy transfers to all the electrons in all the atoms with $b_{\text{min}} < b < b_{\text{max}}$.

- Since b_{max} is large compared to atomic dimensions. So in dense media there are many atoms lying between the incident particle's trajectory and the typical atom if b is comparable to b_{max}.

- These atoms, influenced by the fast particle, will produce perturbing fields at its position, modifying its response to the fields of the fast particle.

- So in dense media the dielectric polarization of the material alters the particle's fields from their free-space values to those characteristic of macroscopic fields in a dielectric.

- For close collisions the incident particle interacts with only one atom at a time. Then the free-particle calculation without polarization effects will apply.

- The dividing impact parameter between close and distant collisions is of atomic dimensions. Since the joining of 2 logarithms is involved in calculating the sum, the dividing value of b need not be specified with great precision.

- To determine the energy loss in distant collisions ($b \geq a$), assuming that the fields in the medium can be calculated in the continuum approximation of $\epsilon(\omega)$.

- This approximation will not be good for the closest of the distant collisions, but will be valid for the great bulk of the collisions.

- To find the electric field in the medium due to the incident fast particle can be solved by Fourier transforms: $F(\mathbf{x}, t) = \dfrac{1}{(2\pi)^2} \int \mathrm{d}^3 k \int F(\boldsymbol{k}, \omega)\, e^{i(\boldsymbol{k}\cdot\mathbf{r} - \omega t)}\, \mathrm{d}\omega$

- $\rho(\mathbf{r}, t) = \mathbb{z}\, e\, \delta(\mathbf{r} - \mathbf{v}\, t) \;\; \Rightarrow \;\; \rho(\boldsymbol{k}, \omega) = \dfrac{\mathbb{z}\, e}{2\pi} \delta(\omega - \boldsymbol{k}\cdot\mathbf{v}) \quad + \quad \boldsymbol{J} = \rho\, \mathbf{v}$

$$\Rightarrow \left(k^2 - \frac{\omega^2}{c^2}\epsilon(\omega)\right) \begin{bmatrix} \Phi(\boldsymbol{k}, \omega) \\ \mathbf{A}(\boldsymbol{k}, \omega) \end{bmatrix} = 4\pi \begin{bmatrix} \dfrac{\rho(\boldsymbol{k}, \omega)}{\epsilon(\omega)} \\ \dfrac{\boldsymbol{J}(\boldsymbol{k}, \omega)}{c} \end{bmatrix} = 2\,\mathbb{z}\, e \begin{bmatrix} \dfrac{1}{\epsilon} \\ \dfrac{\mathbf{v}}{c} \end{bmatrix} \delta(\omega - \boldsymbol{k}\cdot\mathbf{v})$$

$$\Rightarrow \quad \Phi(\boldsymbol{k},\omega) = \frac{2\,\mathbb{Z}\,e}{\epsilon}\,\frac{c^2\,\delta(\omega - \boldsymbol{k}\cdot\mathbf{v})}{c^2 k^2 - \epsilon\,\omega^2} \qquad \Rightarrow \quad \mathbf{E}(\boldsymbol{k},\omega) = i\left(\frac{\omega\,\epsilon\,\mathbf{v}}{c^2} - \boldsymbol{k}\right)\Phi(\boldsymbol{k},\omega)$$

$$\mathbf{A}(\boldsymbol{k},\omega) = \epsilon\,\frac{\mathbf{v}}{c}\,\Phi(\boldsymbol{k},\omega) \qquad \mathbf{B}(\boldsymbol{k},\omega) = i\,\epsilon\,\boldsymbol{k}\times\frac{\mathbf{v}}{c}\,\Phi(\boldsymbol{k},\omega)$$

$$\Rightarrow \quad \Delta\mathcal{E} = -2\,e\int_{-\infty}^{+\infty}\mathbf{v}\cdot\mathbf{E}\,\mathrm{d}\,t = 2\,e\,\Re\int_{0}^{\infty} i\,\omega\,\mathbf{r}\cdot\mathbf{E}^{\star}(\omega)\,\mathrm{d}\,\omega$$

energy loss to an electron in an atom at impact parameter b

- Consider $\mathbf{E}$ at a perpendicular distance b from the path of the particle moving along the x axis, ie, at $(0,b,0)$

$$\mathbf{E}(\omega) = \frac{1}{(2\,\pi)^{3/2}}\int\mathbf{E}(\boldsymbol{k},\omega)\,e^{i\,b\,k_y}\,\mathrm{d}^3 k \quad \text{for } \mathbf{v} = v\,\hat{\mathbf{x}} \qquad \Downarrow\; \lambda^2 = \frac{\omega^2}{v^2} - \frac{\omega^2}{c^2} = \omega^2\,\frac{1-\epsilon\,\beta^2}{v^2}$$

$$\Rightarrow \quad E_x(\omega) = \frac{2\,i\,\mathbb{Z}\,e}{\epsilon\,(2\,\pi)^{3/2}}\int\frac{\omega\,\epsilon\,v - c^2 k_x}{c^2 k^2 - \epsilon\,\omega^2}\,\delta(\omega - v\,k_x)\,e^{i\,b\,k_y}\,\mathrm{d}^3 k \qquad \Leftarrow\; = \frac{\pi}{\sqrt{k_y^2 + \lambda^2}}$$

$$= -\frac{2\,i\,\mathbb{Z}\,e\,\omega}{(2\,\pi)^{3/2}\,v^2}\,\frac{1-\epsilon\,\beta^2}{\epsilon}\int_{-\infty}^{+\infty} e^{i\,b\,k_y}\,\mathrm{d}\,k_y\int_{-\infty}^{+\infty}\frac{\mathrm{d}\,k_z}{k_y^2 + k_z^2 + \lambda^2}$$

$$= -\frac{i\,\mathbb{Z}\,e\,\omega}{\sqrt{(2\,\pi)}\,v^2}\,\frac{1-\epsilon\,\beta^2}{\epsilon}\int_{-\infty}^{+\infty}\frac{e^{i\,b\,k_y}}{k_y^2 + \lambda^2}\,\mathrm{d}\,k_y \quad \Leftarrow \quad \text{modified Bessel function}$$

$$= -\frac{i\,\mathbb{Z}\,e\,\omega}{v^2}\sqrt{\frac{2}{\pi}}\,\frac{1-\epsilon\,\beta^2}{\epsilon}\,K_0(\lambda\,b) \quad \Leftarrow \quad \lambda \text{ is in the } 4^{\text{th}} \text{ quadrant}$$

$$\Rightarrow \quad E_y(\omega) = \frac{\mathbb{Z}\, e}{v}\sqrt{\frac{2}{\pi}}\,\frac{\lambda}{\epsilon}\,K_1(\lambda\, b)\,, \quad B_z(\omega) = \epsilon\,\beta\,E_y(\omega) \qquad ⇍ \quad \xi = \frac{\omega\, b}{\gamma\, v} \quad \text{Prob13.3}$$

$$\epsilon(\omega) \to 1 \quad \Rightarrow \quad E_{\parallel}(\omega) = -i\,\frac{\mathbb{Z}\, e}{\gamma\, b\, v}\sqrt{\frac{2}{\pi}}\,\xi\, K_0(\xi)\,, \quad E_{\perp}(\omega) = \frac{\mathbb{Z}\, e}{b\, v}\sqrt{\frac{2}{\pi}}\,\xi\, K_1(\xi)$$

- $$\Delta\,\mathcal{E}(b) = 2\, e \sum f_j\, \Re \int_0^{\infty} i\,\omega\,\mathbf{r}_j \cdot \mathbf{E}^*(\omega)\,\mathrm{d}\,\omega \quad \text{in general} \qquad \Downarrow \quad (7.50)$$

$$= \frac{1}{2\,\pi\, N}\,\Re \int_0^{\infty} -i\,\omega\,\epsilon\,|\mathbf{E}(\omega)|^2\,\mathrm{d}\,\omega \quad \Leftarrow \quad -e \sum f_j\, \mathbf{r}_j(\omega) = \frac{\epsilon(\omega)-1}{4\,\pi\, N}\,\mathbf{E}(\omega)$$

- The energy loss per unit distance in collisions with impact parameter $b \geq a$

$$\frac{\mathrm{d}\,\mathcal{E}}{\mathrm{d}\,x}_{|b\geq a} = 2\,\pi\, N \int_a^{\infty} \Delta\,\mathcal{E}(b)\, b\,\mathrm{d}\, b$$

$$= \frac{2\,\mathbb{Z}^2\, e^2}{\pi\, v^2}\,\Re \int_0^{\infty} i\,\omega\,\lambda^*\, a\, K_1(\lambda^*\, a)\, K_0(\lambda\, a)\,\frac{1-\epsilon\,\beta^2}{\epsilon}\,\mathrm{d}\,\omega \qquad (!)$$

- This result can be also obtained by calculating the EM energy flow through a cylinder of radius a around the path of the incident particle. By conservation of energy this is the energy lost/time by the incident particle.

$$\frac{\mathrm{d}\,\mathcal{E}}{\mathrm{d}\,x}_{|b\geq a} = \frac{1}{v}\,\frac{\mathrm{d}\,\mathcal{E}}{\mathrm{d}\,t} = -\frac{c}{4\,\pi\, v}\int_{-\infty}^{+\infty} 2\,\pi\, a\, B_z\, E_x\,\mathrm{d}\, x = -\frac{c\, a}{2}\int_{-\infty}^{+\infty} B_z(t)\, E_x(t)\,\mathrm{d}\, t$$

$$= -c\, a\,\Re \int_0^{\infty} B_z^*(\omega)\, E_x(\omega)\,\mathrm{d}\,\omega \quad (\%) \quad \Rightarrow \quad \text{giving } (!)$$

- Under conditions where polarization effects are unimportant, (!) yields the same results as before.

- For nonrelativistic particles $\beta \ll 1 \quad \Rightarrow \quad \lambda \simeq \frac{\omega}{v}$

$$\epsilon = 1 + \frac{4\pi N e^2}{m} \sum \frac{f_j}{\omega_j^2 - \omega^2 - i\,\omega\,\Gamma_j} \quad \Leftarrow \quad \text{Sec. } 4.5 \ \& \ (7.50)$$

- Assuming that the 2nd term is small, its imaginary part can be substituted into (!). Then the integral can be performed in the narrow-resonance approximation.

- If the small-argument limits of the Bessel functions are used, the nonrelativistic form of (@) emerges, with $B_c \simeq \frac{v}{a\langle\omega\rangle}$. Let $\lambda \simeq \frac{\omega}{\gamma v} \quad \Rightarrow$ (@) shows with $B_c = \frac{\gamma v}{a\langle\omega\rangle}$

- The density effect comes from the presence of complex arguments in the modified Bessel functions, corresponding to taking into account $\epsilon(\omega)$ in λ^2.

- Since $\epsilon(\omega)$ there is multiplied by β^2, the density effect can is really important only at high energies.

- Let $\beta \simeq 1\,, \quad |\lambda a| \sim \frac{\omega a}{c} \ll 1\,, \quad x \to 0$ for $K_\nu(x)$

$$\Rightarrow \quad \left.\frac{d\,\mathcal{E}}{d\,x}\right|_{b \ge a} \simeq \frac{2\,\mathbb{z}^2 e^2}{\pi c^2} \Re \int_0^\infty i\,\omega\,\frac{1-\epsilon}{\epsilon} \left(\ln \frac{1.123\,c}{\omega\,a} - \frac{1}{2} \ln(1-\epsilon) \right) d\,\omega \qquad (\ast)$$

- The argument of the 2nd logarithm is actually $1-\beta^2\epsilon(\omega)$. In the limit $\epsilon=1$, this term gives a factor γ in the logarithm, corresponding to the old result (@).

- Provided $\epsilon\neq 1$, we can write this factor as $1-\epsilon$ to remove γ from the logarithm, in agreement with experiment.

- The integral can be performed by using Cauchy's theorem over over a quarter-circle at infinity to get

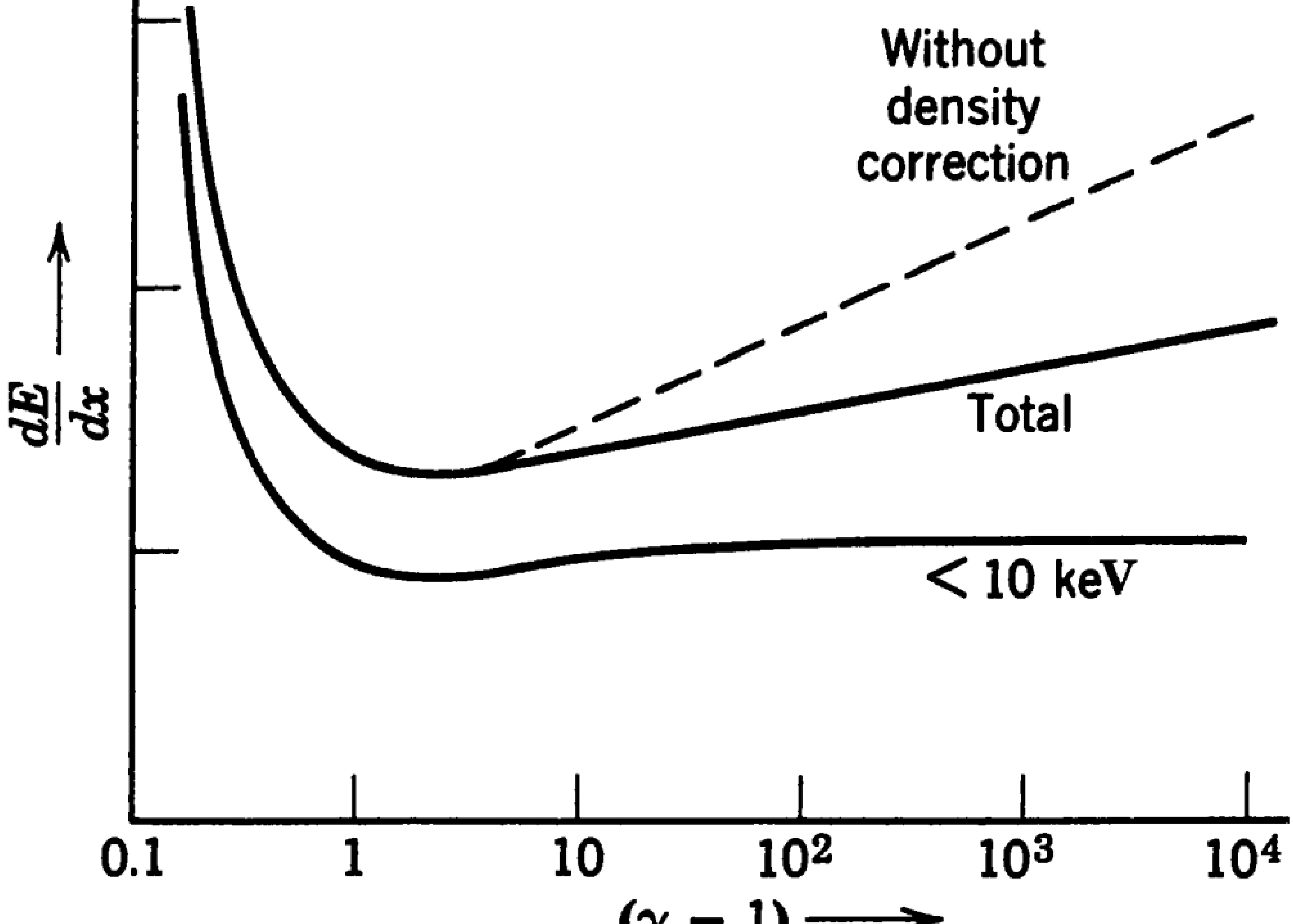
$$\frac{\mathrm{d}\,\mathcal{E}}{\mathrm{d}\,x}_{|b\geq a} = \frac{\mathbb{Z}^2\,e^2\,\omega_p^2}{c^2}\ln\frac{1.123\,c}{a\,\omega_p} \quad \Leftarrow \quad \omega_p^2=\frac{4\,\pi\,N\,Z\,e^2}{m} \quad \text{electronic plasma frequency}$$

- If no density effect $\dfrac{\mathrm{d}\,\mathcal{E}}{\mathrm{d}\,x}_{\mathrm{rel}|b\geq a} \simeq \dfrac{\mathbb{Z}^2\,e^2\,\omega_p^2}{c^2}\ln\dfrac{1.123\,\gamma\,c}{a\,\langle\omega\rangle}$

- The density effect produces a simplification in that the asymptotic energy loss no longer depends on the details of atomic structure through $\langle\omega\rangle$, but only on the number of electrons per unit volume through ω_p.

- 2 substances having very different atomic structures will produce the same energy loss for ultrarelativistic particles provided their densities are such that the density of electrons is the same in each.

- The decrease in energy loss due to the density effect $\displaystyle\lim_{\beta\to 1}\Delta\frac{\mathrm{d}\,\mathcal{E}}{\mathrm{d}\,x}=\frac{\mathrm{d}\,\mathcal{E}}{\mathrm{d}\,x}_{|b\geq a}-\frac{\mathrm{d}\,\mathcal{E}}{\mathrm{d}\,x}_{\mathrm{rel}|b\geq a}\simeq\frac{\mathbb{Z}^2\,e^2\,\omega_p^2}{c^2}\ln\frac{\langle\omega\rangle}{\gamma\,\omega_p}$



- For photographic emulsions with the density correction, this becomes constant at high energies with the value $\frac{d\mathcal{E}(\epsilon)}{dx} \to \frac{\mathbb{Z}^2 e^2 \omega_p^2}{2c^2} \ln \frac{2mc^2\epsilon}{\hbar^2\omega_p^2}$
- For silver bromide, $\hbar\omega_p = 48\,\mathrm{eV}$. Then for singly charged particles (13.44), divided by the density, has the value of approximately $1.02\,\mathrm{MeV}\cdot(\mathrm{cm}^2/\mathrm{g})$, agreed with the experiment, and corresponds to an increase above the minimum value of less than 10%.

Cherenkov Radiation

- The density effect in energy loss is intimately connected to the coherent response of a medium to the passage of a relativistic particle that causes the emission of Cherenkov radiation. They are the same phenomenon in different limiting circumstances.

- Now in (✶) we take the opposite limit, $|\lambda a| \gg 1$, the modified Bessel functions can be approximated by their asymptotic forms.

$$E_x(\omega, b) \to i \frac{\mathbb{z}\, e\, \omega}{c^2} \frac{\beta^2 \epsilon - 1}{\beta^2 \epsilon} \frac{e^{-\lambda b}}{\sqrt{\lambda b}}, \quad E_y(\omega, b) \to \frac{c\, \mathbb{z}\, e\, \lambda}{\beta \epsilon} \frac{e^{-\lambda b}}{\sqrt{\lambda b}}, \quad B_z \to \epsilon \beta E_y \quad (\%2)$$

$$\Rightarrow \quad c\, a\, B_z^{*} E_x \to i \frac{\mathbb{z}^2 e^2 \omega}{c^2} \sqrt{\frac{\lambda^*}{\lambda}} \frac{\beta^2 \epsilon - 1}{\beta^2 \epsilon} e^{-(\lambda^* + \lambda) a} \quad (\%1) \quad \text{in } (\%)$$

The real part of this expression, with the integral of (%), gives the energy deposited far from the path of the particle.

- If λ is positive real (usually true), the exponential factor causes the expression to vanish rapidly at large distances. All the energy is deposited near the path.

- This is not true only when λ *is purely imaginary*. Then the exponential is unity; the expression is independent of a; *some energy escapes to infinity as radiation*.

- From the definition, λ can be purely imaginary if $\epsilon(\omega)$ is real (no absorption) and $\beta^2 \epsilon(\omega) > 1$.

- $\beta^2 \epsilon > 1 \quad \Rightarrow \quad v > \frac{c}{\sqrt{\epsilon}}$ ie, *the speed of the particle must be larger than the phase velocity* of the EM fields at frequency ω in order *to have emission of Cherenkov radiation* of that frequency.

- Consider the phase of λ as $\beta^2\epsilon$ changes from less than unity to greater than unity, assuming that ϵ has an infinitesimal positive imaginary part when $\omega > 0$,

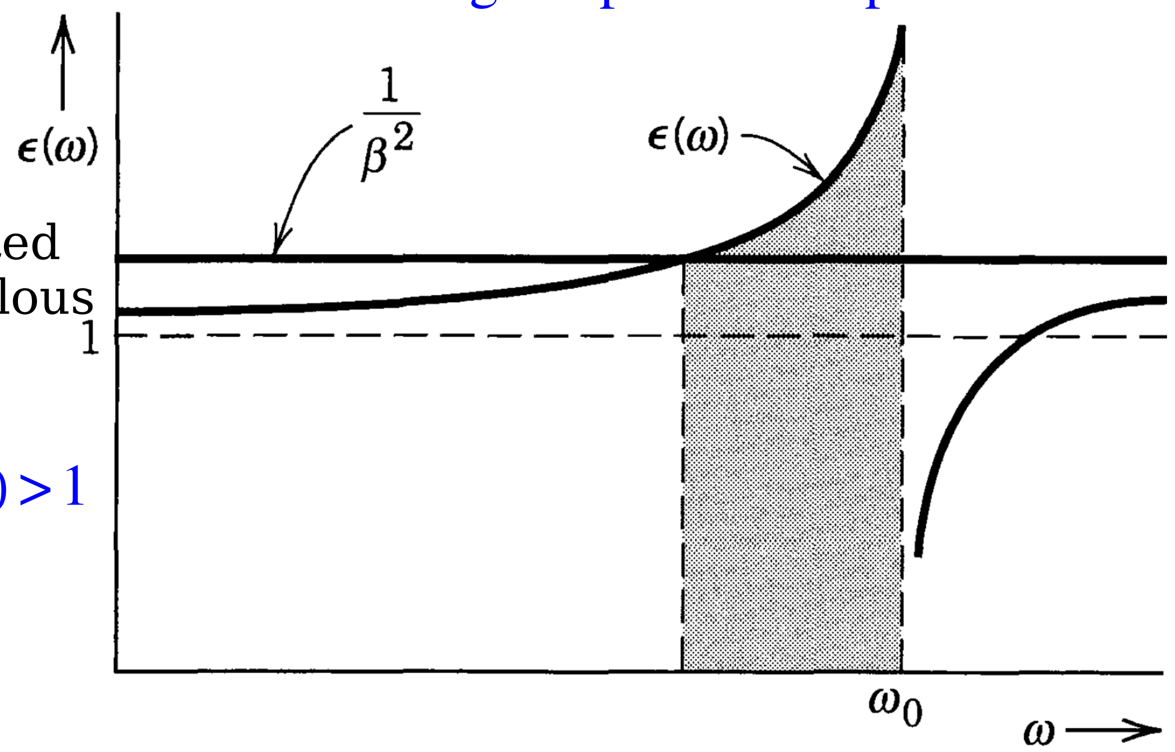
$\Rightarrow \quad \lambda = -i|\lambda| \quad \text{for } \beta^2\epsilon > 1 \quad \Rightarrow \quad \sqrt{\frac{\lambda^*}{\lambda}} = i \quad \Rightarrow \quad (\%1)$ is real and independent of a

$$\Rightarrow \quad (\%) \rightarrow \left.\frac{d\mathcal{E}}{dx}\right|_{rad} = \frac{z^2 e^2}{c^2} \int_{\beta^2\epsilon(\omega)>1} \frac{\beta^2\epsilon - 1}{\beta^2\epsilon} \omega \, d\omega \quad \Rightarrow$$

gives the differential spectrum in frequency the energy radiated as Cherenkov radiation/distance along the path of the particle

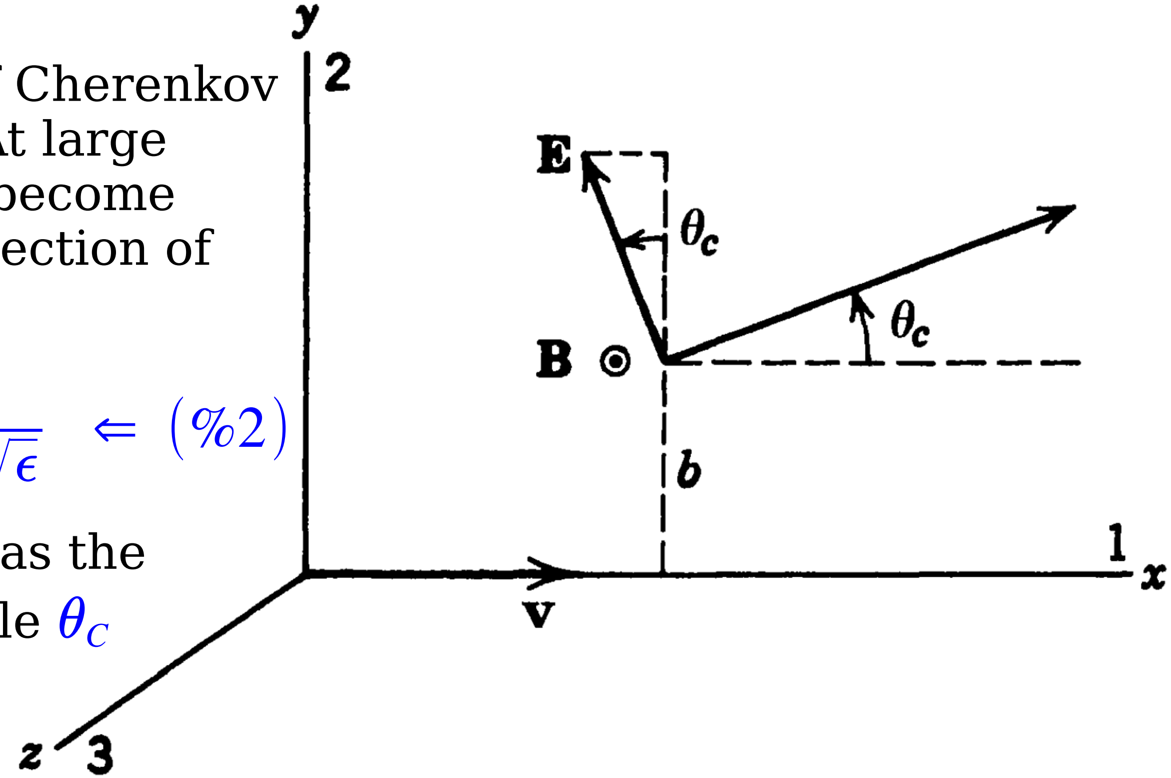
- The Frank-Tamm result shows that the radiation is evidently not emitted uniformly in frequency. It tends to be emitted in bands situated somewhat below regions of anomalous dispersion, where $\beta^2\epsilon(\omega) > 1$.

- If $\beta \simeq 1$, the regions where $\beta^2\epsilon(\omega) > 1$ may be quite extensive.



- Another characteristic feature of Cherenkov radiation is its angle of emission. At large distances from the path the fields become transverse radiation fields. The direction of propagation is given by $\mathbf{E} \times \mathbf{B}$

$$\Rightarrow \quad \tan\theta_C = -\frac{E_x}{E_y} \quad \Rightarrow \quad \cos\theta_C = \frac{1}{\beta\sqrt{\epsilon}} \quad \Leftarrow \quad (\%2)$$



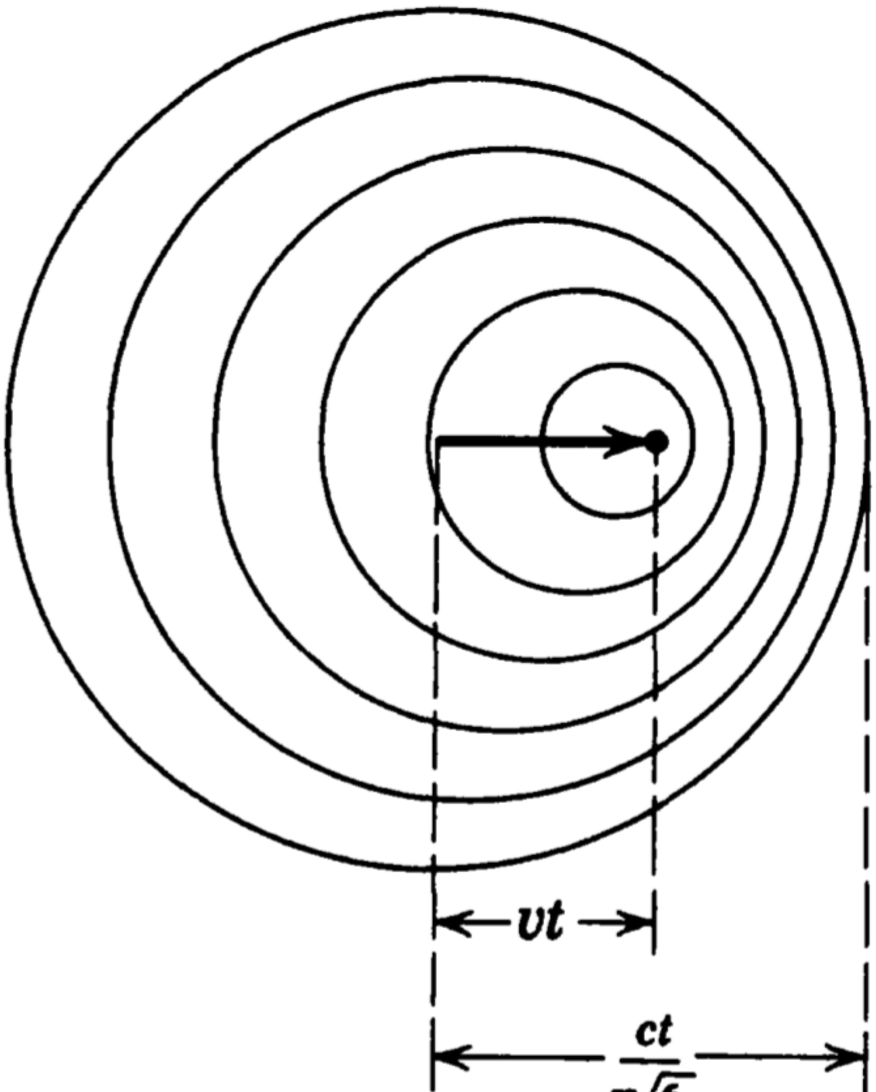
- So $\beta^2 \epsilon > 1$ can now be rephrased as the requirement that the emission angle θ_C be a physical angle with $\cos\theta_C < 1$.

- Cherenkov radiation is completely linearly polarized in the plane containing the direction of observation and the path of the particle.

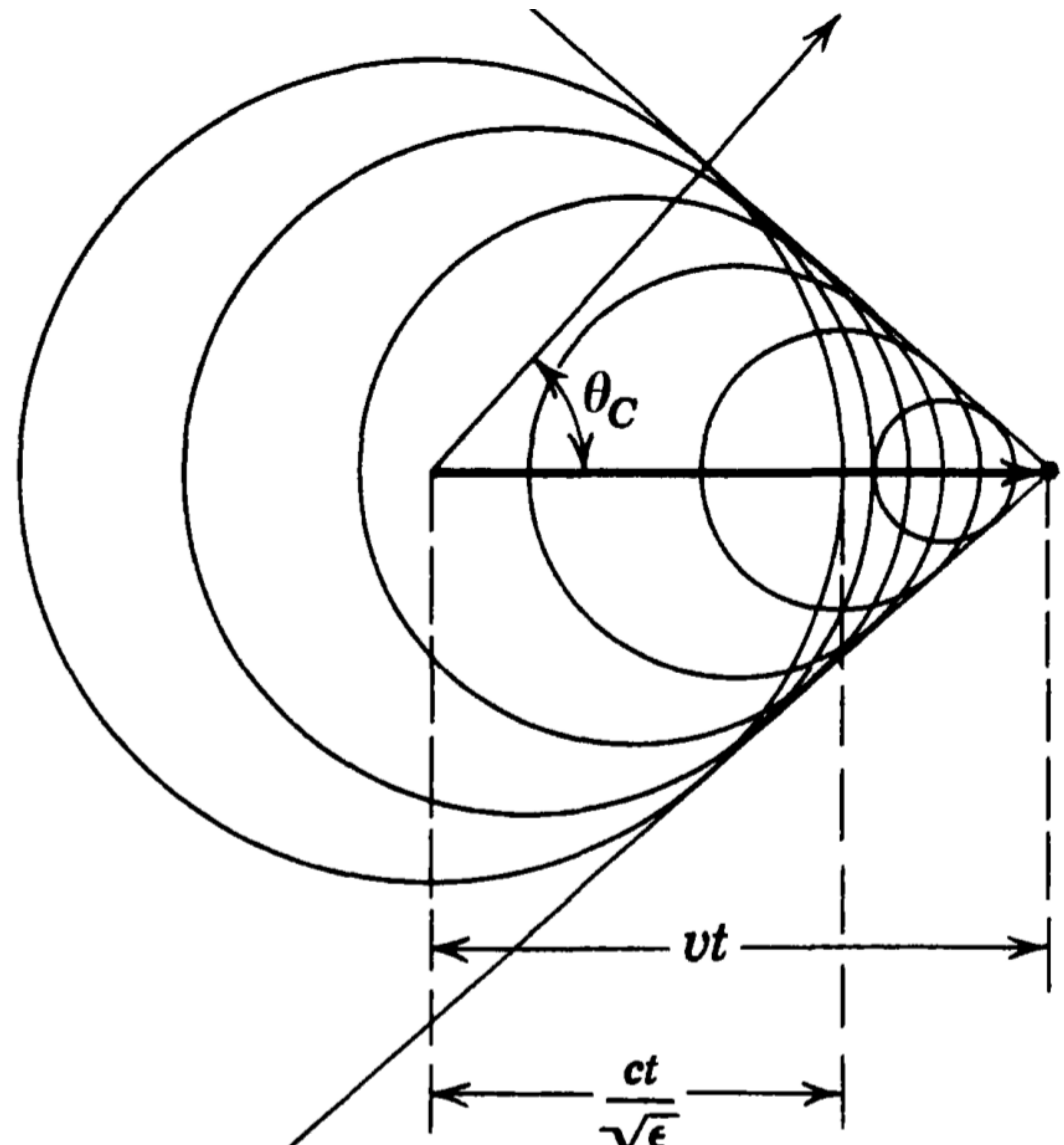
- The emission angle θ_C can be interpreted in terms of a "shock" wave front akin to the shock wave (sonic boom) produced by an aircraft in supersonic flight.

- The “shock wave” behavior can be given quantitative treatment by examining the potentials $\Phi(x,t)$ or $\mathbf{A}(x,t)$, eg,

$$\mathbf{A}(\mathbf{r},t) = \frac{2\mathbb{Z}e}{(2\pi)^2}\beta \int \frac{e^{ik_x(x-vt)}\, e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{\rho}}}{k_x^2(1-\beta^2\epsilon)+k_\perp^2}\,\mathrm{d}^3k \quad \Leftarrow \quad \begin{array}{l} \epsilon = \epsilon(k_x v) \\ \boldsymbol{\rho} \;\&\; \boldsymbol{k}_\perp : \text{transverse coordinates} \end{array}$$



$v < c/\sqrt{\epsilon}$



$v > c/\sqrt{\epsilon}$

- In the Cherenkov regime ($\beta^2\epsilon > 1$) the denominator has poles on the path of integration. Choosing the contour for the k_x integration so that the potential vanishes for points ahead of the particle ($x - v\,t > 0$), the result is

$$\mathbf{A}(\mathbf{r}, t) = \begin{cases} \dfrac{2\,\mathbb{Z}\,e}{\sqrt{(x - v\,t)^2 - (\beta^2\,\epsilon - 1)\,\rho^2}}\,\boldsymbol{\beta} & \text{inside} \\ 0 & \text{outside} \end{cases} \quad \text{the Cherenkov cone}$$

- $\mathbf{A}$ is singular along the shock front, as suggested by the wavelets in the figure.

- In fact the dielectric constant vary with $\omega = k_x v$. This functional dependence will remove the mathematical singularity in the expression.

- The properties of Cherenkov radiation can be utilized to measure velocities of fast particles.

- If the particles of a given velocity pass through a medium of known ϵ, the light is emitted at the Cherenkov angle θ_C. Thus a measurement of the angle allows determination of the velocity.

- Since ϵ in general varies with frequency, light of different colors is emitted at different angles. Narrow-band filters may be employed to select a small interval of frequency and so improve the precision of velocity measurement.

- For very fast particles ($\beta \leq 1$) a gas may be used to provide a dielectric constant differing only slightly from unity and having $\epsilon - 1$ variable over wide limits by varying the gas pressure.

- Counting devices using Cherenkov radiation are employed in high-energy physics, as instruments for velocity measurements, as mass analyzers when combined with momentum analysis, and as discriminators against unwanted slow particles.

Elastic Scattering of Fast Charged Particles by Atoms

- Incident charged particles are elastically scattered by the time-averaged potential created by the atomic nucleus and its associated electrons.

- The potential is Coulombic but is modified at large distances by the screening effect of the electrons and at short distances by the finite size of the nucleus.

- For a pure Coulomb field, the scattering cross section is given by the Rutherford formula, modified at large angles by spin-dependent corrections.

- At small angles, all particles, regardless of spin, scatters according to the small-angle Rutherford expression $\frac{d\sigma}{d\Omega} \approx \left(\frac{2\,\mathbb{z}\,Z\,e^2}{p\,v} \right)^2 \cdot \frac{1}{\theta^4}$

- Even at $\theta = \frac{\pi}{2}$, the small-angle result is within 30% of the exact Rutherford formula. Such accuracy is sufficient for present purposes.

- Because of electronic screening, the differential scattering cross section is finite, not infinite as a pure Coulomb potential, at $\theta = 0$.

- A simple classical impact parameter calculation [Problem 13.l] with a Coulomb force cutoff sharply at $r = a$ gives a small-angle cross section

$$\frac{d\sigma}{d\Omega} \approx \left(\frac{2\,\mathbb{z}\,Z\,e^2}{p\,v} \right)^2 \cdot \frac{1}{\left(\theta^2 + \theta_{\min}^2\right)^2} \quad (\$1) \quad \Leftarrow \quad \begin{matrix}\text{cutoff}\\ \text{angle}\end{matrix} \quad \theta_{\min} \to \theta_{\min}^c = \frac{2\,\mathbb{z}\,Z\,e^2}{p\,v\,a}$$

- A better form of screened Coulomb interaction $\Phi(r) = \frac{\mathbb{Z}\, Z\, e^2}{r} e^{-\frac{r}{a}} \Leftarrow a = 1.4\, a_0 \sqrt[3]{Z}$

  a rough fit to the Thomas-Fermi atomic potential.

- The quantum-mechanical cutoff angle $\theta^q_{\min} = \frac{\hbar}{p\, a} \approx \frac{\sqrt[3]{Z}}{192} \frac{m\, c}{p} \Leftarrow p = \gamma\, M\, v$

- The ratio of classical to quantum-mechanical angles $\theta_{\min}$ is $\eta = \frac{\mathbb{Z}\, Z\, e^2}{\hbar\, v}$, in

agreement with the corresponding ratio of minimum impact parameters.

- For fast particles, $\eta < 1$; the quantum-mechanical cutoff angle should be used.

- At comparatively large angles the scattering cross section departs from ($1) because of the finite size of the nucleus.

- For charged leptons (e, μ, τ) the influence of the finite size is an EM effect, for hadrons $(\pi, \mathrm{K}, \mathrm{p}, \alpha, \ldots)$ specifically strong-interaction effects also arise.

- Since the gross overall effect is to lower the cross section below ($1) at larger angles for whatever reason, we examine only the EM aspect.

- The charge distribution of an atomic nucleus can be approximated by a uniform volume distribution inside a sphere of radius R, falling sharply to zero outside.

- The electrostatic potential $\Phi(r)= \begin{cases} \dfrac{\mathbb{Z}\, Z\, e^2}{R}\left(\dfrac{3}{2}-\dfrac{r^2}{2\,R^2}\right) & \text{for} \quad r\le R \\ \dfrac{\mathbb{Z}\, Z\, e^2}{r} & \text{for} \quad r>R \end{cases}$

- The classical scattering cross section from such a potential exhibits singular behavior at a maximum angle given by the classical formula $\theta^c{}_{\min}$, but with $a\to R$.

- It is a consequence of the scattering angle $\theta(b)=\dfrac{\Delta\, p(b)}{p}$ vanishing at $b=0$, rising to a maximum at just less than $b=R$ and falling again for larger b. The maxi translates into a vanishing $\dfrac{\mathrm{d}\,\theta}{\mathrm{d}\,b}$ and so an infinite differential cross section.

- Quantum mechanically, the wave nature of the incident particle makes the nuclear scattering much like the scattering of EM waves by localized scatterers.

- At short wavelengths, the scattering is diffractive, confined to an angular range

$$\Delta\,\theta=\frac{1}{k\,R} \quad \Leftarrow \quad k=\frac{p}{\hbar}$$

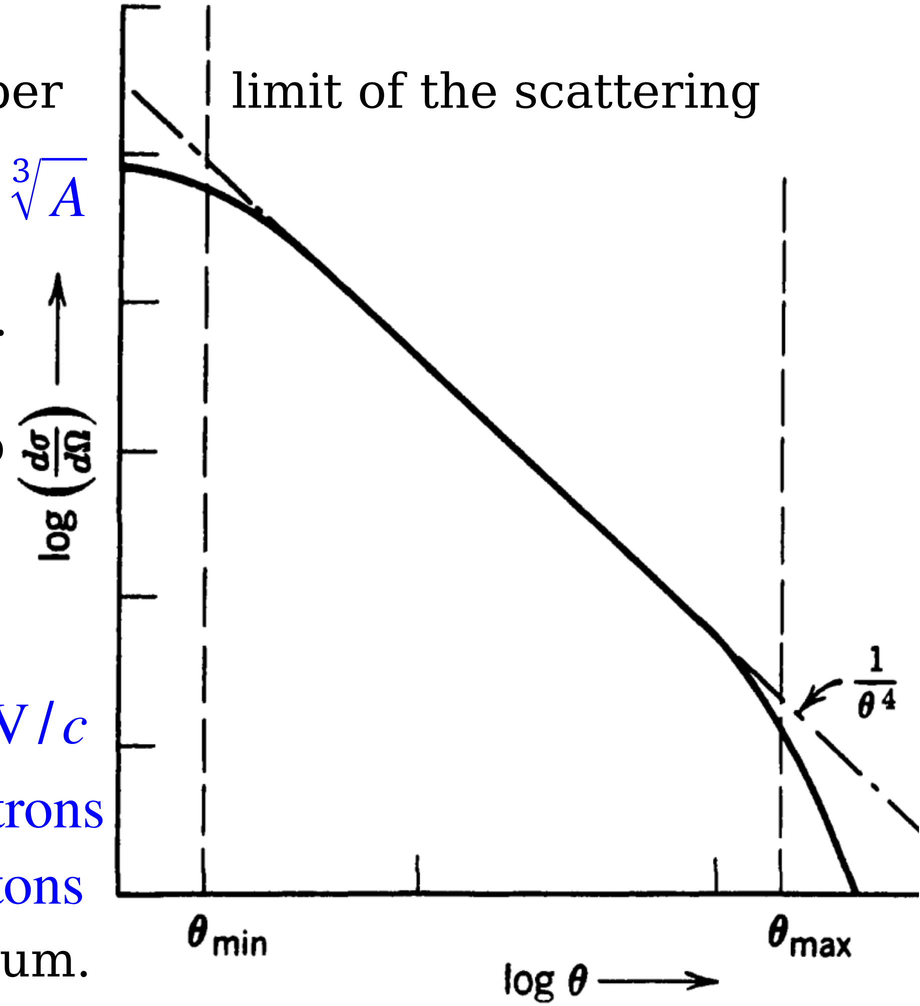
- The scattering cross section will basically fall rapidly below the point Coulomb result at larger angle. Or the scattering amplitude is the product of the Coulomb amplitude for a point charge and a form factor $F(Q^2)$, the spatial Fourier transform of the charge distribution, where $F(0)=1$, and $F\gg 1$ for $Q\,R>1$

- The finite nuclear size sets an effective upper limit of the scattering

$$\theta_{\max} \approx \frac{\hbar}{p\,R} \approx \frac{274}{\sqrt[3]{A}}\,\frac{m\,c}{p} \quad \Leftarrow \quad R = 1.4 \times 10^{-15} \cdot \sqrt[3]{A}$$

- $\theta_{\max} \gg \theta_{\min}$ for all physical values of Z and A.

- p small $\Rightarrow$ $\theta_{\max} \geq 1$ $\Rightarrow$ the nuclear size has no appreciable effect on the scattering. Only at higher energies are nuclear-sized effects important.



- For an aluminum target, $\theta_{\max} = 1 \quad \Rightarrow \quad p \approx 50\ \mathrm{MeV}/c$

$$\Rightarrow \quad T \approx \begin{array}{l} 50\ \text{MeV for electrons} \\ 1.3\ \text{MeV for protons} \end{array}$$

- At p=50 MeV/c, $\theta_{\min}$= 10^{-4} radian in aluminum.

- The total scattering cross section can be obtained by integrating ($1),

$$\sigma = \int \frac{d\,\sigma}{d\,\Omega} \sin\theta\, d\,\theta\, d\,\phi = 2\,\pi \left(\frac{2\,\mathbb{z}\,Z\,e^2}{p\,v}\right)^2 \int_0^\infty \frac{\theta\, d\,\theta}{(\theta^2 + \theta_{\min}^2)^2} \approx \frac{\pi}{\theta_{\min}^{q\,2}} \left(\frac{2\,\mathbb{z}\,Z\,e^2}{p\,v}\right)^2$$

$$= \pi\,a^2 \left(\frac{2\,\mathbb{z}\,Z\,e^2}{\hbar\,v}\right)^2 \quad \Rightarrow$$

At high velocities the total scattering cross section can be far smaller than the classical geometrical area $\pi\,a^2$ of the atom.

Mean Square Angle of Scattering; Angular Distribution of Multiple Scattering

- Rutherford scattering is confined to very small angles even for a point Coulomb field, and for fast particles $\theta_{\max}$ is small compared to 1. There is a very large probability for small-angle scattering.

- A particle traversing a finite thickness of matter will undergo very many small-angle deflections and will generally emerge at a small angle that is the cumulative statistical superposition of a large number of deflections.

- Thus divide the angular range into 2 regions: one region at comparatively large angles, which contains only the single scatterings, and one region at very small angles, which contains the multiple or compound scatterings.

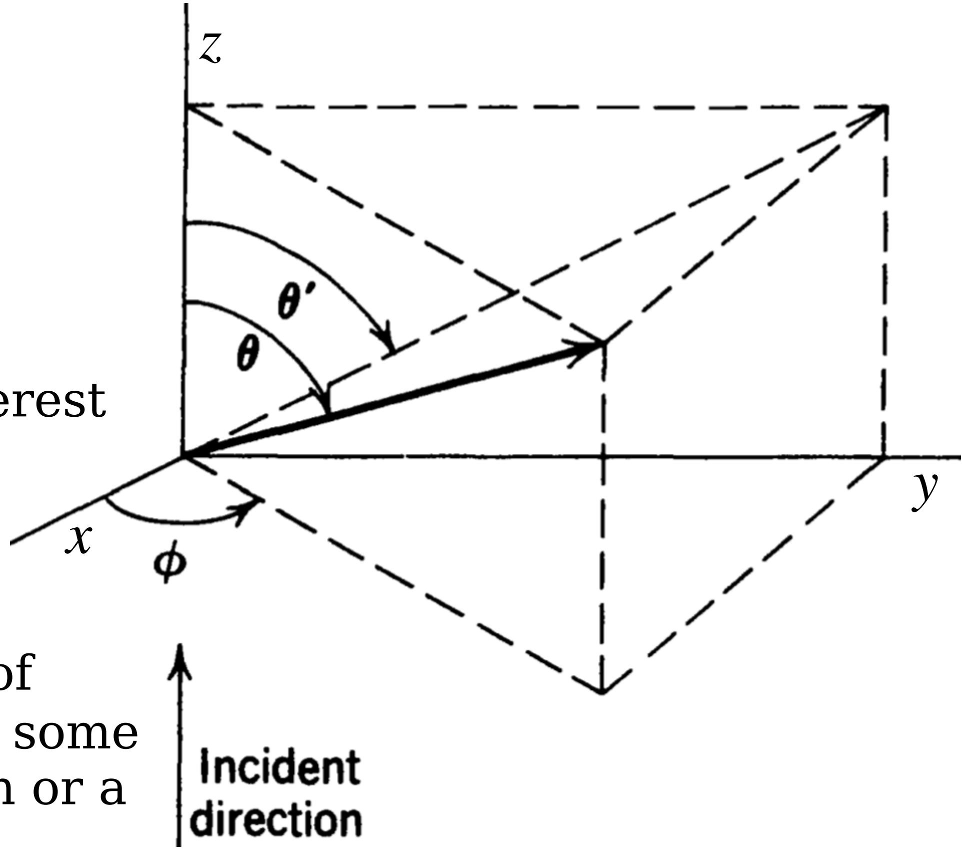
- The complete distribution in angle can be approximated by considering the 2 regions separately. The intermediate region of the plural scattering must allow a smooth transition from small to large angles.

- The important quantity in the multiple-scattering region, a large succession of small-angle deflections symmetrically distributed about the incident direction, is the mean square angle for a single scattering

$$\langle\theta^2\rangle \equiv \frac{\int \theta^2 \frac{d\sigma}{d\Omega}\, d\Omega}{\int \frac{d\sigma}{d\Omega}\, d\Omega} \Rightarrow \langle\theta^2\rangle = 2\,\theta_{\min}^2 \ln\frac{\theta_{\max}}{\theta_{\min}} \Leftarrow \text{the approximations of last section}$$

$$\theta_{\min} \approx \frac{\sqrt[3]{Z}}{192} \frac{m\,c}{p}, \quad \theta_{\max} \approx \frac{274}{\sqrt[3]{A}} \frac{m\,c}{p}$$

$$\Rightarrow \quad \langle \theta^2 \rangle \simeq 4\,\theta_{\min}^2 \ln \frac{204}{\sqrt[3]{Z}} \quad \Leftarrow \quad A \simeq 2\,Z$$



- If nuclear size is unimportant (only of interest for electrons),

$$\theta_{\max} \simeq 1 \quad \Rightarrow \quad \langle \theta^2 \rangle \simeq 2\,\theta_{\min}^2 \ln \frac{192\,p}{\sqrt[3]{Z}\,m\,c}$$

- It is desirable to use the projected angle of scattering θ', the projection being made on some convenient plane for photographic emulsion or a bubble chamber.

- For small angles $\langle \theta'^2 \rangle = \frac{\langle \theta^2 \rangle}{2}$

- In each collision the angular deflections obey the small-angle Rutherford formula suitably cut off at $\theta_{\min}$ & $\theta_{\max}$, with average value 0 and mean square angle $\langle \theta^2 \rangle$.

- Since the successive collisions are independent events, the central-limit theorem of statistics can be used to show that for a large number n of such collisions the distribution will be Gaussian around the forward direction with a mean square angle $\langle \Theta^2 \rangle = n \langle \theta^2 \rangle$.

- The number of collisions occurring as the particle traverses a thickness t of material containing N atoms/volume is $n = N\,\sigma\,t \simeq \pi\,N \left(\frac{2\,\mathbb{z}\,Z\,e^2}{p\,v}\right)^2 \frac{t}{\theta_{\min}^2}$

$$\Rightarrow \quad \langle\Theta^2\rangle \simeq 2\,\pi\,N \left(\frac{2\,\mathbb{z}\,Z\,e^2}{p\,v}\right)^2 t\,\ln\frac{\theta_{\max}}{\theta_{\min}} \quad \text{or} \quad \langle\Theta^2\rangle \simeq 4\,\pi\,N \left(\frac{2\,\mathbb{z}\,Z\,e^2}{p\,v}\right)^2 t\,\ln\frac{204}{\sqrt[3]{Z}}$$

- For reasonable thicknesses without losing appreciable energy, the Gaussian will still be peaked at very small forward angles.

- The multiple-scattering distribution for the projected angle of scattering is

$$P_M(\theta')\,d\,\theta' = \frac{e^{-\frac{\theta'^2}{\langle\Theta^2\rangle}}}{\sqrt{\pi\,\langle\Theta^2\rangle}}\,d\,\theta' \quad \text{for} \quad \theta' = \pm|\theta'| \quad \Rightarrow \quad \frac{d\,\sigma}{d\,\theta'} = \frac{\pi}{2\,\theta'^3}\left(\frac{2\,\mathbb{z}\,Z\,e^2}{p\,v}\right)^2$$

small-angle Rutherford formula

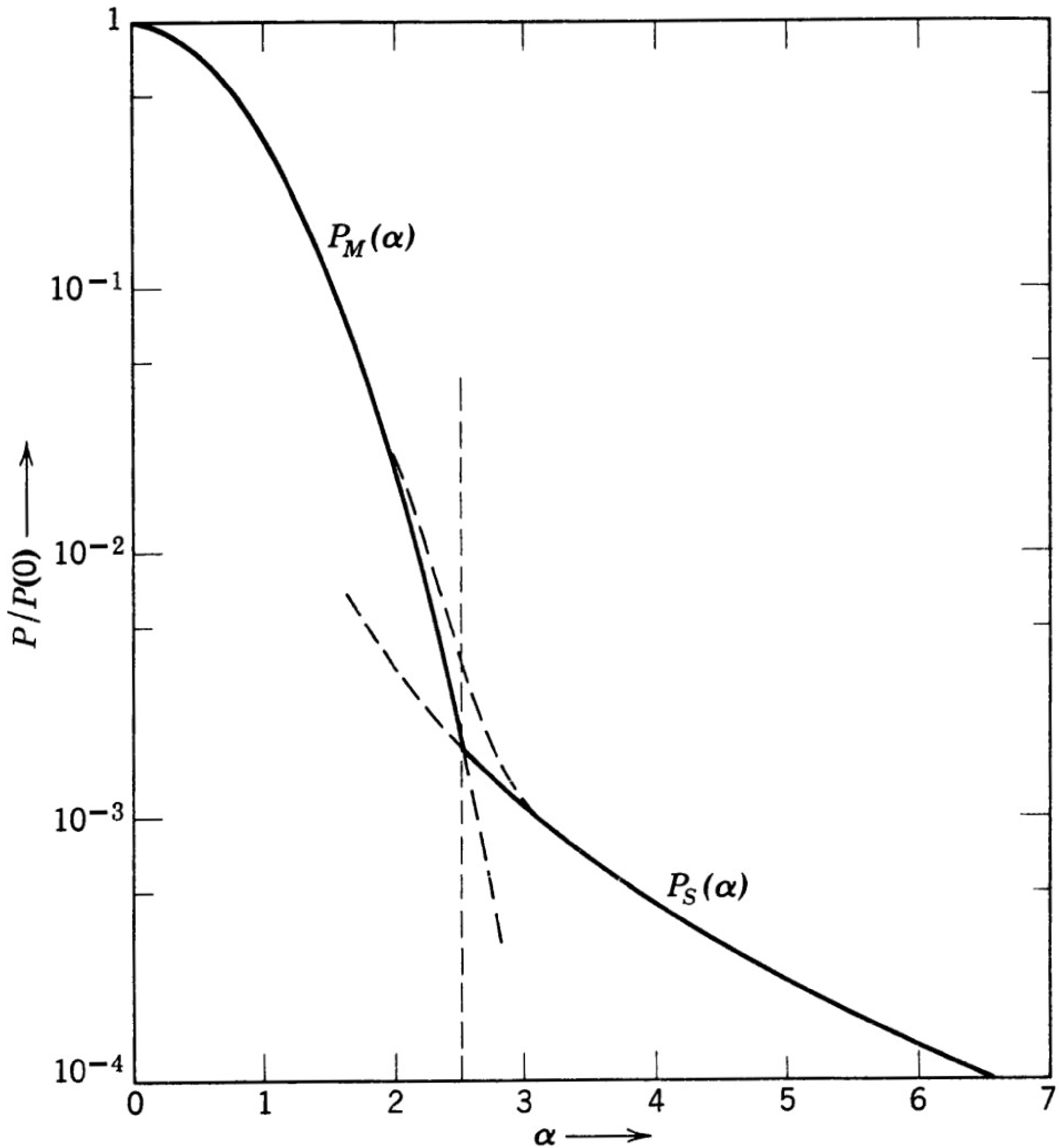
$$\Rightarrow \quad \text{single-scattering distribution} \quad P_S(\theta')\,d\,\theta' = N\,t\,\frac{d\,\sigma}{d\,\theta'}\,d\,\theta' = \frac{\pi\,N\,t}{2}\left(\frac{2\,\mathbb{z}\,Z\,e^2}{p\,v}\right)^2 \frac{d\,\theta'}{\theta'^3}$$

valid only for angles $\geq \sqrt{\langle\Theta^2\rangle}$ and contributes a tail to the Gaussian distribution.

- Express angles in terms of the relative projected angle: $\alpha \equiv \frac{\theta'}{\sqrt{\langle\Theta^2\rangle}} \quad \Rightarrow$

$$P_M(\alpha)\,d\,\alpha = \frac{1}{\sqrt{\pi}}\,e^{-\alpha^2}\,d\,\alpha$$

$$P_S(\alpha)\,d\,\alpha = \frac{1}{8\left(\ln 204 - \ln\sqrt[3]{Z}\right)}\frac{d\,\alpha}{\alpha^3}$$



1
10−1
10−2
10−3
10−4
P/P(0)
0
1
2
3
4
5
6
7
α
P_M(α)
P_S(α)

- The relative amounts of multiple and single scatterings are independent of thickness, and depend only on Z.

- The transition from multiple to single scattering occurs around $\alpha \simeq 2.5$. There the Gaussian has a value of 1/600 times its peak value. Thus the single-scattering distribution gives only a very small tail on the multiple-scattering curve.

- The Gaussian shape is the limiting form of the angular distribution for very large n. If the thickness t makes n not very large (ie, $n \leq 200$), the distribution follows the single-scattering curve to smaller angles than $\alpha \simeq 2.5$, and is more sharply peaked at 0 angle than a Gaussian.

- If the thickness is great enough, $\langle \Theta^2 \rangle$ becomes comparable with $\theta_{\max}$ which limits the angular width of the single-scattering distribution.

- For greater thicknesses the multiple-scattering curve extends in angle beyond the single-scattering region, so that there is no single-scattering tail on the distribution [Problem 13.8].

Transition Radiation

- A charged particle in uniform motion in a straight line in vacuum won’t radiate.

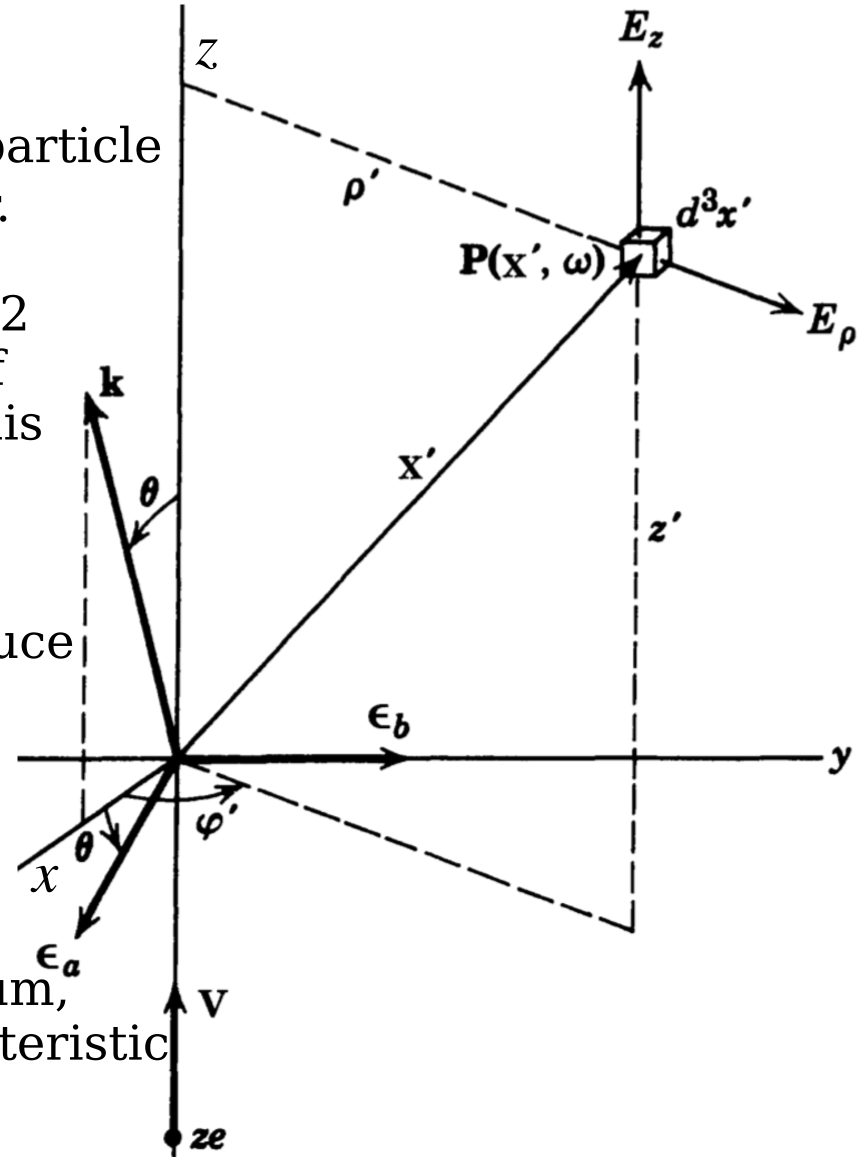
- A particle moving at constant velocity can radiate if it is in a material medium and is moving with a speed > the phase velocity of light in that medium, ie, the Cherenkov radiation, with its characteristic angle of emission, $\sec\theta_C = \beta\sqrt{\epsilon}$.

- There is another type of radiation, *transition radiation*, that is emitted whenever a charged particle passes suddenly from one medium into another.

- Even if the motion is uniform throughout the 2 media, the initial and final fields are different if the 2 media have different EM properties. In this process of reorganization some pieces of the fields are shaken off as transition radiation.

- The moving fields of the charged particle induce a time-dependent polarization $\mathbf{P}(\mathbf{x}', t)$ in the medium. The polarization emits radiation.

- The radiated fields from different points in space combine coherently in the neighborhood of the path and for a certain depth in the medium, giving rise to transition radiation with a characteristic angular distribution and intensity.



- The angular distribution and the *formation length* D are a direct consequence of the requirement of coherence for appreciable radiated intensity.

- $$\mathbf{E}' = \frac{\mathbb{Z}\, e}{\left[\rho'^{\,2} + (z' - v\,t)^2\right]^{3/2}} \begin{bmatrix} \rho' \cos\phi' \\ \rho' \sin\phi' \\ z' - v\,t \end{bmatrix} \quad \& \quad \mathbf{B}' = 0 \quad \begin{array}{l} \Leftarrow \quad \text{Chapter 11} \\ \mathbf{x}' = (z', \rho', \phi') \\ r^2 \equiv \rho'^{\,2} + \gamma^2 (z' - v\,t)^2 \end{array}$$

$$\Rightarrow \quad \mathbf{E} = \begin{bmatrix} \gamma\, E'_x \\ \gamma\, E'_y \\ E'_z \end{bmatrix} = \frac{\gamma\, \mathbb{Z}\, e}{r^3} \begin{bmatrix} \rho' \cos\phi' \\ \rho' \sin\phi' \\ z' - v\,t \end{bmatrix} \quad \& \quad \mathbf{B} = \beta \begin{bmatrix} -E_y \\ E_x \\ 0 \end{bmatrix} = \frac{\mathbb{Z}\, e\, \gamma\, \beta\, \rho'}{r^3} \begin{bmatrix} -\sin\phi' \\ \cos\phi' \\ 0 \end{bmatrix}$$

- The dependence at $\mathbf{x}'$ and on $\dfrac{1}{r^2}$ implies a Fourier component of frequency ω

(a) move in the z direction with $\mathbf{v}$ and so have an amplitude proportional to $e^{i\frac{\omega z'}{v}}$;
(b) have significant magnitude radially from the path only out to distances of the order of $\rho'_{\max} \simeq \dfrac{\gamma\, v}{\omega}$.

- The time-dependent polarization at $\mathbf{x}'$ generates a wave whose form in the radiation zone is $A = \dfrac{e^{i\,k\,(r - z'\cos\theta - \rho'\sin\theta\cos\phi')}}{r} \propto \begin{array}{l}\text{the driving field of} \\ \text{the incident particle}\end{array}$, $\quad k = \dfrac{n(\omega)\,\omega}{c}$

assume that the radiation is observed in the $x\,z$ plane and in the forward hemisphere.

- Appreciable coherent superposition from different points in the medium will occur provided the product of the driving fields of the particle and the generated wave does not change its phase significantly over the region

$$\Rightarrow \quad e^{i\frac{\omega z'}{v}}\, e^{-i\frac{\omega}{c}n(\omega)z'\cos\theta}\, e^{-i\frac{\omega}{c}n(\omega)\rho'\sin\theta\cos\phi'} = e^{i\frac{\omega}{c}\left(\frac{z'}{\beta}-n(z'\cos\theta+\rho'\sin\theta\cos\phi')\right)}$$

- In the radial direction coherence will be maintained only if the phase involving ρ' is unity or less in the region $0<\rho'\leq\rho'_{\max}$ where the exciting field is appreciable.

$$\Rightarrow \quad \frac{\omega}{c}n(\omega)\frac{\gamma v}{\omega}\sin\theta\leq 1 \quad \Rightarrow \quad n(\omega)\gamma\theta\leq 1 \quad \text{for} \quad \gamma\gg 1$$

The angular distribution is therefore confined to the forward cone, $\gamma\theta\leq 1$.

- The z'-dependent factor: $e^{i\frac{\omega}{c}\left(\frac{1}{\beta}-n\cos\theta\right)z'}$ $\Rightarrow \quad \frac{\omega}{c}\left(\frac{1}{\beta}-n\cos\theta\right)d(\omega)\simeq 1$
  Depth of coherence $d(\omega)$:

$$\text{Let} \quad n(\omega)\simeq 1-\frac{\omega_p^2}{2\omega^2}, \quad \frac{1}{\beta}\simeq 1+\frac{1}{2\gamma^2}, \quad \cos\theta\simeq 1, \quad \nu\equiv\frac{\omega}{\gamma\omega_p}$$

$$\Rightarrow \quad d(\nu)\simeq\frac{c}{\omega_p}\frac{2\gamma}{\nu+\nu^{-1}} \quad \Rightarrow \quad \text{formation length} \quad D\equiv d_{\max}(\nu)=d(1)=\frac{\gamma c}{\omega_p}$$

- $\text{density}\simeq 1 \quad \Rightarrow \quad \omega_p\simeq 3\times 10^{16}/\text{s} \quad \Rightarrow \quad \hbar\omega_p\simeq 20\ \text{eV} \quad \Rightarrow \quad \frac{c}{\omega_p}\simeq 10^{-6}\ \text{cm}$

$$\Rightarrow \quad D\simeq 10\ \mu\text{m} \quad \Leftarrow \quad \gamma\geq 10^3 \quad \Rightarrow \quad D_{\text{air}}\simeq 30\ D \quad \text{for the reduced density}$$

- The coherence volume adjacent to the particle's path and the surface from which transition radiation of frequency ω comes

$$V(\omega) \sim \pi \rho_{\max}^2(\omega) d(\omega) \sim \frac{2\pi\gamma}{\nu(\nu^2+1)} \left(\frac{c}{\omega_p}\right)^3 \Rightarrow \quad V \text{ decrease in size rapidly for } \nu > 1$$

- In the absence of compensating factors, the spectrum of transition radiation will extend up to, but not appreciably beyond, $\nu \simeq 1$.

- The radiation is confined to small angles in the forward direction $\gamma\theta \leq 1$. It is produced by coherent radiation of the time-varying polarization in a small volume adjacent to the particle's path and at depths into the medium up to D. Its spectrum extends up to frequencies of the order of $\omega \sim \gamma\omega_p$.

- For frequencies above the optical resonance region, the index of refraction~1. The incident particle's fields at such frequencies are not significantly different in the medium and in vacuum $\Rightarrow \quad \mathbf{P}(\mathbf{r}', \omega) \simeq \dfrac{\epsilon(\omega)-1}{4\pi} \mathbf{E}_i(\mathbf{r}', \omega)$

- The *propagation* of the wave radiated by the polarization must be described properly with the wave number $k = \dfrac{n(\omega)\,\omega}{c}$ appropriate to the medium.

- The dipole radiation field from the polarization $\mathbf{P}(\mathbf{r}',\omega)\,\mathrm{d}^3x'$ in d^3x' at $\mathbf{r}'$ is

$$(9.18) \quad \Rightarrow \quad \mathrm{d}\,\mathbf{E}_{\text{rad}} = \frac{e^{ik\mathbb{r}}}{\mathbb{r}} (\boldsymbol{k} \times \mathbf{P}) \times \boldsymbol{k}\, \mathrm{d}^3x' \quad \Leftarrow \quad \mathbb{r} = r - \hat{\boldsymbol{k}} \cdot \mathbf{r}'$$

$$\Rightarrow \quad \mathbf{E}_{\text{rad}} = \int \mathrm{d}\,\mathbf{E}_{\text{rad}} = \frac{\epsilon(\omega)-1}{4\pi}\frac{e^{ikr}}{r}k^2\int_{z'>0}\left(\hat{\boldsymbol{k}}\times\mathbf{E}_i\right)\times\hat{\boldsymbol{k}}\,e^{-i\boldsymbol{k}\cdot\mathbf{r}'}\,\mathrm{d}^3x'$$

$$\simeq \frac{-\omega_p^2}{4\pi c^2}\frac{e^{ikr}}{r}k^2\int_{z'>0}\left(\hat{\boldsymbol{k}}\times\mathbf{E}_i\right)\times\hat{\boldsymbol{k}}\,e^{-i\boldsymbol{k}\cdot\mathbf{r}'}\,\mathrm{d}^3x' \quad \Leftarrow \quad \epsilon \simeq 1-\frac{\omega_p^2}{\omega^2}, \quad \omega > \omega_p$$

$$\Rightarrow \quad \frac{\mathrm{d}^2 I}{\mathrm{d}\,\omega\,\mathrm{d}\,\Omega} = \frac{c}{32\pi^3}\frac{\omega_p^4}{c^4}\left|\int_{z>0}\left[\hat{\boldsymbol{k}}\times\mathbf{E}_i(\mathbf{r},\omega)\right]\times\hat{\boldsymbol{k}}\,e^{-i\boldsymbol{k}\cdot\mathbf{r}}\,\mathrm{d}^3x\right|^2 \quad \Leftarrow \quad \text{Chapter 14}$$

$$E_\rho = \sqrt{\frac{\pi}{2}}\frac{\mathbb{Z}\,e\,\omega}{\gamma\,v^2}\,e^{i\frac{\omega z}{v}}K_1\left(\frac{\omega\rho}{\gamma v}\right), \quad E_z = -i\sqrt{\frac{\pi}{2}}\frac{\mathbb{Z}\,e\,\omega}{\gamma\,v^2}\,e^{i\frac{\omega z}{v}}K_0\left(\frac{\omega\rho}{\gamma v}\right)$$

$$\Rightarrow \quad \mathbf{F} \equiv \int_{z>0}\left[\hat{\boldsymbol{k}}\times\mathbf{E}_i(\mathbf{r},\omega)\right]\times\hat{\boldsymbol{k}}\,e^{-i\boldsymbol{k}\cdot\mathbf{r}}\,\mathrm{d}^3x$$

$$= \iint\left[\hat{\boldsymbol{k}}\times\mathbf{E}_i\right]_{z=0}\times\hat{\boldsymbol{k}}\,e^{-ikx\sin\theta}\,\mathrm{d}y\,\mathrm{d}x\int_0^\infty e^{iz\frac{\omega-kv\cos\theta}{v}}\,\mathrm{d}z$$

$$= i\,v\,\frac{1-e^{iZ\frac{\omega-kv\cos\theta}{v}}}{\omega-kv\cos\theta}\iint\left[\hat{\boldsymbol{k}}\times\mathbf{E}_i\right]_{z=0}\times\hat{\boldsymbol{k}}\,e^{-ikx\sin\theta}\,\mathrm{d}y\,\mathrm{d}x \quad \Leftarrow \quad Z \geq D$$

For a single interface $\quad \mathbf{F} = \dfrac{i\,v}{\omega-kv\cos\theta}\displaystyle\iint\left[\hat{\boldsymbol{k}}\times\mathbf{E}_i\right]_{z=0}\times\hat{\boldsymbol{k}}\,e^{-ikx\sin\theta}\,\mathrm{d}y\,\mathrm{d}x$

$$(\hat{\boldsymbol{k}} \times \mathbf{E}_i) \times \hat{\boldsymbol{k}} = (E_\rho \cos\theta \cos\phi - E_z \sin\theta)\, \hat{\mathbf{e}}_a + E_\rho \sin\phi\, \hat{\mathbf{e}}_b \quad \Leftarrow \quad \text{figure}$$

$$\Rightarrow \quad \mathbf{F} = \frac{i\, v\, \hat{\mathbf{e}}_a}{\omega - k\, v \cos\theta} \iint \left(\frac{x\, E_\rho}{\sqrt{x^2 + y^2}} \cos\theta - E_z \sin\theta \right)_{z=0} e^{-i\, k\, x \sin\theta}\, \mathrm{d}\, x\, \mathrm{d}\, y$$

$$= \frac{i\, \hat{\mathbf{e}}_a}{\omega - k\, v \cos\theta} \sqrt{\frac{\pi}{2}}\, \frac{\mathbb{Z}\, e\, \omega}{\gamma\, v} \iint \mathrm{d}\, x\, \mathrm{d}\, y\, e^{-i\, k\, x \sin\theta} \times$$

$$\left[\frac{x \cos\theta}{\sqrt{x^2 + y^2}} K_1 \left(\frac{\omega}{\gamma\, v} \sqrt{x^2 + y^2} \right) + i\, \frac{\sin\theta}{\gamma} K_0 \left(\frac{\omega}{\gamma\, v} \sqrt{x^2 + y^2} \right) \right]$$

$$\frac{x}{\sqrt{x^2 + y^2}} K_1 \left(\frac{\omega}{\gamma\, v} \sqrt{x^2 + y^2} \right) = - \frac{\gamma\, v}{\omega} \frac{\partial}{\partial\, x} K_0 \left(\frac{\omega}{\gamma\, v} \sqrt{x^2 + y^2} \right)$$

$$\Rightarrow \quad \mathbf{F} = \hat{\mathbf{e}}_a \sqrt{\frac{\pi}{2}}\, \frac{\mathbb{Z}\, e \sin\theta}{\gamma^2\, v}\, \frac{\gamma^2\, k\, v \cos\theta - \omega}{\omega - k\, v \cos\theta} \iint e^{-i\, k\, x \sin\theta} K_0 \left(\frac{\omega}{\gamma\, v} \sqrt{x^2 + y^2} \right) \mathrm{d}\, x\, \mathrm{d}\, y$$

$$\int_0^\infty K_0 (\beta \sqrt{q^2 + t^2}) \cos(\alpha\, q)\, \mathrm{d}\, q = \frac{\pi}{2 \sqrt{\alpha^2 + \beta^2}}\, e^{-|t| \sqrt{\alpha^2 + \beta^2}}$$

$$\Rightarrow \quad \mathbf{F} = 2 \sqrt{2}\, \pi\, \frac{\mathbb{Z}\, e\, v \sin\theta}{\omega - k\, v \cos\theta}\, \frac{\gamma^2\, k\, v \cos\theta - \omega}{\omega^2 + \gamma^2\, k^2\, v^2 \sin^2\theta}\, \hat{\mathbf{e}}_a$$

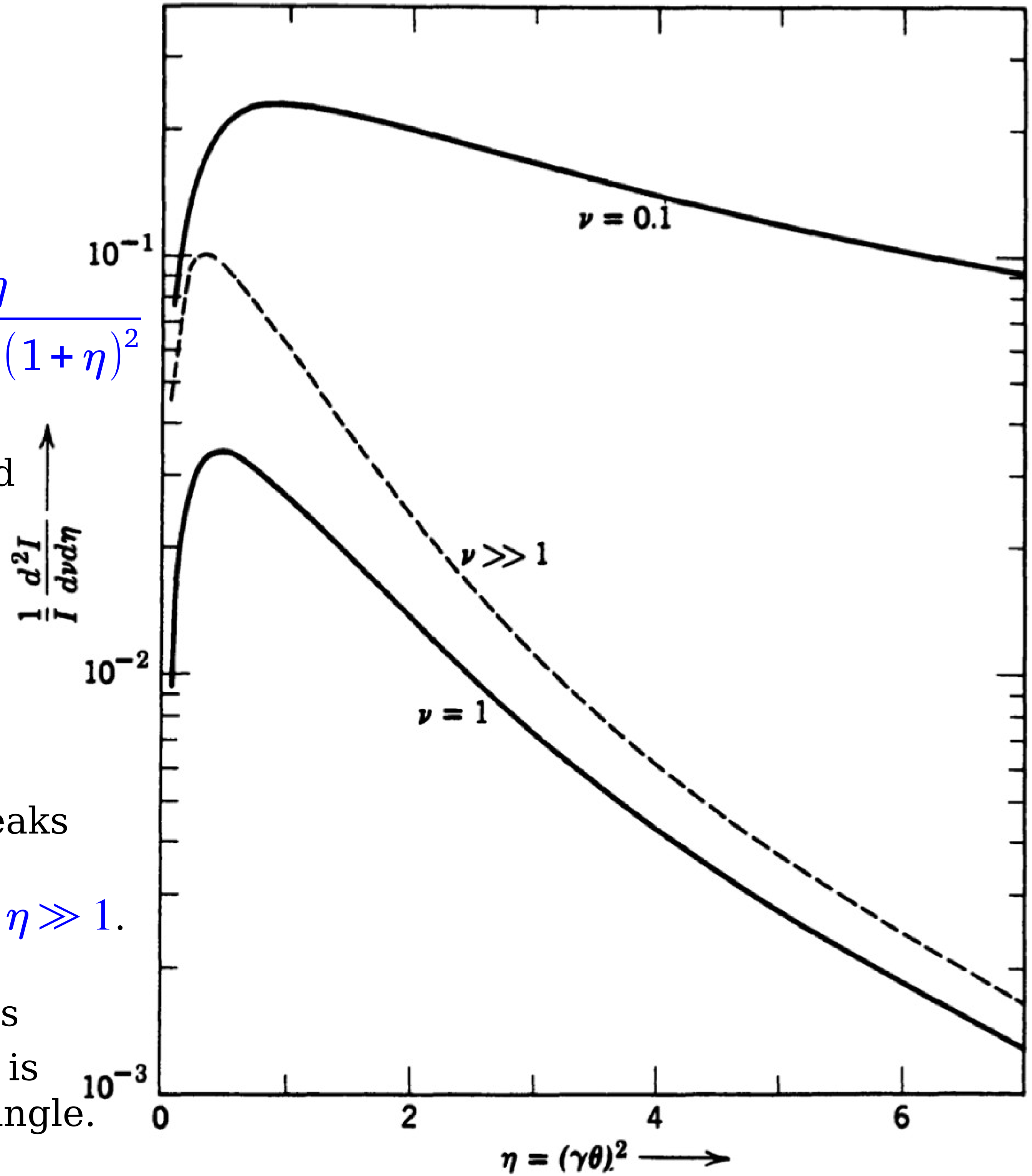
$$\simeq 4 \sqrt{2}\, \pi\, \frac{\mathbb{Z}\, e\, \gamma\, c}{\omega_p^2}\, \frac{\sqrt{\eta}\, \hat{\mathbf{e}}_a}{(1 + \nu^2 + \eta\, \nu^2)(1 + \eta)} \quad \Leftarrow \quad \begin{array}{l} \eta = \gamma^2\, \theta^2 \\ \gamma \gg 1\,, \quad \theta \ll 1\,, \quad \omega \gg \omega_p \end{array}$$

$$\mathrm{d}\,\Omega = \mathrm{d}\,\phi\,\mathrm{d}\cos\theta \simeq \frac{\mathrm{d}\,\phi\,\mathrm{d}\,\eta}{2\,\gamma^2}$$

$$\Rightarrow \quad \frac{\mathrm{d}^2 I}{\mathrm{d}\,\nu\,\mathrm{d}\,\eta} = \frac{\pi\,\omega_p}{\gamma}\,\frac{\mathrm{d}^2 I}{\mathrm{d}\,\omega\,\mathrm{d}\,\Omega}$$

$$\simeq \frac{\mathbb{Z}^2 e^2 \gamma\,\omega_p\,\eta}{\pi\,c\,(1+\nu^2+\eta\,\nu^2)^2\,(1+\eta)^2}$$

- At low frequencies the spectrum peaks at $\eta \simeq 1$ and then falls as $\frac{1}{\eta}$ until the value $\eta = \frac{1}{\nu^2}$ is reached. Then it falls off as $\frac{1}{\eta^3}$.
- For $\nu \geq 1$, the spectrum peaks at $\eta \geq \frac{1}{3}$ and falls at $\frac{1}{\eta^3}$ for $\eta \gg 1$.
- At $\eta = 0$ the denominator is $(1+\nu^2)^2$, so for $\nu \gg 1$ there is negligible intensity at any angle.



- The energy spectrum is

$$\frac{dI}{d\nu} = \frac{\mathbb{Z}^2 e^2 \gamma\, \omega_p}{\pi c}\left(\left(1+2\nu^2\right)\ln\frac{1+\nu^2}{\nu^2} - 2\right)$$

$$\Rightarrow \quad \frac{dI}{d\nu} \simeq \frac{\mathbb{Z}^2 e^2 \gamma\, \omega_p}{\pi c}\begin{cases} -2\ln\nu - 2\,, & \nu \ll 1 \\ \dfrac{1}{6\nu^4}\,, & \nu \gg 1 \end{cases}$$

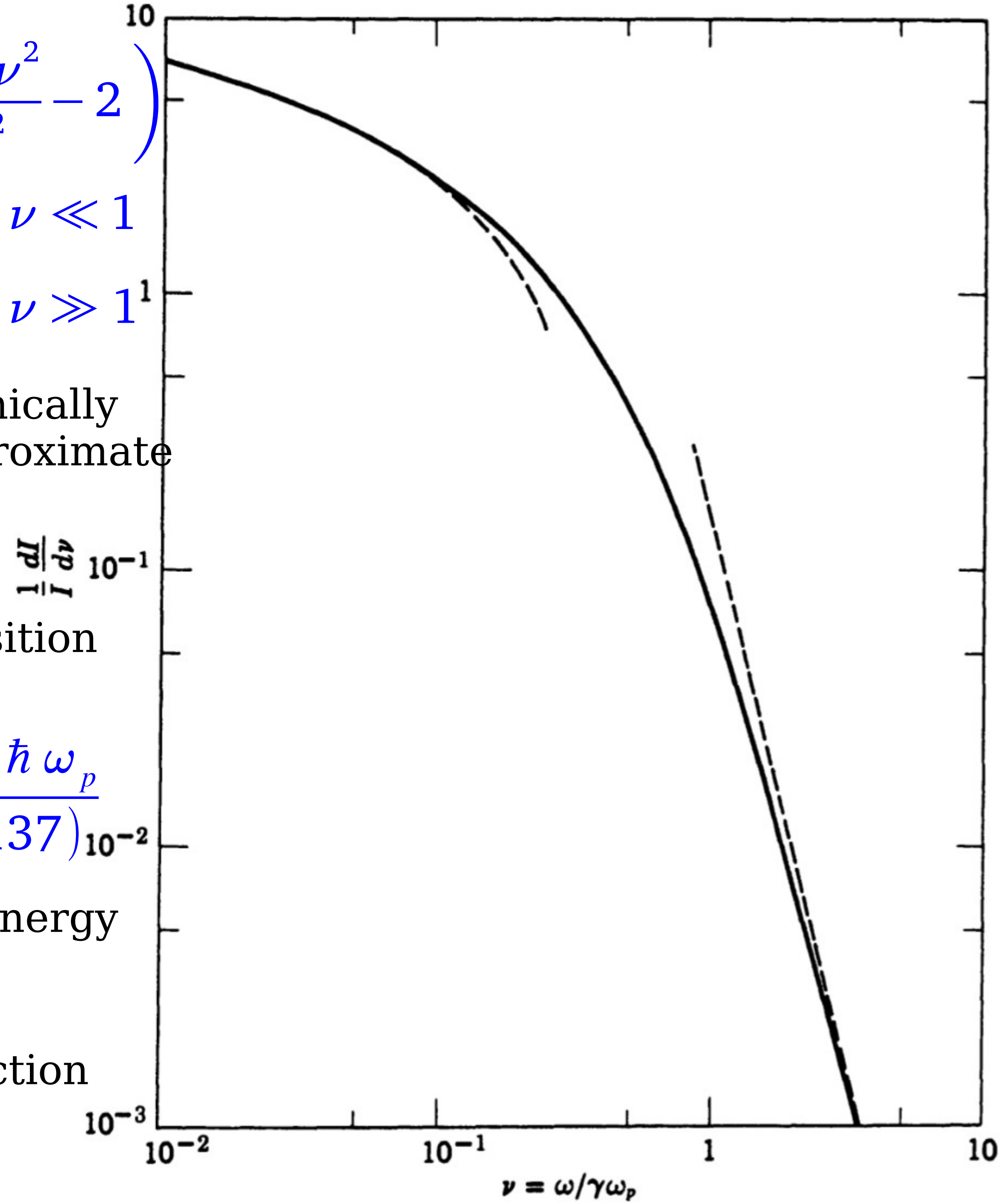
- The spectrum diverges logarithmically at low frequencies, where our approximate treatment fails in any event, but it has a finite integral.

- The total energy emitted in transition radiation per interface

$$I = \int_0^\infty \frac{dI}{d\nu}\, d\nu = \frac{\mathbb{Z}^2 e^2 \gamma\, \omega_p}{3c} = \frac{\mathbb{Z}^2 \gamma\, \hbar\, \omega_p}{3\,(137)}$$

- From the figure, about half the energy is emitted in the range $0.1 \le \nu \le 1$.

- It means that an appreciable fraction of the energy appears as energetic photons.



- With $\gamma = 10^3$ & $\hbar\omega_p = 20\,\mathrm{eV}$, these quanta are in the soft x-ray region of 2-20keV.

- The presence of the factor of γ in the result makes transition radiation attractive as a mechanism for the identification of particles, and perhaps even measurement of their energies, at very high energies where other means are unavailable.

- The numerical factor $\dfrac{1}{3\times 137}$ indicates that the probability of energetic photon emission per transit of an interface is very small.

- So it needs to utilize a stack of many foils with gaps between. The foils can be quite thin, compared to a formation length D. Then a particle traversing each foil will emit twice in transition radiation.

- A typical set-up might involve 200 Mylar foils of thickness 20 μm, with spacings 150-300 μm. The coherent superposition of the fields from the different interfaces, 2 for each foil, causes a modulation of the energy and angular distributions.