Chapter 10 Scattering and Diffraction

- Approaches for the topics of scattering and diffraction differ depending on the relative length scales involved—the wavelength and the size of the target.

- When the wavelength of the radiation is large compared to the dimensions of the target, a simple description in terms of lowest order induced multipoles is appropriate.

- When the wavelength and size are comparable, a more systematic treatment with multipole fields is required.

- In the limit of very small wavelength compared to the size of the target, semi-geometric methods can be utilized to obtain the departures from geometrical optics.

Scattering at Long Wavelengths

A. Scattering by Dipoles Induced in Small Scatterers

- Think of the incident (radiation) fields as inducing electric and magnetic multipoles that oscillate in definite phase relationship with the incident wave and radiate energy in directions other than the direction of incidence.

- The exact form of the angular distribution of radiated energy is governed by the *coherent superposition* of multipoles induced by the incident fields and in general depends on the state of polarization of the incident wave.

- If the wavelength of the radiation is long compared to the size of the scatterer, only the lowest multipoles, usually electric and magnetic dipoles, are important.

- In these circumstances the induced dipoles can be calculated from static or quasi-static boundary-value problems, just as for the small apertures in Chap. 9.

- Consider a plane monochromatic wave to be incident on a scatterer, $\mu_r = \epsilon_r = 1$

$$\mathbf{E}_{\text{inc}} = \hat{\mathbf{e}}_0 E_0 e^{i(\boldsymbol{k}_0\cdot\mathbf{r} - \omega t)} = \hat{\mathbf{e}}_0 E_0 e^{i(k\hat{\boldsymbol{k}}_0\cdot\mathbf{r} - \omega t)}, \quad \mathbf{H}_{\text{inc}} = \frac{\hat{\boldsymbol{k}}_0 \times \mathbf{E}_{\text{inc}}}{Z_0} \quad \Leftarrow \quad \boldsymbol{k}_0 = k\,\hat{\boldsymbol{k}}_0, \quad k = \frac{\omega}{c}$$

- These fields induce dipole moments $\mathbf{p}$ and $\mathbf{m}$ in the small scatterer and these dipoles radiate energy in all directions

$$\Rightarrow \quad \mathbf{E}_{\text{sc}} = \frac{k^2}{4\pi\epsilon_0}\frac{e^{i(kr-\omega t)}}{r}\left(\hat{\mathbf{r}}\times\mathbf{p} + \frac{\mathbf{m}}{c}\right)\times\hat{\mathbf{r}}, \quad \mathbf{H}_{\text{sc}} = \frac{\hat{\mathbf{r}}\times\mathbf{E}_{\text{sc}}}{Z_0} \quad (\$) \quad \Leftarrow \quad \begin{array}{l}\boldsymbol{k} = k\,\hat{\mathbf{r}} \\ \boldsymbol{k}\cdot\mathbf{r} = k\,r\end{array}$$

Differential scattering cross section: the power radiated in the direction $\hat{\mathbf{r}}$ with polarization $\hat{\mathbf{e}}$, per unit solid angle, per unit incident flux (power per unit area) in the direction $\hat{\boldsymbol{k}}_0$ with polarization $\hat{\mathbf{e}}_0$

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{e}};\hat{\boldsymbol{k}}_0,\hat{\mathbf{e}}_0\right) \equiv \frac{\dfrac{dP_{\mathrm{sc}}}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{e}}\right)}{\dfrac{dP_{\mathrm{inc}}}{da}\left(\hat{\boldsymbol{k}}_0,\hat{\mathbf{e}}_0\right)} = \frac{\dfrac{r^2}{2Z_0}\left|\hat{\mathbf{e}}^*\cdot\mathbf{E}_{\mathrm{sc}}\right|^2}{\dfrac{1}{2Z_0}\left|\hat{\mathbf{e}}_0^*\cdot\mathbf{E}_{\mathrm{inc}}\right|^2}$$

$$= \frac{k^4}{\left(4\pi\epsilon_0 E_0\right)^2}\left|\hat{\mathbf{e}}^*\cdot\left(\mathbf{p}+\frac{\mathbf{m}}{c}\times\hat{\mathbf{r}}\right)\right|^2 \quad \Leftarrow \quad \hat{\mathbf{e}}\perp\hat{\mathbf{r}}$$

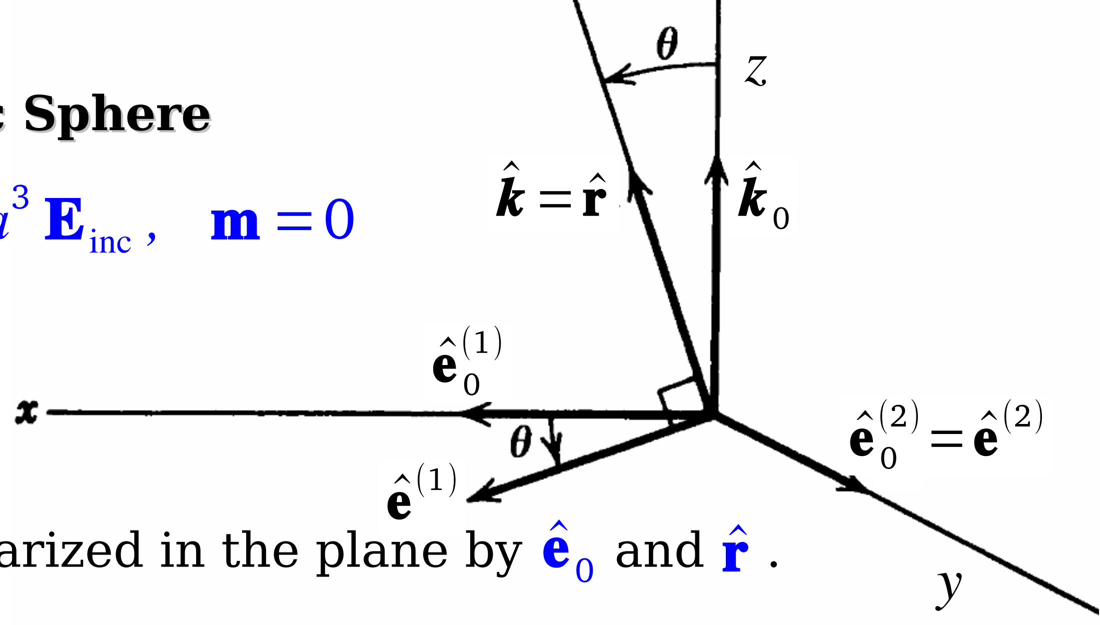
The dependence of the cross section on $\hat{\boldsymbol{k}}_0$ & $\hat{\mathbf{e}}_0$ is implicitly contained in $\mathbf{p}$ & $\mathbf{m}$.

- **Rayleigh's law**: the variation of the differential (and total) scattering cross section with wave number as k^4 (or λ^{-4}) is an almost universal characteristic of the scattering of long-wavelength radiation by any finite system.

- Only if both static dipole moments vanish does the scattering fail to obey Rayleigh's law; the scattering is then via quadrupole or higher multipoles (or frequency-dependent dipole moments) and varies as ω^6 or higher.

- Rayleigh scattering is usually for the *incoherent* scattering by a collection of dipole scatterers.

B. Scattering by a Small Dielectric Sphere



- Sec. 4.4 $\Rightarrow$ $\mathbf{p} = 4\pi\epsilon_0 \dfrac{\epsilon_r(\omega)-1}{\epsilon_r(\omega)+2} a^3 \mathbf{E}_{\text{inc}}$, $\mathbf{m} = 0$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = k^4 a^6 \left| \frac{\epsilon_r - 1}{\epsilon_r + 2} \hat{\mathbf{e}}^* \cdot \hat{\mathbf{e}}_0 \right|^2} \quad (9)$$

The scattered radiation is linearly polarized in the plane by $\hat{\mathbf{e}}_0$ and $\hat{\mathbf{r}}$.

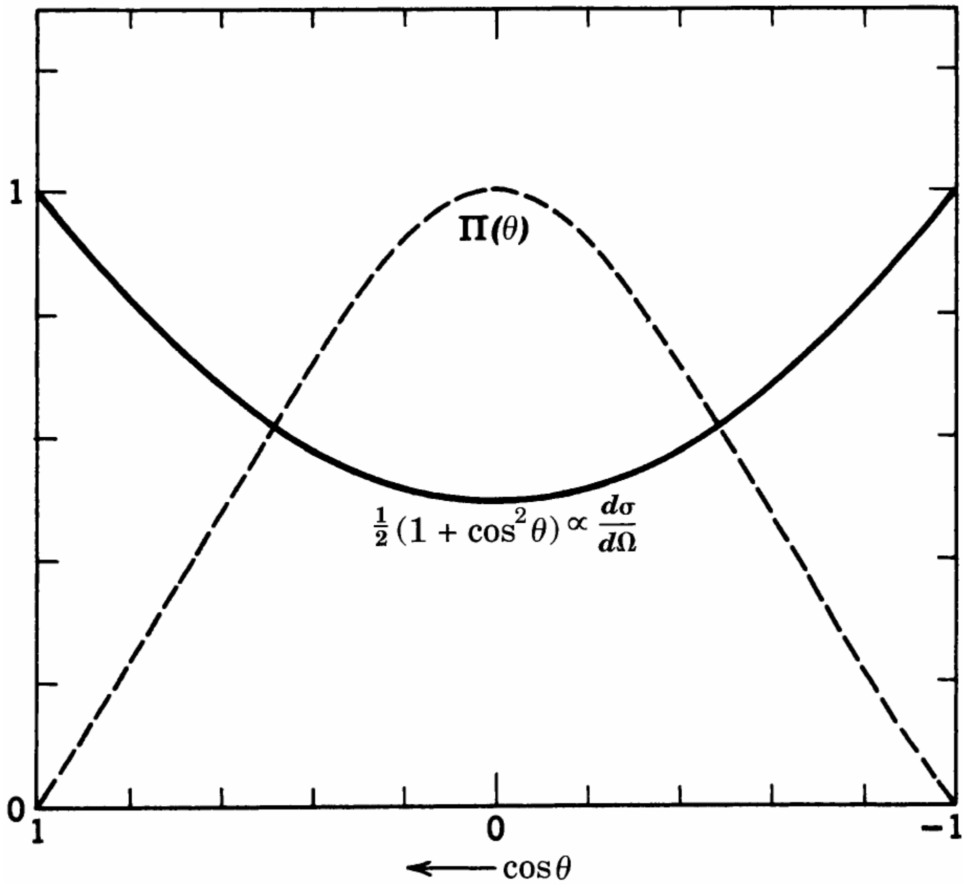
- The incident radiation is unpolarized. So we would like to know the angular distribution of scattered radiation of a definite state of linear polarization. The cross section (9) is averaged over initial polarization $\hat{\mathbf{e}}_0$ for a fixed choice of $\hat{\mathbf{e}}$.

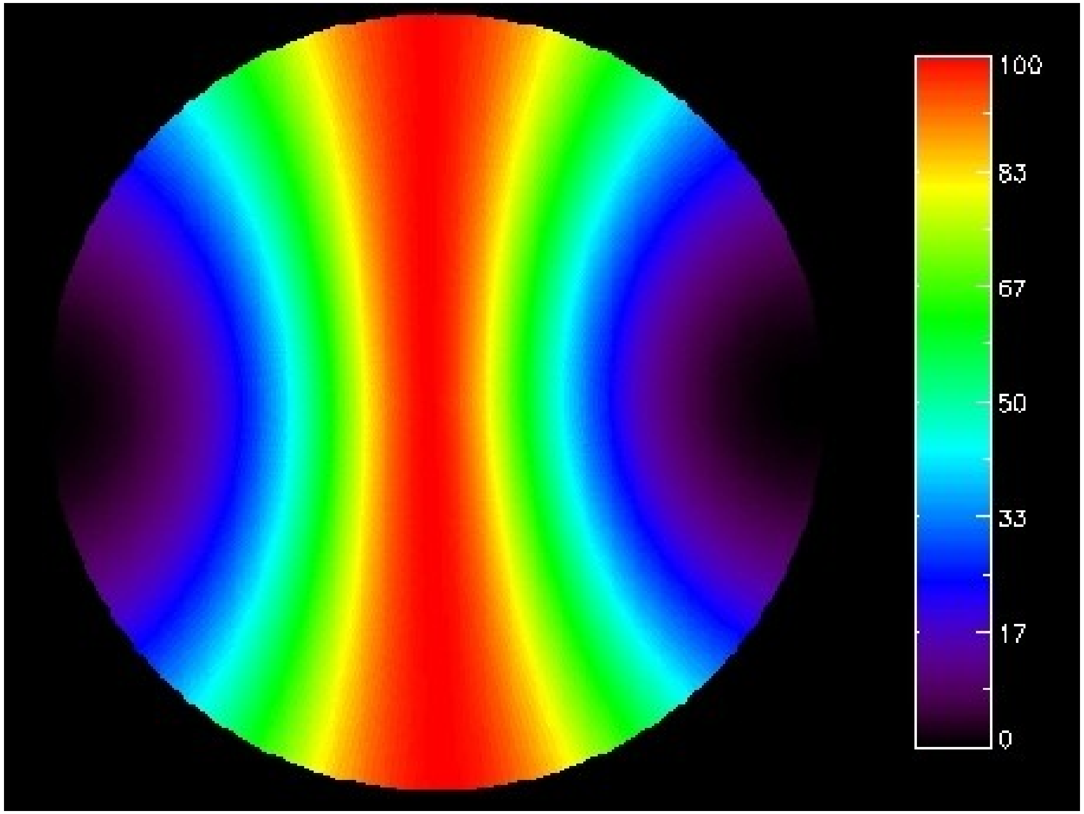
- The scattering plane is defined by $\hat{\boldsymbol{k}}_0$ and $\hat{\mathbf{r}}$. The differential cross sections for an unpolarized incident radiation is averaged to be

$$\Rightarrow \begin{cases} \dfrac{d\sigma_\parallel}{d\Omega} = \dfrac{k^4 a^6}{2} \left| \dfrac{\epsilon_r - 1}{\epsilon_r + 2} \right|^2 \cos^2\theta \\[2ex] \dfrac{d\sigma_\perp}{d\Omega} = \dfrac{k^4 a^6}{2} \left| \dfrac{\epsilon_r - 1}{\epsilon_r + 2} \right|^2 \end{cases} \Rightarrow \text{polarization } \Pi(\theta) \equiv \frac{\dfrac{d\sigma_\perp}{d\Omega} - \dfrac{d\sigma_\parallel}{d\Omega}}{\dfrac{d\sigma_\perp}{d\Omega} + \dfrac{d\sigma_\parallel}{d\Omega}} = \frac{\sin^2\theta}{1+\cos^2\theta}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{k^4 a^6}{2} \left| \frac{\epsilon_r - 1}{\epsilon_r + 2} \right|^2 \left(1+\cos^2\theta\right) \Rightarrow \sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi}{3} k^4 a^6 \left| \frac{\epsilon_r - 1}{\epsilon_r + 2} \right|^2 \quad (**)$$

total scattering cross section





- The polarization Π has its max at $\theta = \frac{\pi}{2}$. At this angle the scattered radiation is 100% linearly polarized $\perp$ to the scattering plane, and for an appreciable range of angles on either side of $\theta = \frac{\pi}{2}$ is quite significantly polarized.
- The polarization characteristics of the blue sky are an illustration of this phenomenon, and can be verified on a sunny day with a linear polarizer or suitable sunglasses.

C. Scattering by a Small Perfectly Conducting Sphere

- Sec . 2.5 $\Rightarrow$ $\mathbf{p} = 4\pi\epsilon_0 a^3 \mathbf{E}_{\text{inc}}$ $\Rightarrow$ $\dfrac{d\sigma}{d\Omega} = \dfrac{k^4 a^6}{4}\left|2\,\hat{\mathbf{e}}^*\cdot\hat{\mathbf{e}}_0 - (\hat{\mathbf{r}}\times\hat{\mathbf{e}}^*)\cdot(\hat{\boldsymbol{k}}_0\times\hat{\mathbf{e}}_0)\right|^2$

  Sec . 5.11 $\Rightarrow$ $\mathbf{m} = -2\pi a^3 \mathbf{H}_{\text{inc}}$

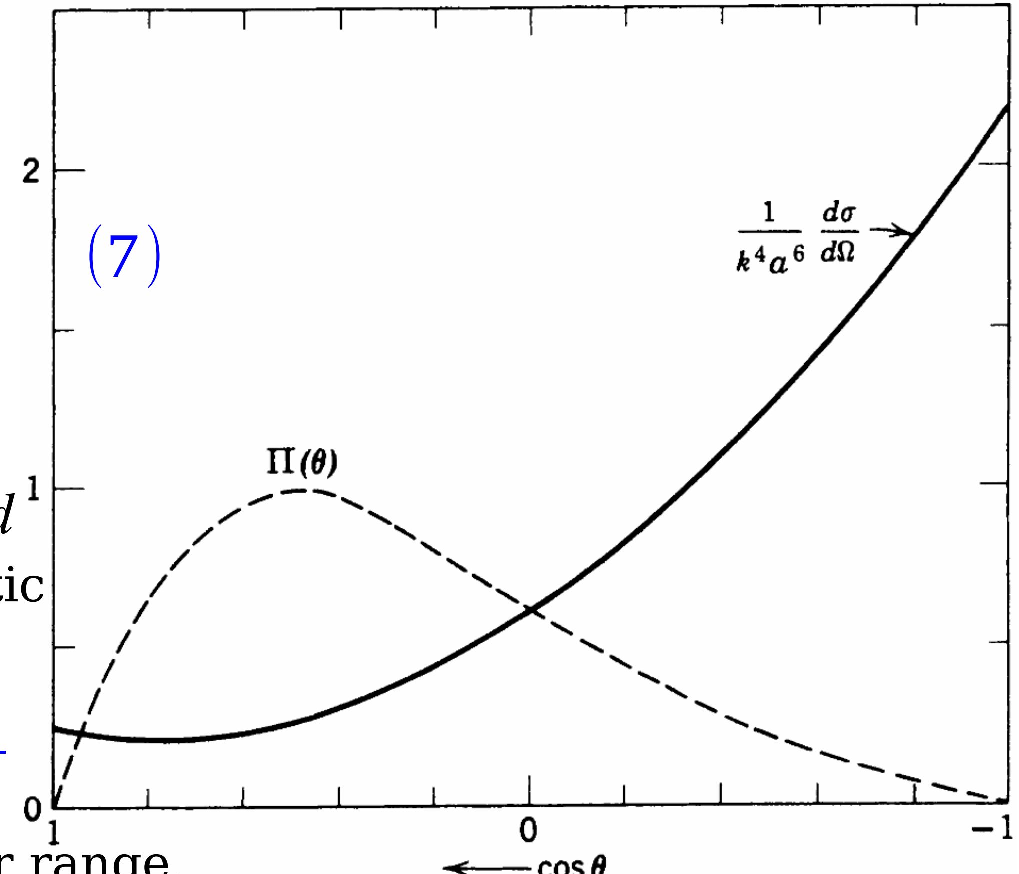
$$(5.115) \;\Rightarrow\; \mathbf{M} = \lim_{\mu\to 0}\frac{3}{\mu_0}\frac{\mu-\mu_0}{\mu+2\mu_0}\mathbf{B} = -\frac{3}{2}\mathbf{H} \;\Rightarrow\; \mathbf{m} = \frac{4\pi}{3}a^3\mathbf{M} = -2\pi a^3\mathbf{H}$$

$$\Rightarrow \frac{d\sigma_\parallel}{d\Omega} = \frac{k^4 a^6}{8}(2\cos\theta - 1)^2,\quad \frac{d\sigma_\perp}{d\Omega} = \frac{k^4 a^6}{8}(2-\cos\theta)^2 \;\Leftarrow$$ with unpolarized radiation incident

$$\Rightarrow \Pi(\theta) = \frac{3\sin^2\theta}{5 - 8\cos\theta + 5\cos^2\theta}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{k^4 a^6}{8}(5 - 8\cos\theta + 5\cos^2\theta) \qquad (7)$$

$$\Rightarrow \sigma = \frac{10\pi}{3}k^4 a^6$$



- The cross section has a *strong backward peaking* caused by electric dipole-magnetic dipole interference.

- The polarization reaches $\Pi = 1$ at $\theta = \dfrac{\pi}{3}$ and is positive through the whole angular range.

- Dipole scattering with its ω^4 dependence on frequency is the lowest order approximation in an expansion in kd.

- In the domain $kd \sim 1$, more than the lowest order multipoles must be considered.

- When $kd \gg 1$, approximation methods of a different sort can be employed, as in Sec. 10.10.

D. Collection of Scatterers

- Consider the scattering system consists of a number of small scatters with fixed spatial separations. The scattering cross section results from a coherent superposition of the individual amplitudes.

- Because the induced dipoles are proportional to the incident fields, evaluated at the position $\mathbf{r}_j$ of the j^{th} scatterer, its moments will possess a phase factor, $e^{i\boldsymbol{k}_0\cdot\mathbf{r}_j}$.

- If the observation point is far from the whole scattering system, the fields for the j^{th} scatterer will have a phase factor $e^{-i\boldsymbol{k}\cdot\mathbf{r}_j}$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\sum \hat{\mathbf{e}}^{\star}\cdot\left(\mathbf{p}_j + \frac{\mathbf{m}_j}{c}\times\hat{\mathbf{r}}\right)e^{i\mathbf{q}\cdot\mathbf{r}_j}\right|^2 \quad \Leftarrow \quad \begin{aligned}\mathbf{q} &= \boldsymbol{k}_0 - \boldsymbol{k}\\ &= k\,(\hat{\boldsymbol{k}}_0 - \hat{\boldsymbol{k}})\end{aligned}$$

- The presence of the phase factors means that the scattering depends sensitively on the exact distribution of the scatterers in space.

- Assume that all the scatterers are identical. Then the cross section is the product of the cross section for one scatterer times a structure factor,

$$\mathscr{F}(\mathbf{q}) = \left|\sum e^{i\mathbf{q}\cdot\mathbf{r}_j}\right|^2 = \sum_{j,\ell} e^{i\mathbf{q}\cdot(\mathbf{r}_j - \mathbf{r}_\ell)}$$

- If the scatterers are randomly distributed, the terms with $j \neq \ell$ give a negligible contribution. Only the terms with $j = \ell$ are significant

$$\Rightarrow \quad \mathscr{F}(\mathbf{q}) = N \quad \Leftarrow \quad \text{incoherent superposition}$$

- If the scatterers are numerous and have a regular distribution in space, the structure factor effectively vanishes everywhere except in the forward direction.

- There is therefore no scattering by a very large regular array of scatterers, of which single crystals of transparent solids like rock salt or quartz are examples.

- For a simple cubic array of scattering centers $N = N_1 N_2 N_3$

$$\mathscr{F}(\mathbf{q}) = N^2 \frac{\sin^2 \frac{N_1 q_1 a}{2}}{N_1^2 \sin^2 \frac{q_1 a}{2}} \frac{\sin^2 \frac{N_2 q_2 a}{2}}{N_2^2 \sin^2 \frac{q_2 a}{2}} \frac{\sin^2 \frac{N_3 q_3 a}{2}}{N_2^2 \sin^2 \frac{q_3 a}{2}}$$

- At short wavelengths $ka > \pi$, $\mathscr{F}$ has peaks when the Bragg scattering condition, $q_i a = 0, 2\pi, 4\pi, \ldots$, is obeyed. This is the situation familiar in x-ray diffraction.

- At long wavelengths only the peak at $q_i a = 0$ is relevant because $(q_i a)_{\max} = 2ka \ll 1$

$$\mathscr{F}(\mathbf{q}) \to \left[N \frac{\sin \frac{N_1 q_1 a}{2}}{\frac{N_1 q_1 a}{2}} \frac{\sin \frac{N_2 q_2 a}{2}}{\frac{N_2 q_2 a}{2}} \frac{\sin \frac{N_3 q_3 a}{2}}{\frac{N_3 q_3 a}{2}} \right]^2$$

- The scattering is thus confined to the region $q_i \leq \frac{2\pi}{N_i a}$, corresponding to angle smaller than $\frac{\lambda}{L}$, where L is a typical overall dimension of the scattering array.

$$\sum_{j=1}^{N} e^{i\,\mathbf{q}\cdot\mathbf{r}_j} = \sum_{j=1}^{N} e^{i\left(q_1 x_j + q_2 y_j + q_3 z_j\right)} = \sum_{\ell=0}^{N_1-1} e^{i\,q_1 \ell a} \sum_{m=0}^{N_2-1} e^{i\,q_2 m a} \sum_{n=0}^{N_3-1} e^{i\,q_3 n a}$$

$$= \frac{e^{i\,q_1 N_1 a}-1}{e^{i\,q_1 a}-1}\,\frac{e^{i\,q_2 N_2 a}-1}{e^{i\,q_2 a}-1}\,\frac{e^{i\,q_3 N_3 a}-1}{e^{i\,q_3 a}-1} \quad \Leftarrow \quad N = N_1 N_2 N_3$$

$$\mathscr{F}\left(\mathbf{q}\right) = \left|\sum e^{i\,\mathbf{q}\cdot\mathbf{r}_j}\right|^2 = \sum_j e^{i\,\mathbf{q}\cdot\mathbf{r}_j} \sum_k e^{-i\,\mathbf{q}\cdot\mathbf{r}_k}$$

$$= \frac{e^{i\,q_1 N_1 a}-1}{e^{i\,q_1 a}-1}\,\frac{e^{-i\,q_1 N_1 a}-1}{e^{-i\,q_1 a}-1}\,\frac{e^{i\,q_2 N_2 a}-1}{e^{i\,q_2 a}-1}\,\frac{e^{-i\,q_2 N_2 a}-1}{e^{-i\,q_2 a}-1}\,\frac{e^{i\,q_3 N_3 a}-1}{e^{i\,q_3 a}-1}\,\frac{e^{-i\,q_3 N_3 a}-1}{e^{-i\,q_3 a}-1}$$

$$= \frac{1-\cos\left(q_1 N_1 a\right)}{1-\cos\left(q_1 a\right)}\,\frac{1-\cos\left(q_2 N_2 a\right)}{1-\cos\left(q_2 a\right)}\,\frac{1-\cos\left(q_3 N_3 a\right)}{1-\cos\left(q_3 a\right)}$$

$$= \frac{\sin^2 \dfrac{q_1 N_1 a}{2}}{\sin^2 \dfrac{q_1 a}{2}}\,\frac{\sin^2 \dfrac{q_2 N_2 a}{2}}{\sin^2 \dfrac{q_2 a}{2}}\,\frac{\sin^2 \dfrac{q_3 N_3 a}{2}}{\sin^2 \dfrac{q_3 a}{2}}$$

Perturbation Theory of Scattering, Rayleigh's Explanation of the Blue Sky, Scattering by Gases and Liquids, Attenuation in Optical Fibers

A. General Theory

- If there are spatial/temporal variations in the EM properties of the medium, the wave is scattered. Some of the energy is deviated from its original course.

- If the variations in the properties are small in magnitude, the scattering is slight and perturbative methods can be employed.

- $\mathbf{D} \neq \bar{\epsilon}\,\mathbf{E}\,, \quad \mathbf{B} \neq \bar{\mu}\,\mathbf{H} \quad \Leftarrow \quad \bar{\epsilon}\,,\ \bar{\mu}$ are the background values

Maxwell's equations $\quad \nabla\cdot\mathbf{D}=0\,, \quad \nabla\cdot\mathbf{B}=0\,, \quad \nabla\times\mathbf{E}=-\dfrac{\partial\,\mathbf{B}}{\partial\,t}\,, \quad \nabla\times\mathbf{H}=\dfrac{\partial\,\mathbf{D}}{\partial\,t}$

$$\nabla\times(\nabla\times\mathbf{D})=\nabla\,(\cancel{\nabla\cdot\mathbf{D}})-\nabla^2\mathbf{D}$$

$$-\bar{\mu}\,\bar{\epsilon}\,\frac{\partial^2}{\partial\,t^2}\mathbf{D}=-\bar{\mu}\,\bar{\epsilon}\,\frac{\partial}{\partial\,t}\nabla\times\mathbf{H}=\bar{\epsilon}\,\frac{\partial}{\partial\,t}\nabla\times(-\bar{\mu}\,\mathbf{H}) \quad \Leftarrow \quad \nabla\times\mathbf{H}=\frac{\partial\,\mathbf{D}}{\partial\,t}$$

$$\bar{\epsilon}\,\frac{\partial}{\partial\,t}\nabla\times\mathbf{B}=\bar{\epsilon}\,\nabla\times\frac{\partial\,\mathbf{B}}{\partial\,t}=-\bar{\epsilon}\,\nabla\times(\nabla\times\mathbf{E}) \quad \Leftarrow \quad \nabla\times\mathbf{E}=-\frac{\partial\,\mathbf{B}}{\partial\,t}$$

$$\Rightarrow \quad \left(\nabla^2-\bar{\mu}\,\bar{\epsilon}\,\frac{\partial^2}{\partial\,t^2}\right)\mathbf{D}=-\nabla\times\nabla\times(\mathbf{D}-\bar{\epsilon}\,\mathbf{E})+\bar{\epsilon}\,\frac{\partial}{\partial\,t}\nabla\times(\mathbf{B}-\bar{\mu}\,\mathbf{H})$$

$$\Rightarrow \quad (\nabla^2+k^2)\,\mathbf{D}=-\nabla\times\nabla\times(\mathbf{D}-\bar{\epsilon}\,\mathbf{E})-i\,\bar{\epsilon}\,\omega\,\nabla\times(\mathbf{B}-\bar{\mu}\,\mathbf{H}) \quad \Leftarrow \quad k^2=\bar{\mu}\,\bar{\epsilon}\,\omega^2$$

$$\Rightarrow\quad \mathbf{D} = \bar{\mathbf{D}} + \frac{1}{4\pi}\int \frac{e^{ik\mathbb{r}}}{\mathbb{r}}\left(\nabla'\times\nabla'\times(\mathbf{D}-\bar{\epsilon}\,\mathbf{E}) + i\,\bar{\epsilon}\,\omega\,\nabla'\times(\mathbf{B}-\bar{\mu}\,\mathbf{H})\right)\mathrm{d}^3x'$$

$$\Rightarrow\quad \mathbf{D}(r\to\infty) = \bar{\mathbf{D}} + \mathbf{A}_{\mathrm{sc}}\frac{e^{ikr}}{r} \quad \Leftarrow \quad \bar{\mathbf{D}} \equiv \bar{\epsilon}\,\mathbf{E}\,,\quad \bar{\mathbf{B}} \equiv \bar{\mu}\,\mathbf{H}\quad \text{unperturbated solution}$$

$$\mathbf{A}_{\mathrm{sc}} = \int \frac{e^{-i\boldsymbol{k}\cdot\mathbf{r}'}}{4\pi}\left(\nabla'\times\nabla'\times(\mathbf{D}-\bar{\epsilon}\,\mathbf{E}) + i\,\bar{\epsilon}\,\omega\,\nabla'\times(\mathbf{B}-\bar{\mu}\,\mathbf{H})\right)\mathrm{d}^3x' \quad \text{scattering amplitude}$$

Integration by parts

$$\Rightarrow\quad \boxed{\mathbf{A}_{\mathrm{sc}} = \frac{k^2}{4\pi}\int e^{-i\boldsymbol{k}\cdot\mathbf{r}}\left(\hat{\mathbf{r}}\times(\mathbf{D}-\bar{\epsilon}\,\mathbf{E}) + \frac{\bar{\epsilon}\,\omega}{k}(\mathbf{B}-\bar{\mu}\,\mathbf{H})\right)\times\hat{\mathbf{r}}\,\mathrm{d}^3x} \qquad (8)$$

- Compared with with the scattered dipole field ($), the polarization dependence of the contribution from $(\mathbf{D}-\bar{\epsilon}\,\mathbf{E})$ is that of an electric dipole $\mathbf{p}$, from $(\mathbf{B}-\bar{\mu}\,\mathbf{H})$ a magnetic dipole $\mathbf{m}$.

- $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{|\hat{\mathbf{e}}^{*}\cdot\mathbf{A}_{\mathrm{sc}}|^2}{|\bar{\mathbf{D}}|^2}$$

- With the expression of scattering amplitude, a systematic scheme of successive approximations can be developed in the same way as the Born approximation series of quantum-mechanical scattering.

B. Born Approximation

- (First) Born approximation: lowest order approximation for the scattering amplitude.

- $$\begin{matrix} \mathbf{D}(\mathbf{r}) = [\bar{\epsilon} + \delta\epsilon(\mathbf{r})]\,\mathbf{E}(\mathbf{r}) \\ \mathbf{B}(\mathbf{r}) = [\bar{\mu} + \delta\mu(\mathbf{r})]\,\mathbf{H}(\mathbf{r}) \end{matrix} \quad \Rightarrow \quad \begin{matrix} \mathbf{D} - \bar{\epsilon}\,\mathbf{E} \simeq \dfrac{\delta\epsilon(\mathbf{r})}{\bar{\epsilon}}\,\bar{\mathbf{D}}(\mathbf{r}) \\ \mathbf{B} - \bar{\mu}\,\mathbf{H} \simeq \dfrac{\delta\mu(\mathbf{r})}{\bar{\mu}}\,\bar{\mathbf{B}}(\mathbf{r}) \end{matrix} \quad + \quad \begin{matrix} \bar{\mathbf{D}} = \hat{\mathbf{e}}_0\,\bar{D}\,e^{i\boldsymbol{k}_0\cdot\mathbf{r}} \\ \bar{\mathbf{B}} = \sqrt{\dfrac{\bar{\mu}}{\bar{\epsilon}}}\,\hat{\boldsymbol{k}}_0 \times \bar{\mathbf{D}} \end{matrix}$$

$$\Rightarrow \boxed{\frac{\hat{\mathbf{e}}^*\cdot\mathbf{A}_{\mathrm{sc}}^{(1)}}{\bar{D}} = \frac{k^2}{4\pi}\int e^{i\mathbf{q}\cdot\mathbf{r}}\left(\hat{\mathbf{e}}^*\cdot\hat{\mathbf{e}}_0\,\frac{\delta\epsilon(\mathbf{r})}{\bar{\epsilon}} + (\hat{\boldsymbol{k}}\times\hat{\mathbf{e}}^*)\cdot(\hat{\boldsymbol{k}}_0\times\hat{\mathbf{e}}_0)\,\frac{\delta\mu(\mathbf{r})}{\bar{\mu}}\right)\mathrm{d}^3x} \quad (2)$$

- If $k\,d \ll 1 \quad \Rightarrow \quad e^{i\mathbf{q}\cdot\mathbf{r}} \to 1 \quad \Rightarrow \quad \mathbf{A}_{\mathrm{sc}}$ is then a dipole approximation $\quad \mathbf{q} = \boldsymbol{k}_0 - \boldsymbol{k}$

- Suppose the scattering region is a uniform dielectric sphere of radius a and $\delta\epsilon$ is constant inside the spherical volume

$$\Rightarrow \frac{\hat{\mathbf{e}}^*\cdot\mathbf{A}_{\mathrm{sc}}^{(1)}}{\bar{D}} = k^2\,\frac{\delta\epsilon}{\bar{\epsilon}}\,\hat{\mathbf{e}}^*\cdot\hat{\mathbf{e}}_0\,\frac{\sin(q\,a) - q\,a\cos(q\,a)}{q^3} \quad \Rightarrow \quad \lim_{q\to 0}\frac{\hat{\mathbf{e}}^*\cdot\mathbf{A}_{\mathrm{sc}}^{(1)}}{\bar{D}} = k^2\,\frac{\delta\epsilon}{\bar{\epsilon}}\,\hat{\mathbf{e}}^*\cdot\hat{\mathbf{e}}_0\,\frac{a^3}{3}$$

- At very low frequencies or in the forward direction at all frequencies, the Born approximation to the differential cross section for scattering by a dielectric sphere of radius a is $\displaystyle \lim_{q\to 0}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}_{\mathrm{Born}} = k^4 a^6 \left|\frac{\delta\epsilon}{3\,\bar{\epsilon}}\,\hat{\mathbf{e}}^*\cdot\hat{\mathbf{e}}_0\right|^2 = \frac{k^4 a^6}{9\,\bar{\epsilon}^2}\left|\delta\epsilon\,\hat{\mathbf{e}}^*\cdot\hat{\mathbf{e}}_0\right|^2 \quad \Leftrightarrow \quad (9)$

A uniform dielectric sphere of radius a and $\delta\epsilon = \text{const}$, $\delta\mu = 0$

$$\Rightarrow \frac{\hat{\mathbf{e}}^* \cdot \mathbf{A}_{\text{sc}}^{(1)}}{\bar{D}} = \frac{k^2}{4\pi} \int e^{i\,\mathbf{q}\cdot\mathbf{r}} \frac{\delta\epsilon}{\bar{\epsilon}} \hat{\mathbf{e}}^* \cdot \hat{\mathbf{e}}_0 \, \mathrm{d}^3 x$$

$$= \frac{k^2}{2} \frac{\delta\epsilon}{\bar{\epsilon}} \hat{\mathbf{e}}^* \cdot \hat{\mathbf{e}}_0 \int \left[\cos(q\,r\cos\theta) + i\sin(q\,r\cos\theta)\right] r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \frac{k^2}{2\,q^3} \frac{\delta\epsilon}{\bar{\epsilon}} \hat{\mathbf{e}}^* \cdot \hat{\mathbf{e}}_0 \int_0^{q\,a} p^2 \,\mathrm{d}p \int_{-1}^{1} \left[\cos(p\,s) + i\,\cancel{\sin(p\,s)}\right] \mathrm{d}s \Leftarrow \begin{array}{l} p = q\,r \\ s = \cos\theta \end{array}$$

$$= k^2 \frac{\delta\epsilon}{\bar{\epsilon}} \hat{\mathbf{e}}^* \cdot \hat{\mathbf{e}}_0 \frac{\sin(q\,a) - q\,a\cos(q\,a)}{q^3}$$

$$\sin(q\,a) - q\,a\cos(q\,a) = q\,a - \frac{q^3 a^3}{3!} - q\,a + \frac{q^3 a^3}{2!} + \cdots = \frac{q^3 a^3}{3} + \cdots$$

$$\Rightarrow \lim_{q\to 0} \frac{\hat{\mathbf{e}}^* \cdot \mathbf{A}_{\text{sc}}^{(1)}}{\bar{D}} = k^2 \frac{\delta\epsilon}{\bar{\epsilon}} \hat{\mathbf{e}}^* \cdot \hat{\mathbf{e}}_0 \frac{a^3}{3}$$

C. Blue Sky: Elementary Argument

- Since the magnetic moments of most gas molecules are negligible compared to the electric dipole moments, the scattering is purely electric dipole in character.

- The treatment is in 2 parts:

(1) Elementary argument is adequate for a dilute ideal gas, where the molecules are truly randomly distributed in space relative to each other;
(2) Based on density fluctuations in the gas, which is of more general validity.

- $\mathbf{p}_j = \bar{\epsilon}\,\gamma_{\text{mol}}\,\mathbf{E}(\mathbf{r}_j) \quad \Rightarrow \quad \delta\epsilon(\mathbf{r}) = \bar{\epsilon}\sum \gamma_{\text{mol}}\,\delta(\mathbf{r}-\mathbf{r}_j)$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2}\left|\gamma_{\text{mol}}\,\hat{\mathbf{e}}^*\cdot\hat{\mathbf{e}}_0\right|^2 \mathscr{F}(\mathbf{q}) \quad \Leftarrow \quad \mathscr{F}(\mathbf{q}) = \left|\sum e^{i\,\mathbf{q}\cdot\mathbf{r}_j}\right|^2$$

- For a random distribution of scattering centers $\mathscr{F}$ is an incoherent sum, and the cross section is that for one molecule, times the number of molecules.

- For a dilute gas $\epsilon_r \simeq 1 + N\,\gamma_{\text{mol}} \quad \Rightarrow \quad \sigma \simeq \dfrac{k^4\left|\epsilon_r - 1\right|^2}{6\pi N^2} \simeq \dfrac{2k^4\left|n-1\right|^2}{3\pi N^2} \quad \Leftarrow \quad \begin{array}{l} |n| \sim 1 \\ \epsilon_r \simeq n^2 \end{array}$

represents the power scattered *per molecule* for a unit incident energy flux.

- In traversing a thickness dx of the gas, the fractional loss of flux is $N\sigma\,dx$.

intensity of incident beam $\quad I(x) = I_0\,e^{-\alpha x} \quad \Leftarrow \quad \boxed{\alpha = N\sigma \simeq \dfrac{2k^4}{3\pi N}\left|n-1\right|^2} \quad$ absorption attenuation coefficient extinction $\qquad (1)$

- These results describe the *Rayleigh scattering*, the incoherent scattering by randomly distributed dipoles, each scattering according to Rayleigh's ω^4 law.

- The k^4 dependence means that in the visible spectrum the red is scattered least and the violet most.

- Light received away from the direction of the incident beam is more heavily weighted in blue components than the spectral distribution of the incident beam, while the transmitted beam becomes increasingly red in its spectral composition, as well as diminishing in overall intensity.

- The blue sky, the red sunset, the waneness of the winter sun, and the easy sunburning at midday in summer are all consequences of Rayleigh scattering in the atmosphere.

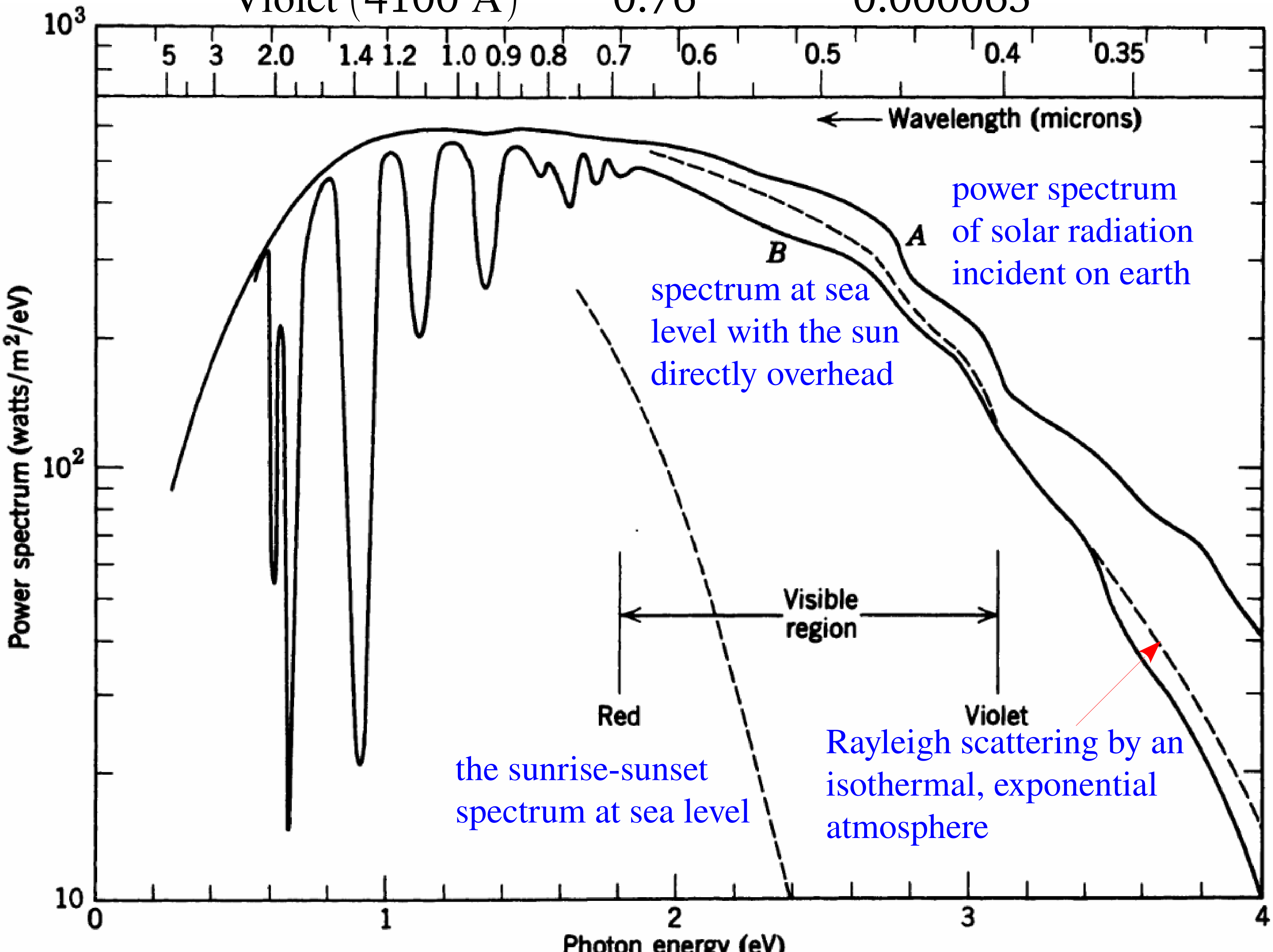
- $n_{\text{air}} - 1 \simeq 2.78 \times 10^{-4}$ in the visible region $(4100 - 6500\ \text{Å})$ & NTP

$$\Rightarrow \begin{array}{c}\text{attenuation}\\ \text{length}\end{array} \quad \Lambda \equiv \frac{1}{\alpha} = \begin{array}{r}30\ \text{km for violet}\ (4100\ \text{Å})\\ 77\ \text{km for green}\ (5200\ \text{Å})\\ 188\ \text{km for}\quad \text{red}\ (6500\ \text{Å})\end{array} \quad \text{with} \quad \begin{array}{c}N = 2.69 \times 10^{19}\\ \text{molecules}/\text{cm}^3\end{array}$$

- With an isothermal model of the atmosphere in which the density varies exponentially with height, the intensities at the earth's surface relative to those incident on the top of the atmosphere.

- It shows strikingly the shift to the red of the surviving sunlight at sunrise and sunset.

Color	*Zenith*	*Sunrise – Sunset*
Red (6500 Å)	0.96	0.21
Green (5200 Å)	0.90	0.024
Violet (4100 Å)	0.76	0.000065



- The real attenuation is greater because of the presence of water vapor, which has strong absorption bands in the infrared, and ozone, which causes absorption of the ultraviolet, as well as other molecular species and dust.

- At $90°$ the polarization of the scattered light is a function of wavelength and reaches a maximum of approximately 75% at 5500Å.

- From (1) if there were no atomicity ($N \to \infty$), there would be no attenuation. Conversely, the observed attenuation can be used to determine N.

D. Density Fluctuations; Critical Opalescence

- A more general approach to the scattering & attenuation of light in gases and liquids is to consider fluctuations in the density and so the index of refraction.

- The volume V of fluid is imagined to be divided into cells small compared to a wavelength. Each cell has volume v with an average number $N_v = v\,N$ of molecules

$$\Rightarrow \quad \delta\epsilon_j = \frac{\Delta N_j}{v}\frac{\partial\epsilon}{\partial N} = \frac{(\epsilon_r - 1)(\epsilon_r + 2)}{3\,N\,v}\,\Delta N_j \quad \Leftarrow \quad \text{Clausius-Mossotti relation}$$

$$\Rightarrow \quad \frac{\hat{\mathbf{e}}^* \cdot \mathbf{A}_{\mathrm{sc}}^{(1)}}{\bar{D}} = \hat{\mathbf{e}}^* \cdot \hat{\mathbf{e}}_0 \frac{k^2}{4\pi} \frac{(\epsilon_r - 1)(\epsilon_r + 2)}{3\,N\,\epsilon_r} \sum \Delta N_j\, e^{i\,\mathbf{q}\cdot\mathbf{r}_j} \quad \Leftarrow \quad (2)$$

Assume the correlation of fluctuations in different cells only extends over a distance small compared to a wavelength $\quad \Rightarrow \quad e^{i\,\mathbf{q}\cdot\mathbf{r}_j} \to 1$

$$\Rightarrow \quad \alpha = \frac{1}{V}\int \frac{\left|\hat{\mathbf{e}}^* \cdot \mathbf{A}_{\mathrm{sc}}^{(1)}\right|^2}{\bar{D}^2}\,\mathrm{d}\,\Omega = \frac{\omega^4}{6\,\pi\,N\,c^4} \frac{\left|(\epsilon_r - 1)(\epsilon_r + 2)\right|^2}{9} \frac{\Delta N_V^2}{N\,V}$$

where $\Delta N_V^2 \equiv \sum \Delta N_j\,\Delta N_{j'}$ mean square number fluctuation

$$\Rightarrow \quad \frac{\Delta N_V^2}{N\,V} = N\,k\,T\,\beta_T \quad \Leftarrow \quad \beta_T \equiv -\frac{1}{V}\frac{\partial V}{\partial P}_{|T} \quad \text{isothermal compressibility}$$

$$\Rightarrow \quad \alpha = \frac{1}{6\,\pi\,N}\frac{\omega^4}{c^4}\frac{\left|(\epsilon_r - 1)(\epsilon_r + 2)\right|^2}{9}\,N\,k\,T\,\beta_T \quad \text{Einstein-Smoluchowski formula}$$

$$\Rightarrow \quad \alpha \simeq \frac{2\,k^4}{3\,\pi\,N}\,|n-1|^2 \quad \Rightarrow \quad (1) \quad \Leftarrow \quad \begin{array}{c} |\epsilon_r - 1| \ll 1 \\ n \simeq \sqrt{\epsilon_r} \end{array} \quad + \quad N\,k\,T\,\beta_T = 1 \quad \text{dilute ideal gas}$$

- As the critical point is approached, β_T becomes infinite, the scattering and attenuation thus become large — *critical opalescence*.

- The large scattering is related to the large fluctuations in density near the critical point. Near the critical point our treatment fails because the correlation length for the density fluctuations becomes greater than a wavelength.

- For large correlation length Λ we must retain the exponential phase factors. The absolute square of the scattering amplitude then involves a sum

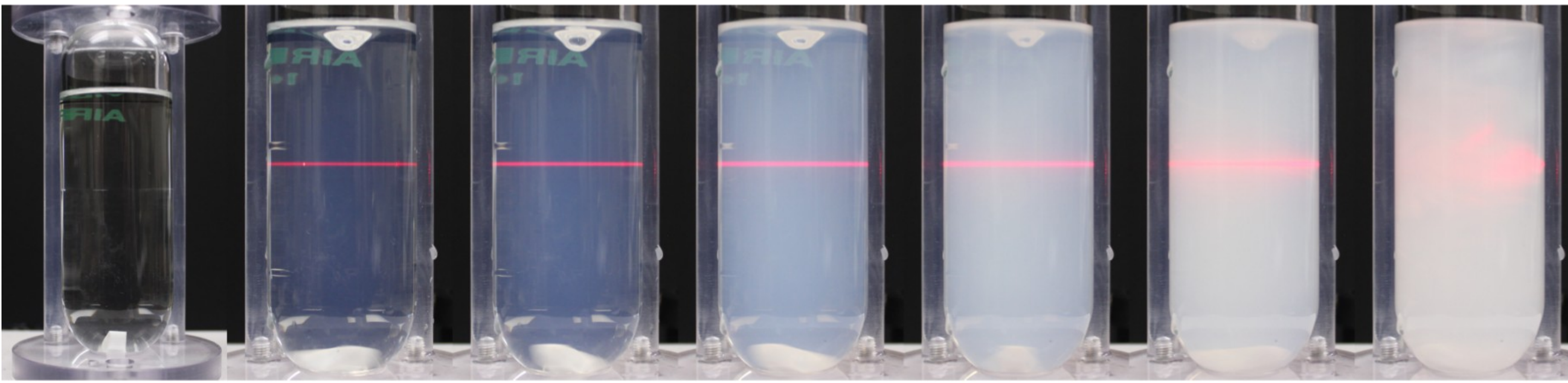
$$\sum_{i,j} \Delta N_i\, \Delta N_j\, e^{i\,\mathbf{q}\cdot(\mathbf{r}_i - \mathbf{r}_j)}$$

- If a correlation function of Yukawa form $\dfrac{e^{-r/\Lambda}}{r}$ is assumed

$$\Rightarrow \quad \frac{d\,\alpha}{d\,\Omega}(\theta) = \frac{\omega^4}{c^4}\,\frac{1+\cos^2\theta}{32\,\pi^2\,N}\,\frac{N\,k\,T\,\beta_T + \Lambda^2\,q^2}{1+\Lambda^2\,q^2}\,\frac{|(\epsilon_r-1)(\epsilon_r+2)|^2}{9} \quad \Leftarrow \quad q^2 = 2\,\omega^2\,\frac{1-\cos\theta}{c^2}$$

$$\Rightarrow \quad \alpha(\Lambda\,q \ll 1) \Rightarrow (1) \;\propto\; \omega^4, \quad \alpha(\Lambda \to \infty) = \frac{c^2}{\Lambda^2\,\omega^2}\,\ln\frac{\Lambda\,\omega}{c} \times (1) \;\propto\; \omega^2$$

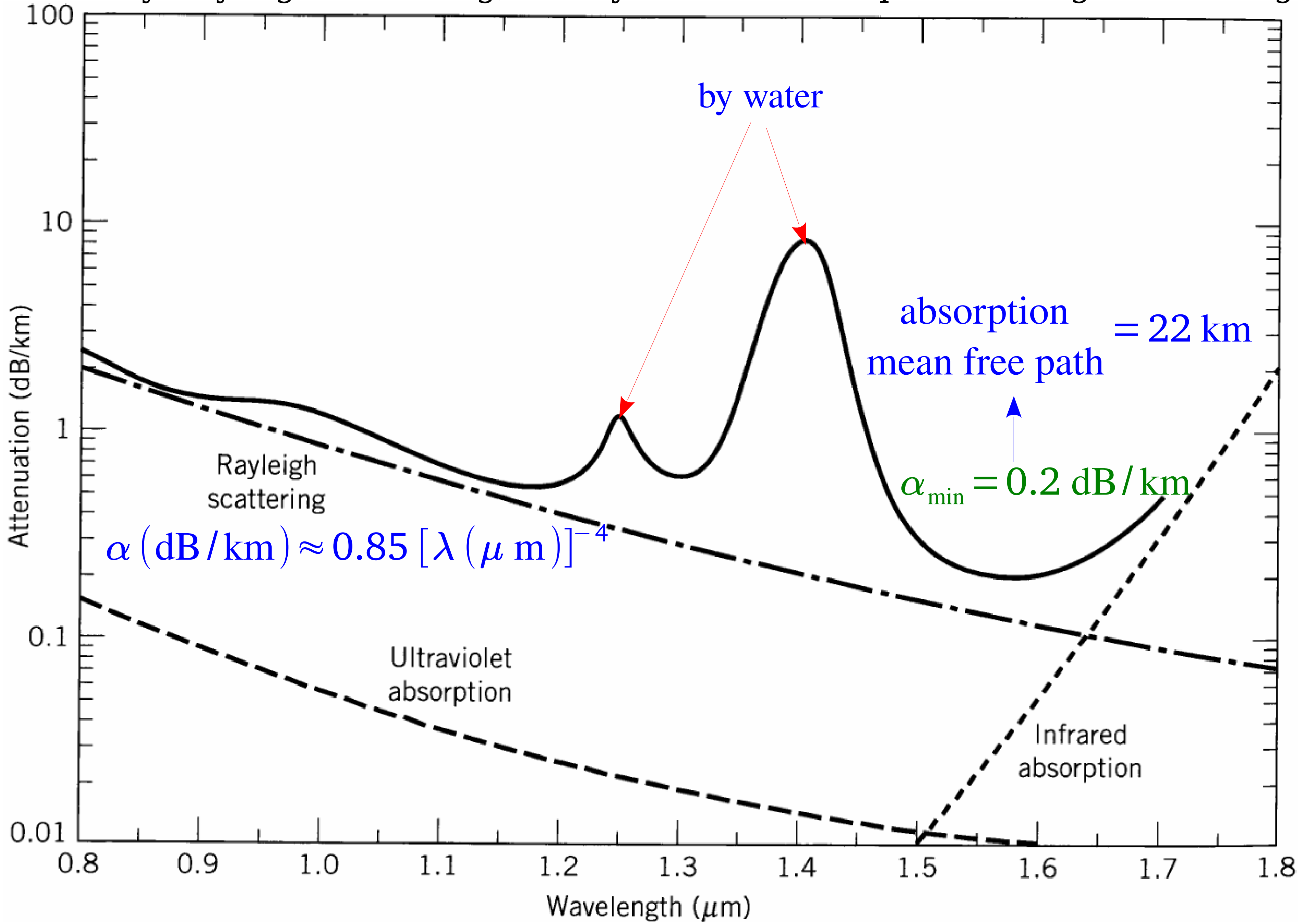
- The frequency dependence as ω^4 away from the critical point is altered to roughly ω^2; the scattered light appears "whiter" close to the critical point.



Room Temperature | 1.0 min (One Phase) | 3.0 min (One Phase) | 5.0 min (One Phase) | 5.5 min (One Phase) | 5.7 min (Two Phases) | 6.0 min (Two Phases)

E. Attenuation in Optical Fibers

- The factor for the max distance in optical fiber transmission is the attenuation caused by Rayleigh scattering, and by infrared absorption at longer wavelengths.



Spherical Wave Expansion of a Vector Plane Wave

- For a scalar field satisfying the wave equation, the expansion can be obtained by using the orthogonality properties of the spherical solutions $j_\ell(k\,r)\,Y_{\ell m}(\theta,\phi)$.

- An alternative derivation makes use of the spherical wave expansion of the Green function

$$r' \to \infty \quad \Rightarrow \quad \mathbb{r} = |\mathbf{r} - \mathbf{r}'| \simeq r' - \hat{\mathbf{r}}' \cdot \mathbf{r}\,, \quad r_> = r'\,, \quad r_< = r\,, \quad h_\ell^+(k\,r') \to (-i)^{\ell+1} \frac{e^{i\,k\,r'}}{k\,r'}$$

$$\Rightarrow \quad G = \frac{e^{i\,k\,\mathbb{r}}}{4\,\pi\,\mathbb{r}} \to \frac{e^{i\,k\,r'}}{4\,\pi\,r'}\, e^{-i\,k\,\hat{\mathbf{r}}'\cdot\mathbf{r}} = i\,k\,\frac{e^{i\,k\,r'}}{k\,r'} \sum_{\ell,m} (-i)^{\ell+1}\, j_\ell(k\,r)\, Y_\ell^{*m}(\theta',\phi')\, Y_\ell^m(\theta,\phi)$$

$$\Rightarrow \quad e^{i\,k\,\hat{\mathbf{r}}'\cdot\mathbf{r}} = 4\,\pi \sum i^\ell\, j_\ell(k\,r)\, Y_\ell^{*m}(\theta,\phi)\, Y_\ell^m(\theta',\phi') \quad \Leftarrow \text{ complex conjugate}$$

$$= \sum i^\ell\,(2\,\ell+1)\, j_\ell(k\,r)\, P_\ell(\cos\gamma) \quad \Leftarrow \quad \text{addition theorem } (3.62)$$

$$= \sum i^\ell \sqrt{4\,\pi\,(2\,\ell+1)}\, j_\ell(k\,r)\, Y_\ell^0(\gamma) \qquad (4) \quad \Leftarrow \quad \hat{\mathbf{r}}'\cdot\hat{\mathbf{r}} = \cos\gamma$$

- Wish to make an equivalent expansion for a circularly polarized plane wave with helicity $\pm$ incident along the z-axis, from (9.122)

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= (\hat{\mathbf{x}} \pm i\,\hat{\mathbf{y}})\, e^{i\,k\,z} = \sum \left(a_{\ell m}^{\pm}\, j_\ell(k\,r)\, \mathbf{X}_\ell^m + \frac{i}{k}\, b_{\ell m}^{\pm}\, \nabla \times j_\ell(k\,r)\, \mathbf{X}_\ell^m \right) \\ c\,\mathbf{B}(\mathbf{r}) &= \hat{\mathbf{z}} \times \mathbf{E} = \mp\, i\,\mathbf{E} = \sum \left(b_{\ell m}^{\pm}\, j_\ell(k\,r)\, \mathbf{X}_\ell^m - \frac{i}{k}\, a_{\ell m}^{\pm}\, \nabla \times j_\ell(k\,r)\, \mathbf{X}_\ell^m \right) \end{aligned} \qquad (3)$$

$$\int [f_{\ell'}(k\,r)\,\mathbf{X}_{\ell'}^{m'}]^{\star}\cdot g_\ell(k\,r)\,\mathbf{X}_\ell^m\,\mathrm{d}\,\Omega = f_{\ell'}^{\star}\,g_\ell \int \mathbf{X}_{\ell'}^{\star\,m'}\cdot\mathbf{X}_\ell^m\,\mathrm{d}\,\Omega = f_\ell^{\star}\,g_\ell\,\delta_{\ell\,\ell'}\,\delta_{m\,m'}$$

$$\nabla = \hat{\mathbf{r}}\,(\hat{\mathbf{r}}\cdot\nabla)+\nabla-\hat{\mathbf{r}}\,(\hat{\mathbf{r}}\cdot\nabla)=\hat{\mathbf{r}}\,(\hat{\mathbf{r}}\cdot\nabla)+(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})\,\nabla-\hat{\mathbf{r}}\,(\hat{\mathbf{r}}\cdot\nabla)=\hat{\mathbf{r}}\,\frac{\partial}{\partial\,r}-\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\nabla)$$

$$\Rightarrow\quad \nabla=\hat{\mathbf{r}}\,\frac{\partial}{\partial\,r}-\frac{i}{r}\,\hat{\mathbf{r}}\times\hat{\mathbf{L}}\quad\Leftarrow\quad \hat{\mathbf{L}}=\frac{\mathbf{r}\times\nabla}{i}$$

$$i\,r\,\nabla\times\hat{\mathbf{L}}=i\,r\left(\cancel{\hat{\mathbf{r}}\,\frac{\partial}{\partial\,r}}-\frac{i}{r}\,\hat{\mathbf{r}}\times\hat{\mathbf{L}}\right)\times\hat{\mathbf{L}}=(\hat{\mathbf{r}}\times\hat{\mathbf{L}})\times\hat{\mathbf{L}}=\sum\epsilon_{j\,k\,\ell}\,\hat{\mathbf{e}}_j\,\epsilon_{k\,m\,p}\,n_m\,\hat{L}_p\,\hat{L}_\ell$$

$$=\sum\hat{\mathbf{e}}_j\,(\delta_{j\,p}\,\delta_{\ell\,m}-\delta_{j\,m}\,\delta_{\ell\,p})\,n_m\,\hat{L}_p\,\hat{L}_\ell=\sum\hat{\mathbf{e}}_j\,n_m\,\hat{L}_j\,\hat{L}_m-\hat{\mathbf{r}}\,\hat{L}^2$$

$$=\sum\hat{\mathbf{e}}_j\,n_m\,(\hat{L}_m\,\hat{L}_j+i\,\epsilon_{j\,m\,k}\,\hat{L}_k)-\hat{\mathbf{r}}\,\hat{L}^2\quad\Leftarrow\quad[\hat{L}_j\,,\,\hat{L}_m]=i\,\epsilon_{j\,m\,k}\,\hat{L}_k$$

$$=\sum\hat{\mathbf{e}}_j\,\cancel{(\hat{\mathbf{r}}\cdot\hat{\mathbf{L}})}\,\hat{L}_j+i\,\hat{\mathbf{r}}\times\hat{\mathbf{L}}-\hat{\mathbf{r}}\,\hat{L}^2\quad\Rightarrow\quad\nabla\times\hat{\mathbf{L}}=\frac{\hat{\mathbf{r}}\times\hat{\mathbf{L}}}{r}+i\,\frac{\hat{\mathbf{r}}}{r}\,\hat{L}^2$$

$$\Rightarrow\quad\nabla\times g_\ell\,\mathbf{X}_\ell^m=\left(\frac{\partial\,g_\ell}{\partial\,r}+\frac{g_\ell}{r}\right)\hat{\mathbf{r}}\times\mathbf{X}_\ell^m+i\,\frac{g_\ell}{r}\,\hat{\mathbf{r}}\,(\hat{\mathbf{L}}\cdot\mathbf{X}_\ell^m)$$

$$=\frac{1}{r}\left(\frac{\partial}{\partial\,r}(r\,g_\ell)\,\hat{\mathbf{r}}\times\mathbf{X}_\ell^m+i\,g_\ell\,(\hat{\mathbf{L}}\cdot\mathbf{X}_\ell^m)\,\hat{\mathbf{r}}\right)$$

$$\Rightarrow\quad\int(f_{\ell'}\,\mathbf{X}_{\ell'}^{m'})^{\star}\cdot\nabla\times g_\ell\,\mathbf{X}_\ell^m\,\mathrm{d}\,\Omega=0\quad\Leftarrow\quad\hat{\mathbf{r}}\cdot\mathbf{X}_\ell^m=0\,,\quad\int\mathbf{X}_{\ell'}^{\star\,m'}\cdot\hat{\mathbf{r}}\times\mathbf{X}_\ell^m\,\mathrm{d}\,\Omega=0$$

$$\int (\nabla\times f_\ell \mathbf{X}_{\ell'}^{m'})^\star\cdot\nabla\times g_\ell \mathbf{X}_\ell^m \,\mathrm{d}\,\Omega=\frac{\delta_{\ell\,\ell'}\delta_{m\,m'}}{r^2}\left(\ell(\ell+1)f_\ell^\star g_\ell+\frac{\partial}{\partial r}(r\,f_\ell^\star)\frac{\partial}{\partial r}(r\,g_\ell)\right)$$

$$\int (\nabla\times f_\ell \mathbf{X}_{\ell'}^{m'})^\star\cdot\nabla\times g_\ell \mathbf{X}_\ell^m \,\mathrm{d}\,\Omega=\frac{1}{r^2}\int[\partial_r(r\,f_{\ell'})\hat{\mathbf{r}}\times\mathbf{X}_{\ell'}^{m'}+i\,f_{\ell'}(\hat{\mathbf{L}}\cdot\mathbf{X}_{\ell'}^{m'})\hat{\mathbf{r}}]^\star$$

$$\cdot[\partial_r(r\,g_\ell)\hat{\mathbf{r}}\times\mathbf{X}_\ell^m+i\,g_\ell(\hat{\mathbf{L}}\cdot\mathbf{X}_\ell^m)\hat{\mathbf{r}}]\,\mathrm{d}\,\Omega=\frac{\delta_{\ell\,\ell'}\delta_{m\,m'}}{r^2}[\ell(\ell+1)f_\ell^\star g_\ell+\partial_r(r\,f_\ell^\star)\partial_r(r\,g_\ell)]$$

$$a_{\ell m}^{\pm} j_\ell(k\,r)=\int \mathbf{X}_\ell^{\star\,m}\cdot\mathbf{E}\,\mathrm{d}\,\Omega=\int \mathbf{X}_\ell^{\star\,m}\cdot(\hat{\mathbf{x}}\pm i\,\hat{\mathbf{y}})\,e^{i\,k\,z}\,\mathrm{d}\,\Omega \quad\Leftarrow\quad (3)$$

$$\Rightarrow\quad =\int e^{i\,k\,z}\frac{(L_\mp Y_\ell^m)^\star}{\sqrt{\ell(\ell+1)}}\,\mathrm{d}\,\Omega=\sqrt{\frac{(\ell\pm m)(\ell\mp m+1)}{\ell(\ell+1)}}\int Y_\ell^{\star\,m\mp1}e^{i\,k\,z}\,\mathrm{d}\,\Omega$$

$$b_{\ell m}^{\pm} j_\ell(k\,r)=c\int \mathbf{X}_\ell^{\star\,m}\cdot\mathbf{B}\,\mathrm{d}\,\Omega=\mp i\int \mathbf{X}_\ell^{\star\,m}\cdot\mathbf{E}\,\mathrm{d}\,\Omega$$

$$\Rightarrow\quad a_{\ell m}^{\pm}=2\,i^\ell\sqrt{\pi(2\,\ell+1)}\,\delta_m^{\pm1}\quad\Leftarrow\quad(4)$$

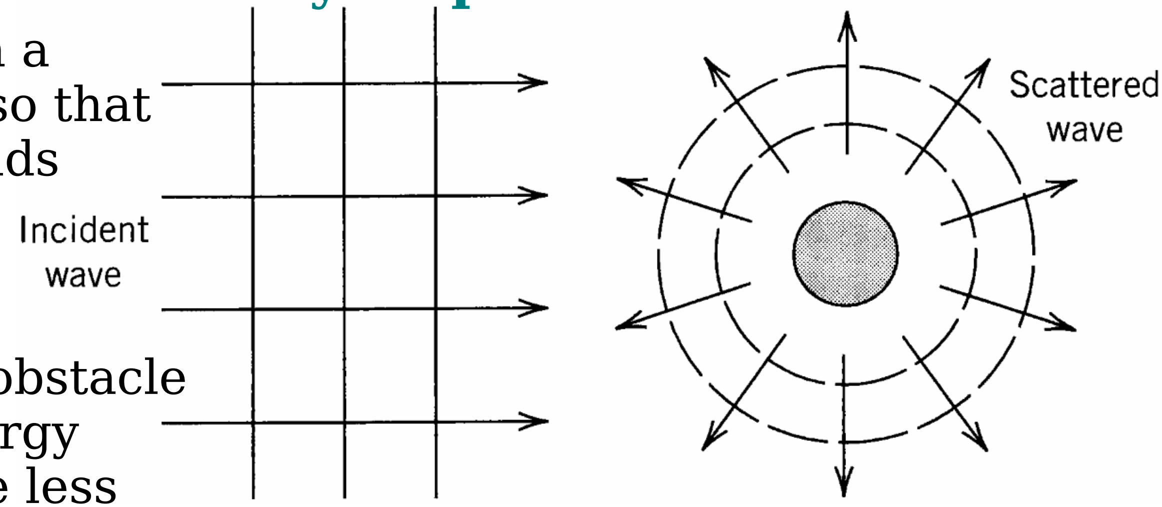
$$b_{\ell m}^{\pm}=\mp i\,a_{\ell m}^{\pm}$$

$$\Rightarrow\quad \boxed{\begin{aligned}\mathbf{E}(\mathbf{r})&=2\sum i^\ell\sqrt{\pi(2\,\ell+1)}\left(+j_\ell(k\,r)\mathbf{X}_\ell^{\pm1}\pm\frac{1}{k}\nabla\times j_\ell(k\,r)\mathbf{X}_\ell^{\pm1}\right)\\ c\,\mathbf{B}(\mathbf{r})&=2\sum i^{\ell-1}\sqrt{\pi(2\,\ell+1)}\left(\pm j_\ell(k\,r)\mathbf{X}_\ell^{\pm1}+\frac{1}{k}\nabla\times j_\ell(k\,r)\mathbf{X}_\ell^{\pm1}\right)\end{aligned}}$$

plane wave

- For such a circularly polarized wave $m=\pm1$ have the obvious interpretation of ±1 unit of angular momentum per photon parallel to the propagation direction.

Scattering of Electromagnetic Waves by a Sphere



- If a plane EM wave is incident on a spherical obstacle, it is scattered, so that far away from the scatterer the fields are represented by a plane wave plus outgoing spherical waves.

- There may be absorption by the obstacle as well as scattering. The total energy flow away from the obstacle will be less than the total energy flow towards it, the difference being absorbed.

- $\mathbf{E}(\mathbf{r}) = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{sc}}, \quad \mathbf{B}(\mathbf{r}) = \mathbf{B}_{\text{inc}} + \mathbf{B}_{\text{sc}}$

$$\Rightarrow \begin{aligned} \mathbf{E}_{\text{sc}} &= \sum_{\ell} i^{\ell} \sqrt{\pi(2\ell+1)} \left(+\alpha_{\ell}^{\pm} h_{\ell}^{+}(kr) \mathbf{X}_{\ell}^{\pm 1} \pm \frac{\beta_{\ell}^{\pm}}{k} \nabla \times h_{\ell}^{+}(kr) \mathbf{X}_{\ell}^{\pm 1} \right) \\ c\,\mathbf{B}_{\text{sc}} &= \sum_{\ell} i^{\ell-1} \sqrt{\pi(2\ell+1)} \left(\pm \beta_{\ell}^{\pm} h_{\ell}^{+}(kr) \mathbf{X}_{\ell}^{\pm 1} + \frac{\alpha_{\ell}^{\pm}}{k} \nabla \times h_{\ell}^{+}(kr) \mathbf{X}_{\ell}^{\pm 1} \right) \end{aligned}$$

- The coefficients $\alpha^{\pm}{}_{\ell}$ and $\beta^{\pm}{}_{\ell}$ will be determined by the boundary conditions on the surface of the scatterer.

- For the spherically symmetric problems considered here, only $m = \pm 1$ occurs.

- The scattered power is the outward component of the Poynting vector formed from the scattered fields, integrated over the spherical surface.

- The absorbed power is the inward component formed from the total fields

$$P_{\mathrm{sc}} = -\frac{a^2}{2\,\mu_0}\,\Re\int \mathbf{E}_{\mathrm{sc}}\cdot\left(\hat{\mathbf{r}}\times\mathbf{B}^*_{\mathrm{sc}}\right)\mathrm{d}\,\Omega$$

$$P_{\mathrm{abs}} = +\frac{a^2}{2\,\mu_0}\,\Re\int \mathbf{E}\cdot\left(\hat{\mathbf{r}}\times\mathbf{B}^*\right)\mathrm{d}\,\Omega \qquad (4) \quad \Leftarrow$$

only the transverse parts of the fields enter these eqns

- $\nabla\times f_\ell\,\mathbf{X}^m_\ell = i\,\dfrac{\sqrt{\ell\,(\ell+1)}}{r}\,f_\ell\,Y^m_\ell\,\hat{\mathbf{r}} + \dfrac{1}{r}\,\dfrac{\partial}{\partial\,r}\left(r\,f_\ell\right)\hat{\mathbf{r}}\times\mathbf{X}^m_\ell$ used in (4)

gives sums of these forms: $\mathbf{X}^{*\,m}_\ell\cdot\mathbf{X}^{m'}_\ell\,,\quad \mathbf{X}^{*\,m}_\ell\cdot\left(\hat{\mathbf{r}}\times\mathbf{X}^{m'}_\ell\right),\quad \left(\hat{\mathbf{r}}\times\mathbf{X}^{*\,m}_\ell\right)\cdot\left(\hat{\mathbf{r}}\times\mathbf{X}^{m'}_\ell\right)$

$$\Rightarrow \quad \sigma_{\mathrm{sc}} \equiv \frac{\text{power scattered}}{\text{incident flux}} = \frac{\pi}{2\,k^2}\sum\left(2\,\ell+1\right)\left(|\alpha_\ell|^2+|\beta_\ell|^2\right) \quad \Leftarrow \quad \text{use Wronskians}$$

$$\sigma_{\mathrm{abs}} \equiv \frac{\text{power absorbed}}{\text{incident flux}} = \frac{\pi}{2\,k^2}\sum\left(2\,\ell+1\right)\left(2-|\alpha_\ell+1|^2-|\beta_\ell+1|^2\right)$$

$\Rightarrow$ total (extinctioin) cross section $\quad \sigma_{\mathrm{total}} = \sigma_{\mathrm{sc}} + \sigma_{\mathrm{abs}} = -\dfrac{\pi}{k^2}\sum\left(2\,\ell+1\right)\Re\left[\alpha_\ell+\beta_\ell\right]$

resemble closely the partial wave expansions of quantum-mechanical scattering.

- $\dfrac{\mathrm{d}\,\sigma_{\mathrm{sc}}}{\mathrm{d}\,\Omega} \equiv \dfrac{\dfrac{\mathrm{d}\,P_{\mathrm{sc}}}{\mathrm{d}\,\Omega}}{\text{incident flux}} = \dfrac{\pi}{2\,k^2}\left|\sum_\ell \sqrt{2\,\ell+1}\left(\alpha^\pm_\ell\,\mathbf{X}^{\pm 1}_\ell \pm i\,\beta^\pm_\ell\,\hat{\mathbf{r}}\times\mathbf{X}^{\pm 1}_\ell\right)\right|^2$ for incident polarization $\hat{\mathbf{e}}_1 \pm i\,\hat{\mathbf{e}}_2$

$$\nabla\times g_\ell\,\mathbf{X}_\ell^m=\frac{\partial_r\, r\, g_\ell}{r}\,\hat{\mathbf{r}}\times\mathbf{X}_\ell^m+i\,\frac{g_\ell}{r}\left(\hat{\mathbf{L}}\cdot\mathbf{X}_\ell^m\right)\hat{\mathbf{r}}\quad\Rightarrow\quad\hat{\mathbf{r}}\times\nabla\times g_\ell\,\mathbf{X}_\ell^m=-\frac{\partial_r\, r\, g_\ell}{r}\,\mathbf{X}_\ell^m$$

$$\Rightarrow\quad\begin{aligned}\mathbf{E}_{\mathrm{sc}}&=\sum i^\ell\sqrt{\pi\left(2\,\ell+1\right)}\left(\alpha_\ell^\pm\, h_\ell^+\,\mathbf{X}_\ell^{\pm1}\pm\beta_\ell^\pm\,\frac{\partial_r\left(r\, h_\ell^+\right)\hat{\mathbf{r}}\times\mathbf{X}_\ell^{\pm1}+i\, h_\ell^+\left(\hat{\mathbf{L}}\cdot\mathbf{X}_\ell^{\pm1}\right)\hat{\mathbf{r}}}{k\, r}\right)\\ \hat{\mathbf{r}}\times\mathbf{B}_{\mathrm{sc}}&=\sum i^{\ell+1}\,\frac{\sqrt{\pi\left(2\,\ell+1\right)}}{c}\left(\frac{\alpha_\ell^\pm}{k\, r}\,\partial_r\left(r\, h_\ell^+\right)\mathbf{X}_\ell^{\pm1}\mp\beta_\ell^\pm\, h_\ell^+\,\hat{\mathbf{r}}\times\mathbf{X}_\ell^{\pm1}\right)\end{aligned}$$

$$\begin{aligned}\Rightarrow\quad P_{\mathrm{sc}}&=-\frac{a^2}{2\,\mu_0}\,\Re\int\mathbf{E}_{\mathrm{sc}}\cdot\left(\hat{\mathbf{r}}\times\mathbf{B}_{\mathrm{sc}}\right)^*\,\mathrm{d}\,\Omega\\ &=-\frac{a}{2\,\mu_0}\,\Re\left[\frac{\pi}{i\, c\, k}\sum\left(2\,\ell+1\right)\left[h_\ell^+\,\partial_r\left(r\, h_\ell^-\right)\left|\alpha_\ell\right|^2-h_\ell^-\,\partial_r\left(r\, h_\ell^+\right)\left|\beta_\ell\right|^2\right]\right]_{r=a}\\ &=\frac{i\,\pi\, a^2}{4\,\mu_0\, c\, k}\sum\left(2\,\ell+1\right)\left[\left(\left|\alpha_\ell\right|^2+\left|\beta_\ell\right|^2\right)\left(h_\ell^+\,\partial_r\, h_\ell^--h_\ell^-\,\partial_r\, h_\ell^+\right)\right]_{r=a}\\ &=\frac{\pi}{2\,\mu_0\, c\, k^2}\sum\left(2\,\ell+1\right)\left(\left|\alpha_\ell\right|^2+\left|\beta_\ell\right|^2\right)\quad\Leftarrow\quad W\left(h_\ell^+\left(x\right),\, h_\ell^-\left(x\right)\right)=\frac{2}{i\, x^2}\end{aligned}$$

$$\mathbf{E}=\left(\hat{\mathbf{x}}\pm i\,\hat{\mathbf{y}}\right)e^{i\,k\,z},\quad c\,\mathbf{B}=\hat{\mathbf{z}}\times\mathbf{E}\quad\Rightarrow\quad S_z\equiv\hat{\mathbf{z}}\cdot\frac{\mathbf{E}\times\mathbf{B}^*}{2\,\mu_0}=\frac{1}{\mu_0\, c}$$

$$\Rightarrow\quad\sigma_{\mathrm{sc}}\equiv\frac{P_{\mathrm{sc}}}{S_z}=\frac{\pi}{2\,k^2}\sum\left(2\,\ell+1\right)\left(\left|\alpha_\ell\right|^2+\left|\beta_\ell\right|^2\right)$$

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{sc}}\,, \quad \mathbf{B}(\mathbf{r}) = \mathbf{B}_{\text{inc}} + \mathbf{B}_{\text{sc}}$$

$$\Rightarrow \quad \mathbf{E}\cdot(\hat{\mathbf{r}}\times\mathbf{B}^*) = \mathbf{E}_{\text{inc}}\cdot(\hat{\mathbf{r}}\times\mathbf{B}^*_{\text{inc}}) + \mathbf{E}_{\text{inc}}\cdot(\hat{\mathbf{r}}\times\mathbf{B}^*_{\text{sc}}) + \mathbf{E}_{\text{sc}}\cdot(\hat{\mathbf{r}}\times\mathbf{B}^*_{\text{inc}}) + \mathbf{E}_{\text{sc}}\cdot(\hat{\mathbf{r}}\times\mathbf{B}^*_{\text{sc}})$$

$$\mathbf{E}_{\text{inc}} = 2\sum i^{\ell}\sqrt{\pi\,(2\,\ell+1)}\left(j_\ell\,\mathbf{X}_\ell^{\pm 1} \pm \frac{\partial_r\, r\, j_\ell}{k\,r}\,\hat{\mathbf{r}}\times\mathbf{X}_\ell^{\pm 1} + i\,\frac{j_\ell}{k\,r}\,(\hat{\mathbf{L}}\cdot\mathbf{X}_\ell^{\pm 1})\,\hat{\mathbf{r}}\right)$$

$$\hat{\mathbf{r}}\times\mathbf{B}_{\text{inc}} = 2\sum i^{\ell+1}\,\frac{\sqrt{\pi\,(2\,\ell+1)}}{c}\left(\frac{\partial_r\, r\, j_\ell}{k\,r}\,\mathbf{X}_\ell^{\pm 1} \mp j_\ell\,\hat{\mathbf{r}}\times\mathbf{X}_\ell^{\pm 1}\right)$$

$$\Re\int \mathbf{E}_{\text{inc}}\cdot(\hat{\mathbf{r}}\times\mathbf{B}_{\text{inc}})^*\,\mathrm{d}\,\Omega = 0$$

$$\Rightarrow \quad \Re\left[\int \mathbf{E}_{\text{inc}}\cdot(\hat{\mathbf{r}}\times\mathbf{B}_{\text{sc}})^* + \mathbf{E}_{\text{sc}}\cdot(\hat{\mathbf{r}}\times\mathbf{B}_{\text{inc}})^*\,\mathrm{d}\,\Omega\right] = -\frac{2\,\pi}{c\,k^2\,a^2}\sum(2\,\ell+1)\,\Re\,[\alpha_\ell+\beta_\ell]$$

$$\Re\int \mathbf{E}_{\text{sc}}\cdot(\hat{\mathbf{r}}\times\mathbf{B}_{\text{sc}})^*\,\mathrm{d}\,\Omega = -\frac{\pi}{c\,k^2\,a^2}\sum(2\,\ell+1)\,(|\alpha_\ell|^2+|\beta_\ell|^2)$$

$$\Rightarrow \quad P_{\text{abs}} = \frac{a^2}{2\,\mu_0}\,\Re\int \mathbf{E}\cdot(\hat{\mathbf{r}}\times\mathbf{B}^*)\,\mathrm{d}\,\Omega = \frac{\pi}{2\,\mu_0\,c\,k^2}\sum(2\,\ell+1)\,(2-|\alpha_\ell+1|^2-|\beta_\ell+1|^2)$$

$$\Rightarrow \quad \sigma_{\text{abs}} \equiv \frac{P_{\text{abs}}}{S_z} = \frac{\pi}{2\,k^2}\sum(2\,\ell+1)\,(2-|\alpha_\ell+1|^2-|\beta_\ell+1|^2)$$

- The scattered radiation is in general elliptically polarized. Only if $\alpha^{\pm}{}_{\ell} = \beta^{\pm}{}_{\ell}$ for all ℓ would it be circularly polarized.

- Therefore if the incident radiation is linearly polarized, the scattered radiation will be elliptically polarized; if the incident radiation is unpolarized, the scattered radiation will exhibit partial polarization depending on the angle of observation.

- If the scatterer is a sphere of radius a whose EM properties can be described by a *surface impedance* Z_s independent of position, the boundary conditions are

$$\mathbf{E}_{\|}(r=a) = \frac{Z_s}{\mu_0}\,\hat{\mathbf{r}} \times \mathbf{B}\,(r=a) \quad \text{where with } \; x = k\,r$$

$$\mathbf{E}_{\|} = \sum_{\ell,m} i^{\ell} \sqrt{\pi\,(2\,\ell+1)} \left(+(2\,j_\ell + \alpha_\ell^{\pm}\,h_\ell^{+})\,\mathbf{X}_\ell^m \pm \frac{1}{x}\frac{\partial\, x\,(2\,j_\ell + \beta_\ell^{\pm}\,h_\ell^{+})}{\partial\, x}\,\hat{\mathbf{r}} \times \mathbf{X}_\ell^m \right)$$

$$c\,\hat{\mathbf{r}} \times \mathbf{B} = \sum_{\ell,m} i^{\ell-1} \sqrt{\pi\,(2\,\ell+1)} \left(-\frac{1}{x}\frac{\partial\, x\,(2\,j_\ell + \alpha_\ell^{\pm}\,h_\ell^{+})}{\partial\, x}\,\mathbf{X}_\ell^m \pm (2\,j_\ell + \beta_\ell^{\pm}\,h_\ell^{+})\,\hat{\mathbf{r}} \times \mathbf{X}_\ell^m \right)$$

$$\Rightarrow \quad \begin{aligned} 2\,j_\ell + \alpha_\ell^{\pm}\,h_\ell^{+} &= i\,\frac{Z_s}{Z_0}\,\frac{1}{x}\,\frac{\mathrm{d}}{\mathrm{d}\,x}\left[x\,(2\,j_\ell + \alpha_\ell^{\pm}\,h_\ell^{+})\right] \\ 2\,j_\ell + \beta_\ell^{\pm}\,h_\ell^{+} &= i\,\frac{Z_0}{Z_s}\,\frac{1}{x}\,\frac{\mathrm{d}}{\mathrm{d}\,x}\left[x\,(2\,j_\ell + \beta_\ell^{\pm}\,h_\ell^{+})\right] \end{aligned}$$

$$2\,j_\ell = h_\ell^{+} + h_\ell^{-} \quad \Leftarrow \quad h_\ell^{\pm} = j_\ell \pm i\,n_\ell$$

$$x = k\,a$$

For perfect conductors: $\mathbf{E}_{\parallel}=0\,,\quad B_{\perp}=0$

For conductors only: $\mathbf{E}_{\parallel}=\dfrac{1-i}{\delta\,\sigma}\,\hat{\mathbf{n}}\times\mathbf{H}_{\parallel}\quad\Leftarrow\quad(8.11)$

For generalization: $\mathbf{E}_{\parallel}=Z_s\,\hat{\mathbf{n}}\times\mathbf{H}_{\parallel}=\dfrac{Z_s}{\mu_0}\,\hat{\mathbf{n}}\times\mathbf{B}$

$$2\,j_\ell+\alpha_\ell^{\pm}\,h_\ell^{+}=i\,\frac{Z_s}{Z_0}\,\frac{1}{x}\,\frac{\mathrm{d}}{\mathrm{d}\,x}\left[x\left(2\,j_\ell+\alpha_\ell^{\pm}\,h_\ell^{+}\right)\right]\qquad ⇦\quad 2\,j_\ell=h_\ell^{+}+h_\ell^{-}$$

$$\Rightarrow\quad x\,h_\ell^{+}+x\,h_\ell^{-}+\alpha_\ell^{\pm}\,x\,h_\ell^{+}=i\,\frac{Z_s}{Z_0}\,\frac{\mathrm{d}}{\mathrm{d}\,x}\left(x\,h_\ell^{+}+x\,h_\ell^{-}+\alpha_\ell^{\pm}\,x\,h_\ell^{+}\right)$$

$$\Rightarrow\quad \alpha_\ell^{\pm}\left(x\,h_\ell^{+}-i\,\frac{Z_s}{Z_0}\,\frac{\mathrm{d}\,x\,h_\ell^{+}}{\mathrm{d}\,x}\right)=\left(-\,x\,h_\ell^{+}+i\,\frac{Z_s}{Z_0}\,\frac{\mathrm{d}\,x\,h_\ell^{+}}{\mathrm{d}\,x}-x\,h_\ell^{-}+i\,\frac{Z_s}{Z_0}\,\frac{\mathrm{d}\,x\,h_\ell^{-}}{\mathrm{d}\,x}\right)$$

$$\Rightarrow\quad \alpha_\ell^{\pm}=-1-\frac{x\,h_\ell^{-}-i\,\dfrac{Z_s}{Z_0}\,\dfrac{\mathrm{d}\,x\,h_\ell^{-}}{\mathrm{d}\,x}}{x\,h_\ell^{+}-i\,\dfrac{Z_s}{Z_0}\,\dfrac{\mathrm{d}\,x\,h_\ell^{+}}{\mathrm{d}\,x}}\,.\quad \text{Similar to }\ \beta_\ell^{\pm}$$

$$\Rightarrow \quad \alpha_\ell^\pm = -1 - \frac{x\,h_\ell^- - i\,\dfrac{Z_s}{Z_0}\,\dfrac{d\,x\,h_\ell^-}{d\,x}}{x\,h_\ell^+ - i\,\dfrac{Z_s}{Z_0}\,\dfrac{d\,x\,h_\ell^+}{d\,x}}\,, \quad \beta_\ell^\pm = -1 - \frac{x\,h_\ell^- - i\,\dfrac{Z_0}{Z_s}\,\dfrac{d\,x\,h_\ell^-}{d\,x}}{x\,h_\ell^+ - i\,\dfrac{Z_0}{Z_s}\,\dfrac{d\,x\,h_\ell^+}{d\,x}} \quad (5)$$

- With the surface impedance boundary condition the coefficients are the same for both states of circular polarization.

- For a given Z_s, the multipole coefficients are decided & the scattering is known.

- If $\begin{matrix} Z_s \in \mathbf{I} \\ Z_s = 0 \text{ or } Z_s \to \infty \end{matrix} \Rightarrow \begin{matrix} |\alpha_\ell^\pm + 1| = 1 \\ |\beta_\ell^\pm + 1| = 1 \end{matrix} \Rightarrow \begin{matrix} \alpha_\ell^\pm = e^{2\,i\,\delta_\ell} - 1 \\ \beta_\ell^\pm = e^{2\,i\,\delta_\ell'} - 1 \end{matrix} \Leftarrow \begin{matrix} \delta_\ell \\ \delta_\ell' \end{matrix} : \begin{matrix} \text{scattering} \\ \text{phase shifts} \end{matrix}$

$$\Rightarrow \quad \tan\delta_\ell = \frac{j_\ell(k\,a)}{n_\ell(k\,a)}\,, \quad \tan\delta_\ell' = \left.\frac{\dfrac{d\,x\,j_\ell}{d\,x}}{\dfrac{d\,x\,n_\ell}{d\,x}}\right|_{x=k\,a} \quad \text{and} \quad \begin{matrix} \tan\delta_\ell = \tan\delta'_{\ell|Z_s=0} \\ \tan\delta_\ell' = \tan\delta_{\ell|Z_s=0} \\ \text{for } Z_s \to \infty \end{matrix}$$

for $Z_s = 0$, perfectly conducting sphere

- (5) $\Rightarrow$ $$\begin{matrix} \alpha_\ell^\pm \simeq -\dfrac{2\,i\,(k\,a)^{2\ell+1}}{(2\ell+1)[(2\ell-1)!!]^2}\,\dfrac{Z_0\,x - i\,(\ell+1)\,Z_s}{Z_0\,x + i\,\ell\,Z_s} \\ \beta_\ell^\pm \simeq -\dfrac{2\,i\,(k\,a)^{2\ell+1}}{(2\ell+1)[(2\ell-1)!!]^2}\,\dfrac{Z_s\,x - i\,(\ell+1)\,Z_0}{Z_s\,x + i\,\ell\,Z_0} \end{matrix} \quad \text{for } k\,a \ll \ell \quad (6)$$

- $\alpha_\ell^{\pm} \simeq \frac{Z_0 - Z_s}{Z_0 + Z_s}(-1)^\ell e^{-2ika} - 1\,, \quad \beta_\ell^{\pm} = -\alpha_\ell^{\pm}$ for $ka \gg \ell$
- In the long-wavelength limit, $\alpha^{\pm}{}_\ell$ and $\beta^{\pm}{}_\ell$ become small very rapidly as ℓ increases. Only the lowest term ($\ell = 1$) need be retained for each multipole series.

- In the short-wavelength limit, the successive coefficients have comparable magnitudes, but phases that fluctuate widely.

- For $\ell \sim \ell_{\max} = ka$, there is a transition region and for $\ell \gg \ell_{\max}$, (6) holds.

- For the long-wavelength limit ($ka \ll 1$) and a perfectly conducting sphere ($Z_s = 0$), only the $\ell = 1$ terms are important

$$\Rightarrow \quad \alpha_1^{\pm} = -\frac{\beta_1^{\pm}}{2} = -\frac{2i}{3}(ka)^3 \quad \Rightarrow \quad \frac{d\sigma_{sc}}{d\Omega} \simeq \frac{2\pi}{3} k^4 a^6 \left|\mathbf{X}_1^{\pm 1} \mp 2i\,\hat{\mathbf{r}} \times \mathbf{X}_1^{\pm 1}\right|^2$$

$$\left|\hat{\mathbf{r}} \times \mathbf{X}_1^{\pm 1}\right|^2 = \left|\mathbf{X}_1^{\pm 1}\right|^2 = \frac{3}{16\pi}\left(1 + \cos^2\theta\right) \quad + \quad \pm i\left(\hat{\mathbf{r}} \times \mathbf{X}_1^{\pm 1}\right)^* \cdot \mathbf{X}_1^{\pm 1} = -\frac{3}{8\pi}\cos\theta$$

$$\Rightarrow \quad \frac{d\sigma_{sc}}{d\Omega} \simeq \frac{k^4 a^6}{8}\left(5 - 8\cos\theta + 5\cos^2\theta\right) = (7)$$

- The result is valid for either state of circular polarization incident, or for an unpolarized incident beam.

Scalar Diffraction Theory

- Diffraction is associated with departures from geometrical optics caused by the finite wavelength of the waves. Thus diffraction involves apertures or obstacles whose dimensions are large compared to a wavelength.

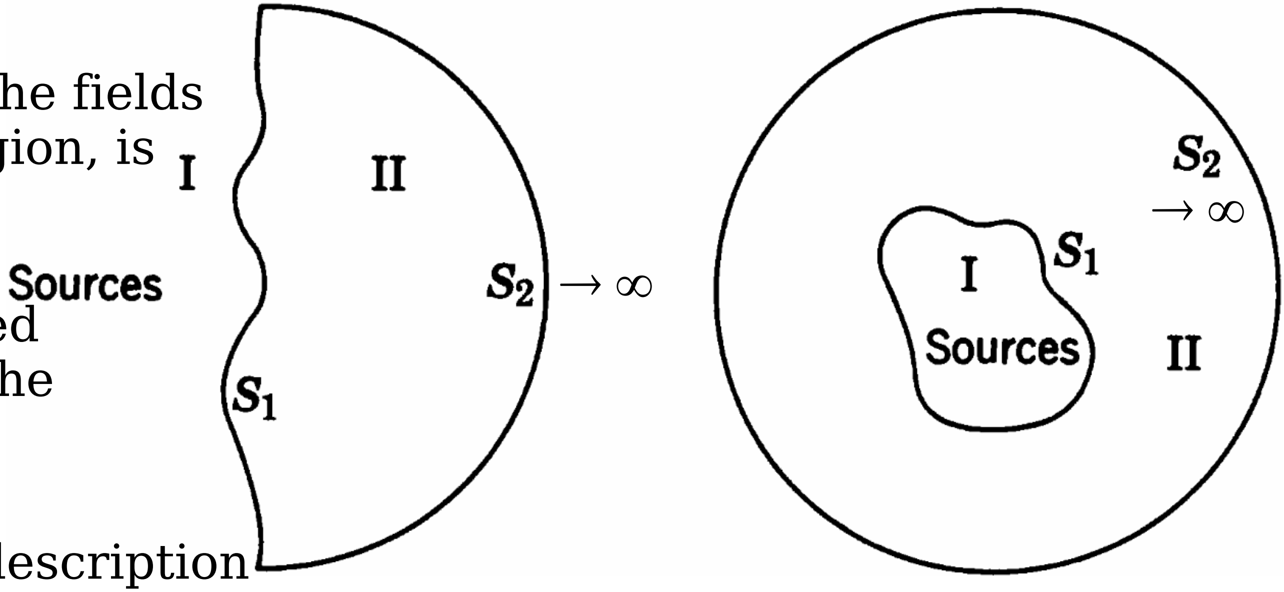
- To lowest approximation EM waves' interaction is described by ray tracing (geometrical optics). The next approximation involves the diffraction of the waves around the obstacles or through the apertures with a spreading of the waves.

- Simple arguments based on Fourier transforms show that the angles of deflection of the waves are confined to the region $\theta \leq \dfrac{\lambda}{d}$.

- The various approximations all work best for $\lambda \ll d$, and fail for $\lambda \sim d$ or $\lambda > d$.

- The angular distribution of the fields in region II, the diffraction region, is called the *diffraction pattern*.

- Wish to express the diffracted fields in region II in terms of the fields on the surface S_1.



- The geometry and mode of description for diffraction is equally applicable to scattering, with the sources in region I replaced by a scatterer.

- Scalar Helmholtz wave equation $\left(\nabla^2+k^2\right)\psi(\mathbf{r})=0 \;\Rightarrow\; \left(\nabla^2+k^2\right)G(\mathbf{r},\mathbf{r}')=-\delta(\mathbf{r}-\mathbf{r}')$

$$\Rightarrow \quad \psi(\mathbf{r})=\oint_S\left[\psi\,\nabla' G(\mathbf{r},\mathbf{r}')-G(\mathbf{r},\mathbf{r}')\,\nabla'\psi\right]\cdot \mathrm{d}\boldsymbol{a}' \quad \Leftarrow \quad (1.36)$$

$\mathrm{d}\boldsymbol{a}'$: inwardly

$$=-\frac{1}{4\pi}\oint_S\frac{e^{ik\mathscr{r}}}{\mathscr{r}}\left[\nabla'\psi+\left(ik-\frac{1}{\mathscr{r}}\right)\hat{\mathscr{r}}\,\psi\right]\cdot \mathrm{d}\boldsymbol{a}' \quad \Leftarrow \quad \vec{\mathscr{r}}=\mathbf{r}-\mathbf{r}', \quad G(\mathbf{r},\mathbf{r}')=\frac{e^{ik\mathscr{r}}}{4\pi\mathscr{r}}$$

$$=\int_{S_1}+\cancel{\int_{S_2\to\infty}} \quad \Leftarrow \quad \psi_\infty\to f(\theta,\phi)\frac{e^{ikr}}{r}, \quad \left.\frac{1}{\psi}\frac{\partial\psi}{\partial r}\right|_\infty\to ik-\frac{1}{r}$$

$$\Rightarrow \quad \nabla'\psi+\left(ik-\frac{1}{\mathscr{r}}\right)\hat{\mathscr{r}}\,\psi \propto \frac{1}{r^2} \quad \text{radiation condition}$$

$$\Rightarrow \quad \boxed{\psi(\mathbf{r})=-\frac{1}{4\pi}\int_{S_1}\frac{e^{ik\mathscr{r}}}{\mathscr{r}}\left[\nabla'\psi+\left(ik-\frac{1}{\mathscr{r}}\right)\hat{\mathscr{r}}\,\psi\right]\cdot \mathrm{d}\boldsymbol{a}'} \quad \text{Kirchhoff integral} \quad (11)$$

- The *Kirchhoff approximation* consists of the assumptions:

(1) ψ and $\dfrac{\partial\psi}{\partial n}$ vanish everywhere on S_1 except in the openings.

(2) The values of ψ and $\dfrac{\partial\psi}{\partial n}$ in the openings are equal to the values of the incident wave in the absence of any screen or obstacles.

- The standard diffraction calculations of classical optics are all based on the Kirchhoff approximation. However, the recipe can have only limited validity.

- A mathematical inconsistency in the Kirchhoff assumptions: for the Helmholtz wave equation, if $\psi = \frac{\partial \psi}{\partial n} = 0$ on any finite surface, then $\psi = 0$ everywhere.

- The only mathematically correct result of the 1st Kirchhoff assumption is that the diffracted field vanishes everywhere, inconsistent with the 2nd assumption.

- The mathematical inconsistencies in the Kirchhoff approximation can be removed by the choice of a proper Green function.

- If ψ is known or approximated on S_1, a Dirichlet Green function is required,

$$G_D = 0 \text{ for } \mathbf{r}' \text{ on } \mathcal{S} \quad \Rightarrow \quad \boxed{\psi(\mathbf{r}) = \int_{\mathcal{S}_1} \psi \nabla' G_D \cdot \mathrm{d}\boldsymbol{a}'} \quad (12)$$

- If $\frac{\partial \psi}{\partial n}$ is known or approximated on S_1, a Neumann Green function is required,

$$\frac{\partial G_N}{\partial n'} = 0 \text{ for } \mathbf{r}' \text{ on } \mathcal{S} \quad \Rightarrow \quad \boxed{\psi(\mathbf{r}) = -\int_{S_1} G_N \nabla' \psi \cdot \mathrm{d}\boldsymbol{a}'} \quad (13)$$

- If S_1 is an infinite plane screen at $z=0$, the method of images can be used

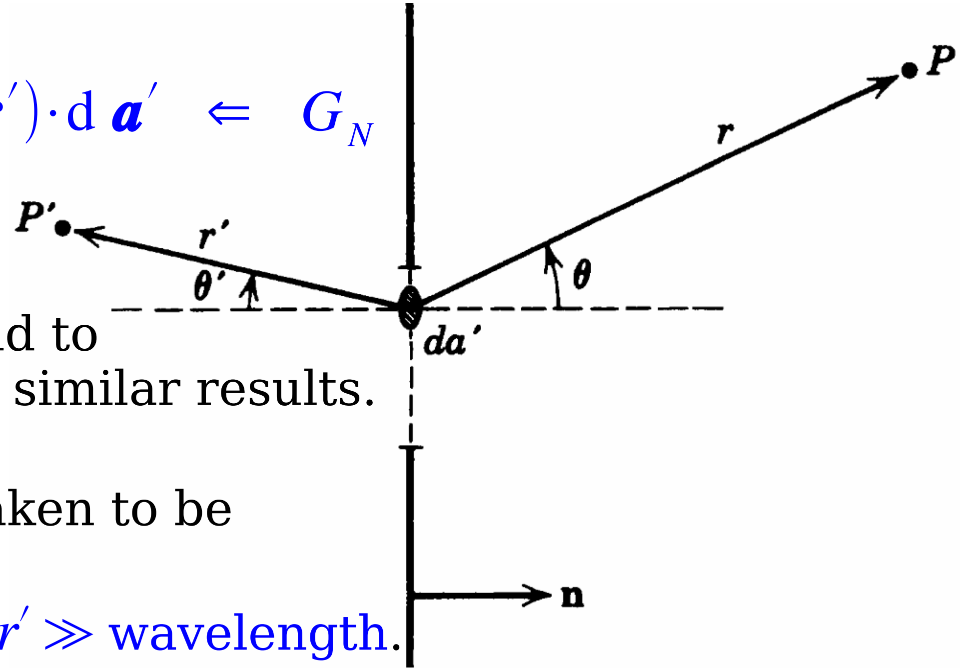
$$\Rightarrow \quad G_{D,N}(\mathbf{r},\mathbf{r}')=\frac{1}{4\pi}\left(\frac{e^{ik\mathfrak{r}}}{\mathfrak{r}}\mp\frac{e^{ik\mathfrak{r}'}}{\mathfrak{r}'}\right) \quad \Leftarrow \quad \mathfrak{r}=|\mathbf{r}-\mathbf{r}'|, \quad \mathfrak{r}'=|\mathbf{r}-\mathbf{r}''|$$

$\mathbf{r}''$: image of $\mathbf{r}'$

then $\mathfrak{r}=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}, \quad \mathfrak{r}'=\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}$

$$\Rightarrow \quad \psi(\mathbf{r})=\frac{k}{2\pi i}\int_{S_1}\left(1+\frac{i}{k\mathfrak{r}}\right)\frac{e^{ik\mathfrak{r}}}{\mathfrak{r}}\psi(\mathbf{r}')\,\hat{\mathfrak{r}}\cdot\mathrm{d}\boldsymbol{a}' \quad \Leftarrow \quad G_D$$

$$-\frac{1}{2\pi}\int_{S_1}\frac{e^{ik\mathfrak{r}}}{\mathfrak{r}}\nabla'\psi(\mathbf{r}')\cdot\mathrm{d}\boldsymbol{a}' \quad \Leftarrow \quad G_N$$



- It might appear that the 3 approximate formulas (11)-(13) are different and will lead to different results. But in fact they yield very similar results.

- The amplitude of the point source P' is taken to be spherically symmetric and equal to $\dfrac{e^{ikr'}}{r'}$, $r, r' \gg$ wavelength.

- The diffracted fields for all 3 approximations can be written in the common form, $\psi(P)=\dfrac{k}{2\pi i}\displaystyle\int_{\text{apertures}}\frac{e^{ikr}}{r}\frac{e^{ikr'}}{r'}\mathcal{O}(\theta,\theta')\,\mathrm{d}a' \quad (10)$

where obliquity factor $\mathcal{O}(\theta, \theta') = \left[\begin{array}{ll} \cos\theta & \psi \text{ approximated on } \mathcal{S}_1 \\ \cos\theta' & \dfrac{\partial \psi}{\partial n} \text{ approximated on } \mathcal{S}_1 \\ \dfrac{\cos\theta + \cos\theta'}{2} & \text{Kirchhoff approximation} \end{array} \right.$

- For apertures being large compared to a wavelength, the diffracted intensity is confined to a narrow range of angles and is governed almost entirely by the interferences between the 2 exponential factors in (10).

- If P' & P are far from the screen in terms of the *aperture* dimensions, the obliquity factor can be treated as a constant. Then the relative amplitudes of the different diffracted fields will be the same.

- For normal incidence all obliquity factors are approximately unity where there is appreciable diffracted intensity.

- The discussion above explains why the mathematically inconsistent Kirchhoff approximation has any success at all.

- The use of Dirichlet/Neumann Green functions gives a better logical structure, but provides little practical improvement without elaboration of the physics.

- An important deficiency of the discussion so far is its scalar nature. EM fields have vector character.

Vector Equivalents of the Kirchhoff Integral

- The vectorial equivalent to a Kirchhoff integral

$$\mathbf{E}(\mathbf{r}) = \oint_S \left(\mathbf{E}(\nabla' G \cdot \mathrm{d}\boldsymbol{a}') - G(\mathrm{d}\boldsymbol{a}' \cdot \nabla')\mathbf{E} \right) \quad \Leftarrow \quad (\nabla^2 + k^2) E_j = 0 \quad \text{components}$$

$$= \oint_S \left(2\mathbf{E}(\nabla' G \cdot \mathrm{d}\boldsymbol{a}') - (\mathrm{d}\boldsymbol{a}' \cdot \nabla) G\mathbf{E} \right)$$

$$= \oint_S 2\mathbf{E}(\nabla' G \cdot \mathrm{d}\boldsymbol{a}') + \int_V \nabla'^2 (G\mathbf{E})\, \mathrm{d}^3 x' \quad \Downarrow \quad \nabla' \cdot \mathbf{E} = 0, \quad \nabla' \times \mathbf{E} = i\omega \mathbf{B}$$

$$\nabla^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}) \quad + \quad \begin{array}{l} \int_V \nabla \phi\, \mathrm{d}^3 x = \oint_S \phi\, \mathrm{d}\boldsymbol{a} \\ \int_V \nabla \times \mathbf{F}\, \mathrm{d}^3 x = \oint_S \mathrm{d}\boldsymbol{a} \times \mathbf{F} \end{array} \quad \Leftarrow \quad \begin{array}{c} \mathrm{d}\boldsymbol{a} \text{ is} \\ \text{outward} \end{array}$$

$$\Rightarrow \quad \mathbf{E} = \oint_S \left(2\mathbf{E}(\nabla' G \cdot \mathrm{d}\boldsymbol{a}') - \nabla' \cdot (G\mathbf{E})\, \mathrm{d}\boldsymbol{a}' + \mathrm{d}\boldsymbol{a}' \times (\nabla' \times G\mathbf{E}) \right)$$

$$= \oint_S \left(2\mathbf{E}(\nabla' G \cdot \mathrm{d}\boldsymbol{a}') - \nabla' G \cdot \mathbf{E}\, \mathrm{d}\boldsymbol{a}' + \mathrm{d}\boldsymbol{a}' \times (i\omega G\mathbf{B} + \nabla' G \times \mathbf{E}) \right)$$

$$\Rightarrow \quad \begin{array}{l} \mathbf{E}(\mathbf{r}) = \oint_S \left(\nabla' G \times (\mathbf{E} \times \mathrm{d}\boldsymbol{a}') + (\mathbf{E} \cdot \mathrm{d}\boldsymbol{a}') \nabla' G - i\omega G\mathbf{B} \times \mathrm{d}\boldsymbol{a}' \right) \\ \mathbf{B}(\mathbf{r}) = \oint_S \left(\nabla' G \times (\mathbf{B} \times \mathrm{d}\boldsymbol{a}') + (\mathbf{B} \cdot \mathrm{d}\boldsymbol{a}') \nabla' G + i\omega G \dfrac{\mathbf{E}}{c^2} \times \mathrm{d}\boldsymbol{a}' \right) \end{array}$$

$$\mathbf{E}(\mathbf{r}) = \oint_{\mathcal{S}_1+\mathcal{S}_2} \left(\nabla' G \times (\mathbf{E}\times \mathrm{d}\boldsymbol{a}') + (\mathbf{E}\cdot \mathrm{d}\boldsymbol{a}')\nabla' G - i\,\omega\, G\,\mathbf{B}\times \mathrm{d}\boldsymbol{a}' \right)$$

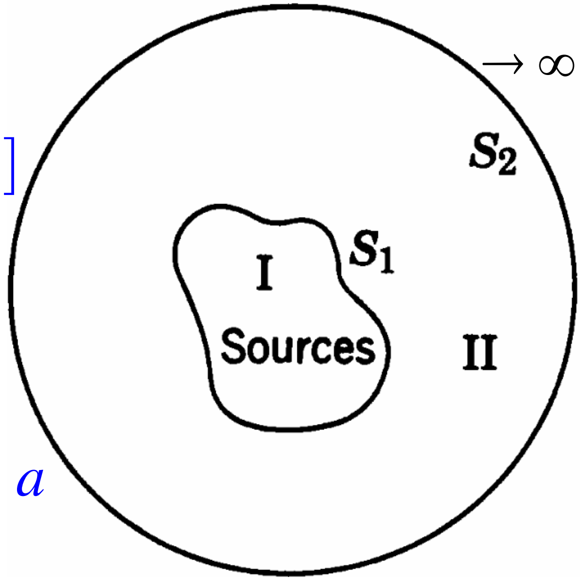
$$G \to \frac{e^{i k r'}}{4\pi r'}\, e^{-i k \hat{\mathbf{r}}'\cdot\mathbf{r}}, \quad \nabla' G \to i k \hat{\mathbf{r}}' G \quad \text{for} \quad r' \to \infty$$

$$\Rightarrow \oint_{S_2} = i k \oint_{S_2} G\,[\hat{\mathbf{r}}' \times (\mathbf{E}\times \mathrm{d}\boldsymbol{a}') + (\mathbf{E}\cdot \mathrm{d}\boldsymbol{a}')\,\hat{\mathbf{r}}' - c\,\mathbf{B}\times \mathrm{d}\boldsymbol{a}']$$

$$= i k \oint_{S_2} G\,[\mathbf{E}\,(\hat{\mathbf{r}}'\cdot \mathrm{d}\boldsymbol{a}') - c\,\mathbf{B}\times \mathrm{d}\boldsymbol{a}' + \mathbf{E}\times(\cancel{\hat{\mathbf{r}}'\times \mathrm{d}\boldsymbol{a}'})]$$

$$= -i k \oint_{S_2} G\,(c\,\mathbf{B}\times \mathrm{d}\boldsymbol{a}' + \mathbf{E}\,\mathrm{d}a') \quad \nwarrow \quad \mathrm{d}\boldsymbol{a}' \equiv -\hat{\mathbf{r}}'\,\mathrm{d}a$$

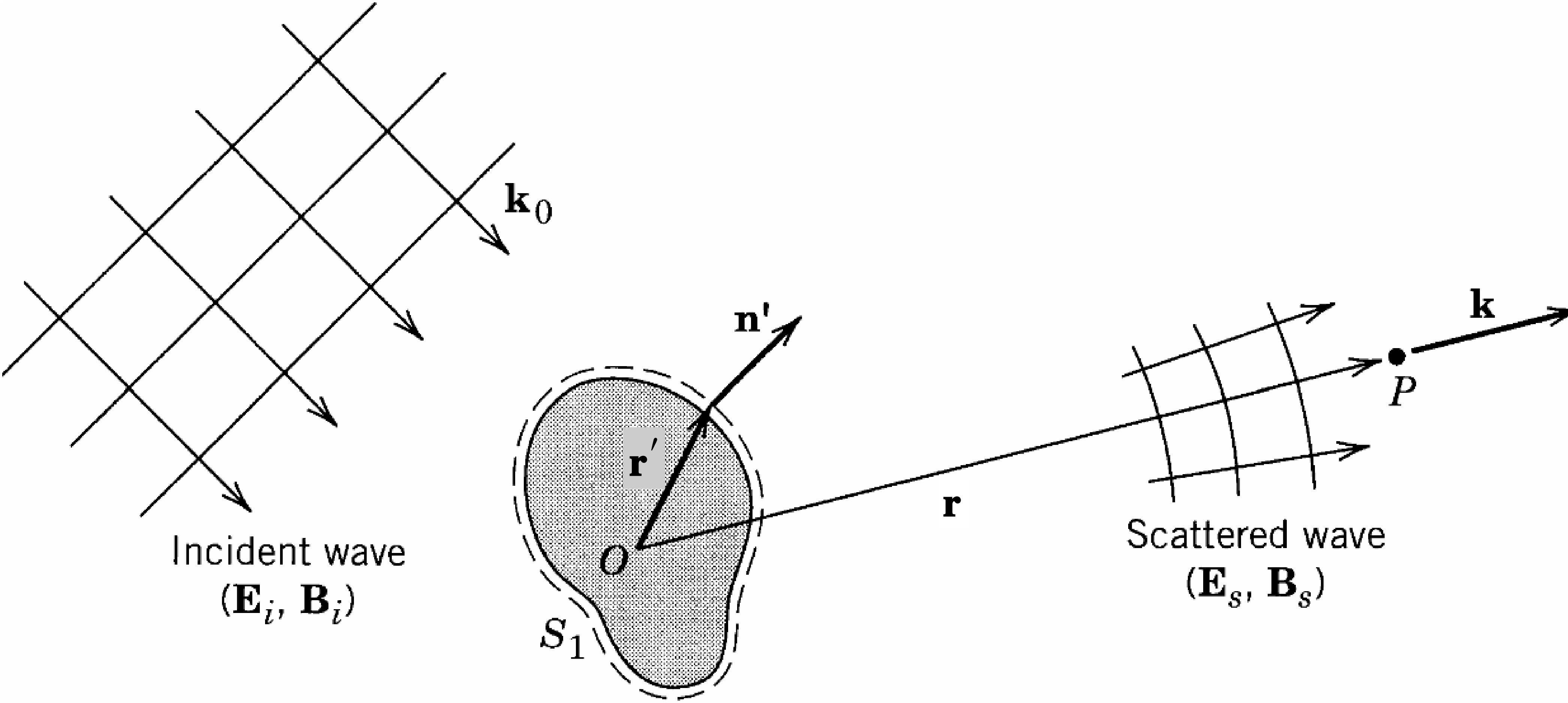
$$\to O\left(\frac{1}{r_0}\right) \to 0 \quad \Leftarrow \quad \mathbf{E} = -\hat{\mathbf{r}}' \times c\,\mathbf{B} + O\left(\frac{1}{r_0^2}\right), \quad r_0 \to \infty$$



$$\Rightarrow \boxed{\mathbf{E}(\mathbf{r}) = \oint_{S_1} \left(\nabla' G \times (\mathbf{E}\times \mathrm{d}\boldsymbol{a}') + (\mathbf{E}\cdot \mathrm{d}\boldsymbol{a}')\nabla' G - i\,\omega\, G\,\mathbf{B}\times \mathrm{d}\boldsymbol{a}' \right)}$$

vector Kirchhoff integral relation, where $G = \dfrac{e^{i k \mathbb{r}}}{4\pi \mathbb{r}}$ at ∞

- On both sides of the relation, (scattered fields) = (total fields)−(incident wave).



- $r \to \infty \;\Rightarrow\; G(\mathbf{r}, \mathbf{r}') \to \dfrac{e^{ikr}}{4\pi r} e^{-i\boldsymbol{k}\cdot\mathbf{r}'}, \;\; \nabla' G \to -i\boldsymbol{k}\, G, \;\; \mathbf{E}_s(\mathbf{r}) \to \dfrac{e^{ikr}}{r} \mathbf{F}(\boldsymbol{k}, \boldsymbol{k}_0)$

$$\Rightarrow \;\; \mathbf{F}(\boldsymbol{k}, \boldsymbol{k}_0) = \frac{1}{4\pi i} \oint_{S_1} e^{-i\boldsymbol{k}\cdot\mathbf{r}'} \Big(\omega\, \mathbf{B}_s \times \mathrm{d}\boldsymbol{a}' + \boldsymbol{k} \times (\mathbf{E}_s \times \mathrm{d}\boldsymbol{a}') + \boldsymbol{k}\, (\mathbf{E}_s \cdot \mathrm{d}\boldsymbol{a}') \Big)$$

scattering amplitude

- $\mathbf{F}(\boldsymbol{k}, \boldsymbol{k}_0)$ depends explicitly on the outgoing direction of $\boldsymbol{k}$, implicitly on the incident direction $\boldsymbol{k}_0$ via the scattered fields $\mathbf{E}_s$ and $\mathbf{B}_s$.

- $\boldsymbol{k}\cdot\mathbf{F}=0 \quad \Leftarrow \quad \omega\left(\hat{\mathbf{n}}'\times\mathbf{B}_s\right)\cdot\hat{\boldsymbol{k}}=k\,\hat{\mathbf{n}}'\cdot\left(c\,\mathbf{B}\times\hat{\boldsymbol{k}}\right)=k\,\hat{\mathbf{n}}'\cdot\mathbf{E}_s \quad \Leftarrow \quad \mathrm{d}\,\boldsymbol{a}'=\hat{\mathbf{n}}'\,\mathrm{d}\,a'$

$$\Rightarrow \quad \mathbf{F}\left(\boldsymbol{k},\boldsymbol{k}_0\right)=\frac{\boldsymbol{k}}{4\pi i}\times\oint_{\mathcal{S}_1} e^{-i\boldsymbol{k}\cdot\mathbf{r}'}\left(\mathbf{E}_s\times\mathrm{d}\,\boldsymbol{a}'-c\,\hat{\boldsymbol{k}}\times\left(\mathbf{B}_s\times\mathrm{d}\,\boldsymbol{a}'\right)\right) \quad \Leftarrow \quad \omega=c\,k$$

$$\mathbf{B}\times\mathrm{d}\,\boldsymbol{a}'-\hat{\boldsymbol{k}}\cdot\left(\mathbf{B}\times\mathrm{d}\,\boldsymbol{a}'\right)\hat{\boldsymbol{k}}=-\hat{\boldsymbol{k}}\times\left[\hat{\boldsymbol{k}}\times\left(\mathbf{B}\times\mathrm{d}\,\boldsymbol{a}'\right)\right]$$

$$\Rightarrow \quad \hat{\mathbf{e}}^*\cdot\mathbf{F}\left(\boldsymbol{k},\boldsymbol{k}_0\right)=\frac{1}{4\pi i}\oint_{\mathcal{S}_1} e^{-i\boldsymbol{k}\cdot\mathbf{r}'}\left(\omega\,\hat{\mathbf{e}}^*\cdot\left(\mathbf{B}_s\times\mathrm{d}\,\boldsymbol{a}'\right)+\hat{\mathbf{e}}^*\cdot\boldsymbol{k}\times\left(\mathbf{E}_s\times\mathrm{d}\,\boldsymbol{a}'\right)\right) \quad (\$)$$

- The terms in big brackets can be interpreted as effective electric and magnetic surface currents on $\mathcal{S}_1$ acting as sources for the scattered fields.

Vectorial Diffraction Theory

- Consider a thin, perfectly conducting, plane screen with apertures at $z=0$, with the sources in the region $z<0$, the diffracted fields in the region $z>0$

$$\Rightarrow \quad \begin{array}{l} \mathbf{E}=\mathbf{E}_0+\mathbf{E}' \\ \mathbf{B}=\mathbf{B}_0+\mathbf{B}' \end{array} \quad \Leftarrow \quad \begin{array}{l} \mathbf{E}_0\,,\ \mathbf{B}_0 : \text{fields produced by the sources without screen/obstacle} \\ \mathbf{E}'\,,\ \mathbf{B}' : \text{fields caused by plane screen} \quad \begin{array}{r} \text{diffracted} \quad z>0 \\ \text{reflected} \quad z<0 \end{array} \end{array}$$

- The scattered fields $\mathbf{E}'$ & $\mathbf{B}'$ have their origin in the surface-current density and surface-charge density that are necessarily produced on the screen to satisfy the boundary conditions.

- $J_z=0 \quad \Rightarrow \quad \begin{array}{l} A'_z=0 \\ \Phi',\ A'_x,\ A'_y \text{ are even in } z \end{array} \quad \Rightarrow \quad \begin{array}{l} E'_x,\ E'_y,\ B'_z \text{ are even in } z \\ B'_x,\ B'_y,\ E'_z \text{ are odd in } z \end{array} \quad (\#)$

- The fields that are odd in z are not necessarily 0 over the whole plane $z=0$.

- The conducting surface exists, $E'_z \neq 0$ implies an associated surface-charge density, equal on the 2 sides of the surface. Similarly, nonvanishing $\mathbf{B}_\parallel$ imply a surface-current density, equal in magnitude & direction on 2 sides of the screen.

- Only in the aperture does continuity require E'_z, B'_x, B'_y vanish. This leads to the statement that in the apertures of a perfectly conducting plane screen $E_\perp$ & $\mathbf{B}_\parallel$ are the same as in the absence of the screen.

- $\dfrac{\partial A_i'}{\partial z} = \hat{\mathbf{z}} \cdot \nabla A_i' = \hat{\mathbf{z}} \cdot \nabla A_i' - \hat{\mathbf{z}} \cdot \dfrac{\partial \mathbf{A}'}{\partial x^i} = [(\nabla \times \mathbf{A}') \times \hat{\mathbf{z}}]_i = (\mathbf{B}' \times \hat{\mathbf{z}})_i \quad \Leftarrow \quad A_z' = 0$

$$\Rightarrow \quad \mathbf{A}'(\mathbf{r}) = \frac{1}{2\pi} \int_{\text{screen}} \frac{e^{ik\mathbb{r}}}{\mathbb{r}} \, \mathrm{d}\boldsymbol{a}' \times \mathbf{B}' \quad \Leftarrow \quad (13) \quad \Leftarrow \quad \mathrm{d}\boldsymbol{a}' = \pm \hat{\mathbf{z}} \, \mathrm{d}a' \ \text{ for } \ z \gtrless 0$$

$$\Rightarrow \quad \mathbf{B}'(\mathbf{r}) = \frac{1}{2\pi} \nabla \times \int_{\text{screen}} \frac{e^{ik\mathbb{r}}}{\mathbb{r}} \, \mathrm{d}\boldsymbol{a}' \times \mathbf{B}' \quad (14)$$

with $\mathbf{B}_{\parallel}' = 0$ in the aperture

$\mathcal{S}_1 = \text{screen} + \text{apertures}$

$$\Rightarrow \quad \mathbf{E}'(\mathbf{r}) = \frac{i}{\omega \mu \epsilon} \nabla \times \mathbf{B}'(\mathbf{r})$$

- (14) is most useful when the diffracting obstacles consist of one or more finite flat segments, eg, a circular disc. Then the surface current on the obstacles can be approximated by using the incident field $\mathbf{B}_0$ in the integrand $\Leftarrow \mathbf{B}(z=0) = 0$.

- Similar argument $\quad \Rightarrow \quad \mathbf{E}'(\mathbf{r}) = \pm \dfrac{1}{2\pi} \nabla \times \displaystyle\int_{\mathcal{S}_1} \dfrac{e^{ik\mathbb{r}}}{\mathbb{r}} \, \mathrm{d}\boldsymbol{a}' \times \mathbf{E}'_{|z'=0^+} \quad (\ast) \quad \Leftarrow \quad \pm \ \text{for} \ z \gtrless 0$

- ($\ast$) satisfies the Maxwell equations and yields consistent boundary values at $z=0$. The reason for the difference in sign for $z \gtrless 0$ is the opposite reflection properties of $\mathbf{E}'$ compared to $\mathbf{B}'$.

- Cannot exploit $\mathbf{E}_{\parallel}=0$ on the metallic portions of the screen in ($\ast$) because it is the *total* electric field, not $\mathbf{E}'$.

$$
\begin{aligned}
&-\frac{\partial \psi'}{\partial z} = -\hat{\mathbf{z}}\cdot\nabla\psi' = \hat{\mathbf{z}}\cdot\left(-\nabla\psi' - \frac{\partial \mathbf{A}'}{\partial t}\right) = \hat{\mathbf{z}}\cdot\mathbf{E}' \quad \Leftarrow \quad A'_z = 0 \\
&\Rightarrow \quad \psi'(\mathbf{r}) = \int \frac{e^{ik\mathbb{r}}}{2\pi\mathbb{r}}\,\mathbf{E}'\cdot \mathrm{d}\,\boldsymbol{a}' \qquad + \qquad \mathbf{A}'(\mathbf{r}) = \int \frac{e^{ik\mathbb{r}}}{2\pi\mathbb{r}}\,\mathrm{d}\,\boldsymbol{a}'\times\mathbf{B}' \\
&\Rightarrow \quad \mathbf{E}'(\mathbf{r}) = -\nabla\psi' - \partial_t\mathbf{A}' = -\nabla\int \frac{e^{ik\mathbb{r}}}{2\pi\mathbb{r}}\,\mathbf{E}'\cdot\mathrm{d}\,\boldsymbol{a}' - \int \frac{e^{ik\mathbb{r}}}{2\pi\mathbb{r}}\,\mathrm{d}\,\boldsymbol{a}'\times\partial_t\mathbf{B}' \\
&\qquad\qquad = \int\left(\nabla'\frac{e^{ik\mathbb{r}}}{2\pi\mathbb{r}}\right)\mathbf{E}'\cdot\mathrm{d}\,\boldsymbol{a}' - \int\frac{e^{ik\mathbb{r}}}{2\pi\mathbb{r}}\left(\nabla'\times\mathbf{E}'\right)\times\mathrm{d}\,\boldsymbol{a}' \\
&\qquad\qquad = \int\nabla'\left(\cancel{\frac{e^{ik\mathbb{r}}}{2\pi\mathbb{r}}\,\mathbf{E}'}\cdot\mathrm{d}\,\boldsymbol{a}'\right) - \int\frac{e^{ik\mathbb{r}}}{2\pi\mathbb{r}}\left(\mathrm{d}\,\boldsymbol{a}'\cdot\nabla'\right)\mathbf{E}' \\
&\qquad\qquad = \int\frac{e^{ik\mathbb{r}}}{2\pi\mathbb{r}}\left(\left(\nabla'\cdot\mathbf{E}'\right)\mathrm{d}\,\boldsymbol{a}' - \left(\mathrm{d}\,\boldsymbol{a}'\cdot\nabla'\right)\mathbf{E}'\right) \\
&\qquad\qquad = \int\frac{e^{ik\mathbb{r}}}{2\pi\mathbb{r}}\,\nabla'\times\left(\mathrm{d}\,\boldsymbol{a}'\times\mathbf{E}'\right) \\
&\qquad\qquad = \int\nabla'\times\left(\cancel{\frac{e^{ik\mathbb{r}}}{2\pi\mathbb{r}}}\,\mathrm{d}\,\boldsymbol{a}'\times\mathbf{E}'\right) - \int\nabla'\frac{e^{ik\mathbb{r}}}{2\pi\mathbb{r}}\times\left(\mathrm{d}\,\boldsymbol{a}'\times\mathbf{E}'\right) \\
&\qquad\qquad = \nabla\times\int\frac{e^{ik\mathbb{r}}}{2\pi\mathbb{r}}\,\mathrm{d}\,\boldsymbol{a}'\times\mathbf{E}'
\end{aligned}
$$

- This difficulty can be removed by use of linear superposition:

$$\mathbf{E}'(\mathbf{r}) = \pm \frac{1}{2\pi} \nabla \times \int_{\text{aperture}} \frac{e^{ik\mathbb{r}}}{\mathbb{r}} \, d\boldsymbol{a}' \times \mathbf{E} - \mathbf{E}_1(\mathbf{r}) \quad \Leftarrow \quad \mathbf{E}_{\|}(\text{plane screen}) = 0$$

$$\text{where} \quad \mathbf{E}_1(\mathbf{r}) = \pm \frac{1}{2\pi} \nabla \times \int_{S_1} \frac{e^{ik\mathbb{r}}}{\mathbb{r}} \, d\boldsymbol{a}' \times \mathbf{E}_0$$

- $\mathbf{E}_1$ is equal to the "source" field $\mathbf{E}_0$ in $z > 0$ to give the extra (diffracted) field $\mathbf{E}'$ in terms of a surface integral.

- Like $\mathbf{E}'$, $E_{1,x}$, $E_{1,y}$ are even in z; $E_{1,z}$ is odd in z $\Rightarrow$ $\mathbf{E}_1(z>0) = \mathbf{E}_0$; $\mathbf{E}_1(z<0)$ reflected field $\Leftarrow$ $\mathbf{E}_0 + \mathbf{E}_1$

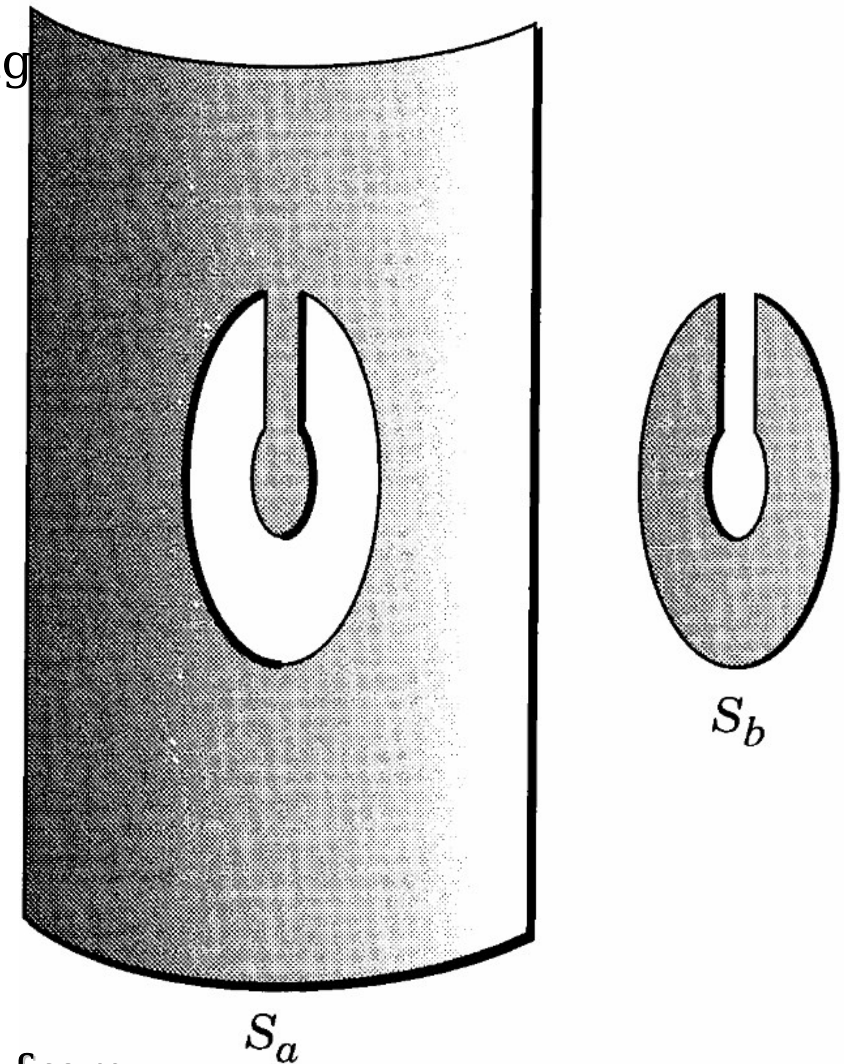
- For $z<0$, $\mathbf{E}_0 + \mathbf{E}_1$ are the fields of the sources in the presence of a perfectly conducting plane (with no apertures) at $z=0$: $\mathbf{E}_1$ (& $\mathbf{B}_1$) are the *reflected* fields!

- $$\mathbf{E}_{\text{diff}}(\mathbf{r}) \equiv \mathbf{E}'(\mathbf{r}) + \mathbf{E}_1(\mathbf{r}) = \frac{1}{2\pi} \nabla \times \int_{\text{apertures}} \frac{e^{ik\mathbb{r}}}{\mathbb{r}} \, d\boldsymbol{a}' \times \mathbf{E} \qquad (16)$$

$$\Rightarrow \quad \begin{array}{l} \mathbf{E}_{\text{diff}}(z>0) = \mathbf{E}(z>0) \quad \text{total electric field} \\ \mathbf{E}(\mathbf{r}) = \mathbf{E}_1(\mathbf{r}) + \mathbf{E}_0(\mathbf{r}) - \mathbf{E}_{\text{diff}}(\mathbf{r}) \quad \text{for} \quad z<0 \quad \text{the illuminated region} \end{array}$$

Babinet's Principle of Complementary Screens

- The complementary screen is the diffracting screen which is obtained by replacing the apertures by screen and the screen by apertures. $\Rightarrow \quad \mathcal{S} = \mathcal{S}_a + \mathcal{S}_b$



- If there are sources inside $\mathcal{S}$ (in region I) giving a field ψ, then in the absence of either screen the field ψ in region II is given by (11) where the surface integral is over the entire surface $\mathcal{S}$.

- With the screen $\mathcal{S}_a$ in position, the field ψ_a in region II is given in (11) with the source field ψ in the integrand and the surface integral only over $\mathcal{S}_b$ (the apertures).

- For the complementary screen $\mathcal{S}_b$, the field ψ_b is given in the same approximation by a surface integral over $\mathcal{S}_a$.

- $\psi = \psi_a + \psi_b$ Babinet's principle

- If ψ stands an incident plane wave, Babinet's principle says that the diffraction patterns away from the incident direction are the same for the original screen and its complement.

- The result also follows from the generalized Kirchhoff integrals if the amplitude or its normal derivative is equal to that of the incident wave in the apertures and zero elsewhere, in the spirit of the Kirchhoff approximation.

- Consider a thin, perfectly conducting plane screen and its complement, and the 2 alternative formulations of this diffraction problem

$$\begin{array}{llll} \text{Original:} & \mathbf{E}_0, & c\,\mathbf{B}_0 & \text{for } \mathcal{S}_a \\ \text{Complement:} & \mathbf{E}_{c0} = c\,\mathbf{B}_0, & c\,\mathbf{B}_{c0} = -\mathbf{E}_0 & \text{for } \mathcal{S}_b \end{array} \qquad (15)$$

- The complementary situation has a screen that is the complement of the original and has source fields with opposite polarization characteristics.

$$\Rightarrow \quad \begin{array}{l} \mathbf{E}(\mathbf{r}) = \dfrac{1}{2\pi}\nabla\times\displaystyle\int_{\mathcal{S}_b}\frac{e^{ik\mathbb{r}}}{\mathbb{r}}\,\mathrm{d}\boldsymbol{a}'\times\mathbf{E} \quad \text{for } \mathcal{S}_a, \quad z>0 \quad \Leftarrow \quad (16) \\ \mathbf{B}'_c(\mathbf{r}) = \dfrac{1}{2\pi}\nabla\times\displaystyle\int_{\mathcal{S}_b}\frac{e^{ik\mathbb{r}}}{\mathbb{r}}\,\mathrm{d}\boldsymbol{a}'\times\mathbf{B}'_c \quad \text{for } \mathcal{S}_b, \quad z>0 \quad \Leftarrow \quad (14) \end{array}$$

- In both equations the integration is over the screen S_b because of the boundary conditions on $\mathbf{E}$ and $\mathbf{B}'_c$ in the 2 cases.

- From the linearity of the Maxwell equations and the relation between the original and complementary source fields,

$$\begin{array}{r} \mathbf{E}(\mathbf{r}) = c\,\mathbf{B}'_c(\mathbf{r}) \\ c\,\mathbf{B}(\mathbf{r}) = -\mathbf{E}'_c(\mathbf{r}) \end{array} \quad \text{for } \mathcal{S}_a, \quad z>0$$

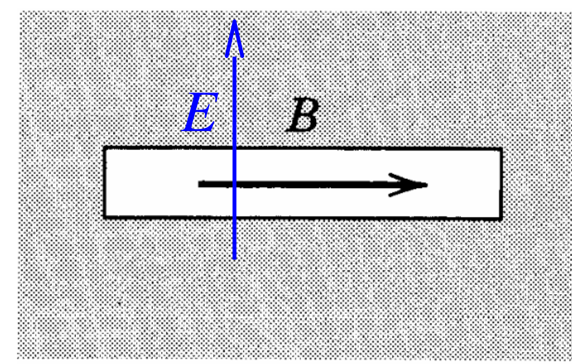
- The **−** sign is a consequence of the requirement of outgoing radiation flux at infinity, just as for the source fields.

- *Babinet's principle* for a perfectly conducting thin screen plan & its complement states that the original fields and the complementary fields are related

$$\begin{aligned} \mathbf{E} - c\,\mathbf{B}_c &= \mathbf{E}_0 \\ c\,\mathbf{B} + \mathbf{E}_c &= c\,\mathbf{B}_0 \end{aligned} \quad \text{for } z > 0 \text{ if } (15) \text{ stands.}$$

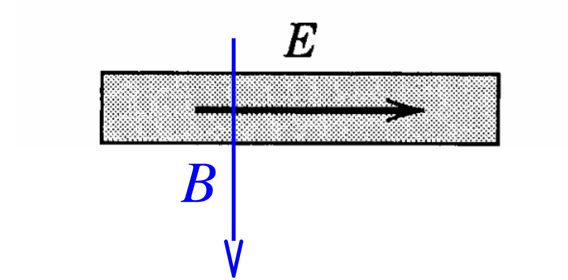
- For practical situations (finite, but large, conductivity; curved screens with large radii of curvature), the vectorial Babinet's principle can hold approximately.

- The vectorial Babinet's principle says that the diffracted intensity in directions other than that of the incident field is the same for a screen and its complement, and the polarization characteristics are rotated.



- The vector formulation of Babinet's principle is useful in microwave problems.

- The radiation pattern from the slot will be the same as that of a thin linear antenna with its driving electric field along the antenna. The polarization of the radiation will be opposite for the 2 systems.



- Elaboration of these ideas makes it possible to design antenna arrays by cutting suitable slots in the sides of waveguides.

Diffraction by a Circular Aperture; Remarks on Small Apertures

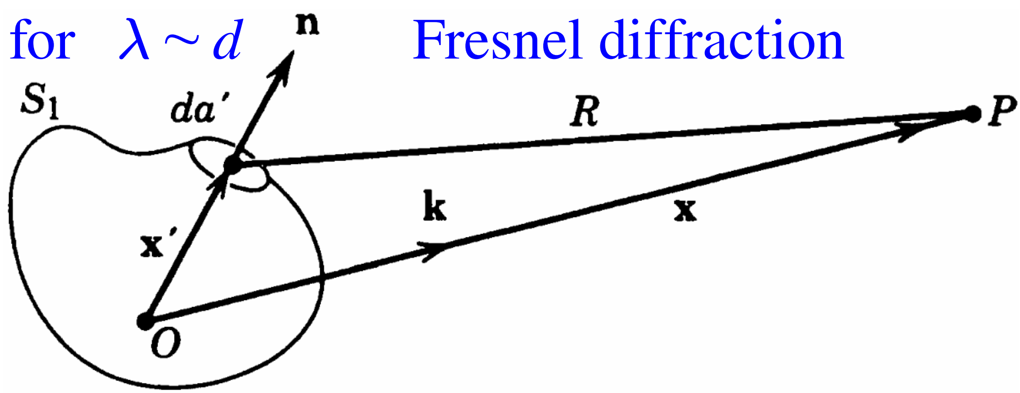
- There are 3 length scales to consider, the size d of the diffracting system, the distance r from the system to the observation point, and the wavelength λ.

- A diffraction pattern only becomes manifest for $r \gg d$. Then in expressions like (10) or (16) slowly varying factors in the integrands can be treated as constants.

- Only the phase factor k in e^{ik} needs to be handled with some care.

$$k\,\mathbb{r} = k\left(r - \hat{\mathbf{r}}\cdot\mathbf{r}' + \frac{r'^2 - (\hat{\mathbf{r}}\cdot\mathbf{r}')^2}{2\,r} + \cdots \right) \quad \text{for} \quad r \gg d$$

$$\simeq \begin{cases} k\,(r - \hat{\mathbf{r}}\cdot\mathbf{r}') & \text{for} \quad kd \ll 1 \quad \text{Fraunhofer diffraction} \\ k\left(r - \hat{\mathbf{r}}\cdot\mathbf{r}' + \dfrac{r'^2 - (\hat{\mathbf{r}}\cdot\mathbf{r}')^2}{2\,r} \right) & \text{for} \quad \lambda \sim d \quad \text{Fresnel diffraction} \end{cases}$$



- In most practical applications the simpler Fraunhofer limit is appropriate. We consider only the Fraunhofer limit.

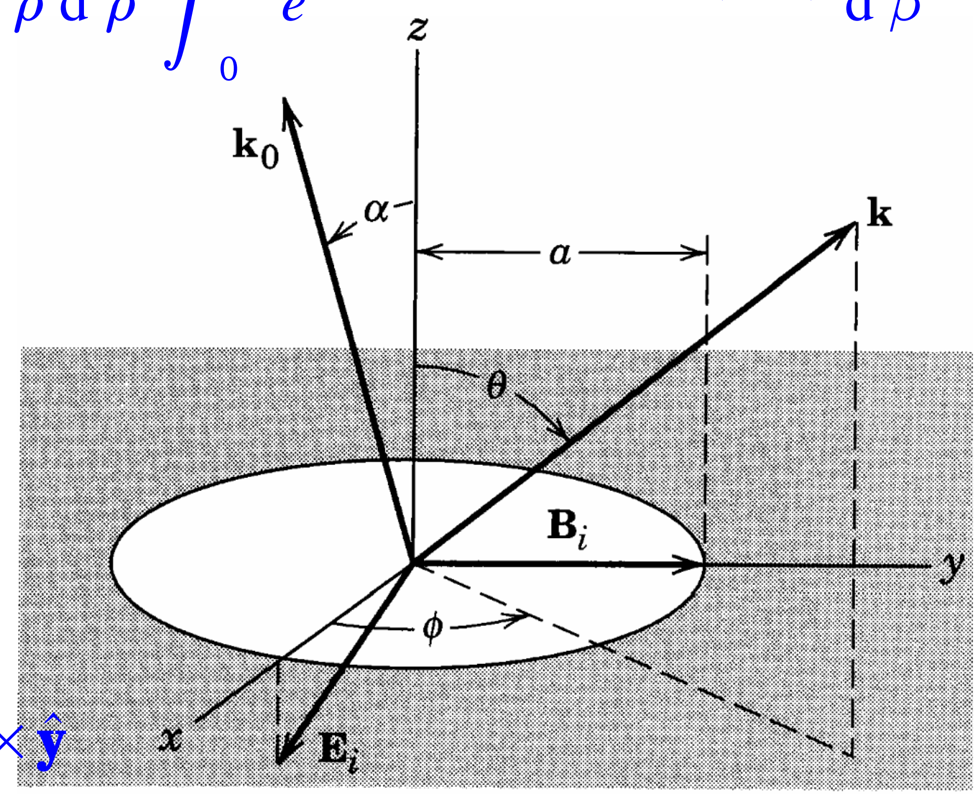
- $$(11) \;\Rightarrow\; \psi(\mathbf{r}) = -\frac{e^{ikr}}{4\pi r}\int_{S_1} e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\left[\nabla'\psi(\mathbf{r}') + i\,\psi(\mathbf{r}')\,\boldsymbol{k}\right]\cdot \mathrm{d}\,\boldsymbol{a}' \qquad (17)$$

$$(16) \;\Rightarrow\; \mathbf{E}(\mathbf{r}) = i\,\frac{e^{ikr}}{2\pi r}\,\boldsymbol{k}\times\int_{S_1} e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\mathrm{d}\,\boldsymbol{a}'\times\mathbf{E}(\mathbf{r}') \qquad (18)$$

- The plane of incidence is the x-z plane,

$$\mathbf{E}_i = E_0 \left(\hat{\mathbf{x}} \cos\alpha - \hat{\mathbf{z}} \sin\alpha\right) e^{ik(z\cos\alpha + x\sin\alpha)} \quad \Rightarrow \quad \left(\hat{\mathbf{n}} \times \mathbf{E}_i\right)_{z=0} = E_0\, \hat{\mathbf{y}} \cos\alpha\, e^{ikx'\sin\alpha}$$

$$(18) \quad \Rightarrow \quad \mathbf{E}(\mathbf{r}) = i\frac{e^{ikr} E_0 \cos\alpha}{2\pi r} \mathbf{k} \times \hat{\mathbf{y}} \int_0^a \rho\, d\rho \int_0^{2\pi} e^{ik\rho[\sin\alpha\cos\beta - \sin\theta\cos(\phi-\beta)]}\, d\beta$$



$$\xi \equiv \sqrt{\sin^2\theta + \sin^2\alpha - 2\sin\theta\sin\alpha\cos\phi}$$

$$\Rightarrow \int_0^{2\pi} e^{ik\rho[\sin\alpha\cos\beta - \sin\theta\cos(\phi-\beta)]}\, d\beta$$

$$= \int_0^{2\pi} e^{-ik\rho\xi\cos\beta'}\, d\beta' = 2\pi J_0(k\rho\xi)$$

$$\Rightarrow \mathbf{E}(\mathbf{r}) = i\frac{e^{ikr}}{r} a^2 E_0 \cos\alpha \frac{J_1(ka\xi)}{ka\xi} \boldsymbol{k} \times \hat{\mathbf{y}}$$

vector Smythe-Kirchhoff approximation (19)

$$\Rightarrow \frac{dP}{d\Omega} = P_i \cos\alpha\left(\cos^2\theta + \cos^2\phi\sin^2\theta\right) \frac{\left|J_1(ka\xi)\right|^2}{\pi\xi^2} \quad (20)$$

where $P_i = \dfrac{\pi a^2}{2Z_0} E_0^2 \cos\alpha$ incident $\perp$ power

- $k\,a \gg 1 \quad \Rightarrow \quad \dfrac{|2\,J_1(k\,a\,\xi)|^2}{k^2\,a^2\,\xi^2} \to \begin{cases} 1 & \text{at} \quad \xi = 0 \\ 0 & \text{for} \quad \xi > \dfrac{1}{k\,a} \end{cases}$

- This means that the main part of the wave passes through the opening in the manner of geometrical optics; only slight diffraction effects occur.

- For $k\,a \sim 1$ the Bessel function varies slowly in angle; the transmitted wave is distributed in directions very different from the incident direction.

- For $k\,a \ll 1$, the angular distribution is determined by the factor $\boldsymbol{k} \times \mathbf{e}_2$. But in this limit the assumption of an unperturbed field in the aperture breaks down badly.

- Total transmitted power $= \displaystyle\int_{\text{forward hemisphere}} \frac{\mathrm{d}\,P}{\mathrm{d}\,\Omega}\,\mathrm{d}\,\Omega$

$$\Rightarrow \quad T \equiv \frac{\text{transmitted power}}{\text{incident power}} \qquad \text{transmission coefficient}$$

$$= \frac{\cos\alpha}{\pi}\int_0^{2\pi}\int_0^{\pi/2}\left(\cos^2\theta + \cos^2\phi\,\sin^2\theta\right)\frac{|J_1(k\,a\,\xi)|^2}{\xi^2}\sin\theta\,\mathrm{d}\,\theta\,\mathrm{d}\,\phi$$

$$\to \begin{cases} 1 & k\,a \gg 1 \\ \dfrac{k^2\,a^2\cos\alpha}{3} & k\,a \ll 1 \quad \Leftarrow \end{cases}$$

the transmission is small
for very small holes

$$T(\alpha=0)=\int_0^{\pi/2}\left(2\csc\theta-\sin\theta\right)J_1^2\left(ka\sin\theta\right)\mathrm{d}\,\theta \qquad \Downarrow \quad + \quad (3.87) \quad \& \quad (3.88)$$

$$=\begin{matrix} 1-\dfrac{1}{ka}\sum J_{2m+1}\left(2ka\right) \\ 1-\dfrac{1}{2ka}\int_0^{2ka}J_0\left(t\right)\mathrm{d}\,t \end{matrix} \quad \Leftarrow \quad \begin{matrix} \displaystyle\int_0^{\pi/2}\frac{J_n^2\left(z\sin\theta\right)}{\sin\theta}\mathrm{d}\,\theta=\int_0^{2z}\frac{J_{2n}\left(t\right)}{t}\mathrm{d}\,t \\ \displaystyle\int_0^{\pi/2}J_n^2\left(z\sin\theta\right)\sin\theta\,\mathrm{d}\,\theta=\frac{1}{2z}\int_0^{2z}J_{2n}\left(t\right)\mathrm{d}\,t \end{matrix}$$

- The transmission coefficient increases monotonically as ka increases, with small oscillations superposed.
- $ka \gg 1 \quad \Rightarrow \quad T \simeq 1-\dfrac{1}{2ka}-\dfrac{1}{2ka\sqrt{\pi ka}}\sin\left(2ka-\dfrac{\pi}{4}\right)+\cdots$

  exhibits the small oscillations explicitly

- For a wave not normally incident, the question immediately arises as to what to choose for the scalar function in (11).

- the most consistent assumption is to take the magnitude of $\mathbf{E}$ or $\mathbf{B}$. Then the diffracted intensity is treated as proportional to the absolute square of (11).

- If a component of $\mathbf{E}$ or $\mathbf{B}$ is chosen for ψ, we must decide to keep or throw away radial components of the diffracted fields in calculating the diffracted power.

- Choosing the magnitude of $\mathbf{E}$ for ψ, then

$$(17) \quad \Rightarrow \quad \psi(\mathbf{r}) = -i \frac{e^{ikr}}{r} k a^2 E_0 \frac{\cos\alpha + \cos\theta}{2} \frac{J_1(k a \xi)}{k a \xi} \quad \text{vs} \quad (19)$$

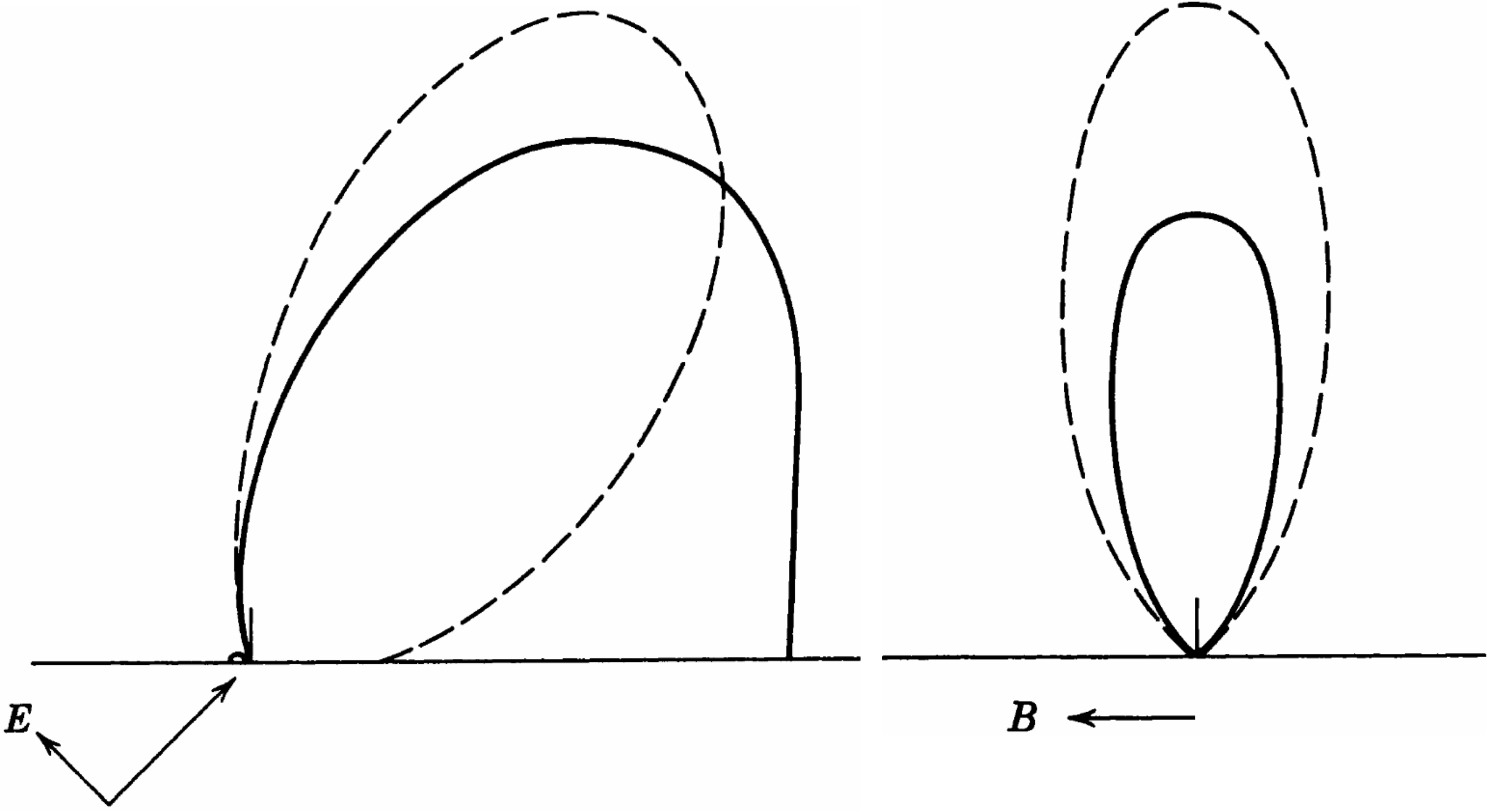
$$\Rightarrow \quad \frac{\mathrm{d}P}{\mathrm{c}\,\Omega} \simeq P_i \frac{k^2 a^2}{\pi} \cos\alpha \left(1 + \frac{\cos\theta}{\cos\alpha}\right)^2 \left|\frac{J_1(k a \xi)}{k a \xi}\right|^2 \quad (21)$$

- The vector Smythe-Kirchhoff result (20) and the scalar result (21) contain the same "diffraction" distribution factor $\left|\frac{J_1(k a \xi)}{k a \xi}\right|^2$ & the same dependence on wave number. But the scalar result has no azimuthal dependence (apart from that contained in ξ), whereas the vector expression does.

- The azimuthal variation comes from the polarization properties of the field, and must be absent in a scalar approximation.

- For normal incidence($\alpha = 0$) & $k a \gg 1$ the polarization dependence is unimportant. The diffraction is confined to very small angles in the forward direction. Then all scalar & vector approximations reduce to the common expression,

$$\frac{\mathrm{d}P}{\mathrm{d}\,\Omega} \simeq P_i \frac{k^2 a^2}{\pi} \left|\frac{J_1(k a \sin\theta)}{k a \sin\theta}\right|^2$$



- For $ka = \pi$ there is a considerable disagreement between the 2 approximations.
- The Smythe-Kirchhoff result is close to the correct one.
- The vector approximation and exact calculations for a rectangular open- opening yield results in surprisingly good agreement, even down to $ka \sim 1$.

Scattering in the Short-Wavelength Limit

- If the wavelength is short compared to the dims of the obstacle, the surface can be divided approximately into an illuminated region and a shadow region.

- The boundary between these regions is only in the limit of geometrical optics.

The transition region has a width of $\sqrt[3]{\dfrac{2\,R^2}{k}}$, R: curvature radius of the surface.

- Since R is of the order of magnitude of the dimensions of the obstacle, the short-wavelength limit will approximately satisfy the geometrical condition.

- In the shadow region the scattered fields on the surface must be very nearly equal and opposite to the incident fields.

- In the illuminated region the scattered fields at the surface will depend on the properties of the obstacle.

- If the wavelength is short compared to the min radius of curvature, then treat the surface as locally flat.

- Eventually specialize to a perfectly conducting obstacle, for which $\mathbf{E}_{s,\parallel}$ and $\mathbf{B}_{s,\perp}$ are equal and opposite to the corresponding incident fields, while $\mathbf{B}_{s,\parallel}$ & $\mathbf{E}_{s,\perp}$ are approximately equal to the incident values ⇒ (#)

- $\mathbf{e}^* \cdot \mathbf{F} = \mathbf{e}^* \cdot \mathbf{F}_{\mathrm{sh}} + \mathbf{e}^* \cdot \mathbf{F}_{\mathrm{ill}} \quad + \quad \mathbf{E}_i = E_0 \mathbf{e}_0 e^{i \boldsymbol{k}_0 \cdot \mathbf{r}}, \quad c \mathbf{B}_i = \hat{\boldsymbol{k}}_0 \times \mathbf{E}_i$

$$(\$) \Rightarrow \mathbf{e}^* \cdot \mathbf{F}_{\mathrm{sh}} = \frac{i E_0}{4\pi} \int_{\mathrm{sh}} e^{i(\boldsymbol{k}_0 - \boldsymbol{k}) \cdot \mathbf{r}'} \mathbf{e}^* \cdot [(\boldsymbol{k}_0 \times \mathbf{e}_0) \times \mathrm{d}\boldsymbol{a}' + \boldsymbol{k} \times (\mathbf{e}_0 \times \mathrm{d}\boldsymbol{a}')] \Leftarrow \begin{matrix} \mathbf{E}_s \simeq -\mathbf{E}_i \\ \mathbf{B}_s \simeq -\mathbf{B}_i \end{matrix}$$

$$= \frac{i E_0}{4\pi} \int_{\mathrm{sh}} e^{i(\boldsymbol{k}_0 - \boldsymbol{k}) \cdot \mathbf{r}'} \mathbf{e}^* \cdot [(\boldsymbol{k} + \boldsymbol{k}_0) \times (\mathbf{e}_0 \times \mathrm{d}\boldsymbol{a}') - (\mathrm{d}\boldsymbol{a}' \cdot \mathbf{e}_0) \boldsymbol{k}_0]$$

$$\lambda \ll 1 \quad \Rightarrow \quad \boldsymbol{k}_0 \cdot \mathbf{r}', \quad \boldsymbol{k} \cdot \mathbf{r}' \gg 1$$

$$\Rightarrow \int e^{i(\boldsymbol{k}_0 - \boldsymbol{k}) \cdot \mathbf{r}'} \mathbf{e}^* [\ \cdots\ ]\, \mathrm{d}a' \to 0 \quad \text{except} \quad \boldsymbol{k}_0 \simeq \boldsymbol{k} \quad \Leftarrow \quad \text{forward direction}$$

$$\text{forward region} \quad \Rightarrow \quad \theta \le \frac{1}{k R} \quad \Rightarrow \quad \mathbf{e}^* \cdot \hat{\boldsymbol{k}}_0 \sim \mathbf{e}_0 \cdot \hat{\boldsymbol{k}} \sim \sin\theta \ll 1 \quad \Leftarrow \quad \mathbf{e}_0 \cdot \hat{\boldsymbol{k}}_0 = \mathbf{e}^* \cdot \hat{\boldsymbol{k}} = 0$$

$$\Rightarrow \mathbf{e}^* \cdot \mathbf{F}_{\mathrm{sh}} \simeq \frac{i E_0}{2\pi} \mathbf{e}^* \cdot \mathbf{e}_0 \int_{\mathrm{sh}} e^{i(\boldsymbol{k}_0 - \boldsymbol{k}) \cdot \mathbf{r}'} \boldsymbol{k}_0 \cdot \mathrm{d}\boldsymbol{a}'$$

the remarkable property of *depending only on the projected area* normal to the incident direction and not at all on the detailed shape of the obstacle.

$$\boldsymbol{k}_0 \cdot \mathrm{d}\boldsymbol{a}' = k\, \mathrm{d}x'\, \mathrm{d}y' = k\, \mathrm{d}^2 x_\perp = k \times \text{projected element of area} \quad \Leftarrow \quad \boldsymbol{k}_0 = k\, \hat{\mathbf{z}}$$

$$(\boldsymbol{k}_0 - \boldsymbol{k}) \cdot \mathbf{r}' = k(1 - \cos\theta) z' - \boldsymbol{k}_\perp \cdot \mathbf{r}_\perp \simeq -\boldsymbol{k}_\perp \cdot \mathbf{r}_\perp \quad \Leftarrow \quad \begin{matrix} \mathbf{r}_\perp = x' \mathbf{e}_1 + y' \mathbf{e}_2 \\ \boldsymbol{k}_\perp = k_x \mathbf{e}_1 + k_y \mathbf{e}_2 \end{matrix}$$

$$k R \gg 1 \quad + \quad \theta \ll 1 \quad \Rightarrow \quad \mathbf{e}^* \cdot \mathbf{F}_{\mathrm{sh}} \simeq \frac{i k}{2\pi} E_0 \mathbf{e}^* \cdot \mathbf{e}_0 \int_{\mathrm{sh}} e^{-i \boldsymbol{k}_\perp \cdot \mathbf{r}_\perp} \mathrm{d}^2 x_\perp$$

- In this limit all scatterers of the same projected area give the same shadow-scattering contribution.

- The polarization character of the scattered radiation is the factor $\mathbf{e}^{*}\cdot\mathbf{e}_0$.

- Since the scattering is at small angles, the dominant contribution has the same polarization as the incident wave ⇒ *no spin flip* in quantum-mechanical language.

- Consider a scatterer whose projected area is a circular disc of radius a

$$\Rightarrow \int_{\text{sh}} e^{-i\boldsymbol{k}_\perp\cdot\mathbf{r}_\perp}\,\mathrm{d}^2x_\perp = 2\pi a^2\,\frac{J_1(k\,a\sin\theta)}{k\,a\sin\theta}$$

$$\Rightarrow \quad \mathbf{e}^{*}\cdot\mathbf{F}_{\text{sh}} = i\,k\,a^2 E_0\,\mathbf{e}^{*}\cdot\mathbf{e}_0\,\frac{J_1(k\,a\sin\theta)}{k\,a\sin\theta} \qquad (22)$$

- The scattering from the illuminated side of the obstacle cannot be calculated without specifying the shape and nature of the surface. Assume the illuminated surface is perfectly conducting

$$\Rightarrow \quad \mathbf{E}_{s,\|}\simeq -\mathbf{E}_i \quad \text{and} \quad \mathbf{B}_{s,\|}\simeq \mathbf{B}_i \quad \text{on} \quad \mathcal{S}_1$$

$$\Rightarrow \quad \mathbf{e}^{*}\cdot\mathbf{F}_{\text{ill}} = \frac{i\,E_0}{4\pi}\int_{\text{ill}} e^{i(\boldsymbol{k}_0-\boldsymbol{k})\cdot\mathbf{r}'}\,\mathbf{e}^{*}\cdot\left[\boldsymbol{k}\times(\mathbf{e}_0\times\mathrm{d}\boldsymbol{a}')-(\boldsymbol{k}_0\times\mathbf{e}_0)\times\mathrm{d}\boldsymbol{a}'\right]$$

$$= \frac{i\,E_0}{4\pi}\int_{\text{ill}} e^{i(\boldsymbol{k}_0-\boldsymbol{k})\cdot\mathbf{r}'}\,\mathbf{e}^{*}\cdot\left[(\boldsymbol{k}-\boldsymbol{k}_0)\times(\mathbf{e}_0\times\mathrm{d}\boldsymbol{a}')+(\mathbf{e}_0\cdot\mathrm{d}\boldsymbol{a}')\,\boldsymbol{k}_0\right] \qquad (23)$$

- Try the similar argument, but the contribution in the forward direction vanishes due to $\boldsymbol{k}_0 - \boldsymbol{k}$ instead of $\boldsymbol{k}_0 + \boldsymbol{k}$ in the shadow amplitude. The illuminated side of the scatterer thus gives only a modest contribution to the scattering at small angles.

- This makes sense since the illuminated side must give the reflected wave, and the reflection is mainly at angles other than forward.

- To go further we need the shape & EM property of the scatter's illuminated part.

- As in Sec. 7.11, the dominant contribution to (23) comes from the region of integration where the phase of the exponential is stationary

$$\begin{aligned} \boldsymbol{k} &= (k, \theta, \phi) \\ \mathbf{n}' &= (1, \alpha, \beta) \end{aligned} \Rightarrow \begin{aligned} f(\alpha, \beta) &= (\boldsymbol{k}_0 - \boldsymbol{k}) \cdot \mathbf{r}' \quad \Leftarrow \quad \text{phase factor} \\ &= k\, a \left[(1 - \cos\theta) \cos\alpha - \sin\theta \sin\alpha \cos(\beta - \phi) \right] \end{aligned}$$

$$a : \text{radius of the spherical surface}$$

$$\Rightarrow \begin{aligned} \alpha_0 &= \frac{\pi + \theta}{2} \\ \beta_0 &= \phi \end{aligned} \quad \text{stationary point} \Leftarrow \begin{array}{c} \text{reflection from the sphere} \\ \text{according to geometrical optics} \end{array} \Rightarrow \mathbf{n}' = \mathbf{n}_r \equiv \frac{\boldsymbol{k} - \boldsymbol{k}_0}{|\boldsymbol{k} - \boldsymbol{k}_0|}$$

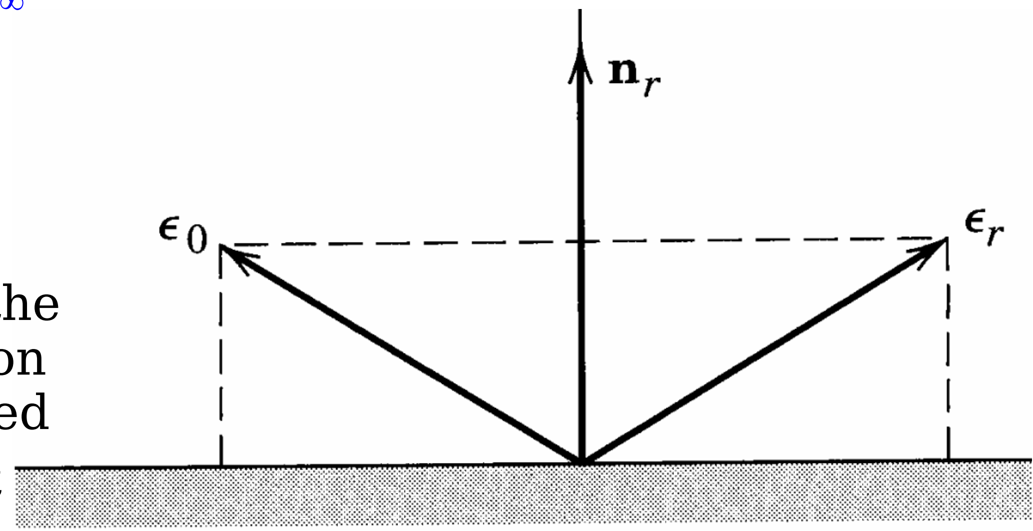
$$\Rightarrow f(\alpha, \beta) = -2\, k\, a \sin\frac{\theta}{2} \left(1 - \frac{x^2}{2} - \frac{y^2}{2} \cos^2\frac{\theta}{2} + \cdots \right) \Leftarrow \begin{aligned} x &= \alpha - \alpha_0 \\ y &= \beta - \beta_0 \end{aligned}$$

Let $\mathbf{e}_r = 2(\mathbf{n}_r \cdot \mathbf{e}_0)\, \mathbf{n}_r - \mathbf{e}_0$

$$\Rightarrow \quad \mathbf{e}^* \cdot \mathbf{F}_{\text{ill}} \simeq \frac{k\,a^2 E_0}{4\,\pi\,i} \sin\theta\, \mathbf{e}^* \cdot \mathbf{e}_r\, e^{-2\,i\,k\,a \sin\frac{\theta}{2}} \int e^{i\,x^2\,k\,a \sin\frac{\theta}{2}}\, \mathrm{d}\,x \int e^{i\,y^2\,k\,a \sin\frac{\theta}{2}\cos^2\frac{\theta}{2}}\, \mathrm{d}\,y \qquad (24)$$

$$\simeq \frac{a}{2} E_0\, e^{-2\,k\,a \sin\frac{\theta}{2}}\, \mathbf{e}^* \cdot \mathbf{e}_r \quad \Leftarrow \int_{-\infty}^{\infty} e^{i\,\alpha\,x^2}\, \mathrm{d}\,x = \sqrt{\frac{\pi\,i}{\alpha}} \quad \text{for} \quad 2\,k\,a \sin\frac{\theta}{2} \gg 1$$

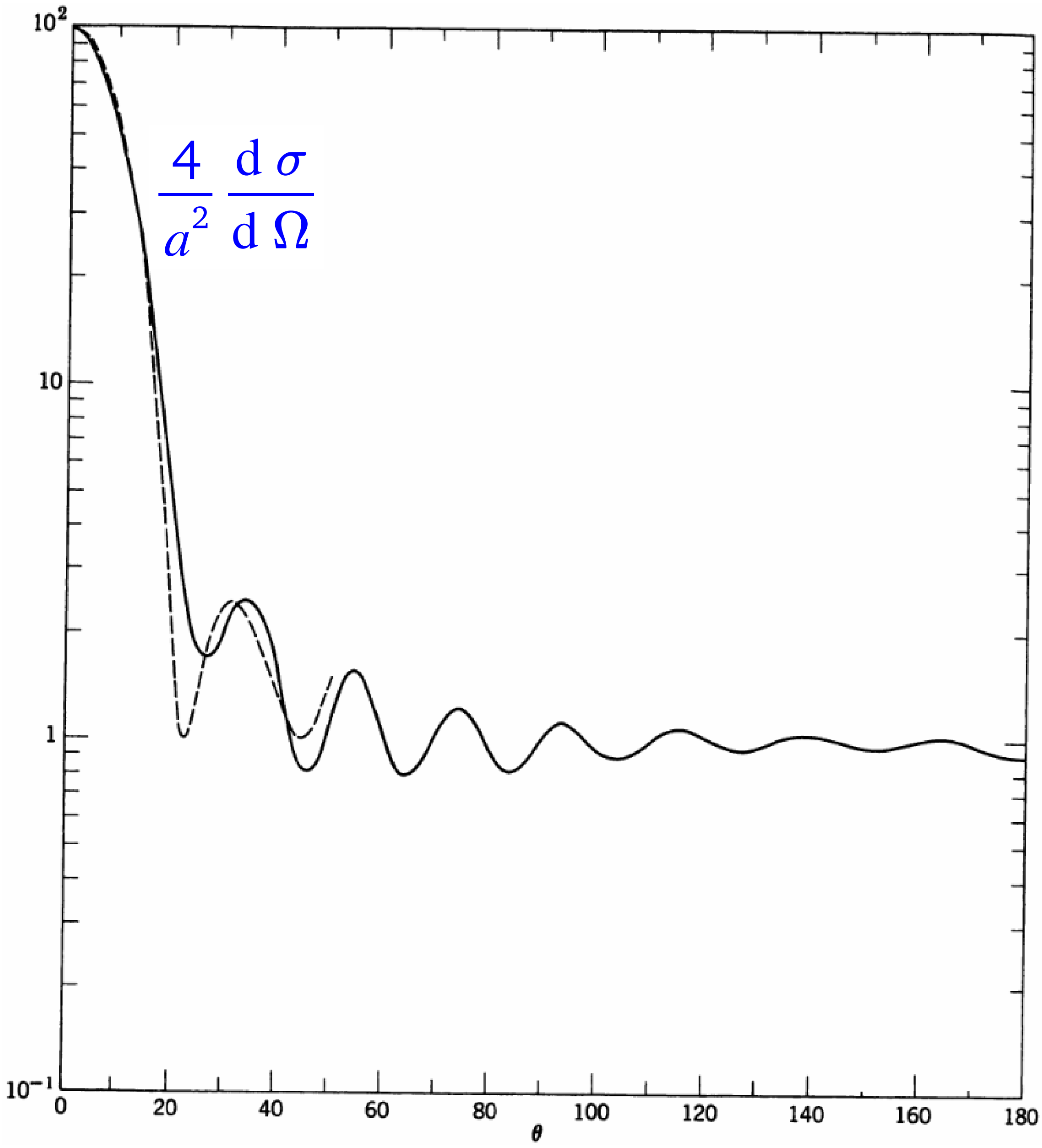
$$\Rightarrow \quad \begin{array}{l} \left|\mathbf{e}^* \cdot \mathbf{F}_{\text{ill}}\right| \to \text{const} \quad \text{as} \quad 2\,k\,a \sin\dfrac{\theta}{2} \gg 1 \\ \left|\mathbf{e}^* \cdot \mathbf{F}_{\text{ill}}\right| \propto \quad \theta^2 \quad \text{as} \quad \theta \to 0 \end{array}$$



- Compare (22) with (24) shows that in the forward direction the shadow contribution dominates in magnitude over the reflected amplitude by a factor of $k\,a \gg 1$, while at angles where $k\,a \sin\theta \gg 1$, the ratio of the magnitudes is of $\dfrac{1}{\sqrt{k\,a \sin^3\theta}}$

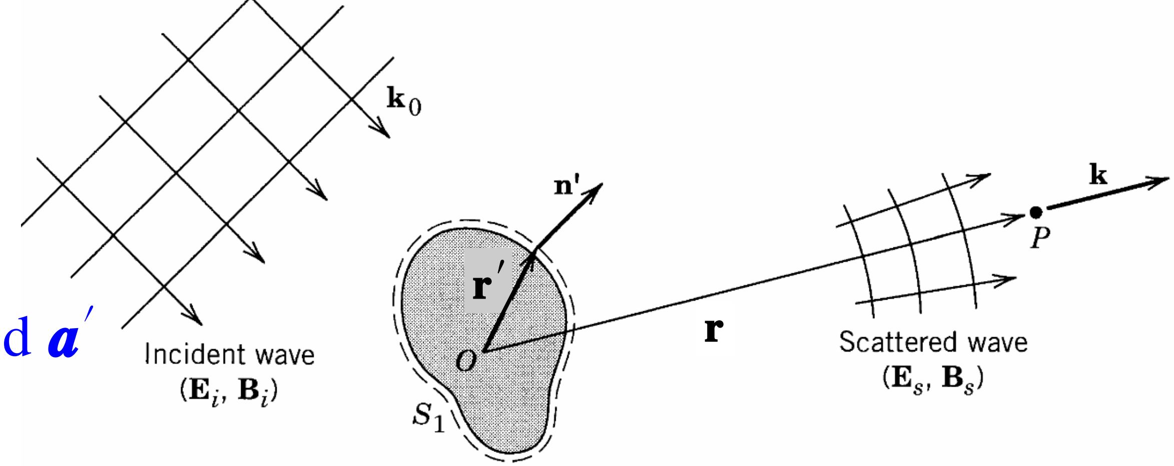
$$\Rightarrow \quad \frac{\mathrm{d}\,\sigma}{\mathrm{d}\,\Omega} = \begin{cases} a^2 \left|\dfrac{J_1(k\,a \sin\theta)}{\sin\theta}\right|^2, & \theta \le \dfrac{10}{k\,a} \\ \dfrac{a^2}{4}, & \theta \gg \dfrac{1}{k\,a} \end{cases}$$

- The scattering in the forward direction is a typical diffraction pattern with a central maximum and smaller secondary maxima, while at larger angles it is isotropic.



- The shadow diffraction peak gives a contribution of πa^2, and so does the isotropic part. The total scattering cross section is thus $2\pi a^2$, one factor of the geometrical projected area coming from direct reflection and the other from the diffraction scattering that must accompany the formation of a shadow behind the obstacle.

Optical Theorem and Related Matters



$$\mathbf{E} = \mathbf{E}_i + \mathbf{E}_s\,, \quad \mathbf{B} = \mathbf{B}_i + \mathbf{B}_s$$

$$\Rightarrow \quad P_{\text{abs}} = -\frac{1}{2\,\mu_0} \oint_{S_1} \Re\left[\mathbf{E} \times \mathbf{B}^*\right] \cdot d\,\boldsymbol{a}'$$

$$P_{\text{scatt}} = \frac{1}{2\,\mu_0} \oint_{S_1} \Re\left[\mathbf{E}_s \times \mathbf{B}_s^*\right] \cdot d\,\boldsymbol{a}' \quad \Leftarrow \quad d\,\boldsymbol{a}' = \mathbf{n}'\, d\,a'$$

$$\mathbf{E}_i = E_0\, e^{i\,\boldsymbol{k}_0 \cdot \mathbf{r}}\, \mathbf{e}_0\,, \quad c\,\mathbf{B}_i = \hat{\boldsymbol{k}}_0 \times \mathbf{E}_i\,, \quad \boldsymbol{k}_0 = k\,\hat{\boldsymbol{k}}_0$$

$$\Rightarrow \quad P = P_{\text{abs}} + P_{\text{scatt}} + P_{\text{incident}} = -\frac{1}{2\,\mu_0} \oint_{S_1} \Re\left[\mathbf{E}_s \times \mathbf{B}_i^* + \mathbf{E}_i^* \times \mathbf{B}_s\right] \cdot d\,\boldsymbol{a}'$$

$$= -\frac{1}{2\,\mu_0\, c} \Re\left[E_0^* \oint_{S_1} e^{-i\,\boldsymbol{k}_0 \cdot \mathbf{r}'}\, \mathbf{e}^* \cdot \left[c\,\mathbf{B}_s \times d\,\boldsymbol{a}' + \hat{\boldsymbol{k}}_0 \times \left(\mathbf{E}_s \times d\,\boldsymbol{a}'\right)\right]\right]$$

$$= \frac{2\,\pi}{k\,Z_0} \Im\left[E_0^*\, \mathbf{e}_0^* \cdot \mathbf{F}\left(\boldsymbol{k} = \boldsymbol{k}_0\right)\right]$$

optical theorem $\Leftarrow$ the forward scattering amplitude $\boldsymbol{k} = \boldsymbol{k}_0\,,\ \mathbf{e} = \mathbf{e}_0$ $\Leftarrow (\$)$

$$\Rightarrow \quad \sigma_t \equiv \frac{P}{P_{\text{incident}} / \text{area}} = \frac{P}{\dfrac{|E_0|^2}{2\,Z_0}}$$

total cross section (extinction cross section)

$$= \frac{4\,\pi}{k} \Im\left[\mathbf{e}^* \cdot \mathbf{f}\left(\boldsymbol{k} = \boldsymbol{k}_0\right)\right]$$

optical theorem $\Leftarrow$ $\mathbf{f}\left(\boldsymbol{k}\,, \boldsymbol{k}_0\right) \equiv \dfrac{\mathbf{F}\left(\boldsymbol{k}\,, \boldsymbol{k}_0\right)}{E_0}$ normalized scattering amplitude

- The notation corresponds to the standard quantum-mechanical conventions.

- For particles with spin the forward scattering amplitude is the one in which none of the particles change their spin state.

- For photons it is indicated by $\mathbf{e}^{*}{}_{0} \cdot \mathbf{f}$ for scattered radiation with the same polarization finally as it was initially.

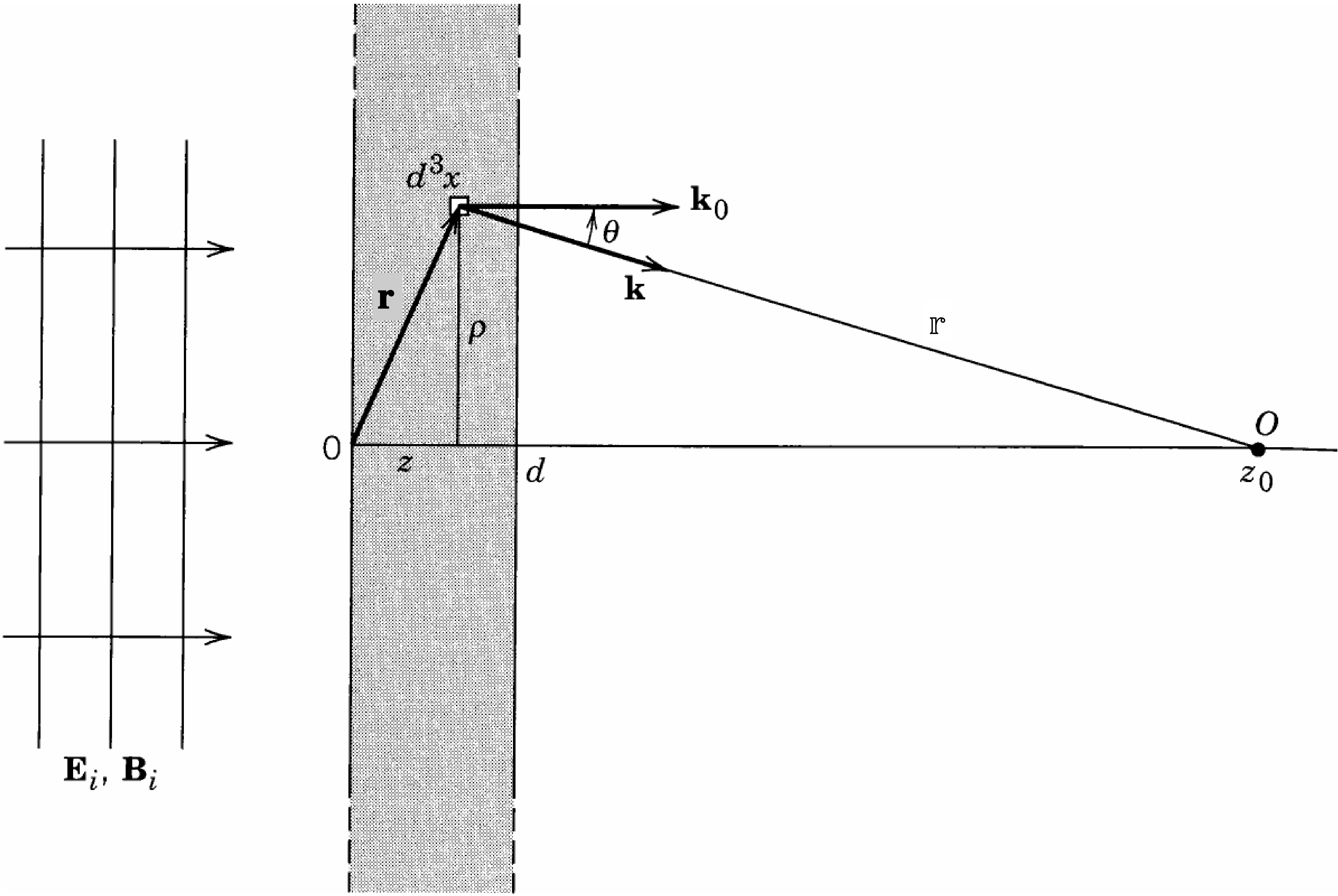
- The optical theorem can connect the forward scattering amplitude for a single scatterer to the macroscopic EM properties, ie, the dielectric constant, of a medium composed of a large number of scatterers.

- Let a plane wave incident normally on a thin slab of uniform material composed of N identical scattering centers per unit volume.

- The incident wave impinges on the scattering centers, causing each to generate a scattered wave. The coherent sum of the incident wave and of all the scattered waves gives a modified wave to the right of the slab.

- Comparison of this modified wave with the expected wave through a slab of a macroscopic, electric susceptibility $\epsilon(\omega)$ leads to a relation between ϵ and $\mathbf{f}$.

- The thickness and the density of the slab are assumed to be so small that only single scatterings in the slab need be considered and thus the effective exciting field at each scatterer is just the incident field itself.



d^3x
$\mathbf{k}_0$
θ
$\mathbf{k}$
$\mathbf{r}$
ρ
$\mathbb{r}$
O
0
z
d
z_0
$\mathbf{E}_i$, $\mathbf{B}_i$

- The scattered field produced at the observation point O is approximately

$$\mathrm{d}\,\mathbf{E}_s = \frac{e^{ik\mathbb{r}}}{\mathbb{r}}\,\mathbf{f}\,(k,\theta,\phi)\,E_0\,e^{i\boldsymbol{k}_0\cdot\mathbf{r}}\,N\,\mathrm{d}^3x \quad \Leftarrow \quad \mathbb{r} = \sqrt{\rho^2 + (z-z_0)^2}\,, \quad \cos\theta = \frac{z_0 - z}{\mathbb{r}}$$

$$\Rightarrow \quad \mathbf{E}_s = N\,E_0 \int_0^{2\pi} \mathrm{d}\,\phi \int_0^{d} e^{ikz}\,\mathrm{d}\,z \int_0^{\infty} \frac{e^{ik\mathbb{r}}}{\mathbb{r}}\,\mathbf{f}\,(k,\theta,\phi)\,\rho\,\mathrm{d}\,\rho$$

$$= N\,E_0 \int_0^{2\pi} \mathrm{d}\,\phi \int_0^{d} e^{ikz}\,\mathrm{d}\,z \int_{|z_0-z|}^{\infty} e^{ik\mathbb{r}}\,\mathbf{f}\,(k,\theta,\phi)\,\mathrm{d}\,\mathbb{r} \quad \Leftarrow \quad \rho\,\mathrm{d}\,\rho = \mathbb{r}\,\mathrm{d}\,\mathbb{r}$$

$$\int_{|z_0-z|}^{\infty} e^{ik\mathbb{r}}\,\mathbf{f}\,(k,\theta,\phi)\,\mathrm{d}\,\mathbb{r} = \frac{1}{ik}\left(e^{ik\mathbb{r}}\,\mathbf{f}\,\Big|_{\mathbb{r}=|z_0-z|}^{\infty} + (z_0 - z)\int_{|z_0-z|}^{\infty} \frac{e^{ik\mathbb{r}}}{\mathbb{r}^2}\,\frac{\mathrm{d}\,\mathbf{f}}{\mathrm{d}\cos\theta}\,\mathrm{d}\,\mathbb{r} \right)$$

$$= \frac{i}{k}\,e^{ik|z-z_0|}\,\mathbf{f}\,(k,0) + O\left(\frac{1}{k\,|z_0 - z|}\right) \quad \Leftarrow \quad k\,|z_0 - z| \to \infty$$

$$\Rightarrow \quad \mathbf{E}_s \approx \frac{2\pi i}{k}\,N\,E_0\,\mathbf{f}\,(k,0)\int_0^{d} e^{ik(z+|z_0-z|)}\,\mathrm{d}\,z = \frac{2\pi i}{k}\,N\,E_0\,\mathbf{f}\,(k,0)\,e^{ikz_0}\,d \quad \Leftarrow \quad z_0 > z$$

$\Rightarrow$ *Total* electric field at O $\quad \mathbf{E} = E_0\,e^{ikz_0}\left(\hat{\mathbf{e}}_0 + \frac{2\pi i}{k}\,N\,d\,\mathbf{f}\,(k,0)\right) + O\,(d^2)$

- The amplitude at O for a wave with the same polarization state as the incident wave is

$$\hat{\mathbf{e}}_0^{*}\cdot\mathbf{E} \approx E_0\,e^{ikz_0}\left(1 + \frac{2\pi i}{k}\,N\,d\,\hat{\mathbf{e}}_0^{*}\cdot\mathbf{f}\,(k,0)\right)$$

- Now consider the slab macroscopically, with its EM properties specified by a dielectric constant $\frac{\epsilon(\omega)}{\epsilon_0}$ appropriate to describe the propagation of the wave of frequency $\omega = ck$ and polarization $\mathbf{e}_0$.

$$\Rightarrow \begin{array}{l}\text{the transmitted}\\ \text{wave at } z = z_0\end{array} \quad \hat{\mathbf{e}}_0^* \cdot \mathbf{E}\,(\text{macroscopic}) = E_0\, e^{ikz_0}\left[1 + i\,\frac{kd}{2}\left(\frac{\epsilon}{\epsilon_0} - 1\right)\right] + O\left(d^2\right)$$

$$\Rightarrow \frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{4\pi N}{k^2}\,\hat{\mathbf{e}}_0^* \cdot \mathbf{f}(k, 0) \qquad (25)$$

- The derivation has a number of simplifying assumptions and the notion of a macroscopic description assumed rather than derived.

- The scattering amplitude should be evaluated at the wave number k' in the medium, not at the free-space wave number k, and there is a multiplier to the 2nd term that gives a measure of the effective exciting field at a scatterer relative to the total coherent field in the medium.

- For the dipole moment of the atom, summed over the various oscillators

$$\mathbf{p} = \frac{e^2}{m}\sum_j \frac{f_j E_0 \hat{\mathbf{e}}_0}{\omega_j^2 - \omega^2 - i\,\omega\,\gamma_j} \quad \Rightarrow \quad \mathbf{f}(\boldsymbol{k}) = \frac{1}{4\pi\epsilon_0}\frac{e^2}{m}\sum_j \frac{f_j\left(\boldsymbol{k}\times\hat{\mathbf{e}}_0\right)\times\boldsymbol{k}}{\omega_j^2 - \omega^2 - i\,\omega\,\gamma_j}$$

$$\Rightarrow \quad \hat{\mathbf{e}}_0^* \cdot \mathbf{f}\left(\boldsymbol{k} = \boldsymbol{k}_0\right) = \frac{e^2 k^2}{4\pi\epsilon_0 m}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\,\omega\,\gamma_j}$$

$$\Rightarrow \quad \frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{N e^2}{\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \quad \text{in agreement with } (7.51)$$

- The attenuation coefficient α is related to the total cross section of a single scatterer through $\alpha = N\sigma_t$ and to the imaginary part of the wave number in the medium through $\alpha = 2\Im[k']$ $\Rightarrow$ $\alpha = N\sigma_t = \frac{4\pi N}{\Re[k']} \Im[\hat{\mathbf{e}}_0^* \cdot \mathbf{f}(\Re[k'], 0)]$ $\Leftarrow \begin{matrix}(25)\\(7.54)\end{matrix}$

- Thus if we consider scattering by a single scatterer embedded in a medium with k', the optical theorem and other relations will appear as before.

- The same situation holds in the scattering of electrons in a solid where the effective mass or other approximation is used to take into account propagation through the lattice.

- The optical theorem is an exact relation. If an approximate expression for $\mathbf{f}$ is employed, a manifestly wrong result for the total cross section may be obtained.

- In the long-wavelength limit, the scattering amplitude for a dielectric sphere of radius a is

$$\begin{matrix}(10.2)\\(10.5)\end{matrix} \quad \Rightarrow \quad \mathbf{f} = \frac{\epsilon_r - 1}{\epsilon_r + 2} a^3 (\boldsymbol{k} \times \hat{\mathbf{e}}_0) \times \boldsymbol{k} \quad \Rightarrow \quad \begin{matrix}\text{forward}\\ \text{amplitude}\end{matrix} \quad \hat{\mathbf{e}}_0^* \cdot \mathbf{f}(\boldsymbol{k} = \boldsymbol{k}_0) = k^2 a^3 \frac{\epsilon_r - 1}{\epsilon_r + 2}$$

- For a lossless dielectric, this amplitude is real; the optical theorem then yields $\sigma_t = 0$.

- On the other hand, the total cross section is in this case equal to the scattering cross section (**):

$$\sigma_{\mathrm{sc}} = \frac{8\pi}{3} k^4 a^6 \left| \frac{\epsilon_r - 1}{\epsilon_r + 2} \right|^2$$

- Even with a lossy dielectric ($\Im[\epsilon] \neq 0$), the optical theorem yields a total cross

$$\sigma_t = \frac{12\pi k a^3 \Im[\epsilon]}{\left|\epsilon_r + 2\right|^2}$$

- The seeming contradictions are reflections of the necessity of different orders of approximation required to obtain consistency between the two sides of the optical theorem.

- In the long-wavelength limit it is necessary to evaluate the forward scattering amplitude to higher order in powers of ω to find the scattering cross section contribution in the total cross section by means of the optical theorem.

- For lossless or nearly lossless scatterers it is simplest to determine the total cross section directly by integration of the differential scattering cross section over angles.

- For dissipative scatterers, the optical theorem yields a nonzero answer that has a different (lower power) dependence on ω and other parameters from that of the scattering cross section.

- This contribution is the absorption cross section to lowest explicit order in ω. It can be calculated from first principles, but the optical theorem provides an elegant and convenient method to do it.