Chapter 10 Potentials and Fields

The Potential Formulation

Scalar and Vector Potentials

- We seek the *general* solution to Maxwell's equations,

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\epsilon_0},\quad \nabla\cdot\mathbf{B}=0,\quad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},\quad \nabla\times\mathbf{B}=\mu_0\boldsymbol{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

ie, given $\rho(\mathbf{r},t)$ and $\boldsymbol{J}(\mathbf{r},t)$, what are the fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$?

- In electrostatics $\nabla\times\mathbf{E}=0$ allowed to write $\mathbf{E}$ as the gradient of a scalar potential: $\mathbf{E}=-\nabla\Phi$. But in electro*dynamics* it's not true since $\nabla\times\mathbf{E}\neq 0$.

- $\mathbf{B}$ remains divergenceless, so $\mathbf{B}=\nabla\times\mathbf{A}$ as in magnetostatics since $\nabla\cdot\mathbf{B}=0$.

$$\Rightarrow\ \nabla\times\mathbf{E}=-\frac{\partial}{\partial t}\nabla\times\mathbf{A}\ \Rightarrow\ \nabla\times\left(\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}\right)=0\ \Rightarrow\ \mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}=-\nabla\Phi$$

$$\Rightarrow\ \mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}\ \Rightarrow\ \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

- This reduces to the old form when $\mathbf{A}$ is constant.

- $$\nabla\cdot\mathbf{E}=\frac{\rho}{\epsilon_0}\ \Rightarrow\ \nabla^2\Phi+\frac{\partial}{\partial t}\nabla\cdot\mathbf{A}=-\frac{\rho}{\epsilon_0}\ \ (1)\ \Rightarrow\ \nabla^2\Phi=-\frac{\rho}{\epsilon_0}\ \Leftarrow\ \mathbf{A}=\text{const}$$

Poisson equation

- $\nabla\times\mathbf{B}=\mu_0\boldsymbol{J}+\mu_0\epsilon_0\dfrac{\partial\mathbf{E}}{\partial t} \quad\Rightarrow\quad \nabla\times(\nabla\times\mathbf{A})=\mu_0\boldsymbol{J}-\mu_0\epsilon_0\nabla\dfrac{\partial\Phi}{\partial t}-\mu_0\epsilon_0\dfrac{\partial^2\mathbf{A}}{\partial t^2}$

  $\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$

  $$\Rightarrow\quad\left(\nabla^2\mathbf{A}-\mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2}\right)-\nabla\left(\nabla\cdot\mathbf{A}+\mu_0\epsilon_0\frac{\partial\Phi}{\partial t}\right)=-\mu_0\boldsymbol{J}\qquad(2)$$

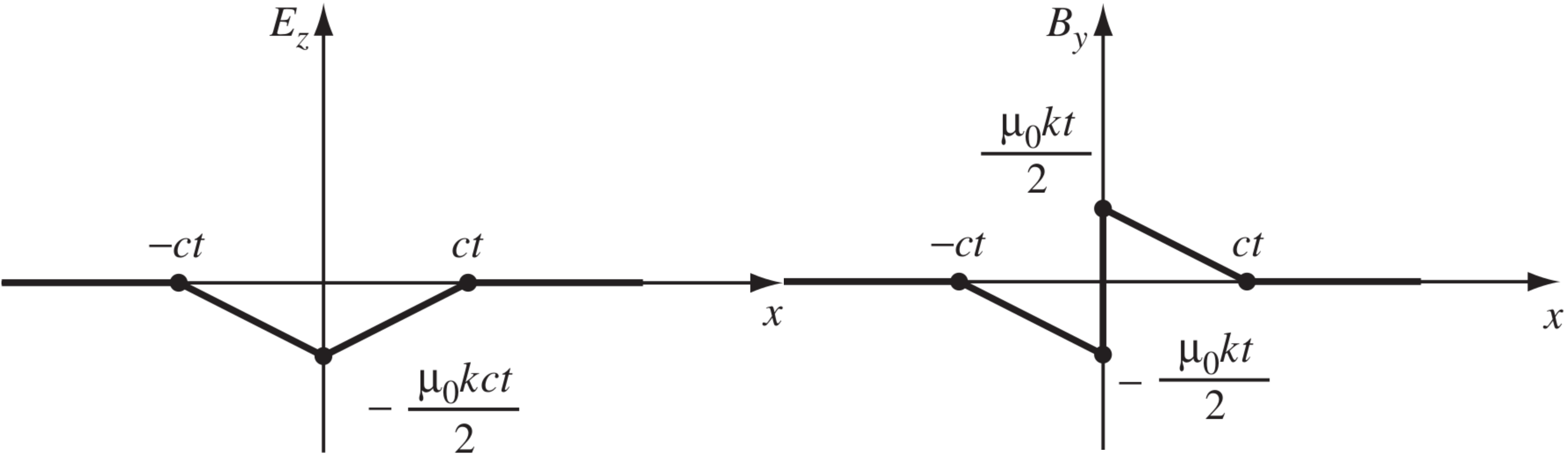
- (1) & (2) contain all the information in Maxwell's equations.

<u>Example 10.1</u>: Find the charge and current distributions that would give rise to the potentials $\Phi=0$, $\quad\mathbf{A}=\begin{cases}\dfrac{\mu_0 k}{4c}(ct-|x|)^2\,\hat{\mathbf{z}}, & \text{for } |x|<ct\\ 0, & \text{for } |x|>ct\end{cases}$

- $|x|<ct \quad\Rightarrow\quad \begin{cases}\mathbf{E}=-\dfrac{\partial\mathbf{A}}{\partial t}=-\dfrac{\mu_0 k}{2}(ct-|x|)\,\hat{\mathbf{z}}\\ \mathbf{B}=\nabla\times\mathbf{A}=-\dfrac{\mu_0 k}{4c}\dfrac{\partial}{\partial x}(ct-|x|)^2\,\hat{\mathbf{y}}=\pm\dfrac{\mu_0 k}{2c}(ct-|x|)\,\hat{\mathbf{y}}\end{cases}$

  $|x|>ct \quad\Rightarrow\quad \mathbf{E}=\mathbf{B}=0$

$$\Rightarrow\ \nabla\cdot\mathbf{E}=0,\ \ \nabla\cdot\mathbf{B}=0,\ \ \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}=\frac{\mp\mu_0 k}{2}\hat{\mathbf{y}},\ \ \nabla\times\mathbf{B}=\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}=-\frac{\mu_0 k}{2c}\hat{\mathbf{z}}$$



$$\Rightarrow \quad \rho = 0 = \boldsymbol{J}$$

- $\mathbf{B}$ has a discontinuity at x=0, and this signals the presence of a surface current

$\boldsymbol{K}$ in the yz plane; $\dfrac{\mathbf{B}_1^{\|} - \mathbf{B}_2^{\|}}{\mu_0} = \boldsymbol{K} \times \hat{\mathbf{n}} \;\Rightarrow\; k\,t\,\hat{\mathbf{y}} = \boldsymbol{K} \times \hat{\mathbf{x}} \;\Rightarrow\; \boldsymbol{K} = k\,t\,\hat{\mathbf{z}}$

- A uniform surface current flows in the z direction over the plane x=0, which starts up at t=0, and increases in proportion to t.

- The news travels out (in both directions) at the speed of light: for points $|x|>ct$ the message ("current is now flowing") has not yet arrived, so the fields are 0.

Gauge Transformations

- Although (1) & (2) are *ugly*, we *have* reduced 6 problems—finding $\mathbf{E}$ & $\mathbf{B}$ (3 components each)—down to 4: Φ (1 component) and $\mathbf{A}$ (3 components).

- Moreover, the potentials have not uniquely been defined ; we are free to impose extra conditions on Φ & $\mathbf{A}$, as long as $\mathbf{E}$ & $\mathbf{B}$ keep the same—**gauge freedom**.

- Suppose we have 2 sets of potentials, $(\Phi, \mathbf{A})$ and $(\Phi', \mathbf{A}')$, which correspond to the same electric and magnetic fields $\mathbf{A}' = \mathbf{A} + \boldsymbol{\alpha}$, $\quad \Phi' = \Phi + \beta$

$$\left.\begin{aligned} &\mathbf{B} = \nabla \times \mathbf{A}' = \nabla \times \mathbf{A} \quad \Rightarrow \quad \nabla \times \boldsymbol{\alpha} = 0 \quad \Rightarrow \quad \boldsymbol{\alpha} = \nabla \lambda \\ &\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi' - \frac{\partial \mathbf{A}'}{\partial t} \quad \Rightarrow \quad \nabla \beta + \frac{\partial \boldsymbol{\alpha}}{\partial t} = 0 \end{aligned}\right] \Rightarrow \quad \nabla \left(\beta + \frac{\partial \lambda}{\partial t} \right) = 0$$

$$\Rightarrow \quad \beta = -\frac{\partial \lambda}{\partial t} + k(t) \quad \Rightarrow \quad \text{redefine } \lambda + \int_0^t k(t')\, \mathrm{d}t' \to \lambda \quad \Rightarrow \quad \boxed{\begin{aligned} \mathbf{A}' &= \mathbf{A} + \nabla \lambda \\ \Phi' &= \Phi - \frac{\partial}{\partial t} \lambda \end{aligned}}$$

- For any scalar function $\lambda(\mathbf{r}, t)$, we can add $\nabla \lambda$ to $\mathbf{A}$ and also subtract $\partial_t \lambda$ from Φ. This will not affect the physical quantities $\mathbf{E}$ & $\mathbf{B}$. Such changes in Φ & $\mathbf{A}$ are called **gauge transformations**.

- They can be exploited to adjust the divergence of $\mathbf{A}$, with a view to simplifying the “ugly” equations (1) & (2).

Coulomb Gauge and Lorenz Gauge

The Coulomb Gauge: As in magnetostatics, we pick $\nabla \cdot \mathbf{A} = 0 = \nabla \cdot \mathbf{A}' - \nabla^2 \lambda$

$$\Rightarrow \quad \nabla^2 \Phi + \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = \nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \quad \text{Poisson's equation}$$

$$\Rightarrow \quad \Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t)}{\mathbb{r}} \, d\tau' \quad \Leftarrow \quad \Phi(\infty, t) = 0$$

- A very peculiar thing about the scalar potential in the Coulomb gauge: it is determined by the distribution of charge *right now* (not at a retarded time).

- This sounds odd to special relativity since no message travels faster than c.

- The point is that Φ *by itself* is not a physically measurable quantity—the physical quantity is $\mathbf{E}$, and that involves $\mathbf{A}$ as well.

- It is built into $\mathbf{A}$ (in the Coulomb gauge) that whereas Φ instantaneously reflects all changes in ρ, the combination $-\nabla\Phi - \partial_t \mathbf{A}$ does not; $\mathbf{E}$ will change only after sufficient time has elapsed for the "news" to arrive.

- The *advantage* of the Coulomb gauge is that the scalar potential is particularly simple to calculate; the *disadvantage* (apart from the acausal appearance of Φ) is that $\mathbf{A}$ is particularly *difficult* to calculate.

- $\nabla\cdot\mathbf{A}=0 \;\Rightarrow\; \nabla^2\mathbf{A}-\mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu_0\boldsymbol{J}+\mu_0\epsilon_0\nabla\frac{\partial\Phi}{\partial t} \;\Leftarrow\; (2)$

$$\boldsymbol{J}=\boldsymbol{J}_\ell+\boldsymbol{J}_t \;\Leftarrow\; \begin{array}{ll}\nabla\times\boldsymbol{J}_\ell=0 & \text{longitudinal, irrotational}\\ \nabla\cdot\boldsymbol{J}_t=0 & \text{transverse, solenoidal}\end{array}$$

$$\Rightarrow\; \nabla\cdot\left(\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}\right)=\nabla^2(\nabla\cdot\mathbf{A})-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\nabla\cdot\mathbf{A})=0 \;\Leftarrow\; c^2=\frac{1}{\sqrt{\mu_0\epsilon_0}}$$

$$=\nabla\cdot\left(-\mu_0\boldsymbol{J}+\frac{1}{c^2}\nabla\frac{\partial\Phi}{\partial t}\right)=\nabla\cdot\left(-\mu_0\boldsymbol{J}_\ell+\frac{1}{c^2}\nabla\frac{\partial\Phi}{\partial t}\right) \;\Leftarrow\; \text{the equation of continuity}$$

$$\Rightarrow\; \mu_0\boldsymbol{J}_\ell-\frac{1}{c^2}\nabla\frac{\partial\Phi}{\partial t}=\text{const}\Rightarrow 0 \;\Rightarrow\; \boldsymbol{J}_\ell=\epsilon_0\frac{\partial}{\partial t}\nabla\Phi$$

$$\Rightarrow\; \nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu_0\boldsymbol{J}_t$$

the source for the wave equation for $\mathbf{A}$ can be expressed entirely in terms of the transverse current.

The Lorenz gauge: $\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial \Phi}{\partial t} \Leftarrow \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\lambda = -\nabla \cdot \mathbf{A}' - \frac{1}{c^2}\frac{\partial \Phi'}{\partial t}$

- Designed to eliminate the middle term in (2) $\Rightarrow \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \boldsymbol{J}$

Meanwhile, (1) $\Rightarrow \nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$

- The virtue of the Lorenz gauge is that it treats Φ & $\mathbf{A}$ on an equal footing.

- d'Alembertian operator $\Box \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \Rightarrow$ $\Box \Phi = -\frac{\rho}{\epsilon_0}$, $\Box \mathbf{A} = -\mu_0 \boldsymbol{J}$ (3)

- This democratic treatment of Φ & $\mathbf{A}$ is nice in special relativity, where the d'Alembertian is the natural generalization of the Laplacian, (3) can be regarded as 4d versions of Poisson's eqn. And the wave eqn $\Box f = 0$, might be regarded as the 4d version of Laplace's equation.

- In the Lorenz gauge, Φ & $\mathbf{A}$ satisfy the **inhomogeneous wave equation**, with a "source" term on the right in (3).

- By the Lorenz gauge, the whole of electrodynamics reduces to the problem of *solving the inhomogeneous wave equation for a specified source*.

- Even for potentials that satisfy the Lorenz condition there is arbitrariness. Evidently the *restricted gauge transformation*,

$$\mathbf{A}'' = \mathbf{A} + \nabla \Lambda \,, \quad \Phi'' = \Phi - \frac{\partial \Lambda}{\partial t} \quad \Leftarrow \quad \Box \Lambda = \left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Lambda = 0$$

$$\Rightarrow \quad \nabla \cdot \mathbf{A}'' + \frac{1}{c^2} \frac{\partial \Phi''}{\partial t} = 0 \quad \text{as long as} \quad \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0$$

- All potentials in this restricted class are said to belong to the *Lorenz gauge*.

- The Lorenz gauge is commonly used, first because it leads to the wave equations, which treat Φ and $\mathbf{A}$ on equivalent footings, and second because it is a concept independent of the coordinate system chosen and so fits naturally into the considerations of special relativity.

Example: **The Lorenz Gauge**: The given **E** and **B** are derivable from 2 pairs of scalar and vector potentials: (Φ, **A**) and (Φ_0, $\mathbf{A}_0$), as

$$\Phi(\mathbf{r},t) = \Phi_0(\mathbf{r},t) + \frac{\omega}{r}\cos(kr-\omega t)$$

$$\mathbf{A}(\mathbf{r},t) = \mathbf{A}_0(\mathbf{r},t) + \hat{\mathbf{r}}\left(\frac{k}{r}\cos(kr-\omega t) - \frac{\sin(kr-\omega t)}{r^2}\right)$$

$$\Rightarrow \quad \Phi(\mathbf{r},t) = \Phi_0(\mathbf{r},t) - \frac{\partial}{\partial t}\frac{\sin(kr-\omega t)}{r}$$

$$\mathbf{A}(\mathbf{r},t) = \mathbf{A}_0(\mathbf{r},t) + \hat{\mathbf{r}}\frac{\partial}{\partial r}\frac{\sin(kr-\omega t)}{r}$$

$$\Rightarrow \quad \Phi = \Phi_0 - \frac{\partial \psi}{\partial t}, \quad \mathbf{A} = \mathbf{A}_0 + \nabla\psi \quad \Leftarrow \quad \psi = \frac{\sin(kr-\omega t)}{r}$$

Lorenz condition: $\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = \nabla\cdot\mathbf{A}_0 + \frac{1}{c^2}\frac{\partial \Phi_0}{\partial t} + \nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$

where $\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -k^2\frac{\sin(kr-\omega t)}{r} + \frac{\omega^2}{c^2}\frac{\sin(kr-\omega t)}{r} = 0$

Hence if Φ_0 and $\mathbf{A}_0$ satisfy the Lorentz condition, then Φ and **A** will also satisfy the condition.

Lorentz Force Law in Potential Form

- Express the Lorentz force law in terms of potentials:

$$\mathbf{F}=\frac{d\,\mathbf{p}}{d\,t}=q\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)=q\left(-\nabla\Phi-\frac{\partial\,\mathbf{A}}{\partial\,t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right)\quad\Leftarrow\quad\mathbf{p}=m\,\mathbf{v}$$

$$\nabla\left(\mathbf{v}\cdot\mathbf{A}\right)=\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)+\left(\mathbf{v}\cdot\nabla\right)\mathbf{A}\quad\Leftarrow\quad\text{particle's velocitiy}\quad\mathbf{v}=\mathbf{v}\left(t\right),\ \text{not}\ \text{a function of position}$$

$$\Rightarrow\quad\frac{d\,\mathbf{p}}{d\,t}=-q\left(\frac{\partial\,\mathbf{A}}{\partial\,t}+\left(\mathbf{v}\cdot\nabla\right)\mathbf{A}+\nabla\left(\Phi-\mathbf{v}\cdot\mathbf{A}\right)\right)$$

- The **convective derivative** of $\mathbf{A}$, $\frac{d\,\mathbf{A}}{d\,t}\equiv\frac{\partial\,\mathbf{A}}{\partial\,t}+\left(\mathbf{v}\cdot\nabla\right)\mathbf{A}$, also total derivative. It represents the time rate of change of $\mathbf{A}$ *at the (moving) location of the particle*.

- The change in $\mathbf{A}$ in $d\,t$ is

$$d\,\mathbf{A}=\mathbf{A}\left(\mathbf{r}+\mathbf{v}\,d\,t\,,\,t+d\,t\right)-\mathbf{A}\left(\mathbf{r}\,,\,t\right)=\frac{\partial\,\mathbf{A}}{\partial\,x}v_x\,d\,t+\frac{\partial\,\mathbf{A}}{\partial\,y}v_y\,d\,t+\frac{\partial\,\mathbf{A}}{\partial\,z}v_z\,d\,t+\frac{\partial\,\mathbf{A}}{\partial\,t}d\,t$$

$$\Rightarrow\quad\frac{d\,\mathbf{A}}{d\,t}=\frac{\partial\,\mathbf{A}}{\partial\,t}+\left(\mathbf{v}\cdot\nabla\right)\mathbf{A}$$

- As the particle moves, the potential it “feels” changes for 2 distinct reasons: one is because the potential varies with *time*, and second, because it is now in a new location, where $\mathbf{A}$ is different because of its variation in *space*.

- With the aid of the convective derivative, the Lorentz force law reads:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}(\mathbf{p}+q\,\mathbf{A})=-\nabla\left[q\,(\Phi-\mathbf{v}\cdot\mathbf{A})\right]$$

- The standard formula from mechanics: $\frac{\mathrm{d}\,\mathbf{p}}{\mathrm{d}\,t}=-\nabla\,U \quad \Leftarrow \quad \mathbf{p}:$ canonical momentum

- $\frac{\mathrm{d}\,\mathbf{p}_{\mathrm{can}}}{\mathrm{d}\,t}=-\nabla\,U_{\mathrm{vel}} \quad \Leftarrow \quad \mathbf{p}_{\mathrm{can}}=\mathbf{p}+q\,\mathbf{A}\,,\quad U_{\mathrm{vel}}=q\,(\Phi-\mathbf{v}\cdot\mathbf{A})$ velocitiy-dependent

- A similar argument gives the rate of change of the particle's *energy*:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}(T+q\,\Phi)=\frac{\partial}{\partial\,t}\left[q\,(\Phi-\mathbf{v}\cdot\mathbf{A})\right] \quad \Leftarrow \quad T=\frac{1}{2}m\,v^2 \text{ kinetic energy},\quad q\,\Phi \text{ potential energy}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,t}(T+q\,\Phi)=\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{1}{2}m\,v^2\right)+q\,\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,t}=m\,\mathbf{v}\cdot\frac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}\,t}+q\left(\frac{\partial\,\Phi}{\partial\,t}+(\mathbf{v}\cdot\nabla)\,\Phi\right)$$

$$=\mathbf{v}\cdot\mathbf{F}+q\left(\frac{\partial\,\Phi}{\partial\,t}+(\mathbf{v}\cdot\nabla)\,\Phi\right)$$

$$=q\,\mathbf{v}\cdot\left(-\nabla\,\Phi-\frac{\partial\,\mathbf{A}}{\partial\,t}+\mathbf{v}\times(\nabla\times\mathbf{A})\right)+q\left(\frac{\partial\,\Phi}{\partial\,t}+(\mathbf{v}\cdot\nabla)\,\Phi\right)$$

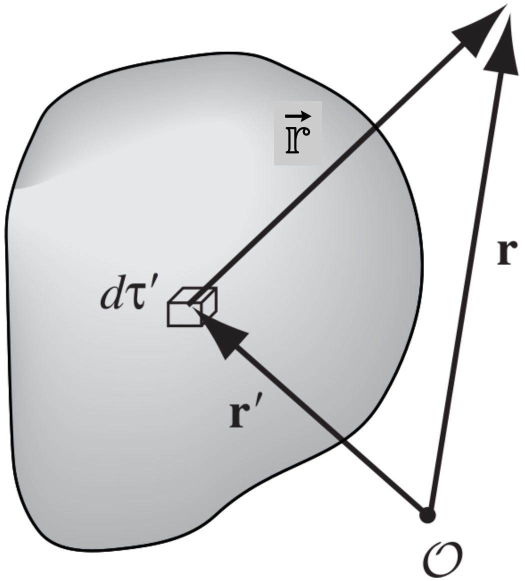
$$=q\left(\frac{\partial\,\Phi}{\partial\,t}-\mathbf{v}\cdot\frac{\partial\,\mathbf{A}}{\partial\,t}\right)=\frac{\partial}{\partial\,t}\left[q\,(\Phi-\mathbf{v}\cdot\mathbf{A})\right]$$

(The derivative on the right acts only on Φ and $\mathbf{A}$, not on $\mathbf{v}$.)

- We can interpret $\mathbf{A}$ as a kind of "potential momentum" per unit charge, just as Φ is potential energy per unit charge.

Continuous Distributions

Retarded Potentials



- In the static case, (3) reduces to 4 copies of Poisson's eqn,

$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}, \quad \nabla^2 \mathbf{A} = -\mu_0 \boldsymbol{J}$, with the familiar solutions

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau', \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau' \quad (4)$$

- Since EM "news" travels at the speed of light. In the *nonstatic* case, it's not the status of the source *right now* that matters, but its condition at some earlier time t_r (called the **retarded time**) when the "message" left: $t_r = t_r(t, \mathbf{r}, \mathbf{r}')$

- Since this message must travel a distance $\mathscr{r}$, the delay is $\frac{\mathscr{r}}{c}$: $t_r = t - \frac{\mathscr{r}}{c}$

- The natural generalization of the solution for *nonstatic* sources is therefore

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}}\,d\tau', \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\mathbf{r}', t_r)}{\mathscr{r}}\,d\tau'$$

- The integrands evaluated at the retarded time are called **retarded potentials**.

- The retarded potentials reduce properly to (4) in the static case, for which ρ and $\boldsymbol{J}$ are independent of time.

- To *prove* the generalized solution, we must show that they satisfy the inhomogeneous wave equation and meet the Lorenz condition.

- Don't apply the same logic to the *fields* you'll get entirely the *wrong* answer:

$$\mathbf{E}(\mathbf{r},t) \neq \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau', \quad \mathbf{B}(\mathbf{r},t) \neq \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t_r)\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

- In calculating the Laplacian of $\Phi(\mathbf{r},t)$, the crucial point to notice is that the integrand depends on $\mathbf{r}$ in 2 places: *explicitly*, in the denominator ($\mathscr{r}=|\mathbf{r}-\mathbf{r}'|$), and *implicitly*, through $t_r = t - \frac{\mathscr{r}}{c}$, in the numerator.

$$\nabla\Phi = \frac{1}{4\pi\epsilon_0}\int\left(\frac{\nabla\rho}{\mathscr{r}} + \rho\nabla\frac{1}{\mathscr{r}}\right)d\tau' \quad \Leftarrow \quad \nabla\rho = \dot{\rho}\,\nabla t_r = -\frac{\dot{\rho}}{c}\nabla\mathscr{r} = -\frac{\dot{\rho}}{c}\hat{\mathscr{r}}$$

$$= \frac{1}{4\pi\epsilon_0}\int\left(-\frac{\dot{\rho}}{c}\frac{\hat{\mathscr{r}}}{\mathscr{r}} - \rho\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau' \quad \Leftarrow \quad \nabla\frac{1}{\mathscr{r}} = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}, \quad \Uparrow \quad \dot{\rho} \equiv \frac{\partial\rho}{\partial t_r}$$

$$\Rightarrow \quad \nabla^2\Phi = \nabla\cdot\nabla\Phi = -\frac{1}{4\pi\epsilon_0}\int\left(\frac{\hat{\mathscr{r}}}{c\,\mathscr{r}}\cdot\nabla\dot{\rho} + \frac{\dot{\rho}}{c}\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}} + \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\nabla\rho + \rho\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

$$\nabla\dot{\rho} = -\frac{\ddot{\rho}}{c}\nabla\mathscr{r} = -\frac{\ddot{\rho}}{c}\hat{\mathscr{r}}, \quad \nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}} = \frac{1}{\mathscr{r}^2}, \quad \nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 4\pi\delta^3(\vec{\mathscr{r}}), \quad \ddot{\rho} \equiv \frac{\partial^2\rho}{\partial t_r^2}$$

$$\Rightarrow \quad \nabla^2\Phi = \frac{1}{4\pi\epsilon_0}\int\left(\frac{\ddot{\rho}}{c^2\,\mathscr{r}} - 4\pi\rho\,\delta^3(\vec{\mathscr{r}})\right)d\tau' = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \frac{1}{\epsilon_0}\rho(\mathbf{r},t) \quad \Leftarrow \quad \frac{\partial t_r}{\partial t} = 1$$

$$\Rightarrow \quad \nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\nabla A_i = \frac{\mu_0}{4\pi}\int\left(\frac{\nabla J_i}{\mathbb{r}} + J_i \nabla \frac{1}{\mathbb{r}}\right) d\tau' \quad \Leftarrow \quad \nabla J_i = \dot{J}_i \nabla t_r = -\frac{\dot{J}_i}{c}\nabla \mathbb{r} = -\frac{\dot{J}_i}{c}\hat{\mathbb{r}}$$

$$= \frac{\mu_0}{4\pi}\int\left(-\frac{\dot{J}_i}{c}\frac{\hat{\mathbb{r}}}{\mathbb{r}} - J_i \frac{\hat{\mathbb{r}}}{\mathbb{r}^2}\right) d\tau' \quad \Leftarrow \quad \nabla \frac{1}{\mathbb{r}} = -\frac{\hat{\mathbb{r}}}{\mathbb{r}^2}, \quad \Uparrow \quad \dot{J}_i \equiv \frac{\partial J_i}{\partial t_r}$$

$$\Rightarrow \quad \nabla^2 A_i = -\frac{\mu_0}{4\pi}\int\left(\frac{\hat{\mathbb{r}}}{c\,\mathbb{r}}\cdot\nabla \dot{J}_i + \dot{J}_i \nabla\cdot\frac{\hat{\mathbb{r}}}{c\,\mathbb{r}} + \frac{\hat{\mathbb{r}}}{\mathbb{r}^2}\cdot\nabla J_i + J_i \nabla\cdot\frac{\hat{\mathbb{r}}}{\mathbb{r}^2}\right) d\tau'$$

$$\nabla \dot{J}_i = -\frac{\ddot{J}_i}{c}\nabla \mathbb{r} = -\frac{\ddot{J}_i}{c}\hat{\mathbb{r}}, \quad \nabla\cdot\frac{\hat{\mathbb{r}}}{\mathbb{r}} = \frac{1}{\mathbb{r}^2}, \quad \nabla\cdot\frac{\hat{\mathbb{r}}}{\mathbb{r}^2} = 4\pi\,\delta^3(\vec{\mathbb{r}}), \quad \ddot{J}_i \equiv \frac{\partial^2 J_i}{\partial t_r^2}$$

$$\Rightarrow \quad \nabla^2 A_i = \frac{\mu_0}{4\pi}\int\left(\frac{\ddot{J}_i}{c^2\,\mathbb{r}} - 4\pi J_i\,\delta^3(\vec{\mathbb{r}})\right) d\tau' = \frac{1}{c^2}\frac{\partial^2 A_i}{\partial t^2} - \mu_0 J_i(\mathbf{r}, t)$$

$$\Rightarrow \quad \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \boldsymbol{J}$$

- This proof applies equally well to the **advanced potentials**,

$$\Phi_a(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}', t_a)}{\mathbb{r}}\,d\tau', \quad \mathbf{A}_a(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int\frac{\boldsymbol{J}(\mathbf{r}', t_a)}{\mathbb{r}}\,d\tau'$$

the charge & the current densities are evaluated at the **advanced time** $t_a = t + \frac{\mathbb{r}}{c}$

- Although the advanced potentials are entirely consistent with Maxwell's eqns, they violate the principle of **causality**. They suggest that the potentials now depend on what the charge & the current distribution will be at in the future.

- In a more general way,

$$\Phi(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{\mathbb{r}}\,\mathrm{d}\tau'=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t')}{\mathbb{r}}\,\delta(t'-t_r)\,\mathrm{d}\tau'\,\mathrm{d}t'$$

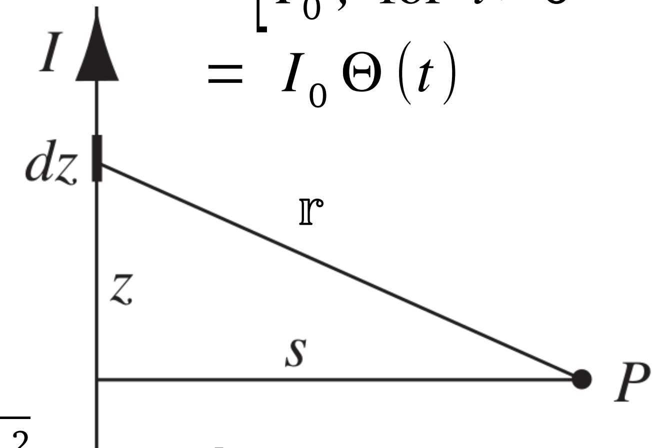
$$\Phi(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t')}{\mathbb{r}}\,\delta(t'-t_r)\,\mathrm{d}^4x' \quad\Leftarrow\quad t_r=t-\frac{\mathbb{r}}{c}=t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}$$

Similarly,

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\frac{\boldsymbol{J}(\mathbf{r}',t')}{\mathbb{r}}\,\delta(t'-t_r)\,\mathrm{d}^4x'$$

- The retarded potentials (automatically) satisfy the Lorenz gauge condition. [Problem 10.10]

Example 10.2: An infinite straight wire carries the current $I(t)=\begin{cases}0, & \text{for } t\le 0\\ I_0, & \text{for } t>0\end{cases} = I_0\,\Theta(t)$
Find the resulting electric and magnetic fields.



- The retarded vector potential at P is

$$\mathbf{A}(s,t)=\frac{\mu_0}{4\pi}\hat{\mathbf{z}}\int_{-\infty}^{+\infty}\frac{I(t_r)}{\mathscr{r}}\,\mathrm{d}z$$

- For $t<\frac{s}{c}$, the "news" has not yet reached P, and the potential is 0. For $t>\frac{s}{c}$, only the segment $|z|\le\sqrt{c^2t^2-s^2}$ contributes,

$$\Rightarrow\quad \mathbf{A}(s,t)=\hat{\mathbf{z}}\frac{\mu_0 I_0}{4\pi}2\int_0^{\sqrt{c^2t^2-s^2}}\frac{\mathrm{d}z}{\sqrt{s^2+z^2}}=\hat{\mathbf{z}}\frac{\mu_0 I_0}{2\pi}\ln\frac{\sqrt{s^2+z^2}+z}{s}\Bigg|_0^{\sqrt{c^2t^2-s^2}}$$

$$=\frac{\mu_0 I_0}{2\pi}\ln\frac{ct+\sqrt{c^2t^2-s^2}}{s}\hat{\mathbf{z}},\quad \text{for } s<ct \quad \Leftarrow$$

doesn't deal with the change at $t=0$

$$\Rightarrow\quad \mathbf{E}(s,t)=-\frac{\partial\mathbf{A}}{\partial t}=-\frac{c\,\mu_0 I_0}{2\pi\sqrt{c^2t^2-s^2}}\hat{\mathbf{z}}\quad(\text{for } \Phi=0)$$

$$\mathbf{B}(s,t)=\nabla\times\mathbf{A}=-\frac{\partial A_z}{\partial s}\hat{\boldsymbol{\phi}}=\frac{\mu_0 I_0}{2\pi s}\frac{ct}{\sqrt{c^2t^2-s^2}}\hat{\boldsymbol{\phi}}$$

$\Rightarrow$ return to static

$$\mathbf{E}=0,\quad \mathbf{B}=\frac{\mu_0 I_0}{2\pi s}\hat{\boldsymbol{\phi}}\quad \text{at } t\to\infty$$

Jefimenko's Equations

- $\Phi(\mathbf{r},t) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{\rho(\mathbf{r}',t_r)}{\mathbb{r}}\,d\tau'$, $\mathbf{A}(\mathbf{r},t) = \dfrac{\mu_0}{4\pi}\displaystyle\int \dfrac{\boldsymbol{J}(\mathbf{r}',t_r)}{\mathbb{r}}\,d\tau'$

- $\mathbf{E} = -\nabla\Phi - \dfrac{\partial\mathbf{A}}{\partial t}$, $\mathbf{B} = \nabla\times\mathbf{A}$. Since $t_r = t - \dfrac{\mathbb{r}}{c} = t - \dfrac{|\mathbf{r}-\mathbf{r}'|}{c}$, $\rho = \rho(\mathbf{r}',t_r)$

$$\nabla\Phi = -\frac{1}{4\pi\epsilon_0}\int\left(\frac{\dot{\rho}}{c}\frac{\hat{\mathbb{r}}}{\mathbb{r}} + \rho\frac{\hat{\mathbb{r}}}{\mathbb{r}^2}\right)d\tau', \quad \frac{\partial\mathbf{A}}{\partial t} = \frac{\mu_0}{4\pi}\int\frac{\dot{\boldsymbol{J}}}{\mathbb{r}}\,d\tau'$$

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\left(\frac{\rho(\mathbf{r}',t_r)}{\mathbb{r}^2}\hat{\mathbb{r}} + \frac{\dot{\rho}(\mathbf{r}',t_r)}{c\,\mathbb{r}}\hat{\mathbb{r}} - \frac{\dot{\boldsymbol{J}}(\mathbf{r}',t_r)}{c^2\,\mathbb{r}}\right)d\tau' \quad (5)$$

- This is the time-dependent generalization of Coulomb's law, to which it reduces in the static case.

- $\nabla\times\mathbf{A} = \dfrac{\mu_0}{4\pi}\displaystyle\int\left(\dfrac{\nabla\times\boldsymbol{J}}{\mathbb{r}} - \boldsymbol{J}\times\nabla\dfrac{1}{\mathbb{r}}\right)d\tau'$ ⬃ $t_r = t - \dfrac{\mathbb{r}}{c}$, ⇩ $\nabla\mathbb{r} = \hat{\mathbb{r}}$

$$\nabla\times\boldsymbol{J} = \sum_{i,j,k}\epsilon^{ijk}\hat{\mathbf{x}}_i\,\partial_j J_k = \sum_{i,j,k}\epsilon^{ijk}\hat{\mathbf{x}}_i\,\dot{J}_k\,\partial_j t_r = -\frac{1}{c}\sum_{i,j,k}\epsilon^{ijk}\hat{\mathbf{x}}_i\,\dot{J}_k\,\partial_j\mathbb{r} = \frac{\dot{\boldsymbol{J}}\times\hat{\mathbb{r}}}{c}$$

$$\Rightarrow \mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\left(\frac{\boldsymbol{J}(\mathbf{r}',t_r)}{\mathbb{r}^2} + \frac{\dot{\boldsymbol{J}}(\mathbf{r}',t_r)}{c\,\mathbb{r}}\right)\times\hat{\mathbb{r}}\,d\tau' \quad (6) \quad \Leftarrow \quad \nabla\frac{1}{\mathbb{r}} = -\frac{\hat{\mathbb{r}}}{\mathbb{r}^2}$$

- This is the time-dependent generalization of the Biot-Savart law, to which it reduces in the static case.

- (5) & (6) are the (causal) solutions to Maxwell's equations —**Jefimenko's equations**

- Jefimenko's equations are of limited utility, since it is easier to calculate the retarded potentials and differentiate them, rather than going directly to the fields.

- They help to clarify the puzzle: To get to the retarded *potentials*, all you do is replace t by t_r in the electrostatic and magnetostatic formulas, but in the case of the *fields* not only is t replaced by t_r, but completely new terms appear.

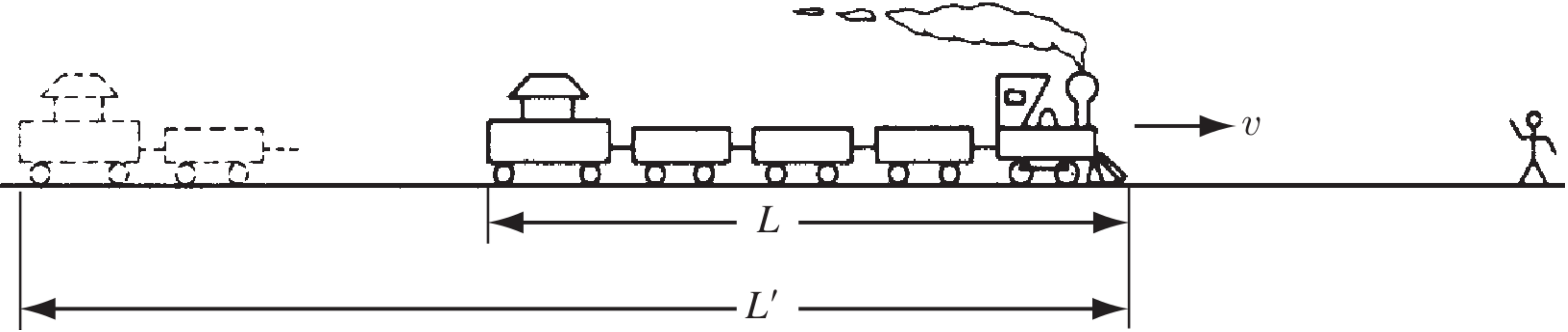
- They provide strong support for the quasistatic approximation.

Point Charges

Liénard-Wiechert Potentials

- To calculate the (retarded) potentials, $\Phi(\mathbf{r}, t)$ & $\mathbf{A}(\mathbf{r}, t)$, of a point charge q that is moving on a specified trajectory $\mathbf{x}(t_r) \equiv \text{position of } q \text{ at time } t_r \Leftarrow \text{trajectory}$
- A naïve reading of the formula $\Phi(\mathbf{r}, t) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{\rho(\mathbf{r}', t_r)}{\mathscr{r}}\, d\tau'$ suggests that the potential is $\Phi(\mathbf{r}, t) = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{\mathscr{r}}$. But this is wrong.
- It is true that for a point source the denominator $\mathscr{r}$ comes outside the integral, but what remains, $\displaystyle\int \rho(\mathbf{r}', t_r)\, d\tau'$, is *not* equal to the charge of the particle.
- To calculate the total charge of a configuration, you must integrate ρ over the entire distribution at *one instant of time*, but here the retardation, $t_r = t - \dfrac{\mathscr{r}}{c}$, obliges us to evaluate ρ at *different times* for different parts of the configuration.
- If the source is moving, this will give a distorted picture of the total charge, and this problem would disappear for *point* charges.
- A point charge is regarded as the limit of an extended charge, when the size goes to 0. For an extended particle, no matter how small, the retardation gives

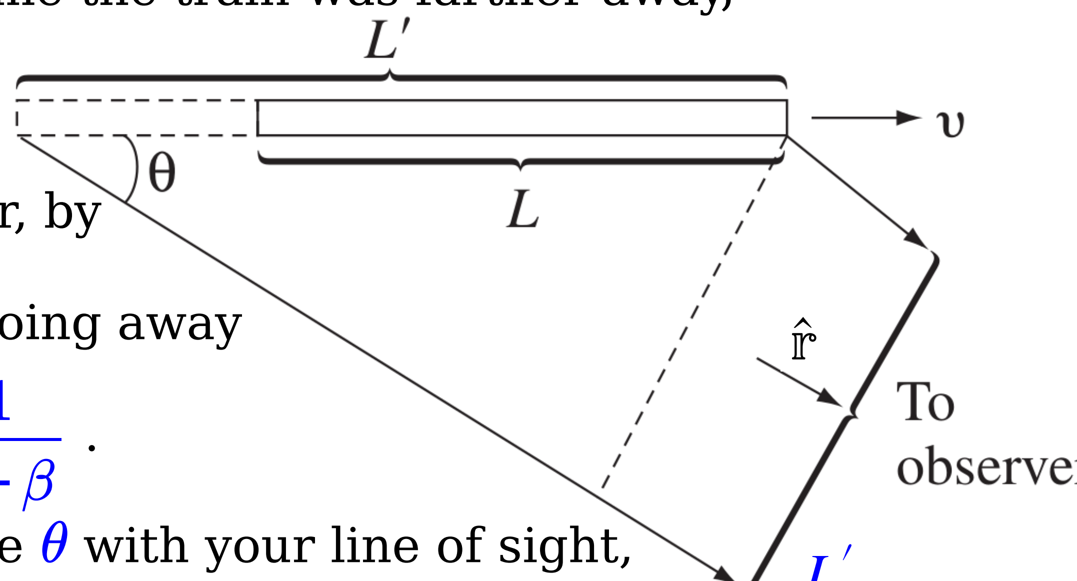
$$\int \rho(\mathbf{r}', t_r)\, d\tau' = \frac{q}{1 - \hat{\mathscr{r}} \cdot \mathbf{v}/c} = \frac{q}{1 - \boldsymbol{\beta} \cdot \hat{\mathscr{r}}} \Leftarrow \boldsymbol{\beta} = \frac{\mathbf{v}}{c}, \quad \beta = \frac{v}{c}$$



Proof: This is a purely *geometrical* effect.

- A train coming towards you looks longer than it really is, because the light you receive from the caboose left earlier than the light you receive simultaneously from the engine, and at that earlier time the train was farther away,

$$\frac{L'}{c} = \frac{L' - L}{v} \quad \Rightarrow \quad L' = \frac{L}{1-\beta}$$



- So approaching trains appear longer, by a factor $\dfrac{1}{1-\beta}$. By contrast, a train going away from you looks shorter, by a factor $\dfrac{1}{1+\beta}$.

- If the train's velocity makes an angle θ with your line of sight, the extra distance light from the caboose cover is $L'\cos\theta$. In the time $\dfrac{L'}{c}\cos\theta$, the train moves a distance $L'-L$: $\dfrac{L'\cos\theta}{c} = \dfrac{L'-L}{v} \quad \Rightarrow \quad L' = \dfrac{L}{1-\beta\cos\theta}$

- This effect does not distort the dimensions ⊥ the motion (the train's height & width) since there's no *motion* in the direction, they look the same distance apart.

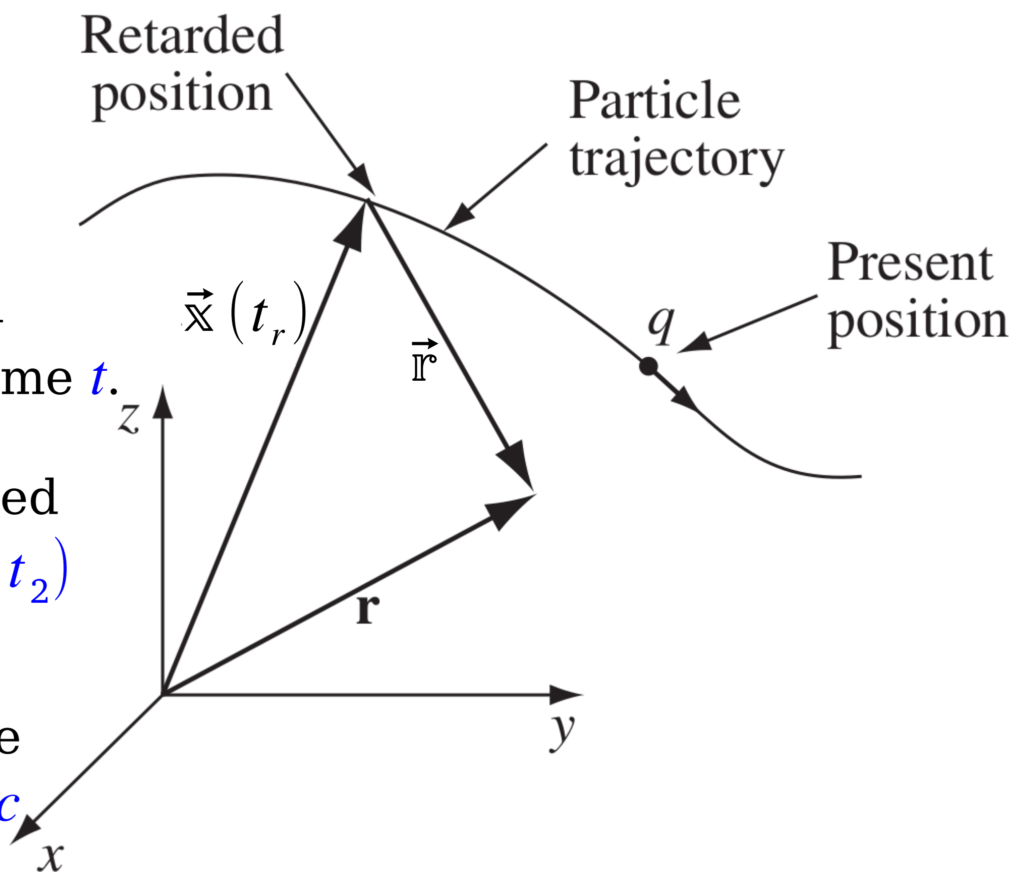
- The *apparent* volume τ' is related to the *actual* volume τ by $\tau' = \dfrac{\tau}{1 - \hat{\mathbb{r}} \cdot \boldsymbol{\beta}}$

- Whenever you do a retarded integral, in which the integrand is evaluated at the retarded time, the effective volume is modified by the factor $\dfrac{1}{1 - \hat{\mathbb{r}} \cdot \boldsymbol{\beta}}$.

- This correction factor makes no reference to the size of the particle, it is every bit as significant for a point charge as for an extended charge.

- For a point charge the retarded time is determined implicitly by

$$\mathbb{r} = c\left(t - t_r\right) \quad \Leftarrow \quad \vec{\mathbb{r}} = \mathbf{r} - \vec{\mathbb{x}}\left(t_r\right)$$



- At most *one* point on the trajectory is in communication with $\mathbf{r}$ at any particular time t.

- If there were 2 such points, with retarded times t_1 and t_2: $\mathbb{r}_1 = c\left(t - t_1\right), \quad \mathbb{r}_2 = c\left(t - t_2\right)$

$$\Rightarrow \quad \mathbb{r}_1 - \mathbb{r}_2 = c\left(t_2 - t_1\right)$$

so the average speed of the particle in the direction of the point $\mathbf{r}$ would have to be c

- Since no charged particle can travel at the speed of light, it follows that *only one retarded point contributes to the potentials, at any given moment*.

- $$\Phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 \mathbb{r}} \frac{1}{1 - \hat{\mathbb{r}} \cdot \boldsymbol{\beta}} = \frac{q}{4\pi\epsilon_0} \frac{1}{\mathbb{r} - \vec{\mathbb{r}} \cdot \boldsymbol{\beta}} \quad \Leftarrow \quad \vec{\mathbb{r}} = \mathbb{r}\, \hat{\mathbb{r}}$$

$$\boldsymbol{J} = \rho\, \mathbf{v} \quad \Rightarrow \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\rho(\mathbf{r}', t_r)\, \mathbf{v}(t_r)}{\mathbb{r}}\, \mathrm{d}\tau' = \frac{\mu_0}{4\pi} \frac{\mathbf{v}}{\mathbb{r}} \int \rho(\mathbf{r}', t_r)\, \mathrm{d}\tau'$$

$$\Rightarrow \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{q\, \mathbf{v}}{\mathbb{r}} \frac{1}{1 - \hat{\mathbb{r}} \cdot \boldsymbol{\beta}} = \frac{\boldsymbol{\beta}}{c} \Phi(\mathbf{r}, t) \quad \Leftarrow \quad \mu_0 \epsilon_0 = \frac{1}{c^2}$$

Liénard-Wiechert potentials for a moving point charge.

Selected problems: 3, 12, 18, 23, 28, 31

$$\Phi(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{\mathbb{r}}\,d\tau'=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t')}{\mathbb{r}}\,\delta(t'-t_r)\,d\tau'\,dt'$$

$$=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t')}{\mathbb{r}}\,\delta(t'-t_r)\,d^4x' \quad\Leftarrow\quad t_r=t-\frac{\mathbb{r}}{c}=t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}$$

For point charge $\rho(\mathbf{r}',t')=q\,\delta^3(\mathbf{r}'-\vec{\mathbb{x}}(t')) \quad\Leftarrow\quad \vec{\mathbb{x}}(t')$: trajectory at t'

$$\Rightarrow\quad \Phi(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int\frac{q}{\mathbb{r}}\,\delta^3(\mathbf{r}'-\vec{\mathbb{x}})\,\delta(t'-t_r)\,d^4x'=\frac{q}{4\pi\epsilon_0}\int\frac{\delta(t'-t_r)}{|\mathbf{r}-\vec{\mathbb{x}}|}\,dt'$$

$$=\frac{q}{4\pi\epsilon_0}\int\frac{\delta(t'-t+\mathbb{r}/c)}{\mathbb{r}}\,dt' \quad\Leftarrow\quad \vec{\mathbb{r}}=\mathbf{r}-\vec{\mathbb{x}}(t'),\quad \mathbb{r}=|\vec{\mathbb{r}}| \ \text{ now}$$

$$f(t')\equiv t'-t+\frac{\mathbb{r}}{c} \quad\Rightarrow\quad \frac{df}{dt'}=1-\hat{\mathbb{r}}\cdot\frac{\mathbf{v}}{c}=1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta} \quad\Leftarrow\quad \mathbf{v}=\frac{d\vec{\mathbb{x}}}{dt'},\quad \boldsymbol{\beta}=\frac{\mathbf{v}}{c}$$

$$\delta(f)=\frac{\delta(t'-t_s)}{\sum|df/dt'|_{t'=t_s}} \quad\Leftarrow\quad f(t_s)=0 \quad\Rightarrow\quad t_s=t_r=t-\frac{\mathbb{r}}{c}=t-\frac{|\mathbf{r}-\vec{\mathbb{x}}(t_s)|}{c}$$

$$\Rightarrow\quad \Phi(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0}\int\frac{\delta(t'-t+\mathbb{r}/c)}{\mathbb{r}}\,dt'=\frac{q}{4\pi\epsilon_0\mathbb{r}}\,\frac{1}{1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta}}$$

Similarly, $\mathbf{A}(\mathbf{r},t)=\dfrac{\mu_0}{4\pi}\,\dfrac{q\,\mathbf{v}}{\mathbb{r}}\,\dfrac{1}{1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta}}=\dfrac{1}{4\pi\epsilon_0 c}\,\dfrac{q\,\boldsymbol{\beta}}{\mathbb{r}-\vec{\mathbb{r}}\cdot\boldsymbol{\beta}}$

Definition of the Dirac delta function: $\int_{-\infty}^{+\infty} f(x)\,\delta(x-y)\,\mathrm{d}x = f(y)$

$$\int_{-\infty}^{+\infty} g(x)\,\delta(f(x))\,\mathrm{d}x = \int g\,\delta(f)\,\frac{\mathrm{d}x}{\mathrm{d}f}\,\mathrm{d}f \quad \Leftarrow \quad \text{if } u = f(x), \text{ then } x = f^{-1}(u) = f^{-1}(f)$$

$$= \int_{-}^{+} g\,\delta(f)\left|\frac{\mathrm{d}x}{\mathrm{d}f}\right|\mathrm{d}f = \int_{-}^{+} \frac{g\,\delta(f)}{\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|}\,\mathrm{d}f$$

$$f(x) = 0 \text{ if } x = z_i \text{ the root} \quad \Rightarrow \int_{-}^{+} \frac{g\,\delta(f)}{\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|}\,\mathrm{d}f = \int_{z_i-\epsilon}^{z_i+\epsilon} \frac{g(x)\,\delta(x-z_i)}{\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=z_i}}\,\mathrm{d}x$$

$$\Rightarrow \int_{-\infty}^{+\infty} g(x)\,\delta(f(x))\,\mathrm{d}x = \int_{-\infty}^{+\infty} g(x) \sum_i \frac{\delta(x-z_i)}{\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=z_i}}\,\mathrm{d}x \quad \Leftarrow \quad \text{if more than 1 root}$$

$$\Rightarrow \delta(f(x)) = \sum_i \frac{\delta(x-z_i)}{\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=z_i}} \quad \text{where } z_i\text{'s are roots.}$$

Example 10.3: Find the potentials of a point charge moving with constant velocity

$$\mathbf{v} = \text{constant} \quad \Rightarrow \quad \vec{\mathbb{x}}(t_r) = \mathbf{v}\, t_r \quad \Leftarrow \quad \mathbf{a} = 0$$

$$\Rightarrow \quad \mathbb{r} = |\mathbf{r} - \mathbf{v}\, t_r| = c\,(t - t_r) \;\Rightarrow\; r^2 - 2\, t_r\, \mathbf{r}\cdot\mathbf{v} + v^2\, t_r^2 = c^2\left(t^2 - 2\, t\, t_r + t_r^2\right)$$

$$\Rightarrow \quad t_r = \frac{c^2\, t - \mathbf{r}\cdot\mathbf{v} \pm \sqrt{\left(c^2\, t - \mathbf{r}\cdot\mathbf{v}\right)^2 + \left(c^2 - v^2\right)\left(r^2 - c^2\, t^2\right)}}{c^2 - v^2} \;\Rightarrow\; \lim_{v\to 0} t_r \to t \pm \frac{r}{c} \quad \text{choose } - \text{ sign}$$

$$\Rightarrow \quad t_r = \frac{c\, t - \mathbf{r}\cdot\boldsymbol{\beta} - \sqrt{\left(c\, t - \mathbf{r}\cdot\boldsymbol{\beta}\right)^2 + \left(1 - \beta^2\right)\left(r^2 - c^2\, t^2\right)}}{c\left(1 - \beta^2\right)}$$

$$\Rightarrow \quad \mathbb{r}\left(1 - \boldsymbol{\beta}\cdot\hat{\mathbb{r}}\right) = c\,(t - t_r)\left(1 - \boldsymbol{\beta}\cdot\frac{\mathbf{r} - \mathbf{v}\, t_r}{c\,(t - t_r)}\right) \quad \Leftarrow \quad \mathbb{r} = c\,(t - t_r), \quad \hat{\mathbb{r}} = \frac{\mathbf{r} - \mathbf{v}\, t_r}{c\,(t - t_r)}$$

$$= c\,(t - t_r) - \boldsymbol{\beta}\cdot\mathbf{r} + c\, \beta^2\, t_r = c\, t - \boldsymbol{\beta}\cdot\mathbf{r} - c\left(1 - \beta^2\right) t_r$$

$$= \sqrt{\left(c\, t - \boldsymbol{\beta}\cdot\mathbf{r}\right)^2 + \left(1 - \beta^2\right)\left(r^2 - c^2\, t^2\right)} \quad \Leftarrow \text{ using } t_r\text{'s expression.}$$

$$\Rightarrow \quad \begin{aligned} \Phi(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0}\,\frac{q}{\sqrt{\left(c\, t - \boldsymbol{\beta}\cdot\mathbf{r}\right)^2 + \left(1 - \beta^2\right)\left(r^2 - c^2\, t^2\right)}} \\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi}\,\frac{q\,\mathbf{v}}{\sqrt{\left(c\, t - \boldsymbol{\beta}\cdot\mathbf{r}\right)^2 + \left(1 - \beta^2\right)\left(r^2 - c^2\, t^2\right)}} \end{aligned}$$

The Fields of a Moving Point Charge

- Calculate the electric and magnetic fields of a point charge in arbitrary motion:

$$\Phi(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\frac{q}{\mathbb{r}-\boldsymbol{\beta}\cdot\vec{\mathbb{r}}},\quad \mathbf{A}(\mathbf{r},t)=\frac{\boldsymbol{\beta}}{c}\Phi(\mathbf{r},t)\quad\Rightarrow\quad \begin{array}{l}\mathbf{E}=-\nabla\Phi-\partial_t\mathbf{A}\\ \mathbf{B}=\nabla\times\mathbf{A}\end{array}$$

$$c\,\boldsymbol{\beta}(t_r)=\mathbf{v}(t_r)=\dot{\vec{\mathbb{x}}}(t_r)=\frac{d\,\vec{\mathbb{x}}}{d\,t_r},\quad \mathbb{r}=c\,(t-t_r)\quad\Leftarrow\quad \vec{\mathbb{r}}=\mathbf{r}-\vec{\mathbb{x}}(t_r)\quad\Rightarrow\quad \nabla\,\mathbb{r}=-c\,\nabla\,t_r$$

$$\Rightarrow\quad \nabla\Phi=-\frac{q}{4\pi\epsilon_0}\frac{\nabla(\mathbb{r}-\boldsymbol{\beta}\cdot\vec{\mathbb{r}})}{(\mathbb{r}-\boldsymbol{\beta}\cdot\vec{\mathbb{r}})^2}=\frac{q}{4\pi\epsilon_0\,\mathbb{r}^2}\frac{c\,\nabla\,t_r+\nabla(\boldsymbol{\beta}\cdot\vec{\mathbb{r}})}{(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})^2}$$

$$\nabla(\boldsymbol{\beta}\cdot\vec{\mathbb{r}})=\nabla\sum_i(\beta^i\,\mathbb{r}_i)=\sum_i(\mathbb{r}_i\,\nabla\,\beta^i+\beta^i\,\nabla\,\mathbb{r}_i)\quad\Leftarrow\quad \vec{\mathbb{r}}=\mathbf{r}-\vec{\mathbb{x}},\quad \mathbf{r}=\sum x^k\,\hat{\mathbf{x}}_k$$

$$=\sum_i\left(\mathbb{r}_i\frac{d\,\beta^i}{d\,t_r}\nabla\,t_r+\beta^i\,\nabla(x_i-\mathbb{x}_i)\right)=(\vec{\mathbb{r}}\cdot\dot{\boldsymbol{\beta}})\,\nabla\,t_r+\boldsymbol{\beta}-(\boldsymbol{\beta}\cdot\mathbf{v})\,\nabla\,t_r$$

$$=\boldsymbol{\beta}+(\vec{\mathbb{r}}\cdot\dot{\boldsymbol{\beta}}-c\,\beta^2)\,\nabla\,t_r\quad\Leftarrow\quad \mathbf{v}=c\,\boldsymbol{\beta},\quad c\,\dot{\boldsymbol{\beta}}=\frac{d\,\mathbf{v}}{d\,t_r}=\mathbf{a}:$$ *acceleration* at the retarded time

$$\Rightarrow\quad \nabla\Phi=\frac{q}{4\pi\epsilon_0\,\mathbb{r}^2}\frac{\boldsymbol{\beta}+(c-c\,\beta^2+\vec{\mathbb{r}}\cdot\dot{\boldsymbol{\beta}})\,\nabla\,t_r}{(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})^2}$$

⇩ according to the above result

$$-c\,\nabla\,t_r=\nabla\,\mathbb{r}=\nabla\sqrt{\vec{\mathbb{r}}\cdot\vec{\mathbb{r}}}=\frac{\nabla(\vec{\mathbb{r}}\cdot\vec{\mathbb{r}})}{2\sqrt{\vec{\mathbb{r}}\cdot\vec{\mathbb{r}}}}=\frac{1}{\mathbb{r}}\sum_i(\mathbb{r}^i\,\nabla\,\mathbb{r}_i)=\frac{\vec{\mathbb{r}}-(\vec{\mathbb{r}}\cdot\mathbf{v})\,\nabla\,t_r}{\mathbb{r}}$$

$$\Rightarrow \boxed{\nabla t_r = \frac{-\hat{\mathbb{r}}}{c\,(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})}} \Rightarrow \nabla\Phi = \frac{q}{4\pi\epsilon_0\mathbb{r}^2}\frac{c\,(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})\,\boldsymbol{\beta}-(c-c\,\beta^2+\vec{\mathbb{r}}\cdot\dot{\boldsymbol{\beta}})\,\hat{\mathbb{r}}}{c\,(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})^3}$$

$$\frac{\partial\vec{\mathbb{r}}}{\partial t} = \frac{\partial}{\partial t}[\mathbf{r}-\vec{\mathbb{x}}(t_r)] = -\mathbf{v}\frac{\partial t_r}{\partial t} \quad \Rightarrow \quad \frac{\partial\mathbb{r}}{\partial t} = \frac{\partial}{\partial t}\sqrt{\vec{\mathbb{r}}\cdot\vec{\mathbb{r}}} = \hat{\mathbb{r}}\cdot\frac{\partial\vec{\mathbb{r}}}{\partial t} = -c\,(\hat{\mathbb{r}}\cdot\boldsymbol{\beta})\frac{\partial t_r}{\partial t}$$

$$\mathbb{r} = c\,(t-t_r) \quad \Rightarrow \quad \frac{\partial\mathbb{r}}{\partial t} = c - c\,\frac{\partial t_r}{\partial t} \quad \Rightarrow \boxed{\frac{\partial t_r}{\partial t} = \frac{1}{1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta}}} \Rightarrow \boxed{\frac{\partial\mathbb{r}}{\partial t} = -\frac{c\,\hat{\mathbb{r}}\cdot\boldsymbol{\beta}}{1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta}}}$$

$$\Rightarrow \boxed{\frac{\partial\vec{\mathbb{r}}}{\partial t} = -\frac{c\,\boldsymbol{\beta}}{1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta}}}, \quad \frac{\partial\boldsymbol{\beta}}{\partial t} = \frac{\mathrm{d}\,\boldsymbol{\beta}}{\mathrm{d}\,t_r}\frac{\partial t_r}{\partial t} = \frac{\dot{\boldsymbol{\beta}}}{1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta}}$$

$$\frac{\partial}{\partial t}\frac{1}{\mathbb{r}-\vec{\mathbb{r}}\cdot\boldsymbol{\beta}} = -\frac{1}{\mathbb{r}^2(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})^2}\left(\frac{\partial\mathbb{r}}{\partial t}-\vec{\mathbb{r}}\cdot\frac{\partial\boldsymbol{\beta}}{\partial t}-\boldsymbol{\beta}\cdot\frac{\partial\vec{\mathbb{r}}}{\partial t}\right) = \frac{c\,\hat{\mathbb{r}}\cdot\boldsymbol{\beta}+\vec{\mathbb{r}}\cdot\dot{\boldsymbol{\beta}}-c\,\beta^2}{\mathbb{r}^2(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})^3}$$

$$\Rightarrow \frac{\partial\mathbf{A}}{\partial t} = \frac{\Phi}{c}\frac{\partial\boldsymbol{\beta}}{\partial t}+\frac{\boldsymbol{\beta}}{c}\frac{\partial\Phi}{\partial t} = \frac{q}{4\pi\epsilon_0}\left(\frac{\dot{\boldsymbol{\beta}}}{c\,\mathbb{r}\,(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})^2}+\frac{\boldsymbol{\beta}}{c}\frac{c\,\hat{\mathbb{r}}\cdot\boldsymbol{\beta}+\vec{\mathbb{r}}\cdot\dot{\boldsymbol{\beta}}-c\,\beta^2}{\mathbb{r}^2(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})^3}\right)$$

$$= \frac{q}{4\pi\epsilon_0\mathbb{r}^2}\frac{(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})(\mathbb{r}\,\dot{\boldsymbol{\beta}}-c\,\boldsymbol{\beta})+(c-c\,\beta^2+\vec{\mathbb{r}}\cdot\dot{\boldsymbol{\beta}})\,\boldsymbol{\beta}}{c\,(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})^3}$$

$$\Rightarrow \boxed{\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0\mathbb{r}^2}\frac{c\,(1-\beta^2)\,(\hat{\mathbb{r}}-\boldsymbol{\beta})+\vec{\mathbb{r}}\times[(\hat{\mathbb{r}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{c\,(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})^3}} \Leftarrow -\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\nabla\times\boldsymbol{\beta}=\sum_{i,j,k}\epsilon^{ijk}\,\hat{\mathbf{x}}_i\,\partial_j\,\beta_k=\sum_{i,j,k}\epsilon^{ijk}\,\hat{\mathbf{x}}_i\,\frac{\partial t_r}{\partial x^j}\,\frac{\mathrm{d}\beta_k}{\mathrm{d}t_r}=\nabla t_r\times\dot{\boldsymbol{\beta}}=\frac{\dot{\boldsymbol{\beta}}\times\hat{\mathbb{r}}}{c\,(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})}$$

$$\Rightarrow\quad\nabla\times\mathbf{A}=\nabla\times\frac{\boldsymbol{\beta}\,\Phi}{c}=\frac{\Phi}{c}\,\nabla\times\boldsymbol{\beta}+\nabla\,\Phi\times\frac{\boldsymbol{\beta}}{c}$$

$$=\frac{q}{4\pi\epsilon_0}\left(\frac{\dot{\boldsymbol{\beta}}\times\hat{\mathbb{r}}}{c^2\,\mathbb{r}\,(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})^2}+\frac{c\,(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})\,\boldsymbol{\beta}-(c-c\,\beta^2+\vec{\mathbb{r}}\cdot\dot{\boldsymbol{\beta}})\,\hat{\mathbb{r}}}{c\,\mathbb{r}^2\,(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})^3}\times\frac{\boldsymbol{\beta}}{c}\right)$$

$$=\frac{q}{4\pi\epsilon_0\,\mathbb{r}^2}\,\frac{[c\,(1-\beta^2)\,\boldsymbol{\beta}+(\vec{\mathbb{r}}\cdot\dot{\boldsymbol{\beta}})\,\boldsymbol{\beta}+\mathbb{r}\,(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})\,\dot{\boldsymbol{\beta}}]\times\hat{\mathbb{r}}}{c^2\,(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})^3}$$

$$[c\,(1-\beta^2)\,\boldsymbol{\beta}+(\vec{\mathbb{r}}\cdot\dot{\boldsymbol{\beta}})\,\boldsymbol{\beta}+\mathbb{r}\,(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})\,\dot{\boldsymbol{\beta}}]\times\hat{\mathbb{r}}$$

$$=\hat{\mathbb{r}}\times[c\,(1-\beta^2)\,(\hat{\mathbb{r}}-\boldsymbol{\beta})+(\vec{\mathbb{r}}\cdot\dot{\boldsymbol{\beta}})\,(\hat{\mathbb{r}}-\boldsymbol{\beta})-\vec{\mathbb{r}}\cdot(\hat{\mathbb{r}}-\boldsymbol{\beta})\,\dot{\boldsymbol{\beta}}]\quad\Leftarrow\quad\mathbb{r}\,(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta})=\vec{\mathbb{r}}\cdot(\hat{\mathbb{r}}-\boldsymbol{\beta})$$

$$=\hat{\mathbb{r}}\times[c\,(1-\beta^2)\,(\hat{\mathbb{r}}-\boldsymbol{\beta})+\vec{\mathbb{r}}\times((\hat{\mathbb{r}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}})]$$

$$\Rightarrow\quad\mathbf{B}(\mathbf{r},t)=\frac{\hat{\mathbb{r}}}{c}\times\mathbf{E}(\mathbf{r},t)\quad\Leftarrow\quad\nabla\times\mathbf{A}$$

- Evidently *the magnetic field of a point charge is always perpendicular to the electric field, and to the vector from the retarded point*.

- The 1st term in $\mathbf{E}$ [the one with $c\left(1-\beta^2\right)\left(\hat{\mathbb{r}}-\boldsymbol{\beta}\right)$] falls off as the inverse square of the distance from the particle.

- If the velocity and acceleration are both 0, this term alone survives and reduces to the old electrostatic result $\mathbf{E}=\dfrac{1}{4\pi\epsilon_0}\dfrac{q}{\mathbb{r}^2}\hat{\mathbb{r}}$. So the 1st term in $\mathbf{E}$ is called the **generalized Coulomb field**.

- It does not depend on the acceleration, it is also known as the **velocity field**.

- The 2nd term [the one with $\vec{\mathbb{r}}\times\left[\left(\hat{\mathbb{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]$] falls off as the inverse 1st power of $\mathbb{r}$ and is therefore dominant at large distances.

- It is this term that is responsible for EM radiation; so it is called the **radiation field**—or, since it is proportional to a, the **acceleration field**.

- With the $\mathbf{E}$ and $\mathbf{B}$ fields, the Lorentz force law determines the resulting force:

$$\Rightarrow\quad \mathbf{F}=\frac{q\,Q}{4\pi\epsilon_0\,\mathbb{r}^2}\,\frac{1}{c^2\left(1-\hat{\mathbb{r}}\cdot\boldsymbol{\beta}\right)^3}\Bigg(c^2\left(1-\beta^2\right)\left(\hat{\mathbb{r}}-\boldsymbol{\beta}\right)+c\,\vec{\mathbb{r}}\times\left[\left(\hat{\mathbb{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]$$

$$+\mathbf{V}\times\left[\vec{\mathbb{r}}\times\left(c\,\left(1-\beta^2\right)\left(\hat{\mathbb{r}}-\boldsymbol{\beta}\right)+\;\vec{\mathbb{r}}\times\left(\left(\hat{\mathbb{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right)\right)\right]\Bigg)$$

where $\mathbf{V}$ is the velocity of Q

Example 10.4: Calculate the electric and magnetic fields of a point charge moving with constant velocity.

- $\dot{\boldsymbol{\beta}} = \frac{\mathbf{a}}{c} = 0 \quad \Rightarrow \quad \mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{(1-\beta^2)(\hat{\mathbb{r}} - \boldsymbol{\beta})}{\mathbb{r}^2 (1 - \hat{\mathbb{r}} \cdot \boldsymbol{\beta})^3} \quad \Leftarrow \quad \boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c}, \quad \beta = \frac{v}{c}, \quad \vec{\mathbb{r}} = \mathbf{r} - \vec{\mathbb{x}}(t_r), \quad \mathbb{r} = |\vec{\mathbb{r}}| = c(t - t_r)$

$$\mathbb{r}(\hat{\mathbb{r}} - \boldsymbol{\beta}) = \vec{\mathbb{r}} - \mathbb{r}\,\mathbf{v}/c = \mathbf{r} - \mathbf{v}\,t_r - (t - t_r)\,\mathbf{v} = \mathbf{r} - \mathbf{v}\,t \quad \Leftarrow \quad \vec{\mathbb{x}}(t_r) = \mathbf{v}\,t_r$$

$$\mathbb{r}(1 - \hat{\mathbb{r}} \cdot \boldsymbol{\beta}) = \vec{\mathbb{r}} \cdot (\hat{\mathbb{r}} - \boldsymbol{\beta}) = \sqrt{(c\,t - \mathbf{r} \cdot \boldsymbol{\beta})^2 + (1 - \beta^2)(r^2 - c^2 t^2)} \quad \Leftarrow \quad \text{using Ex. 10.3}$$

$$= R\sqrt{1 - \beta^2 \sin^2\theta} \quad \Leftarrow \quad \mathbf{R} \equiv \mathbf{r} - \mathbf{v}\,t, \quad \theta: \text{ angle between } \mathbf{R} \text{ \& } \mathbf{v} \quad \Leftarrow \quad \text{Problem 10.16}$$

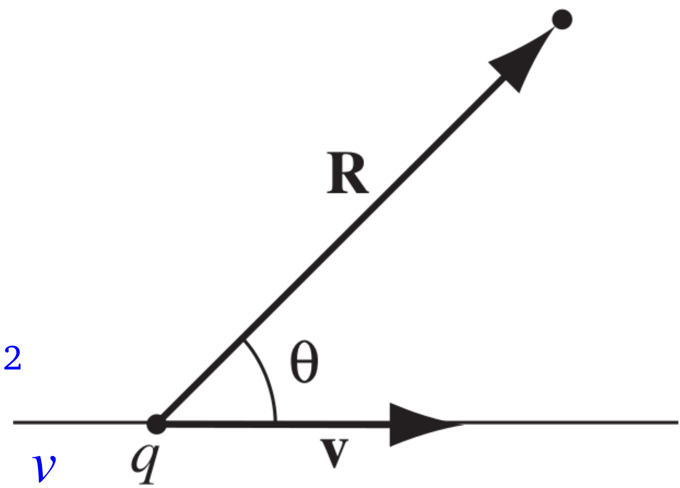
$$\Rightarrow \boxed{\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1 - \beta^2}{(1 - \beta^2 \sin^2\theta)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2}}$$

- Notice that $\mathbf{E}$ points along the line from the *present* position of the particle. This is a real coincidence, since the “message” came from the retarded position.

- Because of $\sin^2\theta$ in the denominator, the field of a fast-moving charge is flattened out like a pancake in the direction ⊥ the motion.

- In the forward/backward directions $\mathbf{E}$ is reduced by a factor $(1-\beta^2)$ relative to the field of a charge at rest; in the perpendicular direction it is enhanced by a factor $\frac{1}{\sqrt{1-\beta^2}}$.

Problem 10.16



$$\mathbf{R} \equiv \mathbf{r} - \mathbf{v}\,t \quad \Rightarrow \quad \mathbf{r} = \mathbf{R} + \mathbf{v}\,t \quad \Rightarrow \quad r^2 = R^2 + 2\,\mathbf{R}\cdot\mathbf{v}\,t + v^2\,t^2$$

$$\Rightarrow \quad (c\,t - \mathbf{r}\cdot\boldsymbol{\beta})^2 + (1-\beta^2)(r^2 - c^2\,t^2) \quad \Leftarrow \quad \boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c}, \quad \beta = \frac{v}{c}$$

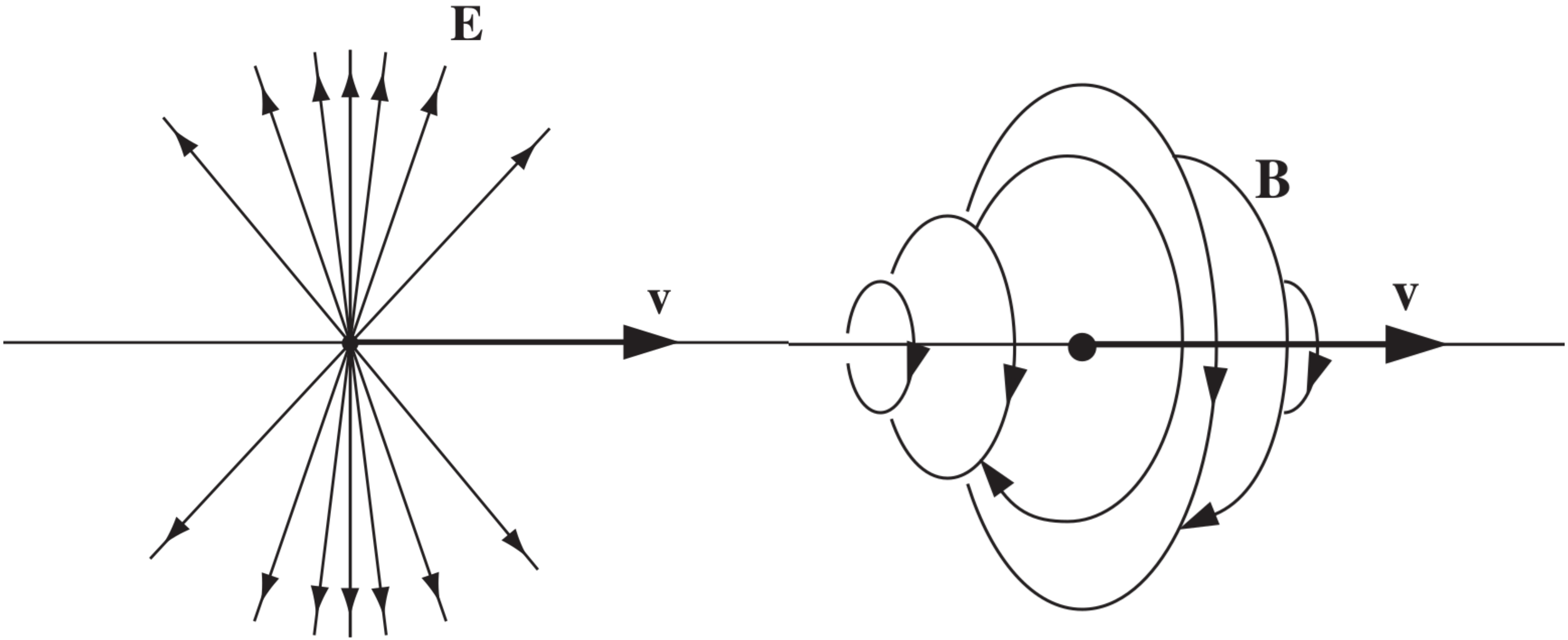
$$= (c\,t - \mathbf{R}\cdot\boldsymbol{\beta} - c\,\beta^2\,t)^2 + (1-\beta^2)(R^2 + 2\,c\,\mathbf{R}\cdot\boldsymbol{\beta}\,t + c^2\,\beta^2\,t^2 - c^2\,t^2)$$

$$= [c\,(1-\beta^2)\,t - \mathbf{R}\cdot\boldsymbol{\beta}]^2 + (1-\beta^2)\,[R^2 + 2\,c\,\mathbf{R}\cdot\boldsymbol{\beta}\,t - c^2\,(1-\beta^2)\,t^2]$$

$$= c^2\,(1-\beta^2)^2\,t^2 - 2\,c\,(1-\beta^2)\,\mathbf{R}\cdot\boldsymbol{\beta}\,t + R^2\,\beta^2\cos^2\theta$$

$$+ (1-\beta^2)\,R^2 + 2\,c\,(1-\beta^2)\,\mathbf{R}\cdot\boldsymbol{\beta}\,t - c^2\,(1-\beta^2)^2\,t^2$$

$$= (1-\beta^2)\,R^2 + R^2\,\beta^2\cos^2\theta = R^2 - \beta^2\,R^2\sin^2\theta = R^2\,(1-\beta^2\sin^2\theta)$$



- $\hat{\mathbb{r}} = \dfrac{\mathbf{r} - \mathbf{v}\, t_r}{\mathbb{r}} = \dfrac{\mathbf{r} - \mathbf{v}\, t + \mathbf{v}\,(t - t_r)}{\mathbb{r}} = \dfrac{\mathbf{R}}{\mathbb{r}} + \boldsymbol{\beta} \;\Rightarrow\; \boxed{\mathbf{B} = \dfrac{\hat{\mathbb{r}}}{c} \times \mathbf{E} = \dfrac{\boldsymbol{\beta}}{c} \times \mathbf{E}}$
- Lines of $\mathbf{B}$ circle around the charge.
- When $v^2 \ll c^2$ they reduce to $\mathbf{E}(\mathbf{r}, t) \approx \dfrac{1}{4\pi\epsilon_0} \dfrac{q}{R^2} \hat{\mathbf{R}}, \quad \mathbf{B}(\mathbf{r}, t) \approx \dfrac{\mu_0}{4\pi} \dfrac{q}{R^2} \mathbf{v} \times \hat{\mathbf{R}}$
- The 1st is essentially Coulomb's law, and the 2nd is the "Biot-Savart law for a point charge."