

# Chapter 29 Magnetic Fields Due to Currents

## Calculating the Magnetic Field Due to a Current

● Wish to calculate the magnetic field produced at  $P$  by a typical *current-length element*  $i d\vec{s}$ .

● Magnetic fields can be superimposed to find a net field.

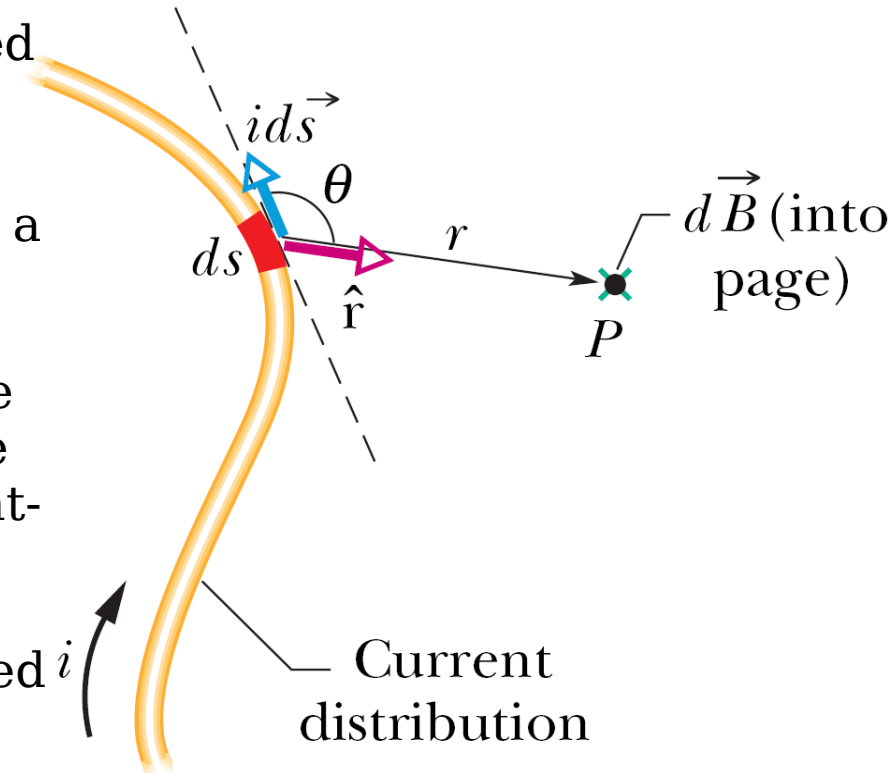
● This calculation is more challenging than the process associated with electric fields because whereas a charge element is a scalar, a current-length element is a vector.

● The magnitude of the magnetic field produced at point  $P$  by a current-length element is

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{i d\vec{s} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi} \frac{i d\vec{s} \times \vec{r}}{r^3} \quad \text{Biot-Savart law}$$

$\mu_0$  is the permeability constant:  $\mu_0 = 4\pi \times 10^{-7} \text{ T}\cdot\text{m}/\text{A} \approx 1.26 \times 10^{-6} \text{ T}\cdot\text{m}/\text{A}$

● This vector equation is **Biot-Savart law**, which is experimentally deduced, an inverse-square law.



## Magnetic Field Due to a Current in a Long Straight Wire

- The magnitude of the magnetic field at a perpendicular distance  $R$  from a long (infinite) straight wire carrying a current  $i$  is

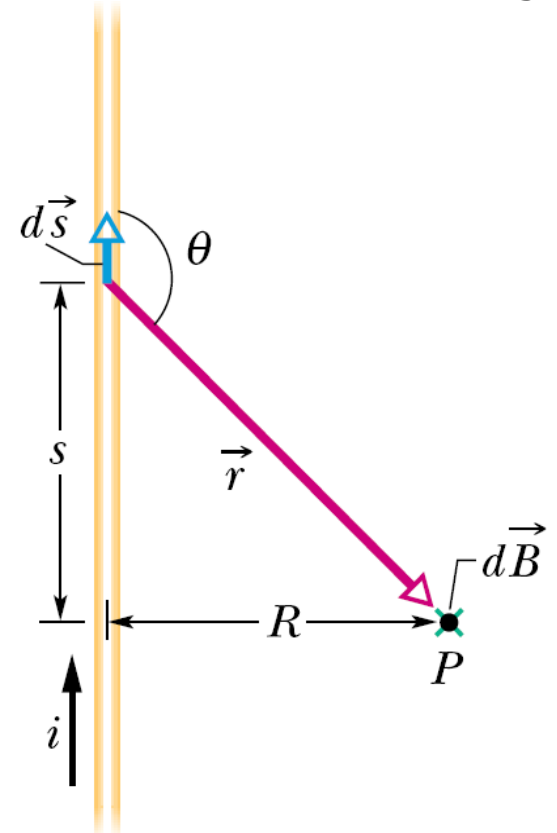
$$B = \frac{\mu_0 i}{2 \pi R} \quad \text{long straight wire}$$

### proof

- The magnitude of the differential magnetic field produced at  $P$  by the current-length element located a distance  $r$  from  $P$  is

$$d B = \frac{\mu_0}{4 \pi} \frac{i d s \sin \theta}{r^2}$$

- The direction of  $d \vec{B}$  is that of the vector  $d \vec{s} \times \vec{r}$ , namely, directly into the page.



- To find the magnitude of the *total* magnetic field at  $P$ ,

$$B = \int_{-\infty}^{\infty} d B = \frac{\mu_0}{4 \pi} \int_{-\infty}^{\infty} \frac{i \sin \theta d s}{r^2} = 2 \frac{\mu_0}{4 \pi} \int_0^{\infty} \frac{i \sin \theta d s}{r^2} = \frac{\mu_0 i}{2 \pi} \int_0^{\infty} \frac{\sin \theta d s}{r^2}$$

$$r = R \csc (\pi - \theta) = R \csc \theta \quad \text{and} \quad s = R \cot (\pi - \theta) = -R \cot \theta$$

$$\Rightarrow d s = -d (R \cot \theta) = R \csc^2 \theta d \theta$$

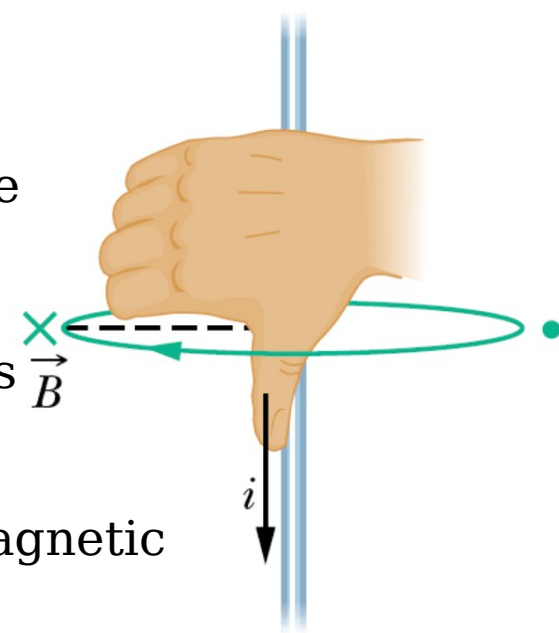
$$\Rightarrow \int_0^{\infty} \frac{\sin \theta d s}{r^2} = \int_{\pi/2}^{\pi} \frac{\sin \theta (R \csc^2 \theta d \theta)}{R^2 \csc^2 \theta} = \frac{1}{R} \int_{\pi/2}^{\pi} \sin \theta d \theta = \frac{1}{R} (-\cos \theta) \Big|_{\pi/2}^{\pi} = \frac{1}{R}$$

$$\Rightarrow B = \frac{\mu_0 i}{2 \pi} \int_0^\infty \frac{\sin \theta \, ds}{r^2} = \frac{\mu_0 i}{2 \pi R}$$

- The field magnitude  $B$  depends only on the current and the perpendicular distance  $R$  of the point from the wire.

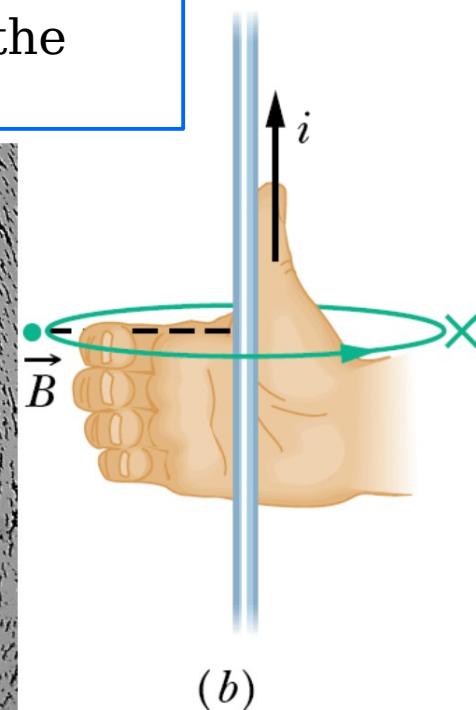
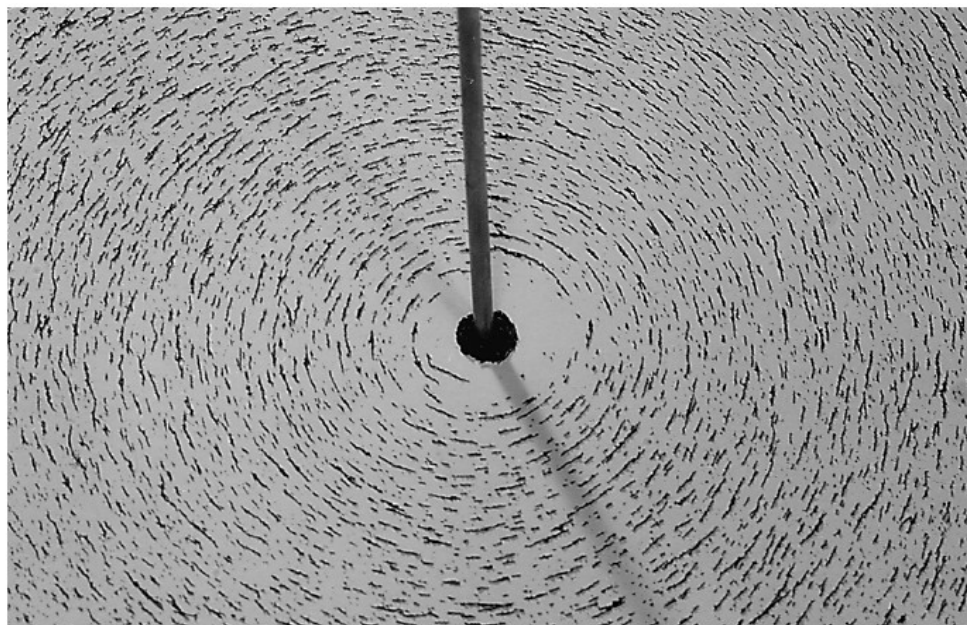
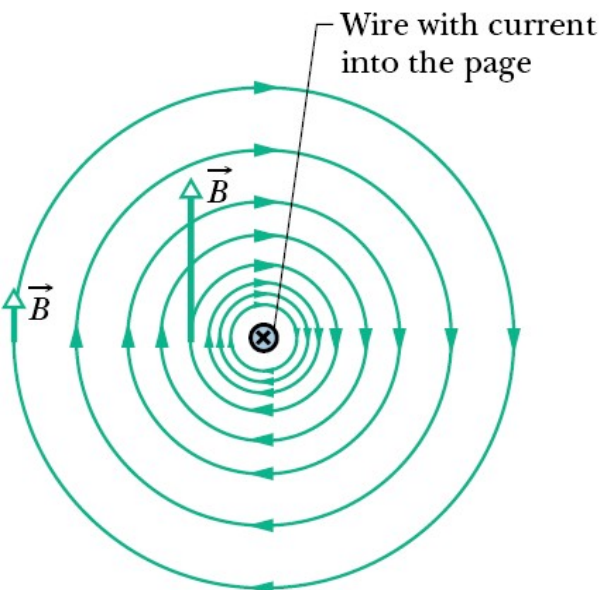
- The field lines of the magnetic field form concentric circles  $\vec{B}$  around the wire.

- A simple right-hand rule for finding the direction of the magnetic field set up by a current-length element,



*Curled-straight Right-hand rule:* Grasp the element in your right hand with your extended thumb pointing in the direction of the current. Your fingers will then naturally curl around in the direction of the magnetic field lines due to that element.

(a)



(b)

- The magnetic field at any point is *tangent to a magnetic field line*; and it is *perpendicular to a dashed radial line connecting the point and the current*.

- Note that the magnetic field at  $P$  due to either the lower half or the upper half of the infinite wire is half this value

$$B = \frac{\mu_0 i}{4 \pi R} \quad \text{semi-infinite straight wire}$$

### Magnetic Field Due to a Current in a Circular Arc of Wire

- As the figure shows, no matter where the element is located on the wire, the angle  $\theta$  between the vectors  $d\vec{s}$  and  $\vec{r}$  is  $\pi/2$ ; also,  $r = R$ . Thus

$$dB = \frac{\mu_0}{4 \pi} \frac{i ds \sin(\pi/2)}{R^2} = \frac{\mu_0}{4 \pi} \frac{i ds}{R^2}$$

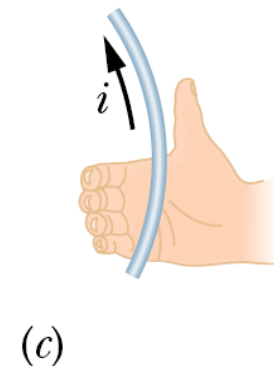
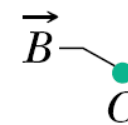
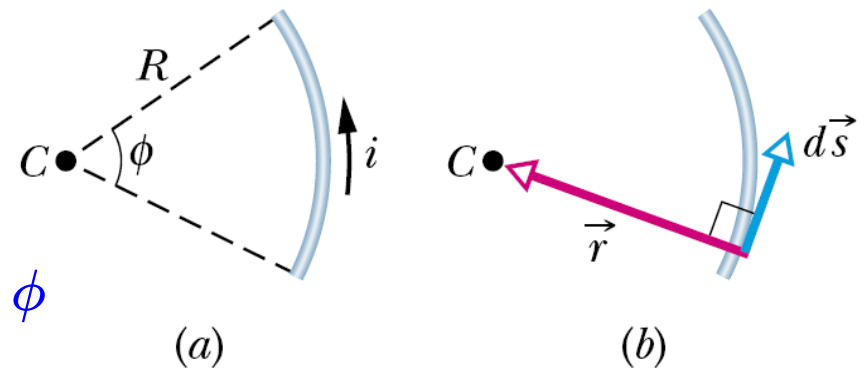
$$ds = R d\phi$$

$$\Rightarrow B = \int dB = \frac{\mu_0}{4 \pi} \int_0^\phi \frac{i R d\phi}{R^2} = \frac{\mu_0}{4 \pi R} \int_0^\phi d\phi$$

$$\Rightarrow B = \frac{\mu_0 i \phi}{4 \pi R} \quad \text{at center of circular arc}$$

- For the magnitude of the magnetic field at the center of a full circle of current,

$$B = \frac{\mu_0 i (2 \pi)}{4 \pi R} = \frac{\mu_0 i}{2 R} \quad \text{at center of full circle}$$



## Magnetic Field Due to Brain Activity

- *Magnetoencephalography* (MEG): a procedure in which magnetic fields of a person's brain are monitored as the person performs a task.

- Let's estimate the magnitude of such a field,

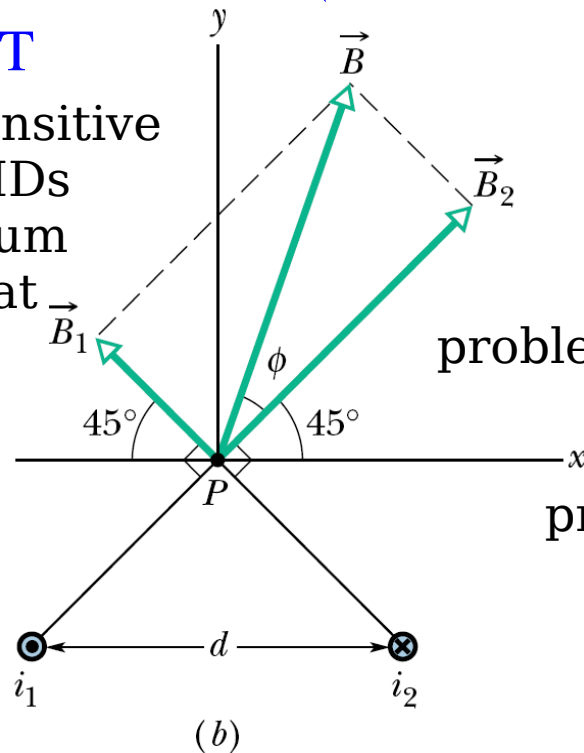
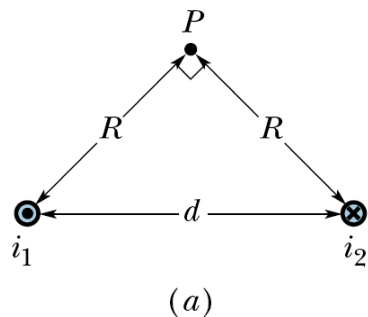
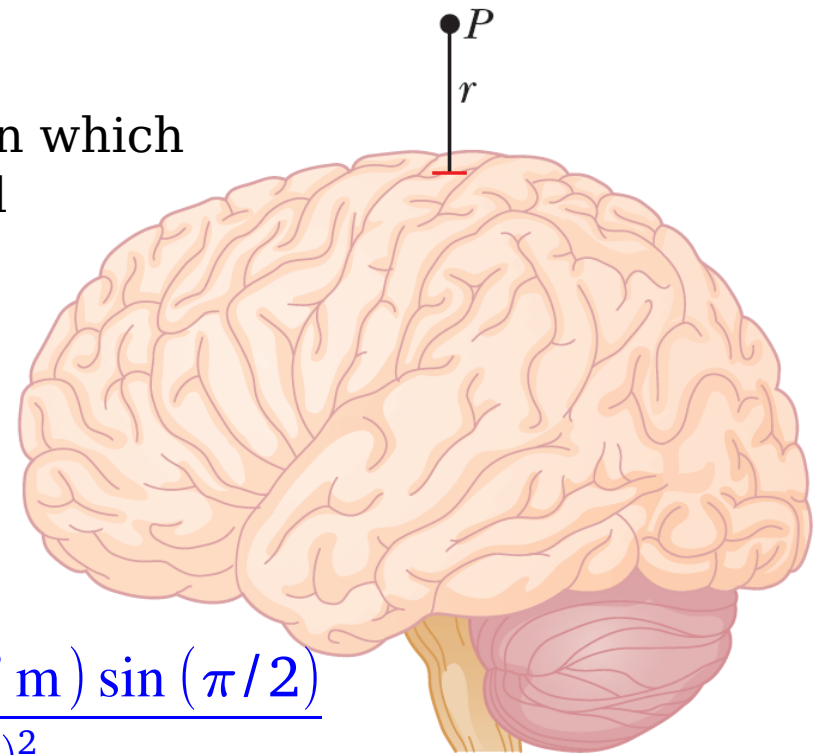
$$r = 2 \text{ cm} , i = 10 \mu \text{ A} , d s = 1 \text{ mm} , \theta = \pi/2$$

$$\Rightarrow B = \frac{\mu_0 i d s \sin \theta}{4 \pi r^2}$$

$$= \frac{4 \pi \times 10^{-7} \text{ T} \cdot \text{m/A} (10 \times 10^{-6} \text{ A}) (1 \times 10^{-3} \text{ m}) \sin (\pi/2)}{4 \pi (2 \times 10^{-2} \text{ m})^2}$$

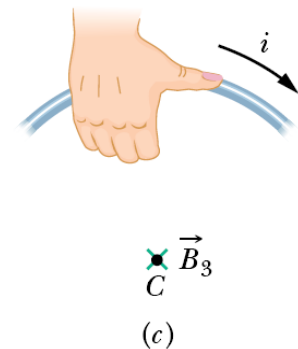
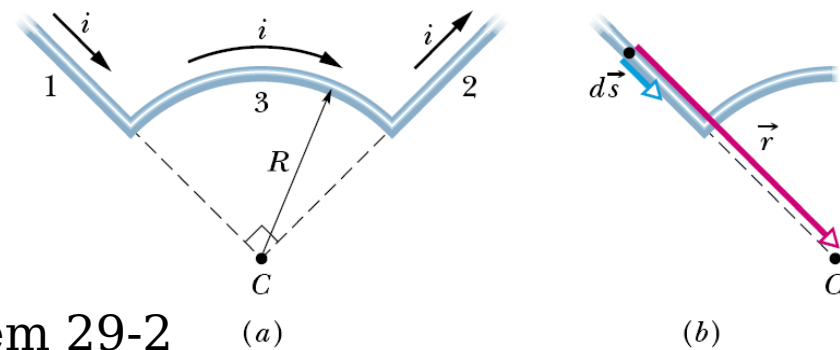
$$= 2.5 \times 10^{-12} \text{ T} \approx 3 \text{ pT}$$

- You need extremely sensitive instruments called SQUIDS (superconducting quantum interference devices) that can measure such small fields.



problem 29-2

problem 29-1





## Force Between 2 Parallel Currents

- 2 long parallel wires carrying currents exert forces on each other.

- The magnetic field produced by the current in wire  $a$  at every point of wire  $b$  is

$$B = \frac{\mu_0 i_a}{2 \pi d}$$

- The direction of the magnetic field at wire  $b$  is down.

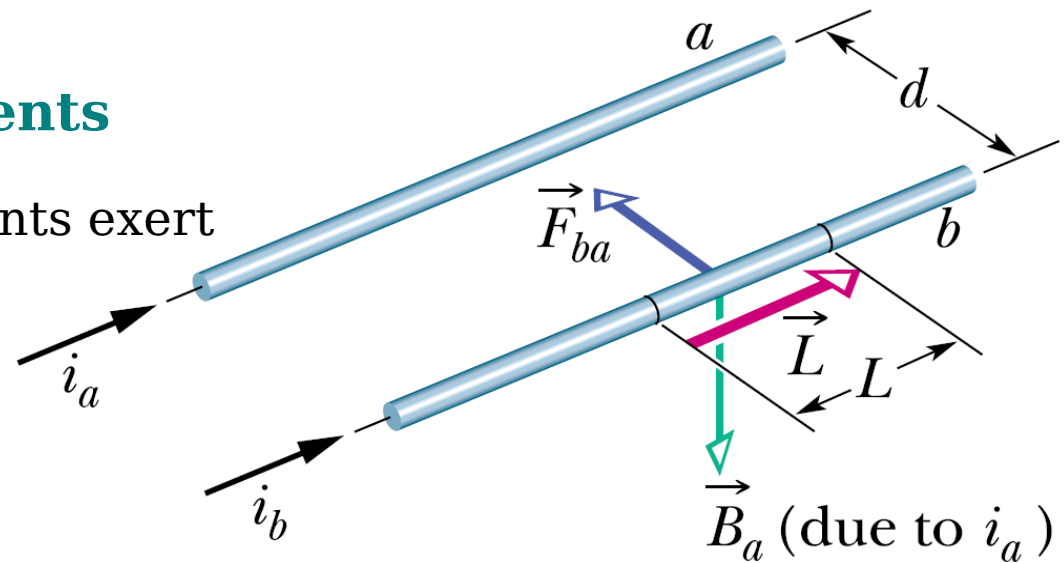
- The force on a length  $L$  of wire  $b$  due to the external magnetic field is

$$\vec{F}_{ba} = i_b \vec{L} \times \vec{B}_a \quad \text{and its magnitude is} \quad F_{ba} = i_b L B_a \sin \frac{\pi}{2} = \frac{\mu_0 L i_a i_b}{2 \pi d}$$

- Applying the right-hand rule, we see that the force's direction is directly toward wire  $a$ .

- The general procedure for finding the force on a current-carrying wire is this:

To find the force on a current-carrying wire due to a 2<sup>nd</sup> current-carrying wire, first find the field due to the 2<sup>nd</sup> wire at the site of the 1<sup>st</sup> wire. Then find the force on the 1<sup>st</sup> wire due to that field.



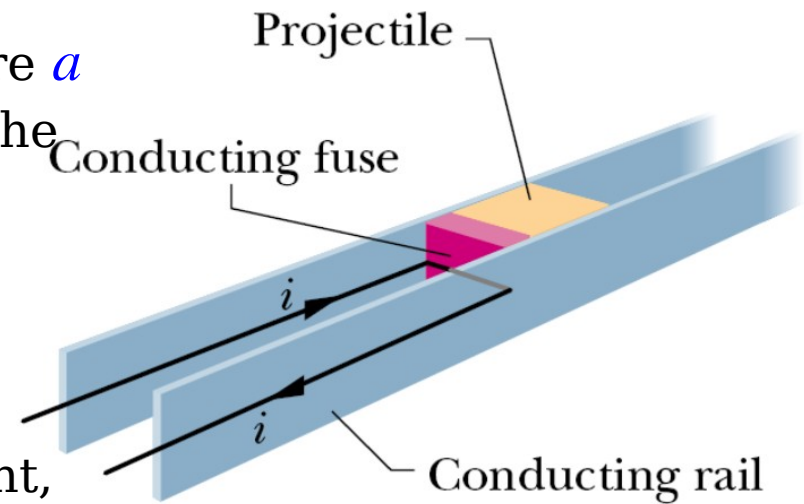
- Use this procedure to compute the force on wire  $a$  due to the current in wire  $b$ , we would find that the force is directly toward wire  $b$ ; hence

Parallel currents attract each other, and anti-parallel currents repel each other.

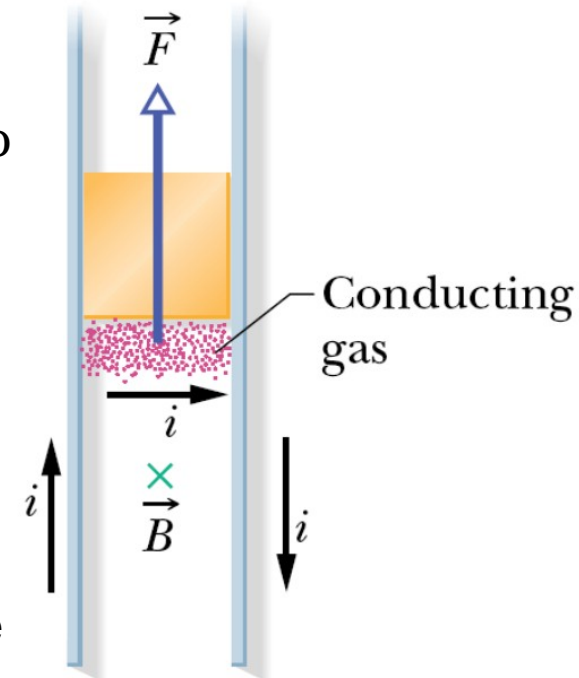
- The definition of ampere: the ampere is that constant current which, if maintained in 2 straight, parallel conductors of infinite length, of negligible circular cross section, and placed 1m apart in vacuum, would produce on each of these conductors a force of magnitude  $2 \times 10^{-7}$  newton per meter of wire length.

## Rail Gun

- In this device, a magnetic force accelerates a projectile to a high speed in a short time.
- The currents in the rails produce magnetic fields that are directed downward between the rails.
- The net magnetic field exerts a force on the gas due to the current through the gas.
- As the gas is forced outward along the rails, it pushes the projectile, accelerating it by as much as  $5 \times 10^6 g$ , and then launches it with a speed of  $10 \text{ km/s}$ , all within  $1 \text{ ms}$ .



(a)



(b)

# Ampere's Law

Coulomb's Law  $\Leftrightarrow$  Gauss's Law

Biot-Savart Law  $\Leftrightarrow$  Ampere's Law

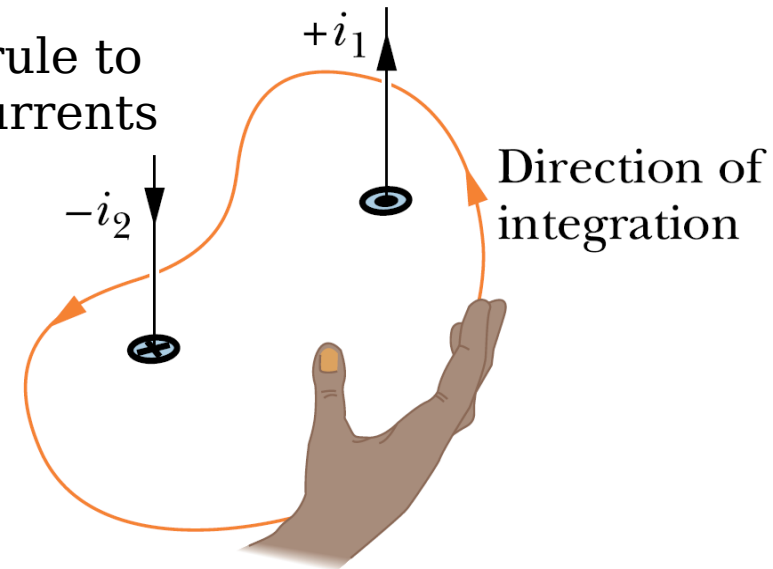
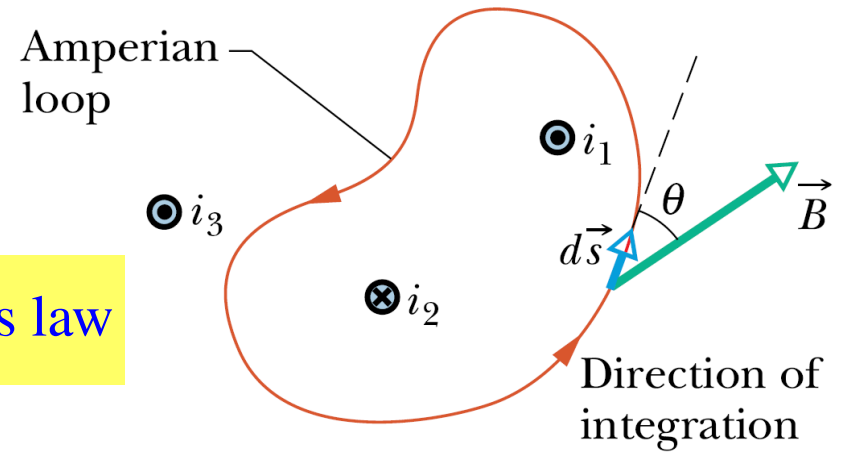
● **Ampere's law** is  $\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc}$  Ampere's law

● The loop on the integral sign means that the scalar product  $\vec{B} \cdot d\vec{s}$  is to be integrated around a closed loop, called an *Amperian loop*. The current  $i_{enc}$  is the *net* current encircled by that closed loop.

● Ampere's law can be written as  $\oint \vec{B} \cdot d\vec{s} = \oint B \cos \theta ds = \mu_0 i_{enc}$

● We use the following curled-straight right-hand rule to assign a plus sign or a minus sign to each of the currents that make up the net encircled current  $i_{enc}$ :

Curl your right hand around the Amperian loop, with the fingers pointing in the direction of integration. A current through the loop in the general direction of your outstretched thumb is assigned a plus sign, and a current generally in the opposite direction is assigned a minus sign.



● With the indicated counterclockwise direction of integration, the net current encircled by the loop is

$$i_{enc} = i_1 - i_2 \Rightarrow \oint B \cos \theta ds = \mu_0 (i_1 - i_2)$$



- The contributions of the current outside the loop to the magnetic field cancel out because the integration is made around the full loop.

## Magnetic Field Outside a Long Straight Wire with Current

- Under the cylindrical symmetry, the magnetic field has the same magnitude  $B$  at every point on the loop.

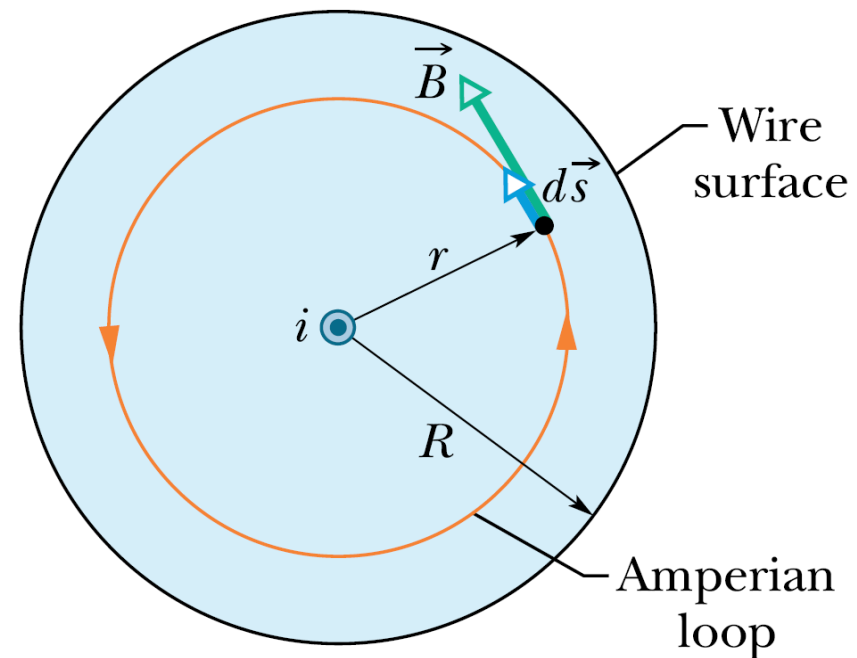
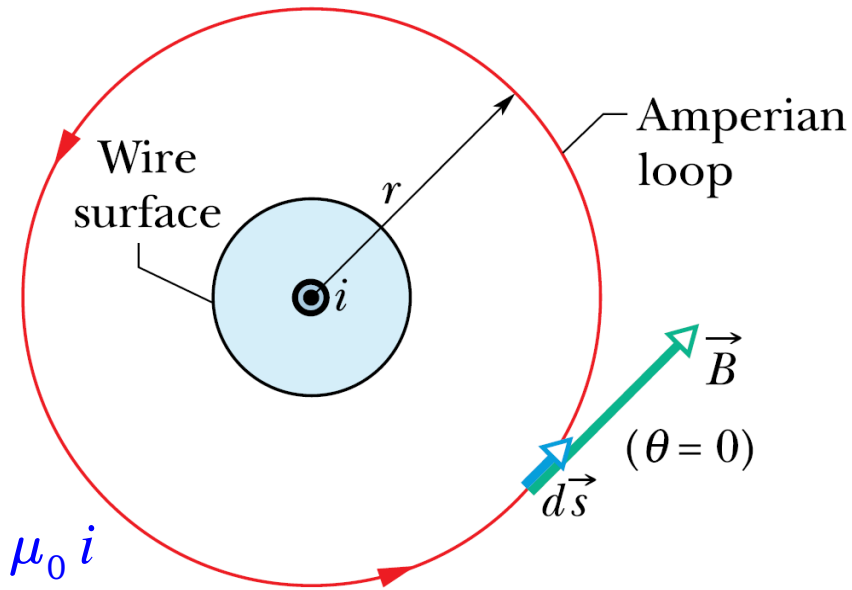
- Since  $\vec{B}$  is tangent to the loop at every point along the loop, as is  $d\vec{s}$ . Thus,  $\vec{B}$  and  $d\vec{s}$  are either parallel or antiparallel at each point of the loop, and we assume they are parallel,

$$\oint \vec{B} \cdot d\vec{s} = \oint B \cos 0 \, ds = B \oint ds = B(2\pi r) = \mu_0 i$$

$$\Rightarrow B = \frac{\mu_0 i}{2\pi r} \quad \text{outside straight wire}$$

## Magnetic Field Inside a Long Straight Wire with Current

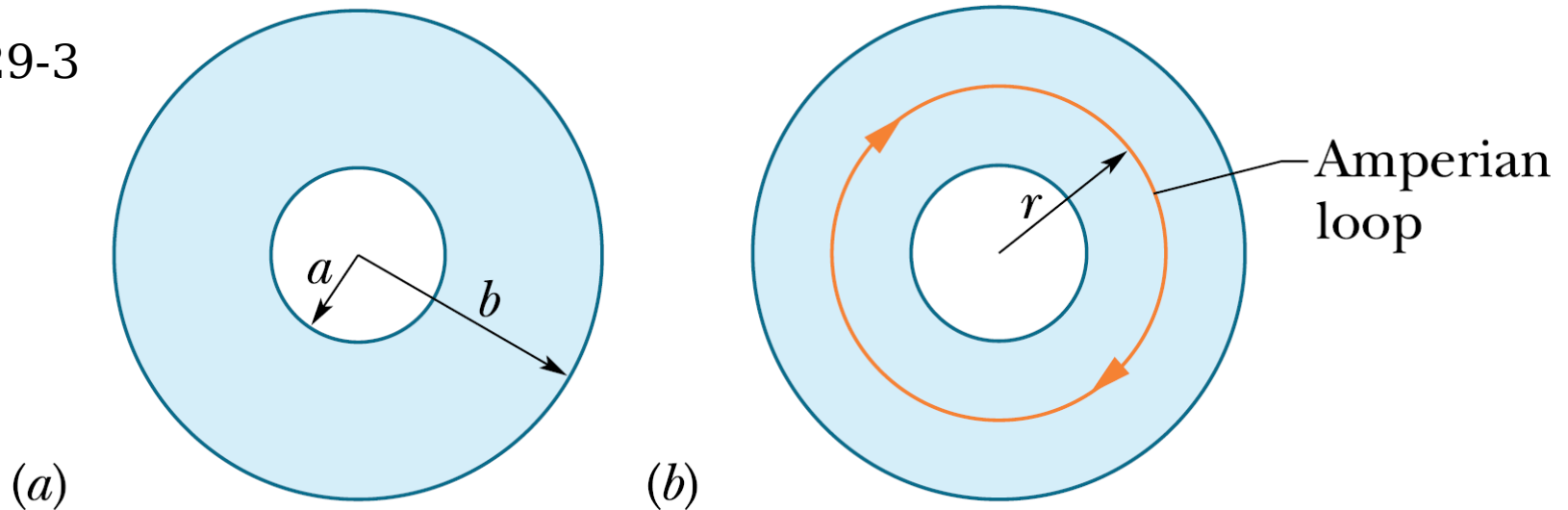
- If the current is uniformly distributed over a cross section of the wire, the magnetic field produced by the current must be cylindrically symmetrical.



- Ampere's law gives  $\oint \vec{B} \cdot d\vec{s} = B \oint ds = B(2\pi r) = \mu_0 i \frac{\pi r^2}{\pi R^2} \Leftarrow \mu_0 i_{\text{enc}}$   
 $\Rightarrow B = \frac{\mu_0 i}{2\pi R^2} r$  inside straight wire

- Inside the wire, the magnitude  $B$  of the magnetic field is proportional to  $r$ ; the magnitude is 0 at the center and a maximum at the surface, where  $r = R$ .

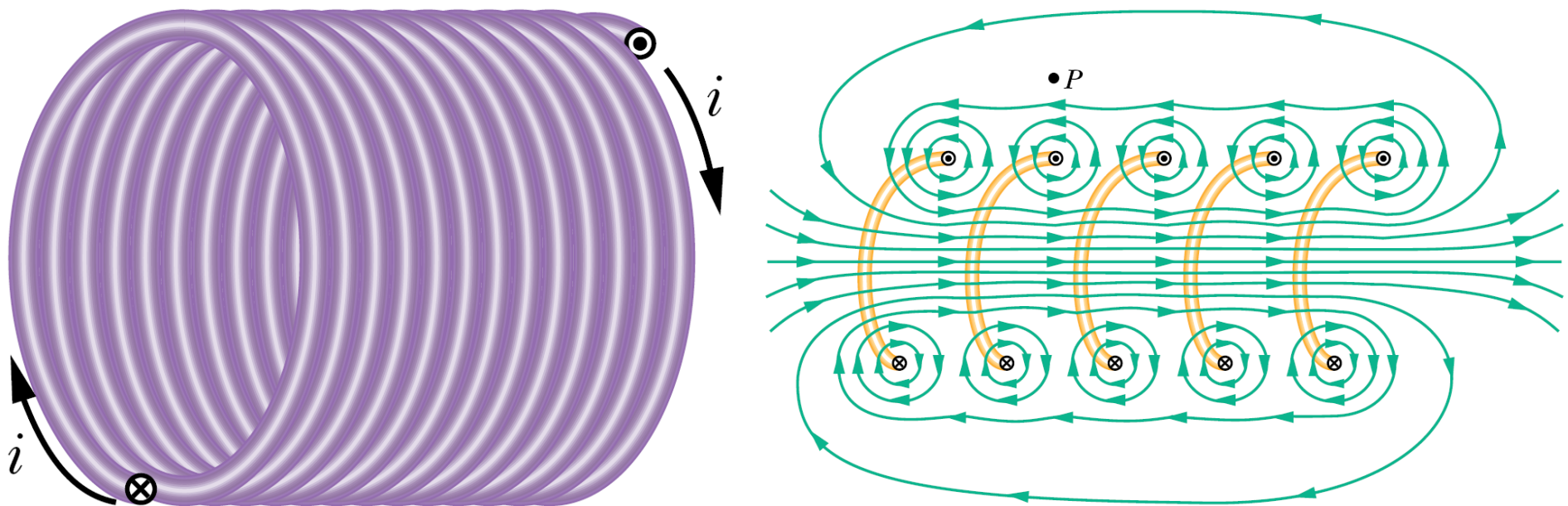
problem 29-3



## Solenoids and Toroids

### Magnetic Field of a Solenoid

- **Solenoid:** a long, tightly wound helical coil of wire.
- Assume that the length of the solenoid is much greater than the diameter.



- The solenoid's magnetic field is the vector sum of the fields produced by the individual turns (*windings*) that make up the solenoid.
- At points inside the solenoid and reasonably far from the wire, the magnetic field is approximately parallel to the (central) solenoid axis.
- In the limiting case of an *ideal solenoid*, which is infinitely long and consists of tightly packed (*close-packed*) turns of square wire, the field inside the coil is uniform and parallel to the solenoid axis.
- In the limiting case of an ideal solenoid, the magnetic field outside the solenoid is 0.
- The direction of the magnetic field along the solenoid axis is given by a curled straight right-hand rule.

- The spacing of these lines in the central region shows that the field inside the coil is fairly strong and uniform over the cross section of the coil. The external field is relatively weak.

- Apply Ampere's law

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}}$$

to the solenoid, then

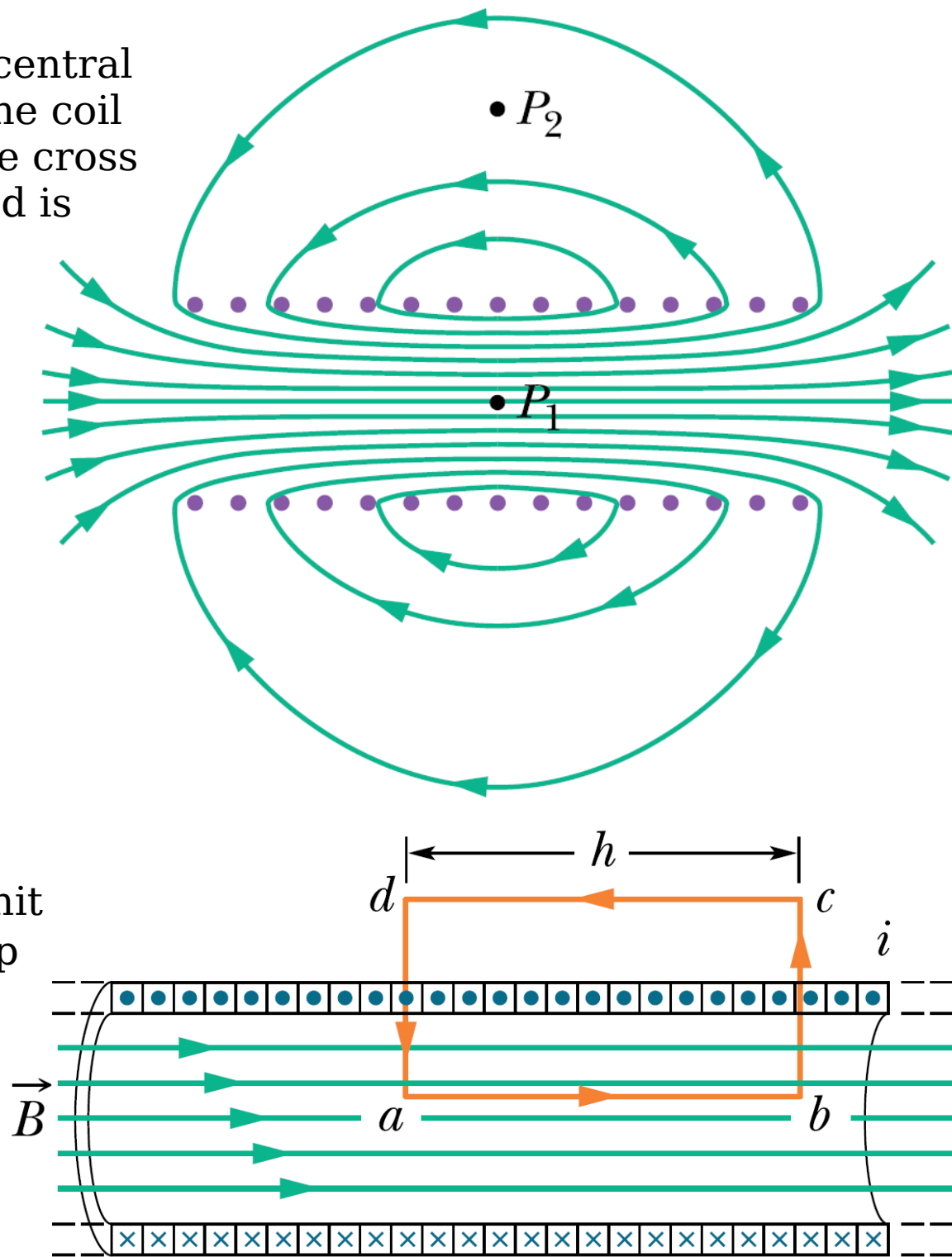
$$\begin{aligned} \oint \vec{B} \cdot d\vec{s} &= \int_a^b \vec{B} \cdot d\vec{s} + \int_b^c \vec{B} \cdot d\vec{s} \\ &+ \int_c^d \vec{B} \cdot d\vec{s} + \int_d^a \vec{B} \cdot d\vec{s} \\ &= B h + 0 + 0 + 0 = B h \end{aligned}$$

Let  $n$  be the number of turns per unit length of the solenoid; then the loop encloses  $nh$  turns and  $i_{\text{enc}} = i(nh)$

- Ampere's law then gives

$$B h = \mu_0 i n h$$

$$\Rightarrow B = \mu_0 i n \quad \text{ideal solenoid}$$

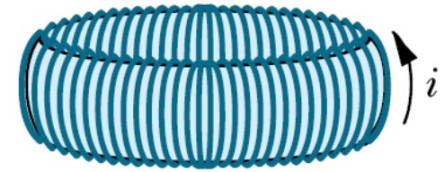


- The magnetic field magnitude  $B$  within a solenoid does not depend on the diameter or the length of the solenoid and that  $B$  is uniform over the solenoidal cross section.

- A solenoid provides a practical way to set up a known uniform magnetic field for experimentation, just as a parallel-plate capacitor provides a practical way to set up a known uniform electric field.

### Magnetic Field of a Toroid

- From the symmetry, we see that the lines of the magnetic field form concentric circles inside the toroid.



(a)

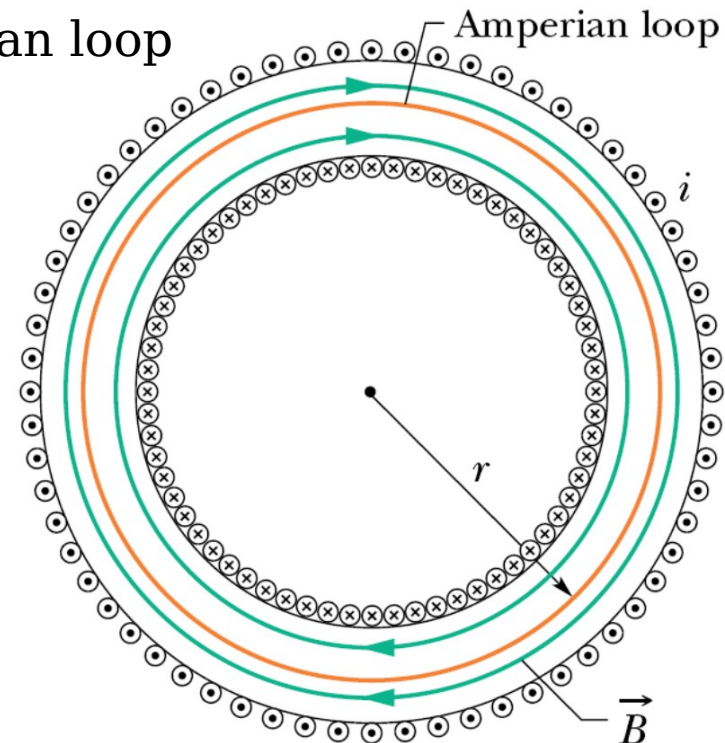
- Choose a concentric circle of radius  $r$  as an Amperian loop and traverse it in the clockwise direction. Ampere's law yields

$$(B)(2\pi r) = \mu_0 i N \Rightarrow B = \frac{\mu_0 i N}{2\pi r} \text{ toroid}$$

- $B$  is not constant over the cross section of a toroid.

- $B = 0$  for points outside an ideal toroid (as if the toroid were made from an ideal solenoid).

- The direction of the magnetic field within a toroid follows from our curled straight right-hand rule.



(b)

## A Current-Carrying Coil as a Magnetic Dipole

- A coil behaves as a magnetic dipole in that, if we place it in an external magnetic field, a torque act on it:

$$\vec{\tau} = \vec{\mu} \times \vec{B} = N i \vec{A} \times \vec{B}$$

### Magnetic Field of a Coil

- Consider a coil with a single circular loop and only points on its perpendicular central axis,  $z$  axis,  $d\vec{B} = d\vec{B}_{\parallel} + d\vec{B}_{\perp}$

- From the symmetry, the vector sum of all the perpendicular components due to all the loop elements  $ds$  is 0, thus

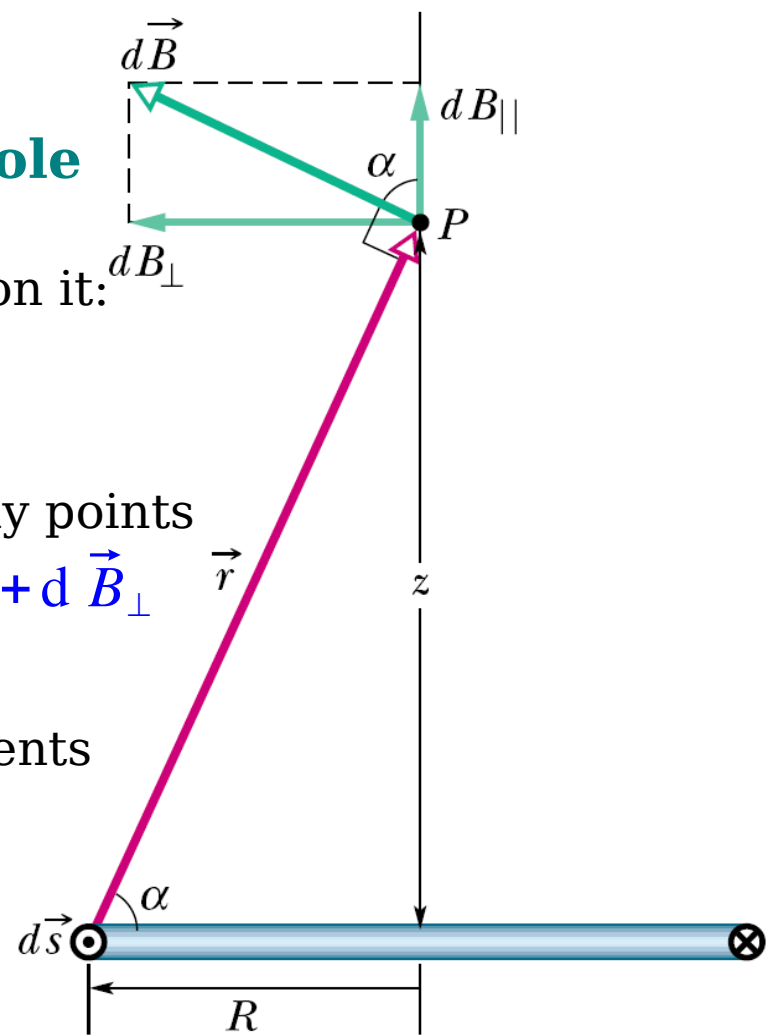
$$B = \int d B_{\parallel}$$

- The Biot-Savart law gives the magnetic field at  $r$

$$d B = \frac{\mu_0}{4 \pi} \frac{i d s \sin(\pi/2)}{r^2} = \frac{\mu_0 i d s}{4 \pi r^2} \Rightarrow d B_{\parallel} = d B \cos \alpha = \frac{\mu_0 i \cos \alpha d s}{4 \pi r^2}$$

$$r = \sqrt{R^2 + z^2} \quad \text{and} \quad \cos \alpha = \frac{R}{r} = \frac{R}{\sqrt{R^2 + z^2}} \Rightarrow d B_{\parallel} = \frac{\mu_0 i R}{4 \pi (R^2 + z^2)^{3/2}} d s$$

$$\Rightarrow B = \int d B_{\parallel} = \frac{\mu_0 i R}{4 \pi (R^2 + z^2)^{3/2}} \int d s = \frac{\mu_0 i R}{4 \pi (R^2 + z^2)^{3/2}} 2 \pi R \Rightarrow B(z) = \frac{\mu_0 i R^2}{2 (R^2 + z^2)^{3/2}}$$





- The direction of the magnetic field is the same as the direction of the magnetic dipole moment of the loop.

- For  $z \gg R$ , then  $B(z) \approx \frac{\mu_0 i R^2}{2 z^3}$

- Extend our result to include a coil of  $N$  turns

$$B(z) = \frac{\mu_0 N i R^2}{2 z^3} = \frac{\mu_0 N i A}{2 \pi z^3} \quad \leftarrow \quad A = \pi R^2$$

- Since the directions of the magnetic field and of the magnetic dipole moment are the same, and  $\mu = NiA$ , thus

$$\vec{B}(z) = \frac{\mu_0 N i \vec{A}}{2 z^3} = \frac{\mu_0}{2 \pi} \frac{\vec{\mu}}{z^3} \quad \text{current-carrying coil}$$

- 2 ways in which we can regard a current-carrying coil as a magnetic dipole:

(1) it experiences a torque when we place it in an external magnetic field;

(2) it generates its own intrinsic magnetic field, given, for distant points along its axis

