Chapter 6 Magnetic Fields in Matter

Magnetization

Diamagnets, Paramagnets, Ferromagnets

- All magnetic phenomena are due to electric charges in motion; you would find tiny currents on an atomic scale in magnetic material: electrons orbiting around nuclei and spinning about their axes.

- For macroscopic purposes, these current loops are so small that we may treat them as magnetic dipoles. Ordinarily, they cancel each other out because of the random orientation of the atoms.

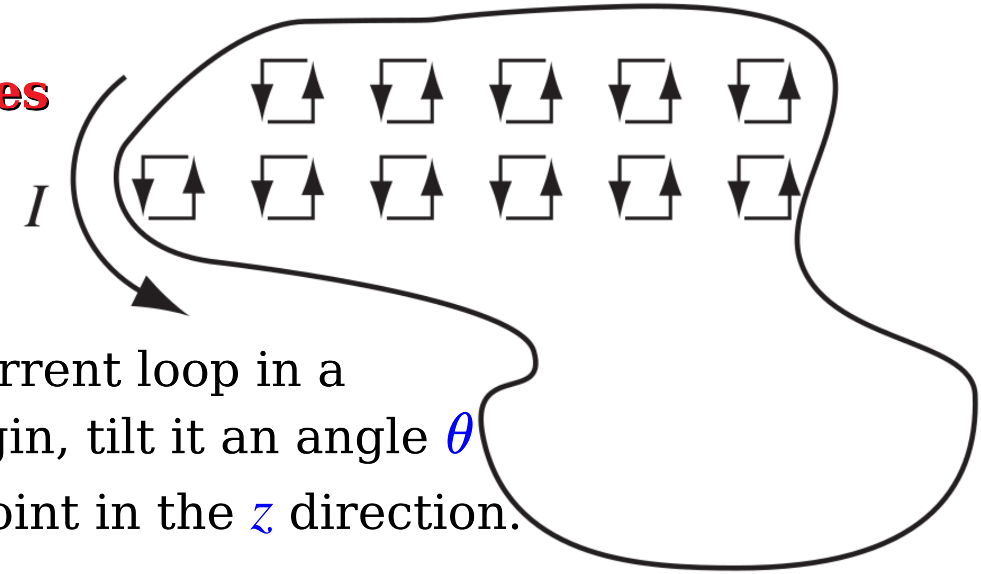
- When a magnetic field is applied, a net alignment of these magnetic dipoles occurs, and the medium becomes magnetically polarized, or **magnetized**.

- Unlike electric polarization, which is almost always in the same direction as $\mathbf{E}$, some materials acquire a magnetization *parallel* to $\mathbf{B}$ (**paramagnets**) and some *opposite* to $\mathbf{B}$ (**diamagnets**).

- Some substances (called **ferromagnets**, eg, iron) retain their magnetization even after the external field has been removed—for these, the magnetization is not only determined by the *present* field but also by the whole magnetic "history" of the object.

Torques and Forces on Magnetic Dipoles

- A magnetic dipole experiences a torque in a magnetic field, just as an electric dipole does in an electric field.



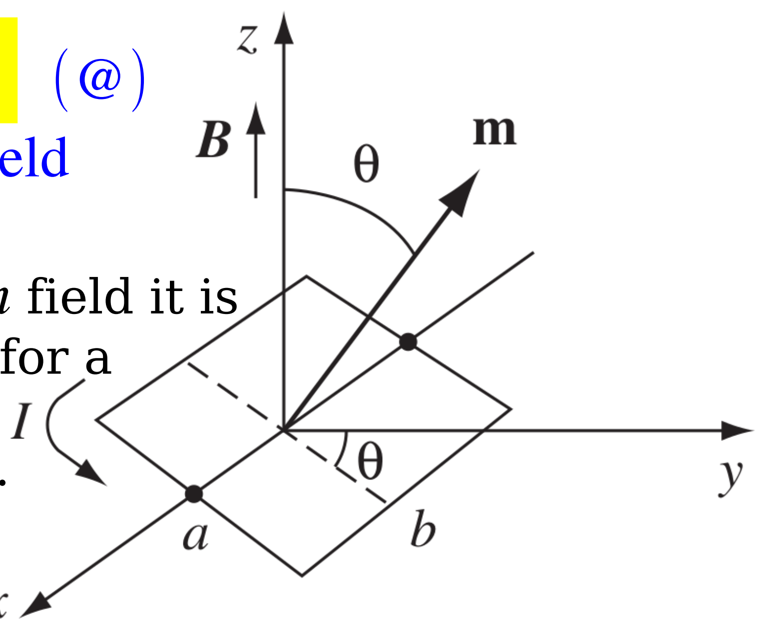
- Calculate the torque on a rectangular current loop in a uniform field $\mathbf{B}$. Center the loop at the origin, tilt it an angle θ from the z axis towards the y axis. Let $\mathbf{B}$ point in the z direction.

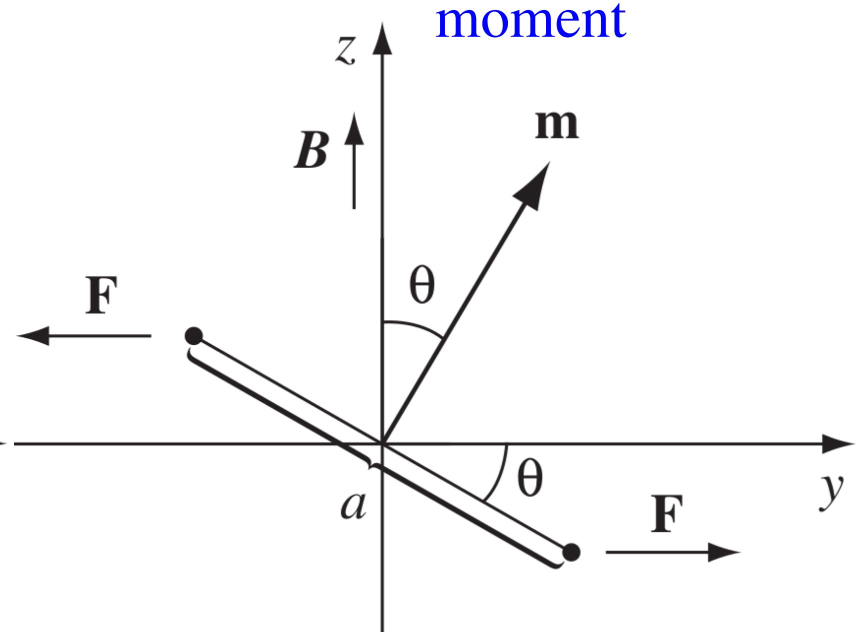
- The forces on the 2 sloping sides cancel, stretching the loop but not rotating.

- The forces on the "horizontal" sides are likewise equal and opposite, so zero net force on the loop, but they do generate a torque: $\mathbf{N} \equiv \mathbf{r} \times \mathbf{F} = a\,F \sin\theta\,\hat{\mathbf{x}}$

- $F = I\,b\,B \quad \Rightarrow \quad \mathbf{N} = I\,a\,b\,B \sin\theta\,\hat{\mathbf{x}} = m\,B \sin\theta\,\hat{\mathbf{x}} \quad \Leftarrow \quad m = I\,a\,b$ magnetic dipole moment

$\Rightarrow$ $\mathbf{N} = \mathbf{m} \times \mathbf{B}$ (@)
in a *uniform* field





- In a *nonuniform* field it is the exact torque for a perfect dipole of infinitesimal size.

- (@) is identical in form to the electrical analog, $\mathbf{N}=\mathbf{p}\times\mathbf{E}$.

- The torque is in such a direction as to line the dipole up *parallel* to the field. It is this torque that accounts for **paramagnetism**.

- Since every electron constitutes a magnetic dipole, you might expect paramagnetism to be a universal phenomenon.

- Actually, quantum mechanics (the Pauli exclusion principle) tends to lock the electrons within a given atom together in pairs with opposing spins, and this effectively neutralizes the torque on the combination.

- So paramagnetism most often occurs in atoms/molecules with an odd number of electrons, then the "extra" unpaired member is subject to the magnetic torque.

- Random thermal collisions tend to destroy the alignment and order.

- In a uniform field, the net *force* on a current loop is 0:

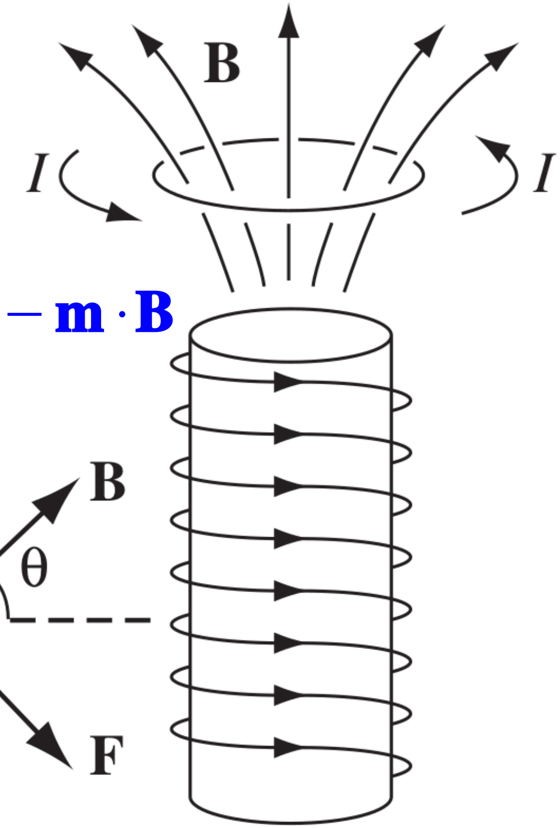
$$\mathbf{F}=I\oint(\mathrm{d}\boldsymbol{\ell}\times\mathbf{B})=I\left(\oint\mathrm{d}\boldsymbol{\ell}\right)\times\mathbf{B}=0$$

- In a *nonuniform* field this is no longer the case.

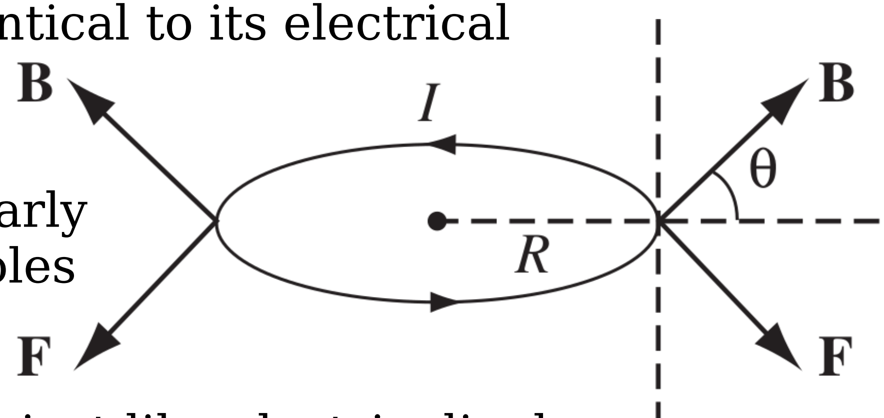
- Suppose a circular wire ring of radius R, carrying a current I, is suspended above a short solenoid in the "fringing" region.

- $\mathbf{B}$ has a radial component, and there is a net downward force on the loop: $F = 2\pi I R B \cos\theta$

- $\mathbf{F} \equiv -\nabla U = \nabla(\mathbf{m}\cdot\mathbf{B})$ $\Leftarrow$ for a *infinitesimal* loop + $U \equiv -\mathbf{m}\cdot\mathbf{B}$



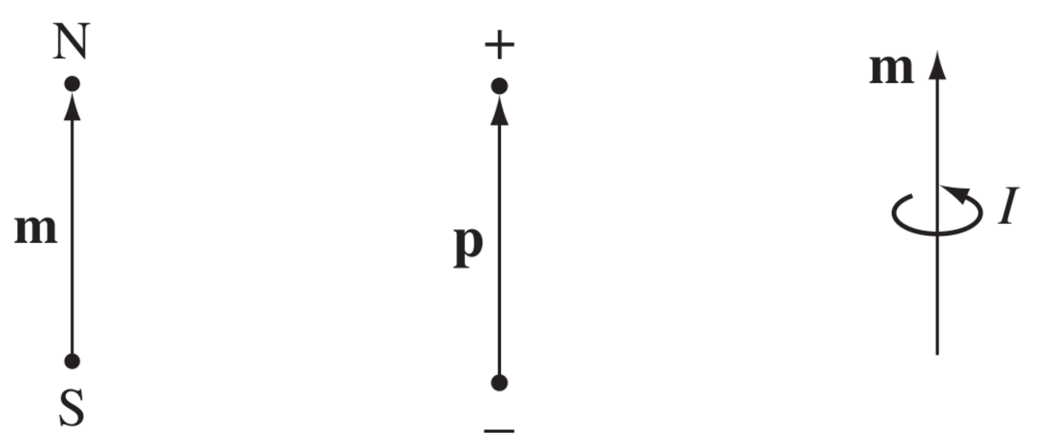
- The magnetic formula is identical to its electrical "twin," for $\mathbf{F}=\nabla(\mathbf{p}\cdot\mathbf{E})$.



- This similarity made some early physicists think magnetic dipoles consisted of positive and negative magnetic "charges," separated by a small distance, just like electric dipoles.

- *There's no* magnetic charge. Magnetism is not due to magnetic monopoles, but rather to *moving electric charges*; magnetic dipoles are tiny current loops.

- Whenever the *close-up* features of the dipole come into play, the Gilbert model and the (correct) Ampere model can yield strikingly different answers.



(a) Magnetic dipole (Gilbert model) (b) Electric dipole (c) Magnetic dipole (Ampère model)

External magnetic field: $\mathbf{B}(\mathbf{r}') = \mathbf{B}(0) + (\mathbf{r}'\cdot\nabla)\mathbf{B}(0) + \cdots \Leftarrow \nabla\cdot\mathbf{B} = 0,\ \nabla\times\mathbf{B} = 0$

$$\int \nabla'\cdot(x_i'\boldsymbol{J})\,\mathrm{d}\tau' = \int \left(\nabla' x_i'\cdot\boldsymbol{J} + x_i'\cancel{\nabla'\cdot\boldsymbol{J}}\right)\mathrm{d}\tau' = \int J_i\,\mathrm{d}\tau' = 0$$

$$\Rightarrow \mathbf{F} = \int \boldsymbol{J}\times\mathbf{B}\,\mathrm{d}\tau' = \int \boldsymbol{J}\cancel{(\mathbf{r}')}\,\mathrm{d}\tau'\times\mathbf{B}(0) + \int \boldsymbol{J}(\mathbf{r}')\times(\mathbf{r}'\cdot\nabla)\mathbf{B}(0)\,\mathrm{d}\tau' + \cdots$$

$$= \int \boldsymbol{J}(\mathbf{r}')\times\nabla[\mathbf{r}'\cdot\mathbf{B}](0)\,\mathrm{d}\tau' + \cdots \Leftarrow \nabla(\mathbf{r}'\cdot\mathbf{B}) = (\mathbf{r}'\cdot\nabla)\mathbf{B} + \cancel{(\mathbf{B}\cdot\nabla)\mathbf{r}'} + \mathbf{r}'\times(\cancel{\nabla\times\mathbf{B}}) + \mathbf{B}\times(\cancel{\nabla\times\mathbf{r}'})$$

$$\Rightarrow \mathbf{F}_{\text{dip}} = -\nabla\times\int(\mathbf{r}'\cdot\mathbf{B})\boldsymbol{J}(\mathbf{r}')\,\mathrm{d}\tau' \Leftarrow \nabla\times(\mathbf{r}'\cdot\mathbf{B})\boldsymbol{J} = \nabla(\mathbf{r}'\cdot\mathbf{B})\times\boldsymbol{J}(\mathbf{r}') + (\mathbf{r}'\cdot\mathbf{B})\cancel{\nabla\times\boldsymbol{J}(\mathbf{r}')}$$

$$\int \nabla'\cdot(x_i'x_j'\boldsymbol{J})\,\mathrm{d}\tau' = \int\left(x_i'J_j + x_j'J_i + x_i'x_j'\cancel{\nabla'\cdot\boldsymbol{J}}\right)\mathrm{d}\tau' = 0 \Leftarrow \text{localized } \boldsymbol{J}$$

$$\Rightarrow \int x_i'J_j\,\mathrm{d}\tau' = -\int x_j'J_i\,\mathrm{d}\tau'$$

$$\Rightarrow \int \mathbf{B}\times(\boldsymbol{J}\times\mathbf{r}')\,\mathrm{d}\tau' = \int[\boldsymbol{J}(\mathbf{r}'\cdot\mathbf{B}) - \mathbf{r}'(\boldsymbol{J}\cdot\mathbf{B})]\,\mathrm{d}\tau' = 2\int \boldsymbol{J}(\mathbf{r}'\cdot\mathbf{B})\,\mathrm{d}\tau'$$

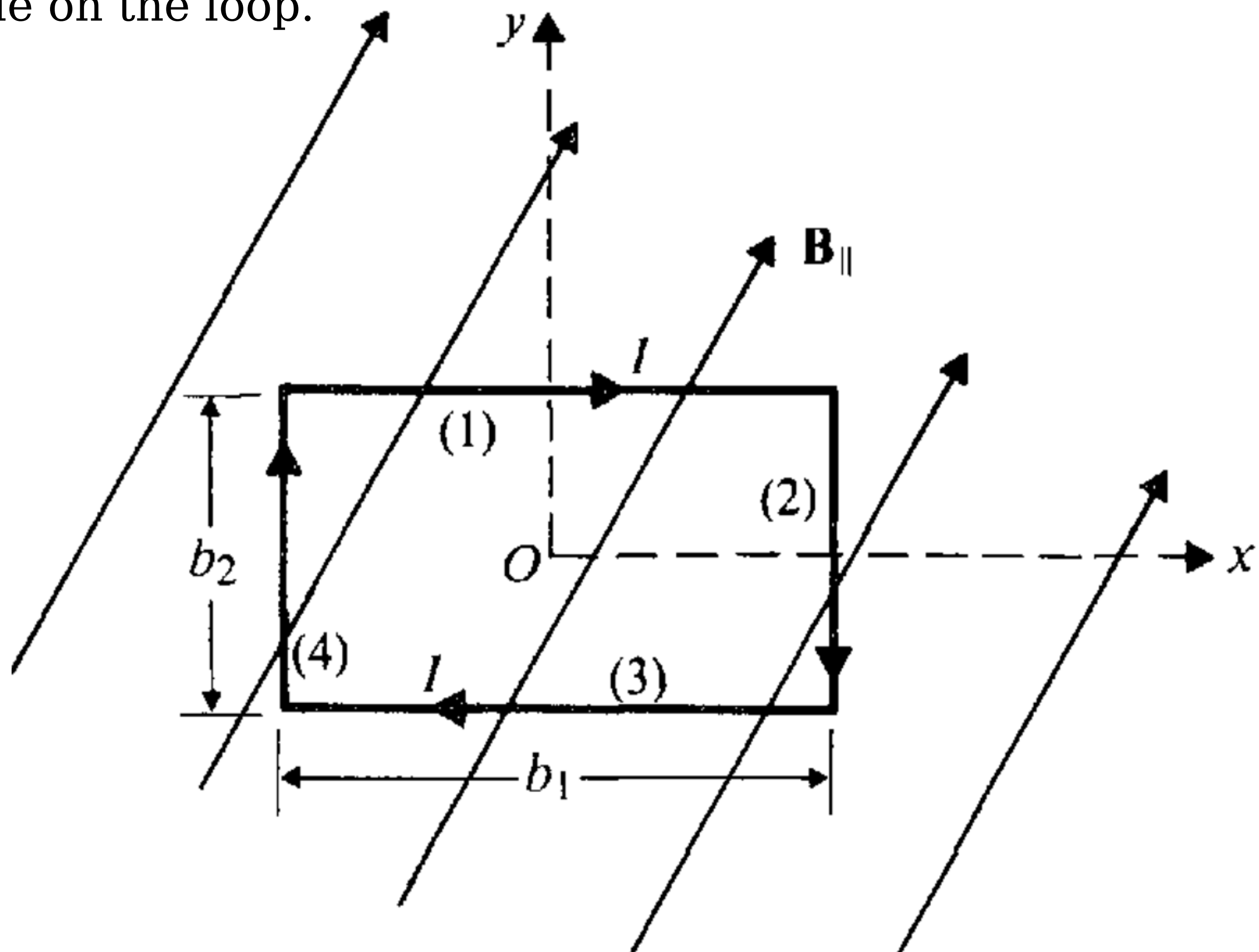
$$\Rightarrow \mathbf{F}_{\text{dip}} = -\nabla\times\int\mathbf{B}\times\left(-\frac{\mathbf{r}'\times\boldsymbol{J}}{2}\right)\mathrm{d}\tau' = \nabla\times(\mathbf{B}\times\mathbf{m}) \Leftarrow \mathbf{m} = \int\frac{\mathbf{r}'\times\boldsymbol{J}}{2}\,\mathrm{d}\tau'$$

$$= \nabla(\mathbf{m}\cdot\mathbf{B}) \Leftarrow \begin{aligned}\nabla\times(\mathbf{B}\times\mathbf{m}) &= (\mathbf{m}\cdot\nabla)\mathbf{B} - \cancel{(\mathbf{B}\cdot\nabla)\mathbf{m}} + \cancel{(\nabla\cdot\mathbf{m})}\mathbf{B} - \cancel{(\nabla\cdot\mathbf{B})}\mathbf{m}\\ \nabla(\mathbf{m}\cdot\mathbf{B}) &= (\mathbf{m}\cdot\nabla)\mathbf{B} + \cancel{(\mathbf{B}\cdot\nabla)\mathbf{m}} + \mathbf{r}\times(\cancel{\nabla\times\mathbf{B}}) + \mathbf{B}\times(\cancel{\nabla\times\mathbf{m}})\end{aligned}$$

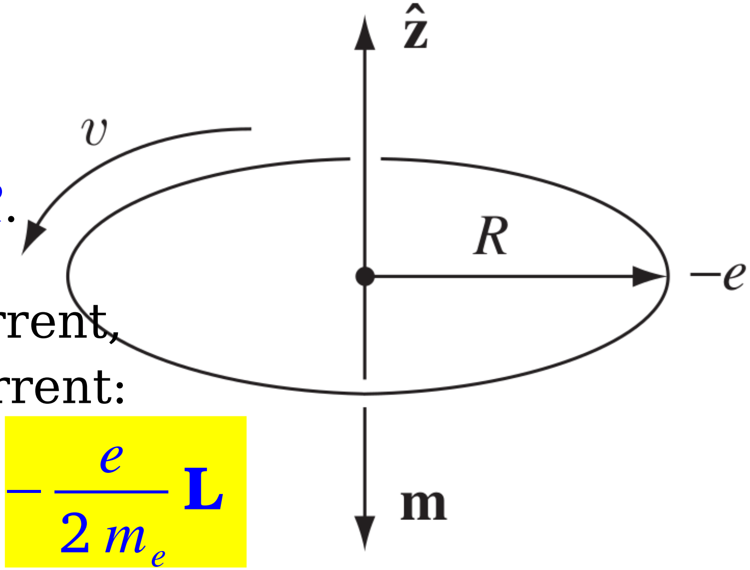
$$\mathbf{N} = \int \mathbf{r}' \times (\boldsymbol{J} \times \mathbf{B})\, \mathrm{d}\tau' \quad \Leftarrow \quad \mathrm{d}\mathbf{N} = \mathbf{r}' \times \mathrm{d}\mathbf{F} = \mathbf{r}' \times (\boldsymbol{J} \times \mathbf{B})\, \mathrm{d}\tau'$$

$$= \int (\mathbf{r}' \cdot \mathbf{B})\, \boldsymbol{J}\, \mathrm{d}\tau' - \mathbf{B} \int \cancel{\mathbf{r}' \cdot \boldsymbol{J}}\, \mathrm{d}\tau' = \mathbf{B} \times \int -\frac{\mathbf{r}' \times \boldsymbol{J}}{2}\, \mathrm{d}\tau' = \mathbf{m} \times \mathbf{B}$$

Example: A rectangular loop in the *xy*-plane with sides b_1 and b_2 carrying a current I lies in a *uniform* magnetic field $\mathbf{B} = B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\,\hat{\mathbf{z}}$. Determine the force and torque on the loop.



Effect of a Magnetic Field on Atomic Orbits



- Electrons not only *spin*; they also *revolve* around the nucleus—assume the orbit is a circle of radius R.

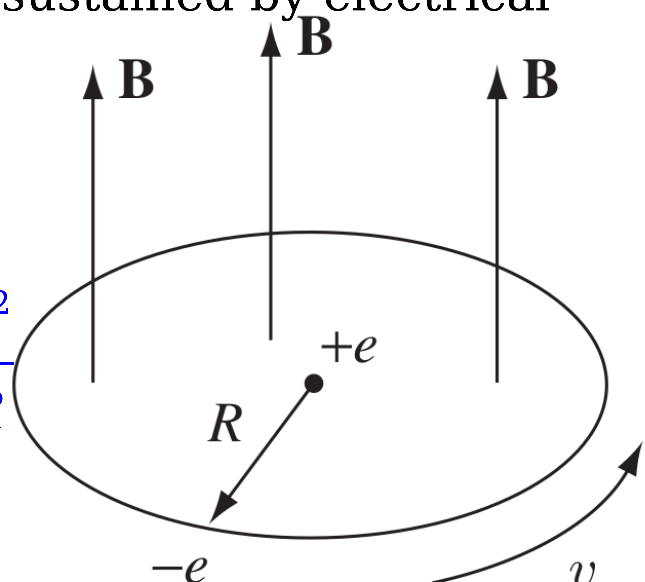
- This orbital motion doesn't constitute a steady current, but the period so short that it *looks* like a steady current:

$$I = \frac{e}{T} = \frac{e(-v)}{2\pi R} \quad \Rightarrow \quad \mathbf{m} = I \cdot \pi R^2 \hat{\mathbf{z}} = -\frac{1}{2} e v R \hat{\mathbf{z}} = -\frac{e}{2m_e}\mathbf{L}$$

- It's harder to tilt the entire orbit than it is the spin, so the orbital contribution to paramagnetism is small.

- This leads to that the electron *speeds up* or *slows down*, depending on the orientation of $\mathbf{B}$.

- Whereas the centripetal acceleration $\frac{v^2}{R}$ is ordinarily sustained by electrical forces alone, $\frac{1}{4\pi\epsilon_0}\frac{-e^2}{R^2}\hat{\mathbf{R}} = -m_e\frac{v^2}{R}\hat{\mathbf{R}}$, in the presence of a magnetic field there is an additional force, $-e\,\mathbf{v}\times\mathbf{B}$.



- Let $\mathbf{B} \perp$ the plane of the orbit, $\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} + e\bar{v}B = m_e\frac{\bar{v}^2}{R}$

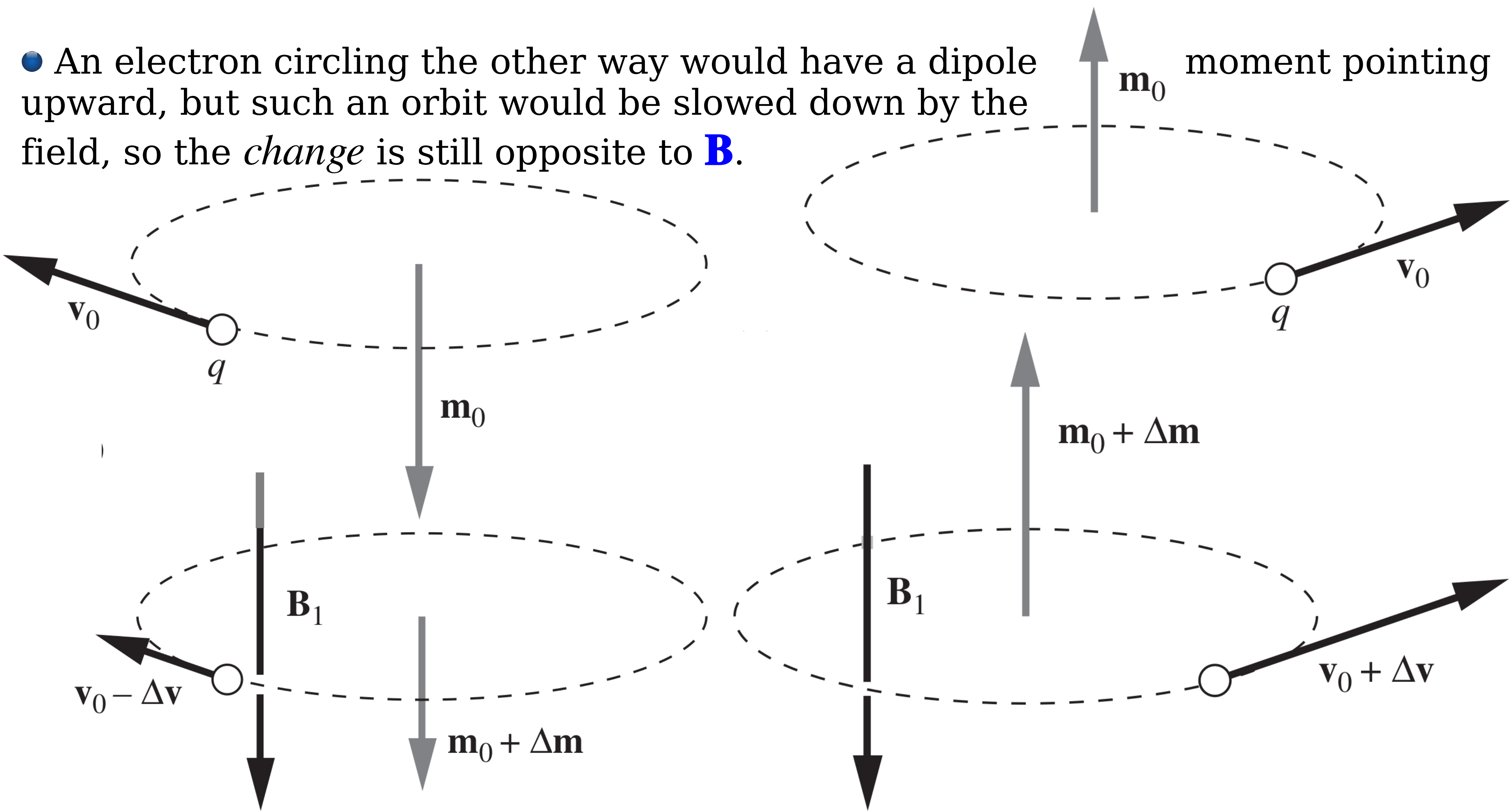
$$\Rightarrow \quad 0 < e\bar{v}B = \frac{m_e}{R}\left(\bar{v}^2 - v^2\right) = \frac{m_e}{R}\left(\bar{v}+v\right)\left(\bar{v}-v\right) \quad \Rightarrow \quad \bar{v} > v$$

Let $\Delta v = \bar{v} - v \ll 1 \quad \Rightarrow \quad \frac{m_e}{R}(\bar{v}+v)(\bar{v}-v) \approx \frac{m_e}{R}(2\bar{v})\Delta v = e\bar{v}B \quad \Rightarrow \quad \Delta v = \frac{eRB}{2m_e}$

- When $\mathbf{B}$ is turned on, the electron speeds up. A change in orbital speed means a change in the dipole moment: $\Delta \mathbf{m} = -\frac{1}{2} e \Delta v R \hat{\mathbf{z}} = -\frac{e^2 R^2}{4 m_e}\mathbf{B}$

- *The change in* $\mathbf{m}$ *is opposite to the direction of* $\mathbf{B}$.

- An electron circling the other way would have a dipole moment pointing upward, but such an orbit would be slowed down by the field, so the *change* is still opposite to $\mathbf{B}$.

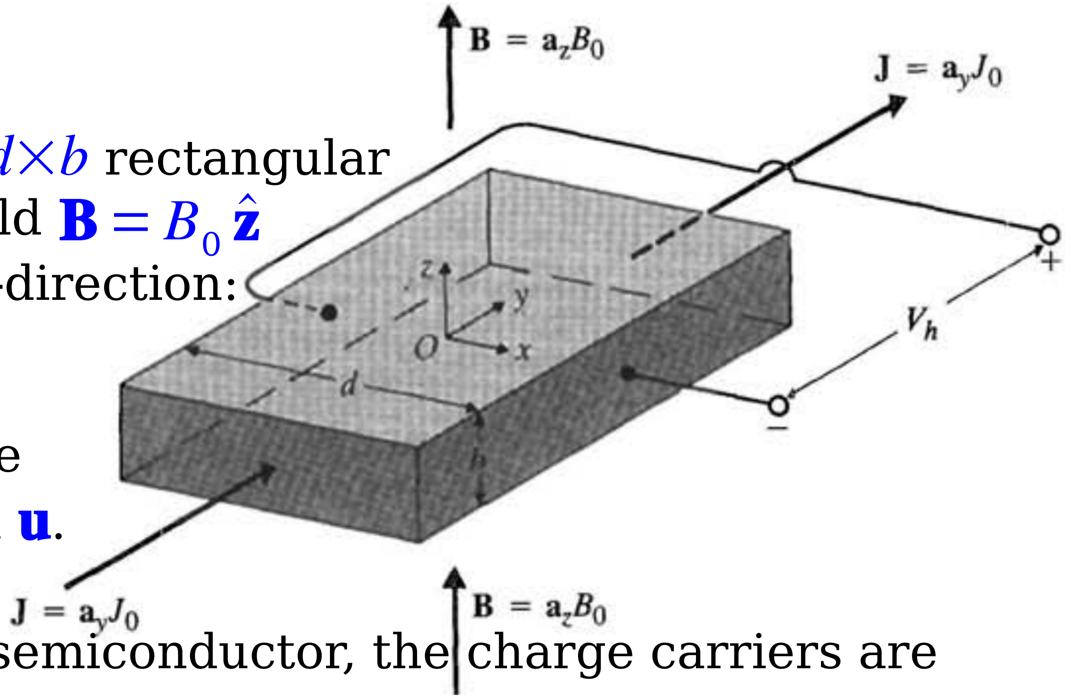


- Ordinarily, the electron orbits are randomly oriented, and the orbital dipole moments cancel out. But with a magnetic field, each atom picks up “extra” dipole moment, and these increments are all *antiparallel* to the field — **diamagnetism**.

- It is a universal phenomenon, affecting all atoms. But it is much weaker than paramagnetism, and thus is observed mainly in atoms with *even* numbers of electrons, where paramagnetism is usually absent.

- This classical model is fundamentally flawed (it’s a true *quantum* phenomenon). What is important is the empirical fact that in diamagnetic materials the induced dipole moments point opposite to the magnetic field.

Hall Effect



- Consider a conducting material of a $d \times b$ rectangular cross section in a uniform magnetic field $\mathbf{B} = B_0 \hat{\mathbf{z}}$
A uniform direct current flows in the y-direction:

$$\boldsymbol{J} = J_0 \hat{\mathbf{y}} = N\, q\, \mathbf{u}$$

- The magnetic force law let the charge carriers experience a force ⊥ both $\mathbf{B}$ & $\mathbf{u}$.

- If the material is a conductor/n-type semiconductor, the charge carriers are electrons, and q is negative. The magnetic force tends to move the electrons in the $+x$-direction, creating a transverse electric field.

- It continues until the transverse field can stop the drift of the charge carriers.

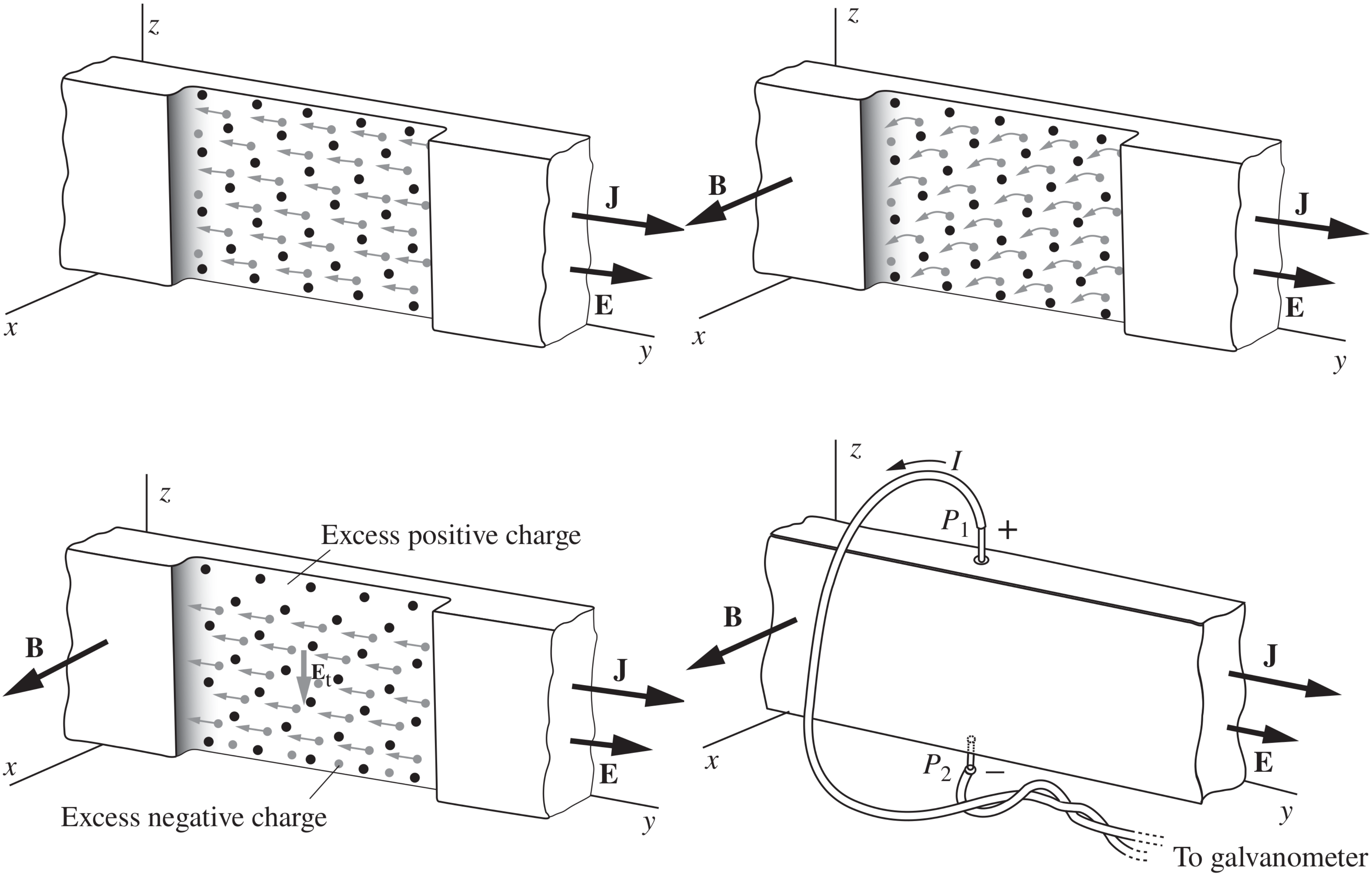
- In the steady state the net force on the charge carriers is 0:

$$\mathbf{F} = q\left(\mathbf{E}_h + \mathbf{u} \times \mathbf{B}\right) = 0 \quad \Leftarrow \quad \mathbf{E}_h : \text{Hall field} \quad \Rightarrow \quad \mathbf{E}_h = -\mathbf{u} \times \mathbf{B} \quad \text{Hall effect}$$

- For conductors/n-type semiconductors and a positive $J_0 > 0$, $\Rightarrow \quad \mathbf{u} = -u_0 \hat{\mathbf{y}}$

$$\Rightarrow \quad \mathbf{E}_h = -\left(-u_0 \hat{\mathbf{y}}\right) \times B_0 \hat{\mathbf{z}} = u_0 B_0 \hat{\mathbf{x}} \quad \Rightarrow \quad \text{Hall voltage } V_h = \int_0^d E_h \, \mathrm{d}x = u_0 B_0 d$$

$$\frac{E_x}{J_y B_z} = \frac{1}{N q} \quad \text{Hall coefficient: a characteristic of the material}$$



z
x
y
J
E
B
Excess positive charge
E_t
Excess negative charge
I
P_1
+
P_2
–
To galvanometer

- If the charge carriers are holes, such as in a *p*-type semiconductor, the Hall field will be reversed, and the Hall voltage will be negative with the reference polarities shown in the figure.

- The Hall effect can be used for measuring the magnetic field and determining the sign of the predominant charge carriers.

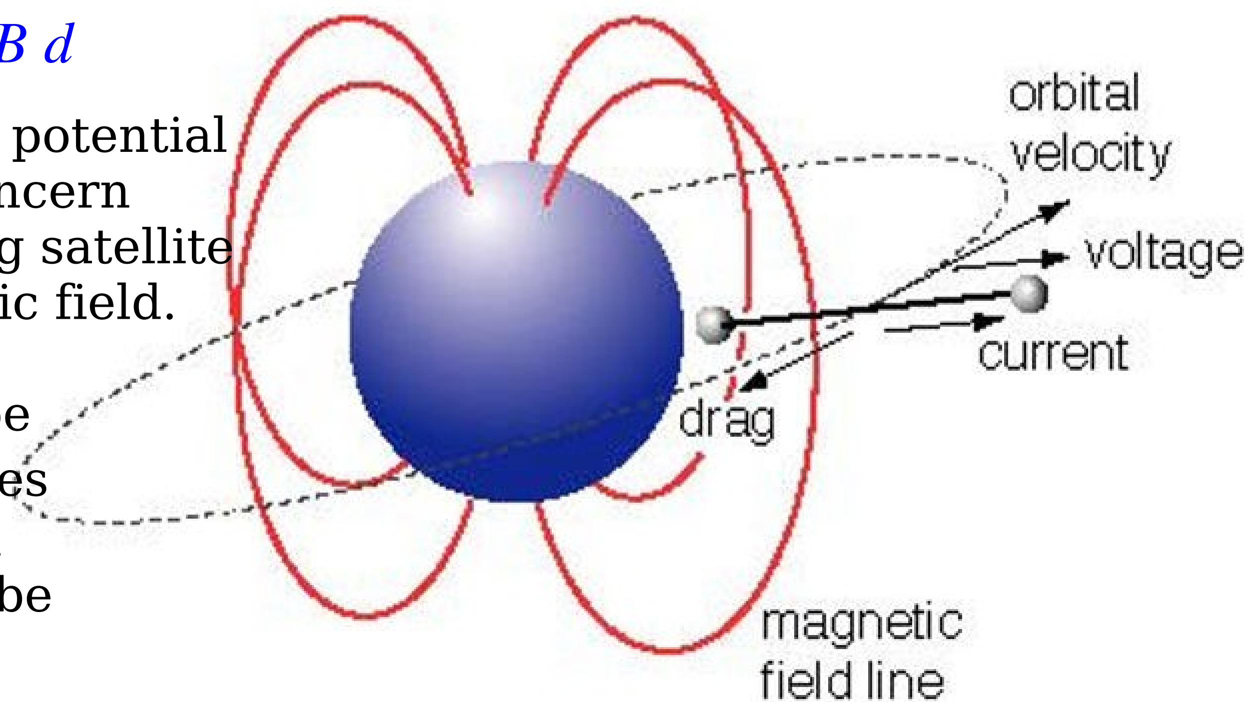
- In actuality it is a complex affair involving quantum theory concepts.

- When a conductor begins to move at speed v through a magnetic field, its conduction electrons do also. And an electric field $\mathbf{E}$ and potential difference V are quickly set up:

$$e\,E = e\,v\,B \quad \Rightarrow \quad V = v\,B\,d$$

- Such a motion-caused circuit potential difference can be of serious concern when a conductor in an orbiting satellite moves through Earth's magnetic field.

- If a conducting line (said to be an *electrodynamic tether*) dangles from the satellite, the potential produced along the line might be used to maneuver the satellite.



Magnetization

- In the presence of a magnetic field, matter becomes magnetized; it contains many tiny dipoles, with a net alignment along some direction.

- 2 mechanisms that account for this magnetic polarization:

  (1) paramagnetism: the dipoles associated with the spins of unpaired electrons experience a torque tending to line them up parallel to the field.

  (2) diamagnetism: the orbital speed of the electrons is altered in such a way as to change the orbital dipole moment in a direction opposite to the field.

- We describe the state of magnetic polarization by the vector quantity

$$\text{magnetization } \mathbf{M} \equiv \frac{\text{magnetic dipole moment}}{\text{volume}} = \lim_{\Delta\tau \to 0} \frac{\sum \mathbf{m}_k}{\Delta\tau}$$

a role analogous to the polarization $\mathbf{P}$ in electrostatics.

- No matter paramagnetism, diamagnetism, or ferromagnetism, we take $\mathbf{M}$ as given and do the consequent calculation.

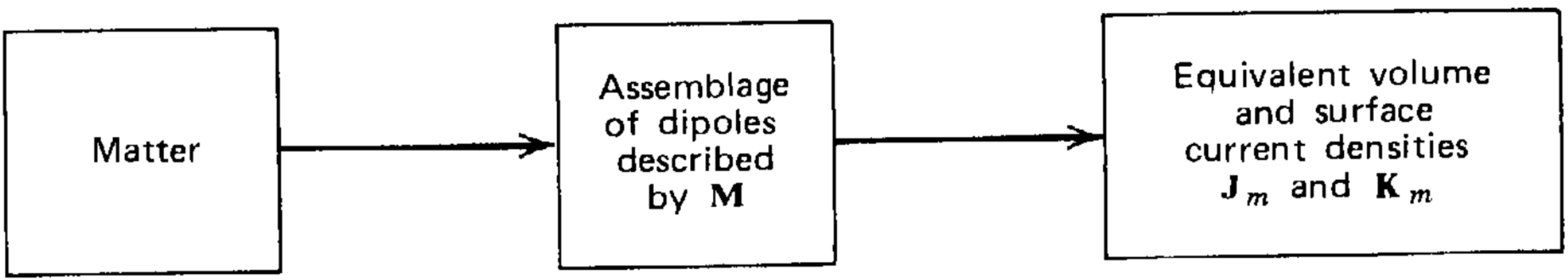
- Except the famous ferromagnetic trio (iron, nickel, and cobalt), few materials are affected by a magnetic field.

- The reason is that diamagnetism and paramagnetism are extremely weak: It takes a delicate experiment and a powerful magnet to detect them at all.

- If you suspend a piece of paramagnetic material above a solenoid, the induced magnetization would be upward, and hence the force downward. By contrast, the magnetization of a diamagnetic object would be downward and the force upward.

- When a sample is placed in a region of nonuniform field, the *paramagnet is attracted into the field*, whereas the *diamagnet is repelled away*.

- But the actual forces are weak—in a typical experimental arrangement the force on iron would be $10^4 - 10^5$ times bigger. That's why we don't worry about the effects of magnetization inside a copper wire most of the time.



The Field of a Magnetized Object

Bound Currents

$$\vec{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}' \quad \Rightarrow \quad \mathscr{r} = |\mathbf{r} - \mathbf{r}'|$$

- Let the magnetic dipole moment per unit volume is $\mathbf{M}$ in magnetized material.
- The vector potential of a single dipole $\mathbf{m}$ is $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi} \mathbf{m} \times \nabla' \frac{1}{\mathscr{r}}$
- In the magnetized object, each volume element $d\tau'$ carries a dipole moment $\mathbf{M}\, d\tau'$, so the total vector potential is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{M}(\mathbf{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau' = \frac{\mu_0}{4\pi} \int \left(\mathbf{M}(\mathbf{r}') \times \nabla' \frac{1}{\mathscr{r}} \right) d\tau' \quad \Leftarrow \quad \nabla' \frac{1}{\mathscr{r}} = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

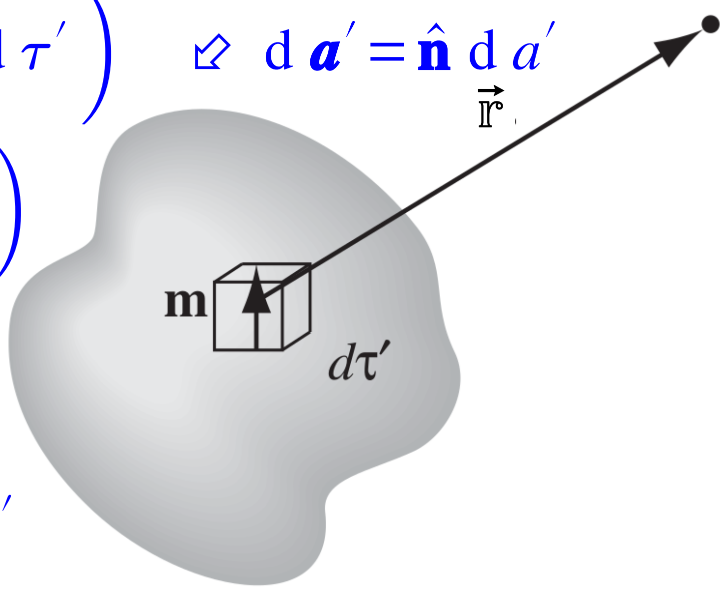
$$= \frac{\mu_0}{4\pi} \left(\int \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{\mathscr{r}} d\tau' - \int \nabla' \times \frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}} d\tau' \right) \quad ⬃ \quad d\boldsymbol{a}' = \hat{\mathbf{n}}\, da'$$

$$= \frac{\mu_0}{4\pi} \left(\int \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{\mathscr{r}} d\tau' + \oint \frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}} \times d\boldsymbol{a}' \right)$$

$$\boldsymbol{J}_b \equiv \nabla \times \mathbf{M}, \qquad \boldsymbol{K}_b \equiv \mathbf{M} \times \hat{\mathbf{n}} \quad \Rightarrow \quad \nabla \cdot \boldsymbol{J}_b = 0$$

volume current　　　surface current

$$\Rightarrow \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\boldsymbol{J}_b(\mathbf{r}')}{\mathscr{r}} d\tau' + \frac{\mu_0}{4\pi} \oint_{\mathcal{S}} \frac{\boldsymbol{K}_b(\mathbf{r}')}{\mathscr{r}} da'$$



- Instead of integrating the contributions of all the infinitesimal dipoles, we first determine the bound currents, and then find the field they produce.

$$\int \nabla\cdot\mathbf{v}\,\mathrm{d}\tau = \oint \mathbf{v}\cdot\mathrm{d}\boldsymbol{a} \quad \Leftarrow \quad \text{divergence theorem}$$

Let $\mathbf{v}\to\mathbf{v}\times\mathbf{c}$ where $\mathbf{c}$ is a constant vector $\Rightarrow \quad \nabla\times\mathbf{c}=0$

$$\nabla\cdot(\mathbf{u}\times\mathbf{w}) = \mathbf{w}\cdot(\nabla\times\mathbf{u}) - \mathbf{u}\cdot(\nabla\times\mathbf{w})$$

$$= \sum_m \hat{\mathbf{x}}^m\,\partial_m \cdot \sum_{ijk}\epsilon^{ijk}\,\hat{\mathbf{x}}_i\,u_j\,w_k = \sum_{ijk}\epsilon^{ijk}\,\partial_i\left(u_j\,w_k\right) = \sum_{ijk}\epsilon^{ijk}\left(u_j\,\partial_i\,w_k + w_k\,\partial_i\,u_j\right)$$

$$= \sum_m w_m\,\hat{\mathbf{x}}^m \cdot \sum_{ijk}\epsilon^{ijk}\,\hat{\mathbf{x}}_i\,\partial_j\,u_k - \sum_m u_m\,\hat{\mathbf{x}}^m \cdot \sum_{ijk}\epsilon^{ijk}\,\hat{\mathbf{x}}_i\,\partial_j\,w_k$$

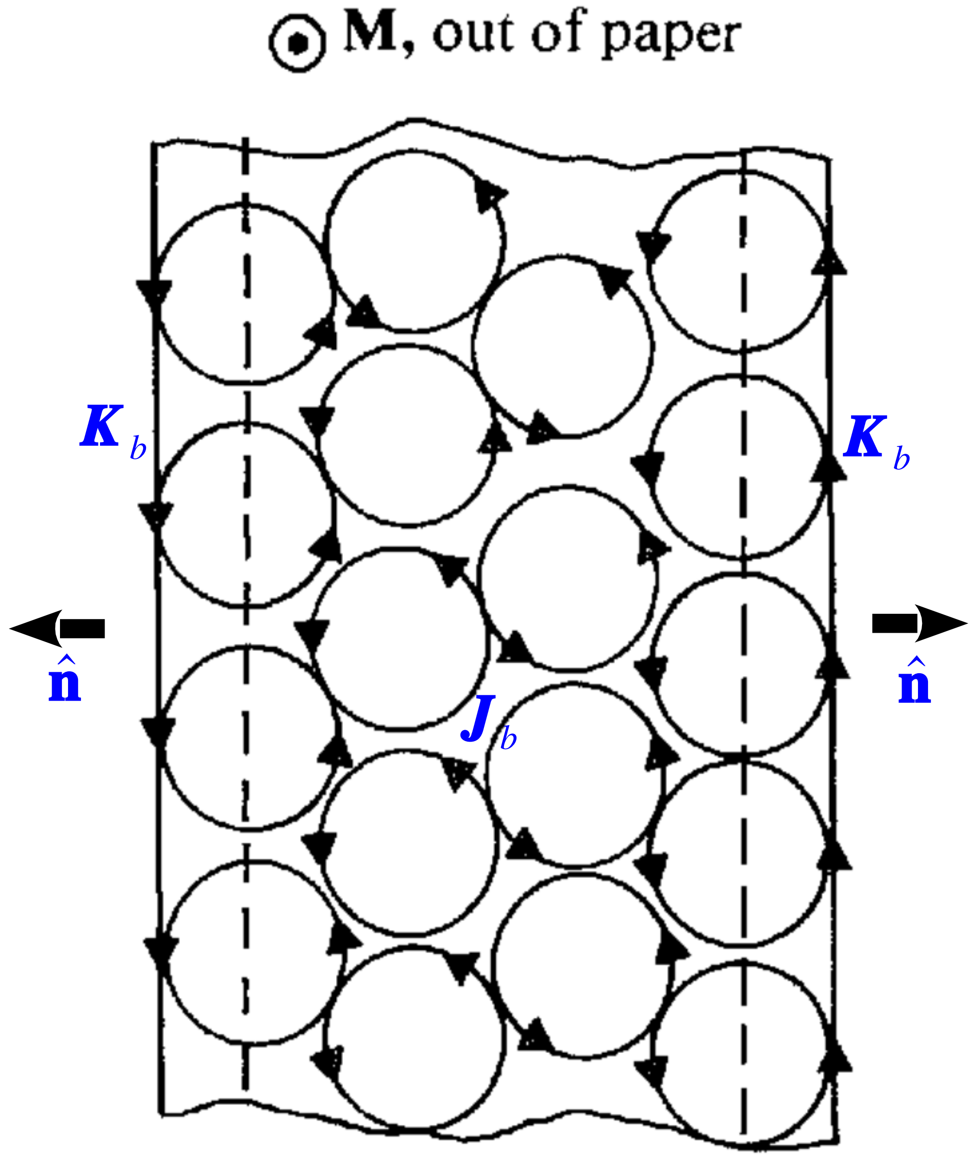
$$\Rightarrow \int \nabla\cdot(\mathbf{v}\times\mathbf{c})\,\mathrm{d}\tau = \int\left[\mathbf{c}\cdot(\nabla\times\mathbf{v}) - \mathbf{v}\cdot(\nabla\times\mathbf{c})\right]\mathrm{d}\tau = \mathbf{c}\cdot\int \nabla\times\mathbf{v}\,\mathrm{d}\tau$$

$$\oint(\mathbf{v}\times\mathbf{c})\cdot\mathrm{d}\boldsymbol{a} = \qquad\qquad \mathbf{c}\cdot\oint \mathrm{d}\boldsymbol{a}\times\mathbf{v}$$

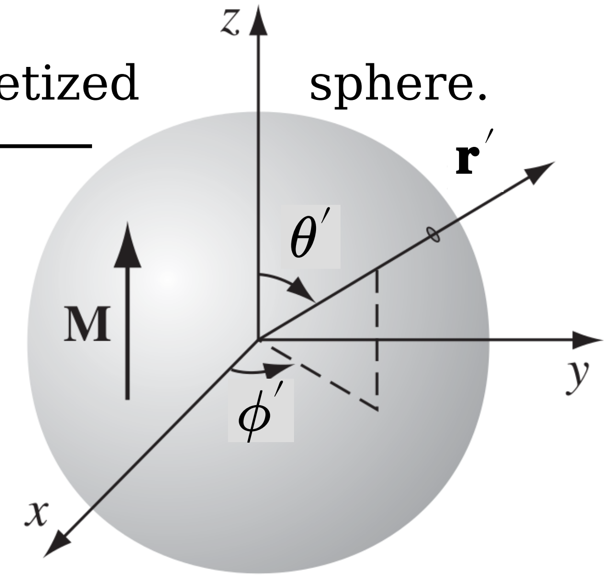
$$\Rightarrow \int_{\mathcal{V}} \nabla\times\mathbf{v}\,\mathrm{d}\tau = \oint_{\mathcal{S}} \mathrm{d}\boldsymbol{a}\times\mathbf{v} = -\oint_{\mathcal{S}} \mathbf{v}\times\mathrm{d}\boldsymbol{a} \quad \Leftarrow \quad \mathbf{c}\ \text{can be any constant.}$$

Check Problem 1.61(b).

$$\Rightarrow \quad \mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\boldsymbol{J}_b(\mathbf{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, \mathrm{d}\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\boldsymbol{K}_b(\mathbf{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, \mathrm{d}a'$$



- Notice the parallel with the electrical case: there the field of a polarized object was the same as that of a bound volume charge $\rho_b = -\nabla \cdot \mathbf{P}$ plus a bound surface charge $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$.



Example 6.1: Find the magnetic field of a uniformly magnetized sphere.

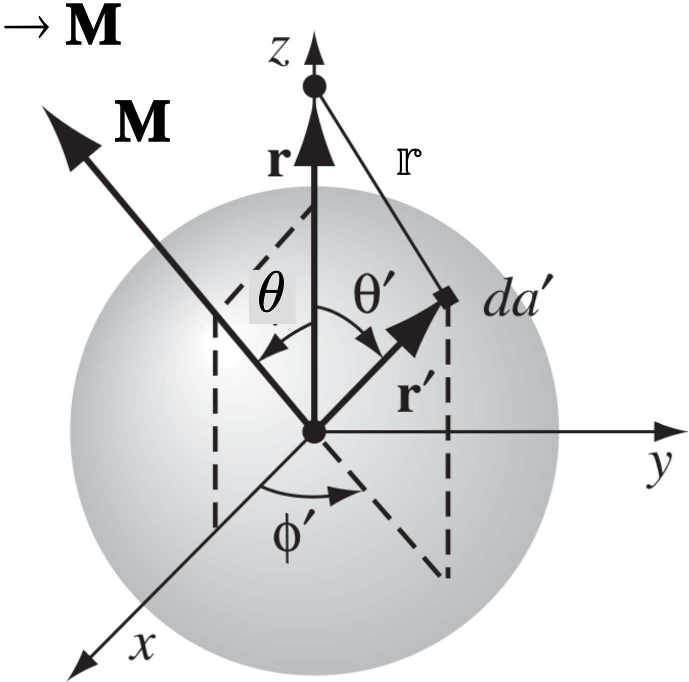
- Choosing the z axis along the direction of $\mathbf{M}$,

$$\boldsymbol{J}_b = \nabla \times \mathbf{M} = 0 , \quad \boldsymbol{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = \mathbf{M} \times \hat{\mathbf{r}}' = M \sin\theta' \, \hat{\boldsymbol{\phi}}'$$

- A rotating spherical shell, of uniform surface charge σ, corresponds to a surface current density (Ex. 5.11)

$$\boldsymbol{K} = \sigma \, \mathbf{v} = \sigma \, \omega \, R \sin\theta' \, \hat{\boldsymbol{\phi}}'$$

- So the field of a uniformly magnetized sphere is identical to the field of a spinning spherical shell, with the identification $\sigma R \boldsymbol{\omega} \to \mathbf{M}$



- The integration is easier if we let $\mathbf{r}$ lie on the z-axis, so that $\mathbf{M}$ is tilted at an angle θ. We orient the x-axis so that $\mathbf{M}$ lies in the xz-plane.

- $$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{K}_b}{\mathbb{r}} \, \mathrm{d}a' \quad \Leftarrow \quad \mathrm{d}a' = R^2 \sin\theta' \, \mathrm{d}\theta' \, \mathrm{d}\phi'$$

$$\mathbf{M} = M \left(\sin\theta \, \hat{\mathbf{x}} + \cos\theta \, \hat{\mathbf{z}} \right) , \quad \mathbb{r} = \sqrt{R^2 + r^2 - 2 r R \cos\theta'}$$

$$\boldsymbol{K}_b = \mathbf{M} \times \hat{\mathbf{r}}' = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ M \sin\theta & 0 & M\cos\theta \\ \sin\theta' \cos\phi' & \sin\theta' \sin\phi' & \cos\theta' \end{vmatrix}$$

$$= M\left[\sin\theta' \left(\sin\theta \sin\phi' \,\hat{\mathbf{z}} - \cos\theta \sin\phi' \,\hat{\mathbf{x}}\right) + \left(\cos\theta \sin\theta' \cos\phi' - \sin\theta \cos\theta'\right)\hat{\mathbf{y}}\right]$$

- $\int_0^{2\pi} \sin\phi' \, d\phi' = \int_0^{2\pi} \cos\phi' \, d\phi' = 0$

$$\Rightarrow \mathbf{A}(\mathbf{r}) = -\frac{\mu_0 M R^2 \sin\theta}{2} \hat{\mathbf{y}} \int_0^{\pi} \frac{\cos\theta' \sin\theta' \, d\theta'}{\sqrt{R^2 + r^2 - 2rR\cos\theta'}}$$

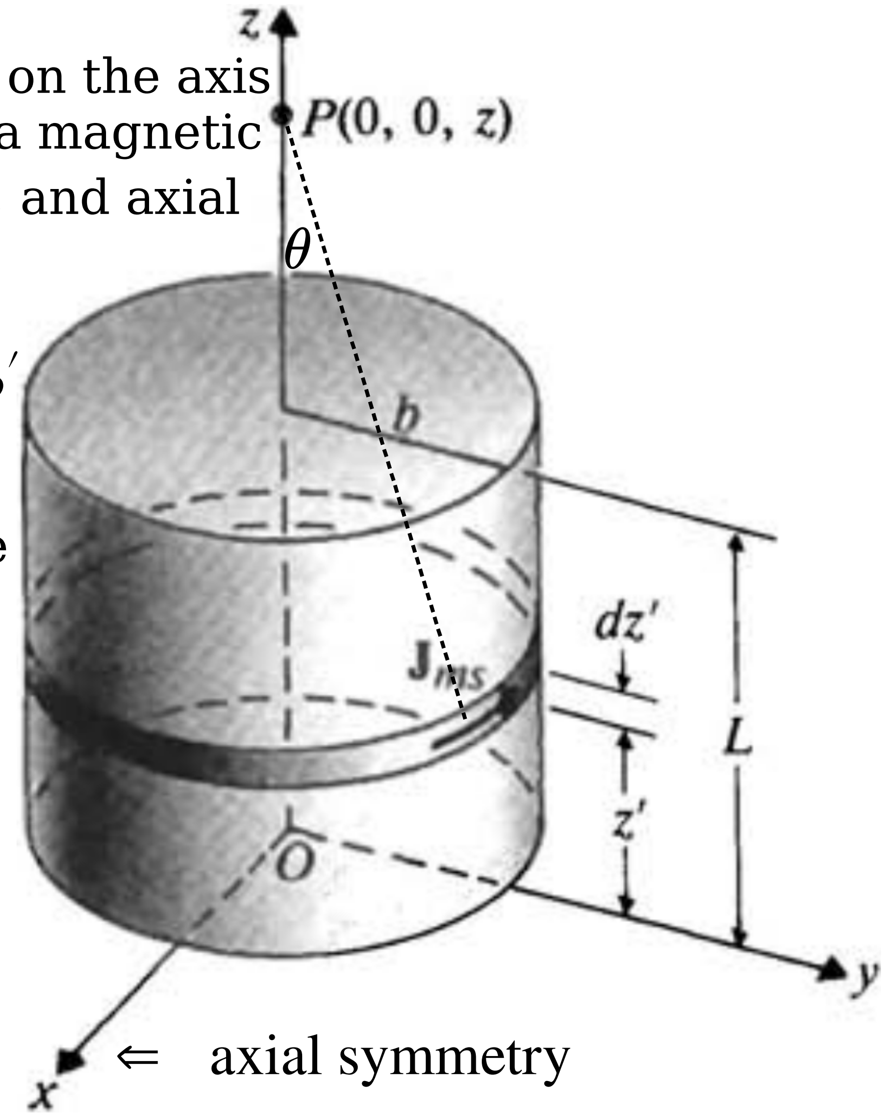
$$= \frac{\mu_0}{3} \frac{r_<^3}{r^3} \mathbf{M} \times \mathbf{r}, \quad r_< = \min(r, R) \quad \Leftarrow \quad \mathbf{M} \times \mathbf{r} = -r M \sin\theta \,\hat{\mathbf{y}}$$

$$= \frac{\mu_0 M}{3} \frac{r_<^3}{r^2} \sin\theta \,\hat{\boldsymbol{\phi}} \quad \Leftarrow \quad \text{revert the coordinates } \mathbf{M} \parallel \hat{\mathbf{z}}, \quad \mathbf{r} = (r, \theta, \phi)$$

$$\Rightarrow \mathbf{B} = \nabla \times \mathbf{A} = \begin{cases} \dfrac{2\mu_0}{3} \mathbf{M} & \textit{uniform} \text{ inside the sphere} \\ \dfrac{\mu_0}{4\pi} \dfrac{3(\hat{\mathbf{r}} \cdot \mathbf{m})\hat{\mathbf{r}} - \mathbf{m}}{r^3} & \text{outside the perfect dipole } \mathbf{m} = \dfrac{4\pi}{3} R^3 \mathbf{M} \end{cases}$$

- The internal field is uniform, like the electric field inside a uniformly polarized sphere, although the factors are different, 2/3 in place of −1/3. The external fields are also analogous: pure dipole in both instances.

Example: Determine the magnetic flux density on the axis of a uniformly magnetized circular cylinder of a magnetic material. The cylinder has a radius b, length L, and axial magnetization $\mathbf{M} = M\,\hat{\mathbf{z}}$.



$$\boldsymbol{J}_b = \nabla \times \mathbf{M} = 0\,, \quad \boldsymbol{K}_b = \mathbf{M} \times \hat{\mathbf{n}} \Rightarrow \mathbf{M} \times \hat{\mathbf{s}}' = M\,\hat{\boldsymbol{\phi}}'$$

The magnet is like a cylindrical sheet with a lineal current density of M. There is no surface current on the top and bottom faces.

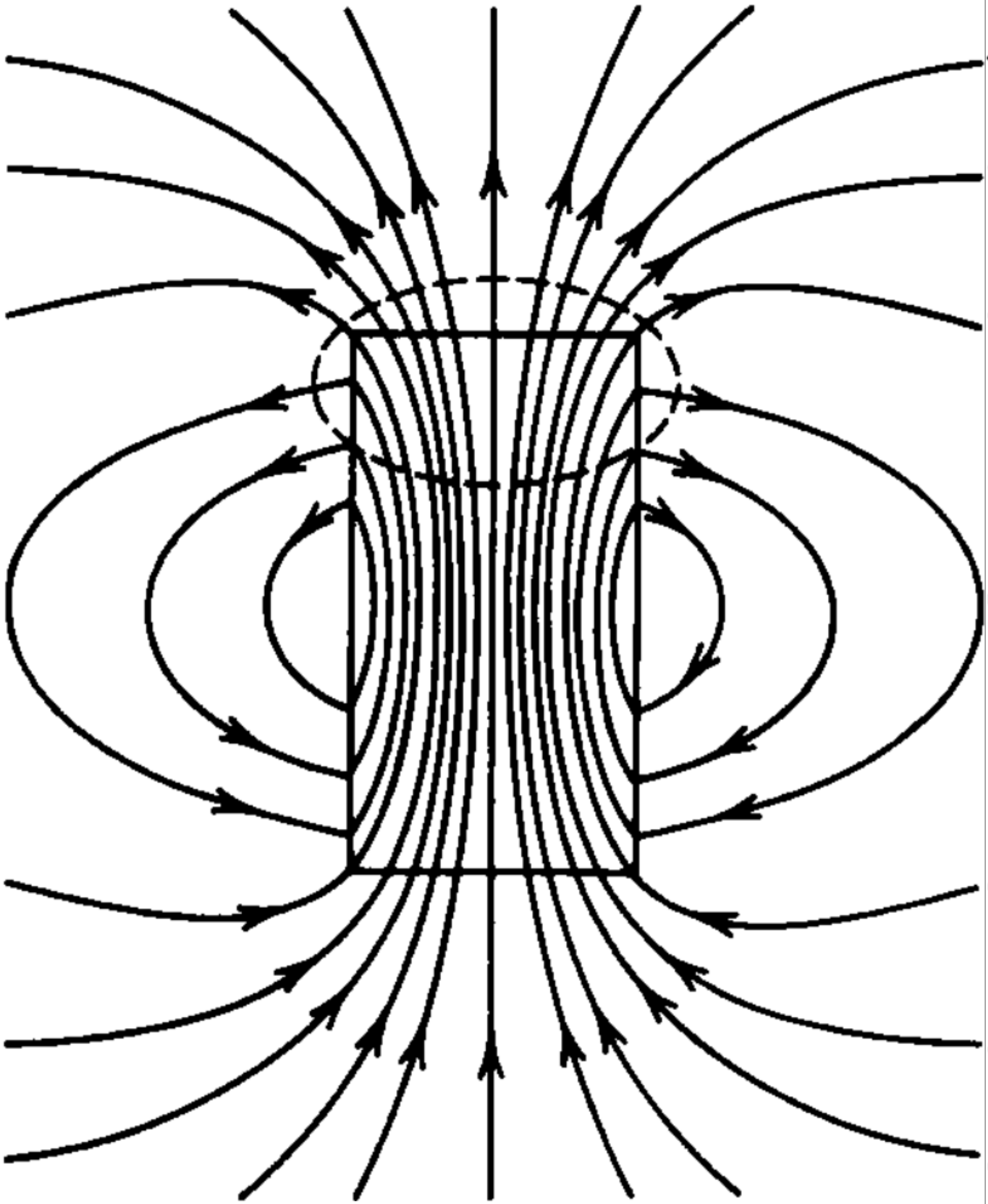
$$\mathrm{d}\,\mathbf{B} = \frac{\mu_0}{4\pi} \frac{M\,\hat{\boldsymbol{\phi}}' \times \left(-b\,\hat{\mathbf{s}}' + (z - z')\,\hat{\mathbf{z}}\right)}{\left[(z' - z)^2 + b^2\right]^{3/2}}\,\mathrm{d}\,a'$$

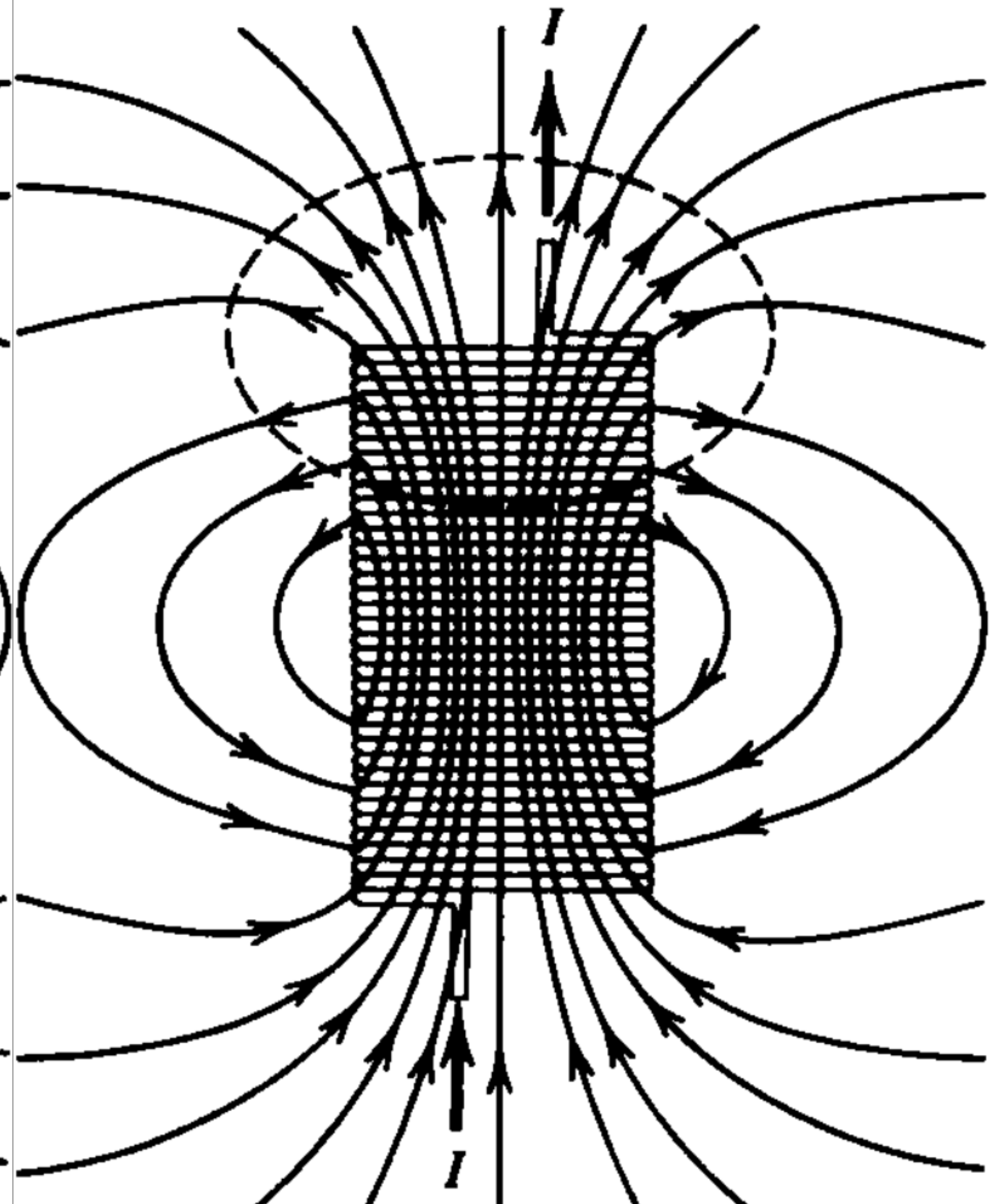
$$= \frac{\mu_0 M}{4\pi} \frac{(z - z')\,\hat{\mathbf{s}}' + b\,\hat{\mathbf{z}}}{\left[(z' - z)^2 + b^2\right]^{3/2}}\,b\,\mathrm{d}\,\phi'\,\mathrm{d}\,z'$$

$$\Rightarrow \quad \mathbf{B} = \int \mathrm{d}\,\mathbf{B} = \frac{\mu_0 M}{4\pi}\,\hat{\mathbf{z}} \int \frac{b^2\,\mathrm{d}\,\phi'\,\mathrm{d}\,z'}{\left[(z' - z)^2 + b^2\right]^{3/2}} \quad \Leftarrow \quad \text{axial symmetry}$$

$$= \frac{\mu_0 M}{2}\,\hat{\mathbf{z}} \int_0^L \frac{-b^2\,\mathrm{d}\,(z - z')}{\left[(z - z')^2 + b^2\right]^{3/2}} = -\frac{\mu_0 M}{2}\,\hat{\mathbf{z}} \int_{\theta_0}^{\theta_L} \sin^3\theta\,\mathrm{d}\cot\theta = \frac{\mu_0 M}{2}\,\hat{\mathbf{z}} \int_{\theta_0}^{\theta_L} \sin\theta\,\mathrm{d}\,\theta$$

$$= -\frac{\mu_0 M}{2} \cos\theta\,\hat{\mathbf{z}}\Big|_{\theta_0}^{\theta_L} = \frac{\mu_0 M}{2} \left(\frac{z}{\sqrt{z^2 + b^2}} - \frac{z - L}{\sqrt{(z - L)^2 + b^2}} \right) \hat{\mathbf{z}}$$

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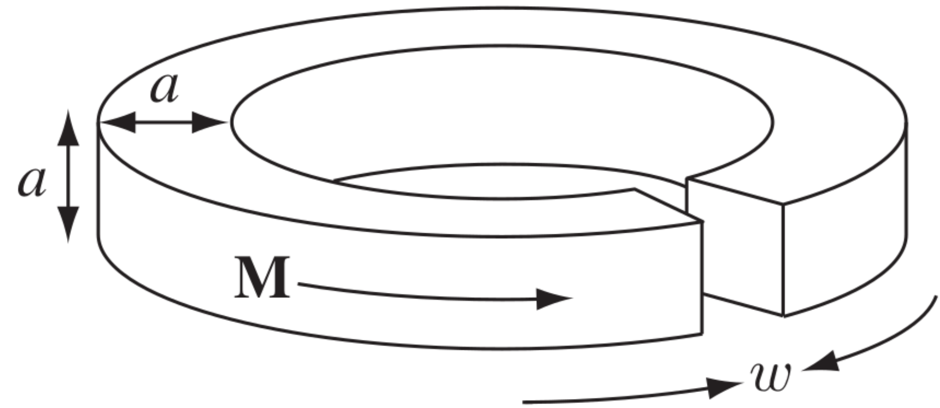
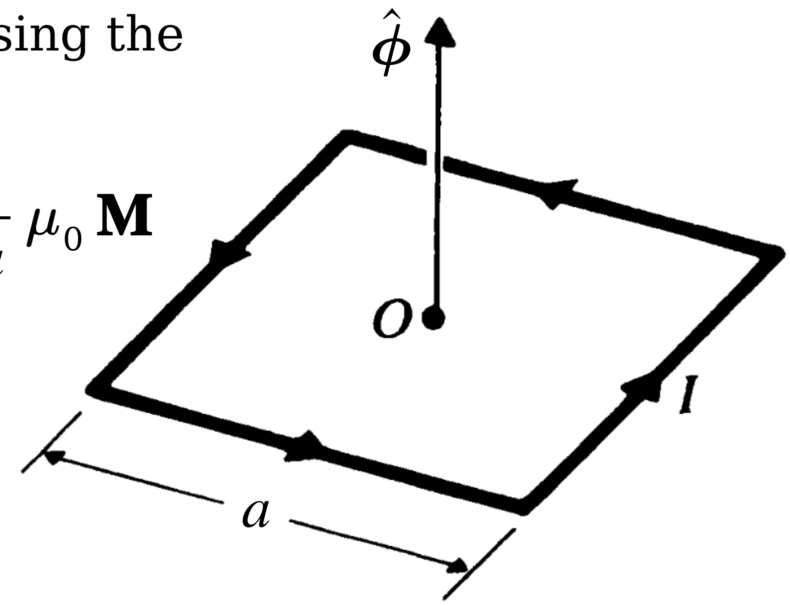
<u>Problem 6.10</u>: An iron rod of length L and square cross section of side a is given a uniform longitudinal magnetization $\mathbf{M}$, then bent around into a circle with a narrow gap of width w. Find the magnetic field at the center of the gap, assuming $w \ll a \ll L$.



$$K_b = M \quad \Rightarrow \quad \mathbf{B}_{\text{inside}} = \mu_0 M \,\hat{\boldsymbol{\phi}} = \mu_0 \mathbf{M} \quad \text{for a } \textit{complete} \text{ ring} \quad \Leftarrow \quad \mathbf{B} = \frac{\mu_0 I_{\text{total}}}{2\pi R}\,\hat{\boldsymbol{\phi}}$$

$$2\pi R \simeq L$$

The magnetic field at the center of a square loop, which carries a steady current $I = w K_b$ (from the quiz, using the result of Ex. 5.5), is



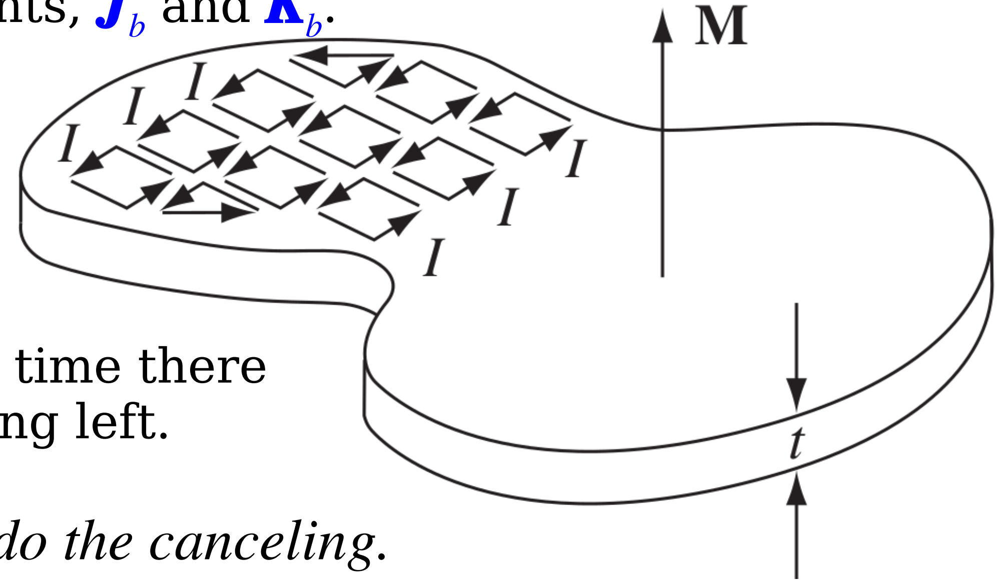
$$\mathbf{B}_{\text{square}} = 2\sqrt{2}\,\frac{\mu_0 I}{\pi a}\,\hat{\boldsymbol{\phi}} = 2\sqrt{2}\,\frac{w}{\pi a}\,\mu_0 M \,\hat{\boldsymbol{\phi}} = 2\sqrt{2}\,\frac{w}{\pi a}\,\mu_0 \mathbf{M}$$

So the net field in the gap is, by superposition,

$$\mathbf{B}_{\text{gap}} = \mathbf{B}_{\text{inside}} - \mathbf{B}_{\text{square}} = \left(1 - 2\sqrt{2}\,\frac{w}{\pi a}\right)\mu_0 \mathbf{M}$$

Physical Interpretation of Bound Currents

- The field of a magnetized object is identical to the field that would be produced by a certain distribution of “bound” currents, $\boldsymbol{J}_b$ and $\boldsymbol{K}_b$.



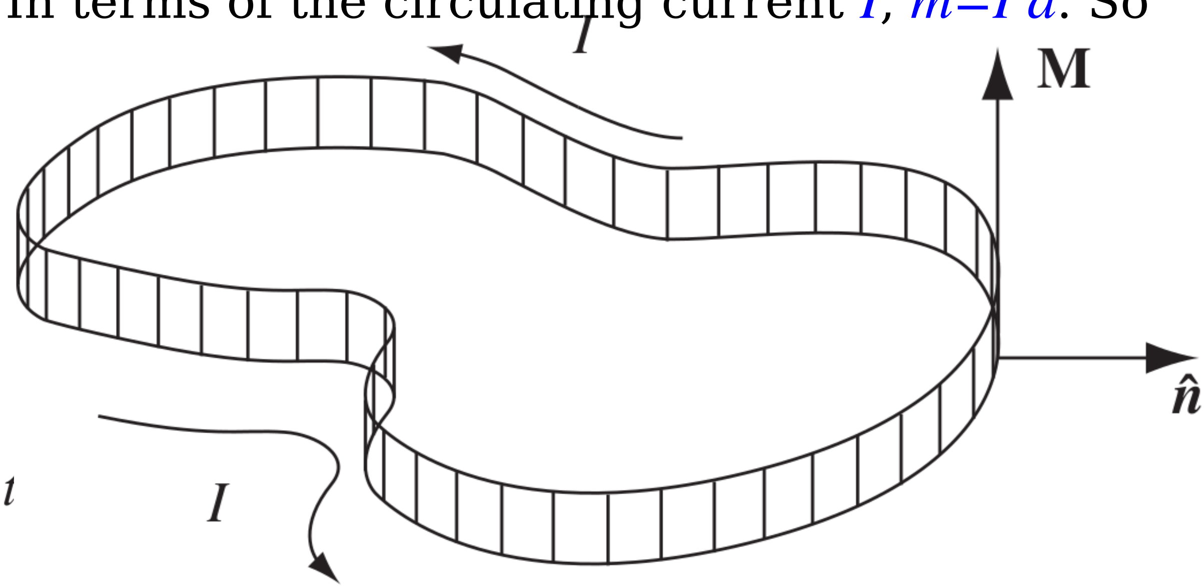
- Let the dipoles be represented by tiny current loops in a thin slab of *uniformly* magnetized material.

- All the “internal” currents cancel: every time there is one going right, a contiguous one is going left.

- At the edge there is *no adjacent loop to do the canceling*. The whole thing is equivalent to a ribbon of current I around the boundary.

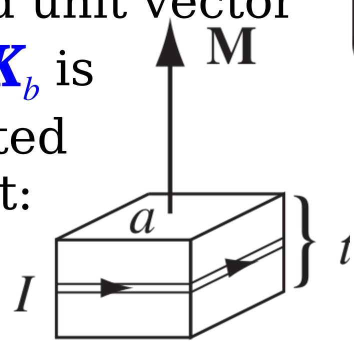
- Each of the tiny loops has area a and thickness t. In terms of the magnetization M, its dipole moment is $m=M\,a\,t$. In terms of the circulating current I, $m=I\,a$. So $I = M\,t$, so the surface current is

$$K_b = \frac{I}{t} = M$$



- Using the outward unit vector $\hat{\mathbf{n}}$, the direction of $\boldsymbol{K}_b$ is conveniently indicated by the cross product:

$$\boldsymbol{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$$



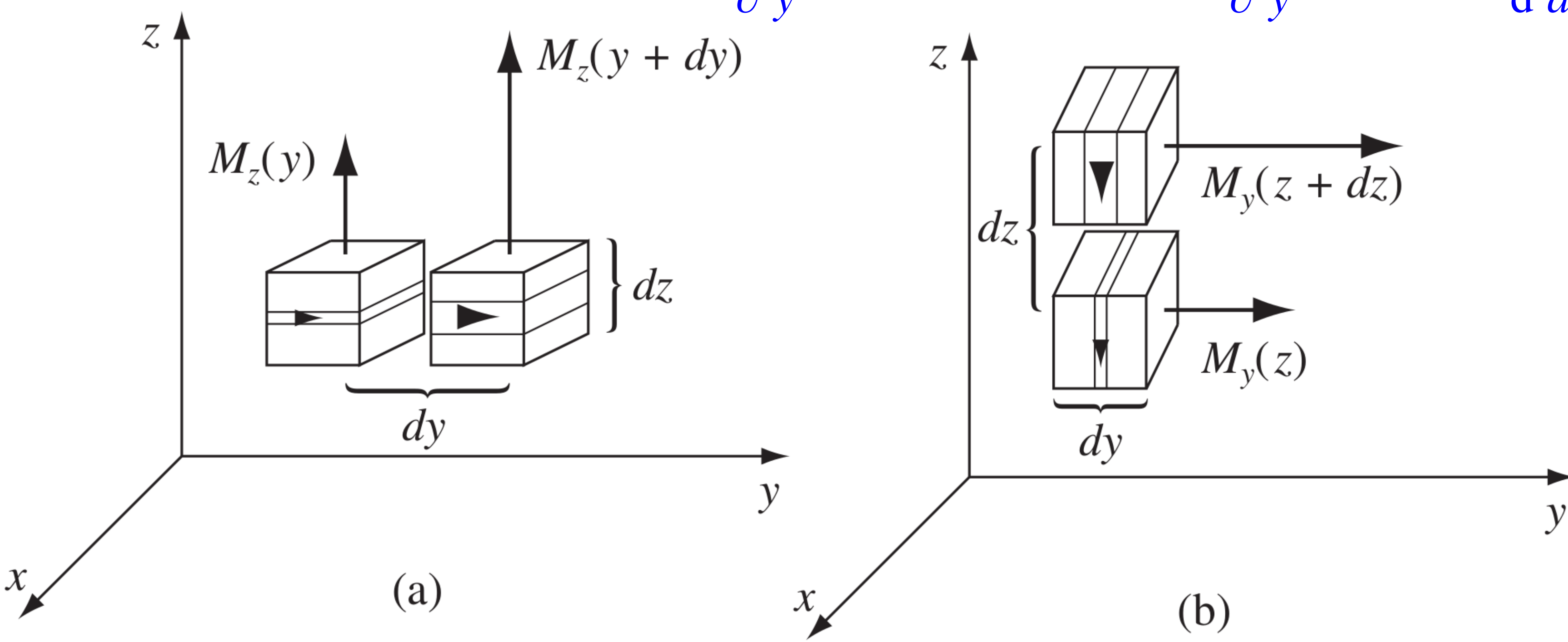
- This expression also indicates that there is *no* current on the top or bottom surface of the slab; here $\mathbf{M}$ is parallel to $\hat{\mathbf{n}}$, so the cross product vanishes.

- It is a *peculiar* kind of current, no single charge makes the whole trip, each charge moves only in a tiny little loop within a single atom. But the net effect is a macroscopic current flowing over the surface of the magnetized object.

- When the magnetization is nonuniform, the internal currents no longer cancel.

- On the surface where they join, there is a net current in the x direction,

$$I_x = [M_z(y + \mathrm{d}y) - M_z(y)]\,\mathrm{d}z = \frac{\partial M_z}{\partial y}\,\mathrm{d}y\,\mathrm{d}z \quad \Rightarrow \quad (J_b)_x = \frac{\partial M_z}{\partial y} \quad \Leftarrow \quad \boldsymbol{J} \equiv \frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}a_\perp}$$



- So a nonuniform magnetization in the y direction would contribute an amount

$$-\frac{\partial M_y}{\partial z} \quad \Rightarrow \quad (J_b)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z} \quad \Rightarrow \quad \boldsymbol{J}_b = \nabla \times \mathbf{M}$$

- Like any other steady current, $\boldsymbol{J}_b$ should obey the conservation law $\nabla \cdot \boldsymbol{J}_b = 0$, for the divergence of a curl is always 0.

The Magnetic Field Inside Matter

- The actual microscopic magnetic field inside matter fluctuates wildly from point to point and instant to instant.

- When we speak of "the" magnetic field in matter, we mean the macroscopic field: the average over regions large enough to contain many atoms. The magnetization $\mathbf{M}$ is "smoothed out" in the same sense.

- It is this macroscopic field that one obtains when the methods in this chapter are applied to points inside magnetized material.

The Auxiliary Field H

Ampère's Law in Magnetized Materials

- The field due to magnetization of the medium is the field produced by these bound currents. The field can come from others called the **free current**.

- In any event, the total current can be written as $\boldsymbol{J} = \boldsymbol{J}_b + \boldsymbol{J}_f$

- The free current is there because a wire is connected to a battery—it involves actual transport of charge; the bound current is there because of magnetization—it results from the conspiracy of many aligned atomic dipoles.

- Ampère's law can be written as

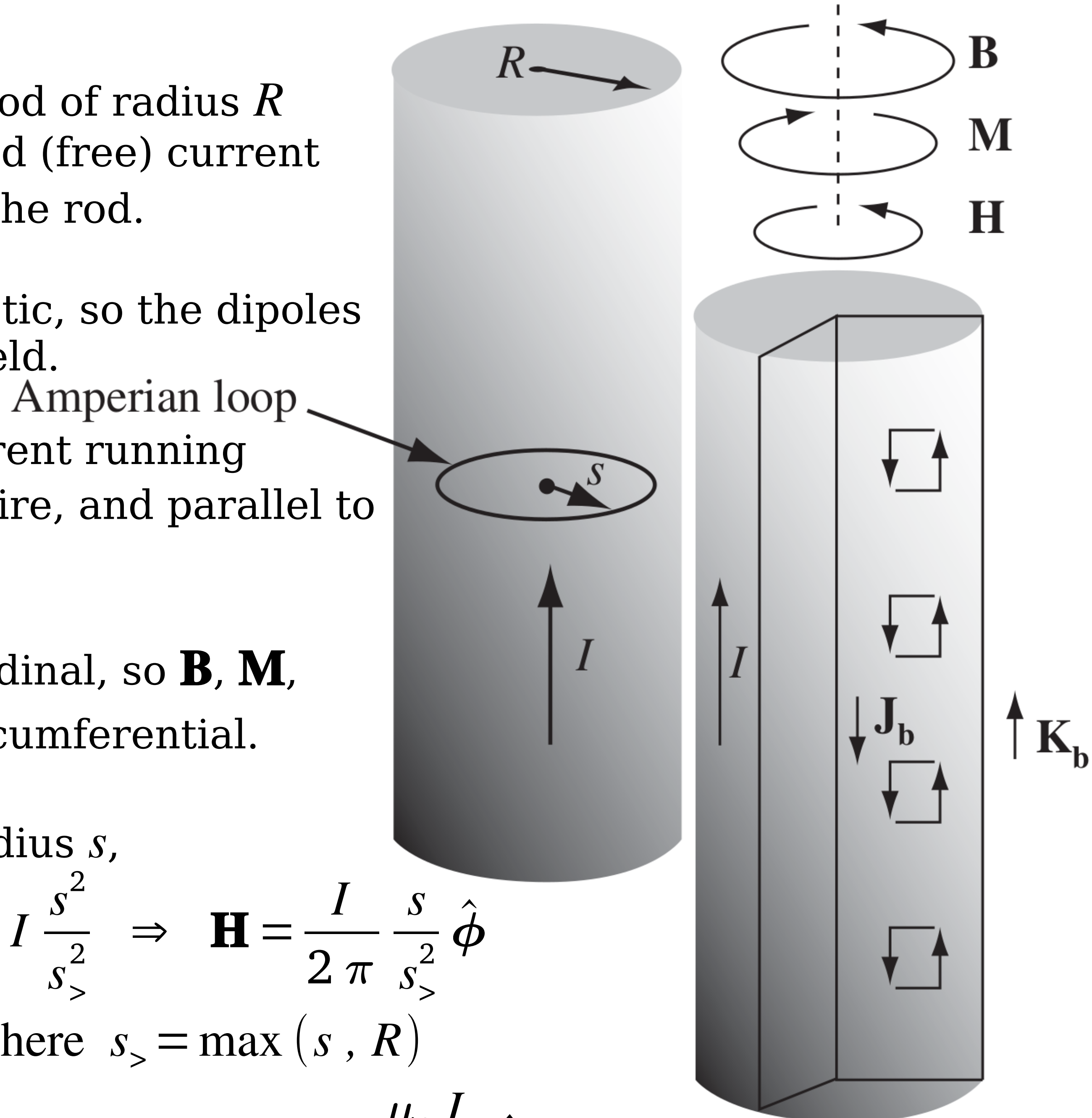
$$\frac{1}{\mu_0}\nabla\times\mathbf{B} = \boldsymbol{J} = \boldsymbol{J}_f + \boldsymbol{J}_b = \mathbf{J}_f + \nabla\times\mathbf{M} \;\Rightarrow\; \nabla\times\left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \boldsymbol{J}_f$$

$$\Rightarrow\; \boxed{\mathbf{H} \equiv \frac{\mathbf{B}}{\mu_0} - \mathbf{M}} \;\Rightarrow\; \boxed{\nabla\times\mathbf{H} = \boldsymbol{J}_f} \;\Rightarrow\; \boxed{\oint \mathbf{H}\cdot d\boldsymbol{\ell} = I_{f_{\text{enc}}}} \;\Leftarrow\; I_{f_{\text{enc}}}: \text{total free current}$$

- **H** plays a role in magnetostatics analogous to **D** in electrostatics: **D** allows to write *Gauss's* law in terms of the free *charge* alone, **H** allows to express *Ampere's* law in terms of the free *current* alone, what we control directly.

- Bound current, like bound charge, comes along for the ride—the material gets magnetized, and this results in bound currents; we cannot turn them on or off independently, as we can free currents.

Example 6.2: A long copper rod of radius R carries a uniformly distributed (free) current I. Find $\mathbf{H}$ inside and outside the rod.



- Copper is weakly diamagnetic, so the dipoles will line up opposite to the field.

- This results in a bound current running antiparallel to I, within the wire, and parallel to I along the surface.

- All the currents are longitudinal, so $\mathbf{B}$, $\mathbf{M}$, and therefore also $\mathbf{H}$, are circumferential.

- For an Amperian loop of radius s,

$$\oint \mathbf{H} \cdot d\boldsymbol{\ell} = H \cdot 2\pi s = I_{f_{\text{enc}}} = I \frac{s^2}{s_>^2} \quad \Rightarrow \quad \mathbf{H} = \frac{I}{2\pi} \frac{s}{s_>^2} \hat{\boldsymbol{\phi}}$$

where $s_> = \max(s, R)$

$$\mathbf{M}(s \geq R) = 0 \quad \Rightarrow \quad \mathbf{B}(s \geq R) = \mu_0 \mathbf{H}(s \geq R) = \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}}$$

same as for a nonmagnetized wire (Ex. 5.7). $\mathbf{B}$ inside cannot be determined yet.

- For linear media (mentioned later) $\mathbf{M} = \chi_m \mathbf{H} = \dfrac{\chi_m I s}{2\pi R^2}\hat{\boldsymbol{\phi}}$ for $s < R$ (and χ_m<0),

$$\Rightarrow \quad \boldsymbol{J}_b = \nabla \times \mathbf{M} = \frac{\chi_m I}{\pi R^2}\hat{\mathbf{z}}, \quad \boldsymbol{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = \begin{cases} -\dfrac{\chi_m I}{2\pi R}\hat{\mathbf{z}} & \text{for } \hat{\mathbf{n}} = \hat{\mathbf{s}} \\ \pm\dfrac{\chi_m I}{2\pi R^2}\mathbf{s} & \text{for } \hat{\mathbf{n}} = \pm\hat{\mathbf{z}} \end{cases}$$

$$\Rightarrow \quad \oint_{\mathcal{C}} \boldsymbol{K}_b \cdot \mathrm{d}\boldsymbol{\ell} = -\chi_m I = -\int_{\mathcal{S}} \boldsymbol{J}_b \cdot \mathrm{d}\boldsymbol{a} \quad \Rightarrow \quad \mathbf{B}_{\text{in}} = \mu_0 (\mathbf{H}_{\text{in}} + \mathbf{M}) = \mu_0 (1 + \chi_m)\mathbf{H}_{\text{in}}$$

- **H** is a more useful quantity than **D**. This is because to build an electromagnet you run a certain (free) current through a coil. The *current* determines **H** (or the line integral of **H**); **B** depends on the specific materials you used and even on the history of the magnet.

- On the other hand, to set up an *electric* field, you do *not* plaster a known free charge on the plates of a parallel plate capacitor; you connect them to a battery of known *voltage*. It's the *potential difference* determines **E** (or the line integral of **E**); **D** depends on the details of the dielectric.

- Theoretically, **D** and **H** are on an equal footing.

- Many authors call **H**, not **B**, the "magnetic field," and call **B** the "flux density," or magnetic "induction." We will continue to call **B** the "magnetic field."

A Deceptive Parallel

- Be aware that $\mu_0 \mathbf{H}$ is "just like $\mathbf{B}$, only its source is $\boldsymbol{J}_f$ instead of $\boldsymbol{J}$." For the curl alone does not determine a vector field—you must also know the divergence.

- For $\nabla \cdot \mathbf{B} = 0$, the divergence of $\mathbf{H}$ is not, in general, 0. In fact, $\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}$. Only when $\nabla \cdot \mathbf{M} = 0$ is the parallel between $\mathbf{B}$ and $\mu_0 \mathbf{H}$ faithful.

- Consider a cylinder of iron that carries a permanent uniform magnetization $\mathbf{M}$ parallel to its axis. In this case there is no free current anywhere, $\oint \mathbf{H} \cdot d\ell = I_f$ might lead you to suppose that $\mathbf{H} = 0$, and hence that $\mathbf{B} = \mu_0 \mathbf{M}$ inside the magnet and $\mathbf{B} = 0$ outside, which is nonsense. It is true that the *curl* of $\mathbf{H}$ vanishes everywhere, but the divergence does not. (Can you see where $\nabla \cdot \mathbf{M} \neq 0$?)

- To find $\mathbf{B}$/$\mathbf{H}$ in a problem involving magnetic materials, first look for symmetry. If the problem exhibits cylindrical, plane, solenoidal, or toroidal symmetry, then you can get $\mathbf{H}$ directly from the above eqn by the usual Ampère's law methods.

- In such cases $\nabla \cdot \mathbf{M} = 0$ since the free current alone determines the answer ($\nabla \cdot \mathbf{H} = 0$). If any symmetry is absent, you have to think of another approach, and in particular you must not assume that $\mathbf{H} = 0$ just because there is no free current.

Boundary Conditions

- The magnetostatic boundary conditions can be rewritten in terms of **H** and the free current:

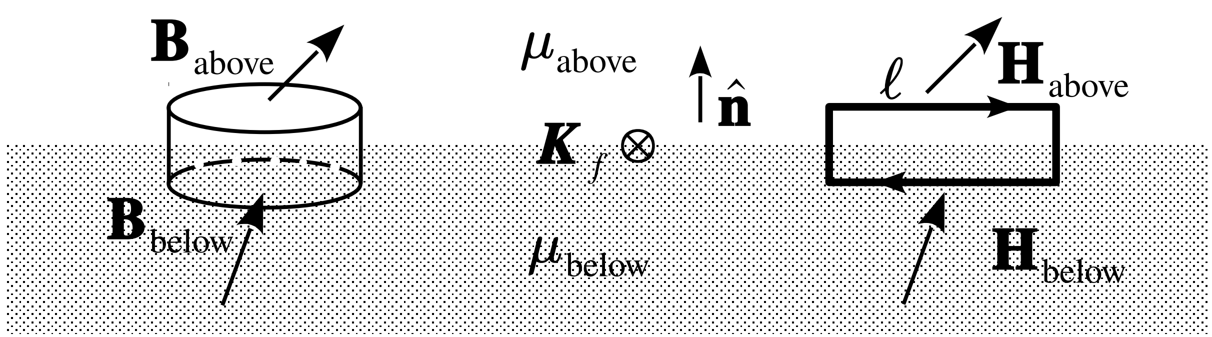
$$\begin{aligned}\nabla\cdot\mathbf{H} &= -\nabla\cdot\mathbf{M}\\ \nabla\times\mathbf{H} &= \mathbf{J}_f\end{aligned} \quad\Rightarrow\quad \begin{aligned}H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} &= -\left(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}}\right)\\ \mathbf{H}^{\parallel}_{\text{above}} - \mathbf{H}^{\parallel}_{\text{below}} &= \mathbf{K}_f\times\hat{\mathbf{n}}\end{aligned}$$

- In the presence of materials, these are sometimes more useful than the corresponding boundary conditions on **B**:

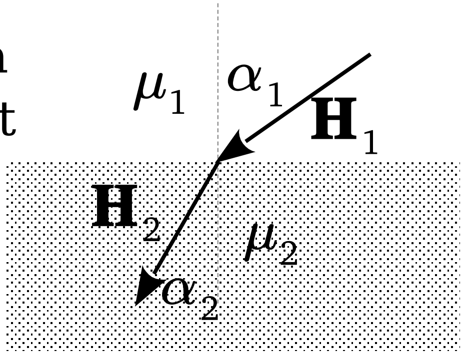
$$\begin{aligned}B^{\perp}_{\text{above}} - B^{\perp}_{\text{below}} &= 0\\ \mathbf{B}^{\parallel}_{\text{above}} - \mathbf{B}^{\parallel}_{\text{below}} &= \mu_0\mathbf{K}\times\hat{\mathbf{n}}\end{aligned} \quad\Leftrightarrow\quad \begin{aligned}\mathbf{A}_{\text{above}} - \mathbf{A}_{\text{below}} &= 0\\ \frac{\partial}{\partial n}\mathbf{A}_{\text{above}} - \frac{\partial}{\partial n}\mathbf{A}_{\text{below}} &= -\mu_0\mathbf{K}\end{aligned}$$

- For linear media (as mentioned later),

$$B^{\perp}_{\text{above}} - B^{\perp}_{\text{below}} = 0 \quad\Rightarrow\quad \mu_{\text{above}}H^{\perp}_{\text{above}} - \mu_{\text{below}}H^{\perp}_{\text{below}} = 0$$



Example: 2 magnetic media with permeabilities μ_1 & μ_2 have a common boundary. The magnetic field intensity in medium 1 at has a magnitude $\mathbf{H}_1$ and makes an angle α_1 with the normal. Determine the magnitude and the direction of the magnetic field intensity in medium 2.



$$\mu_{\text{above}} H_{\text{above}}^{\perp} - \mu_{\text{below}} H_{\text{below}}^{\perp} = 0 \quad \Rightarrow \quad \mu_1 H_1 \cos\alpha_1 = \mu_2 H_2 \cos\alpha_2$$

$$\mathbf{H}_{\text{above}}^{\parallel} - \mathbf{H}_{\text{below}}^{\parallel} = \boldsymbol{K}_f \times \hat{\mathbf{n}} = 0 \qquad H_1 \sin\alpha_1 = H_2 \sin\alpha_2$$

$$\Rightarrow \quad \frac{\tan\alpha_2}{\tan\alpha_1} = \frac{\mu_2}{\mu_1} \quad \Rightarrow \quad \alpha_2 = \tan^{-1}\left(\frac{\mu_2}{\mu_1}\tan\alpha_1\right)$$

$$\Rightarrow \quad H_2 = \sqrt{H_{2\perp}^2 + H_{2\parallel}^2} = \sqrt{(H_2\cos\alpha_2)^2 + (H_2\sin\alpha_2)^2}$$

$$= H_1\sqrt{\left(\frac{\mu_1}{\mu_2}\cos\alpha_1\right)^2 + \sin^2\alpha_1} = \frac{H_1}{\mu_2}\sqrt{\mu_1^2\cos^2\alpha_1 + \mu_2^2\sin^2\alpha_1}$$

Linear and Nonlinear Media

Magnetic Susceptibility and Permeability

- For most substances the magnetization is *proportional* to the field, provided the field is not too strong.

- $\mathbf{M} = \chi_m \mathbf{H}$ (#) $\Leftarrow$ χ_m : magnetic susceptibility

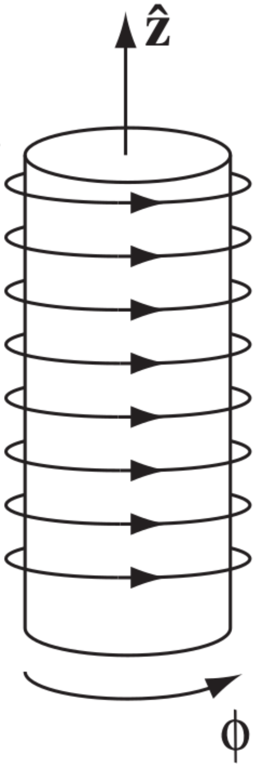
- χ_m is positive (negative) for paramagnets (diamagnets). Typical values$\sim 10^{-5}$.

- Materials that obey (#) are called **linear media**:

$$\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}) = \mu_0(1+\chi_m)\mathbf{H} = \mu\,\mathbf{H} \quad \Leftarrow \quad \text{permeability } \mu \equiv \mu_0(1+\chi_m) = \mu_0\,\mu_r$$

Thus $\mathbf{B}$ is also proportional to $\mathbf{H}$. μ_0 is called the **permeability of free space**.

Material	Susceptibility	Material	Susceptibility
Diamagnetic:		*Paramagnetic:*	
Bismuth	-1.7×10^{-4}	Oxygen (O_2)	1.7×10^{-6}
Gold	-3.4×10^{-5}	Sodium	8.5×10^{-6}
Silver	-2.4×10^{-5}	Aluminum	2.2×10^{-5}
Copper	-9.7×10^{-6}	Tungsten	7.0×10^{-5}
Water	-9.0×10^{-6}	Platinum	2.7×10^{-4}
Carbon Dioxide	-1.1×10^{-8}	Liquid Oxygen ($-200°$ C)	3.9×10^{-3}
Hydrogen (H_2)	-2.1×10^{-9}	Gadolinium	4.8×10^{-1}



Example 6.3: An infinite solenoid (n turns/unit length, current I) is filled with linear material of susceptibility χ_m. Find the magnetic field inside the solenoid.

- This is one of those symmetrical cases in which we can get $\mathbf{H}$ from the free current alone, using Ampère's law

$$\oint \mathbf{H}\cdot d\boldsymbol{\ell} = I_{f_{\text{enc}}} \quad \Rightarrow \quad \mathbf{H} = n\,I\,\hat{\mathbf{z}} \quad \Rightarrow \quad \mathbf{B} = \mu\,n\,I\,\hat{\mathbf{z}} = \mu_0\left(1+\chi_m\right)n\,I\,\hat{\mathbf{z}}$$

- If the matter is paramagnetic/diamagnetic, $\mathbf{B}$ is enhanced/reduced.

- $\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = \chi_m\,\mathbf{H}\times\hat{\mathbf{n}}$ is in the same/opposite direction as I for $\chi_m \gtrless 0$.
$$= \chi_m\,n\,I\,\hat{\boldsymbol{\phi}}$$

- Linear media still does not escape the defect in the parallel between $\mathbf{B}$ and $\mathbf{H}$.

$$\nabla\cdot\mathbf{H} = \nabla\cdot\frac{\mathbf{B}}{\mu} = \frac{1}{\mu}\nabla\cdot\mathbf{B} + \mathbf{B}\cdot\nabla\frac{1}{\mu} = \mathbf{B}\cdot\nabla\frac{1}{\mu} \neq 0 \text{ in general}$$

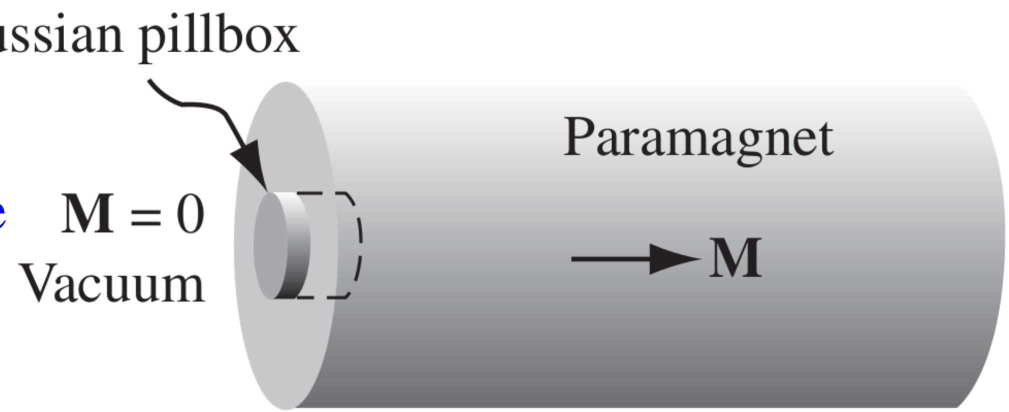
- At the boundary between 2 materials of different permeability, the divergence of $\mathbf{M}$ can actually be ∞.

- At the end of a cylinder of linear paramagnetic material, $\mathbf{M}$ is 0 on one side but not on the other.

- In the plot,

$$\oint \mathbf{M}\cdot d\boldsymbol{a} \neq 0 \quad \Rightarrow \quad \nabla\cdot\mathbf{M} \neq 0 \text{ everywhere}$$

$$\Rightarrow \quad \nabla\cdot\mathbf{H} \neq 0 \text{ everywhere}$$



- Incidentally, the volume bound current density in a homogeneous linear material is proportional to the *free* current density:

$$\boldsymbol{J}_b = \nabla\times\mathbf{M} = \nabla\times(\chi_m\mathbf{H}) = \chi_m \boldsymbol{J}_f$$

- Unless free current actually flows *through* the material, all bound current will be at the surface.

Selected problems: 3, 8, 13, 16, 18, 21, 23

Magnetic Scalar Field

- $\boldsymbol{J}=0 \quad \Rightarrow \quad \nabla\times\mathbf{H}=0 \quad \Rightarrow \quad \mathbf{H}=-\nabla\Phi \quad \Leftarrow \quad \Phi:$ magnetic scalar potential

  $\mathbf{B}=\mu\,\mathbf{H} \quad \Rightarrow \quad \nabla\cdot(\mu\nabla\Phi)=0$

  For $\mu=\text{const} \quad \Rightarrow \quad \nabla^2\Phi=0$ **+** the boundary conditions for $\mathbf{H}$

  $\nabla^2\Psi=0 \quad \Leftarrow \quad \mathbf{B}=-\nabla\Psi$ **+** the boundary conditions for $\mathbf{B}$

- For $\mathbf{M}$ given and $\boldsymbol{J}=0$

$$\nabla\cdot\mathbf{B}=\mu_0\nabla\cdot(\mathbf{H}+\mathbf{M})=0 \quad \Rightarrow \quad \nabla\cdot\mathbf{H}=-\nabla^2\Phi=\rho_M\Leftarrow-\nabla\cdot\mathbf{M} \quad \Rightarrow \quad \nabla^2\Phi=-\rho_M$$

$$\Rightarrow \quad \Phi(\mathbf{r})=\frac{1}{4\pi}\int\frac{\rho_M(\mathbf{r}')}{\mathscr{r}}\,d\tau'=-\frac{1}{4\pi}\int\frac{\nabla'\cdot\mathbf{M}(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$ if no boundary surface

$$=\frac{1}{4\pi}\left(\int\mathbf{M}\cdot\nabla'\frac{1}{\mathscr{r}}\,d\tau'-\oint_{r'\to\infty}\frac{\mathbf{M}}{\mathscr{r}}\cdot d\boldsymbol{a}'\right) \quad \Leftarrow \quad \mathbf{M}$$ well behaved & localized

$$\Rightarrow \quad \Phi(\mathbf{r})=-\frac{1}{4\pi}\nabla\cdot\int\frac{\mathbf{M}}{\mathscr{r}}\,d\tau' \quad (@) \quad \Leftarrow \quad \nabla'\frac{1}{\mathscr{r}}=-\nabla\frac{1}{\mathscr{r}}$$

$$\approx-\frac{1}{4\pi}\nabla\frac{1}{r}\cdot\int\mathbf{M}\,d\tau'=\frac{\mathbf{m}\cdot\hat{\mathbf{r}}}{4\pi r^2} \quad \Leftarrow \quad \mathbf{m}\equiv\int\mathbf{M}\,d\tau\,, \quad r\gg 0$$

- An arbitrary localized distribution of magnetization asymptotically has a dipole field with strength given by the total magnetic moment of the distribution.

- If the magnetized material has a volume and surface, we specify $\mathbf{M}$ inside the volume assuming that it falls suddenly to 0 at the surface, similar to the electric polarized material, and assign an *effective magnetic surface-charge density*, σ_M

$$\Phi(\mathbf{r}) = -\frac{1}{4\pi}\nabla\cdot\int\frac{\mathbf{M}}{\mathbb{r}}\,d\tau' = \frac{1}{4\pi}\int\mathbf{M}\cdot\nabla'\frac{1}{\mathbb{r}}\,d\tau' = \frac{1}{4\pi}\int_{\mathcal{V}}\mathbf{M}\cdot\nabla'\frac{1}{\mathbb{r}}\,d\tau'$$

$$= \frac{1}{4\pi}\oint_{\mathcal{S}}\frac{\mathbf{M}}{\mathbb{r}}\cdot d\boldsymbol{a}' - \frac{1}{4\pi}\int_{\mathcal{V}}\frac{\nabla'\cdot\mathbf{M}}{\mathbb{r}}\,d\tau' \quad \Leftarrow \quad \begin{array}{l}\mathbf{M}(\mathbf{r}'\notin\mathcal{V}) = 0 \\ d\boldsymbol{a}' = \hat{\mathbf{n}}\,da'\end{array}$$

$$\Rightarrow \quad \Phi(\mathbf{r}) = \frac{1}{4\pi}\int_{\mathcal{V}}\frac{\rho_M}{\mathbb{r}}\,d\tau' + \frac{1}{4\pi}\oint_{\mathcal{S}}\frac{\sigma_M}{\mathbb{r}}\,da' \quad \Leftarrow \quad \rho_M = -\nabla\cdot\mathbf{M}, \quad \sigma_M = \mathbf{M}\cdot\hat{\mathbf{n}}$$

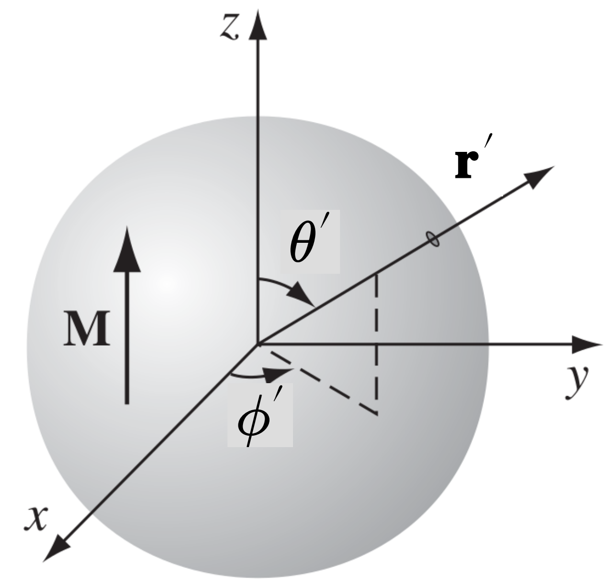
$$\text{if } \mathbf{M} \text{ is uniform} \Rightarrow \quad \Phi(\mathbf{r}) = \frac{1}{4\pi}\oint_{\mathcal{S}}\frac{\mathbf{M}}{\mathbb{r}}\cdot d\boldsymbol{a}'$$

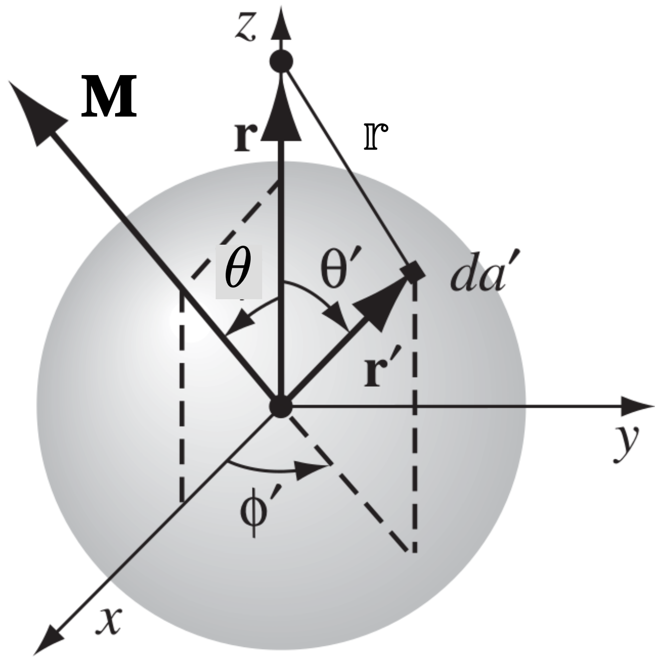
- (@) is generally applicable even for the limit of discontinuous distributions of $\mathbf{M}$. *Never combine the surface integral of* σ_M *with* (@)!

Example 6.1: Find the magnetic field of a uniformly magnetized sphere.

- $\boldsymbol{J}_b = \nabla \times \mathbf{M} = 0\,, \quad \boldsymbol{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = \mathbf{M} \times \hat{\mathbf{r}}' = M \sin\theta' \,\hat{\boldsymbol{\phi}}'$

- Find $\mathbf{A}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} \displaystyle\int \frac{\boldsymbol{K}_b}{\mathbb{r}}\, \mathrm{d}a' \quad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$





Example 6.1: Using a scalar potential as the alternative **1**:

$$\Phi = -\frac{1}{4\pi}\nabla\cdot\int\frac{\mathbf{M}}{\mathbb{r}}\,\mathrm{d}\tau'$$

$$\int\frac{\mathbf{M}}{\mathbb{r}}\,\mathrm{d}\tau' = \mathbf{M}\int\frac{\mathrm{d}\tau'}{\mathbb{r}} = \mathbf{M}\int\frac{r'^2\sin\theta'\,\mathrm{d}r'\,\mathrm{d}\theta'\,\mathrm{d}\phi'}{\sqrt{r^2+r'^2-2\,r\,r'\cos\theta'}} \quad \Leftarrow \quad \text{choose } \mathbf{r} = r\,\hat{\mathbf{z}} \text{ temporarily}$$

$$= \frac{2\pi\mathbf{M}}{r}\int_0^R r'\left(r+r'-|r-r'|\right)\mathrm{d}r' = 2\pi\mathbf{M}\begin{cases} R^2-\dfrac{r^2}{3} & \text{for } r<R \text{ with } \displaystyle\int_0^r + \int_r^R \\ \dfrac{2R^3}{3r} & \text{for } r>R \quad\Rightarrow\quad r>r' \end{cases}$$

$$\Phi = -\frac{1}{4\pi}\nabla\cdot\int\frac{\mathbf{M}}{\mathbb{r}}\,\mathrm{d}\tau' \quad\Rightarrow\quad \Phi_{\mathrm{in}}(r,\theta) = \frac{M}{3}\,z\,, \quad \Phi_{\mathrm{out}}(r,\theta) = \frac{M}{3}\frac{R^3}{r^2}\cos\theta$$

$$\mathbf{H} = -\nabla\Phi \quad\Rightarrow\quad \mathbf{H}_{\mathrm{in}} = -\frac{1}{3}\mathbf{M}\,, \quad \mathbf{H}_{\mathrm{out}} = \frac{M}{3}\frac{R^3}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

$$\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}) \quad\Rightarrow\quad \mathbf{B}_{\mathrm{in}} = \frac{2}{3}\mu_0\mathbf{M}\,, \quad \mathbf{B}_{\mathrm{out}} = \mu_0\frac{M}{3}\frac{R^3}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

Example 6.1: Using a scalar potential as the alternative **2**: $\Phi = \frac{1}{4\pi}\oint_S \frac{\sigma_M}{\mathscr{r}}\, d\, a'$

Use the similar trick like the one with vector potential:

Choose $\mathbf{r} = r\,\hat{\mathbf{z}}$, put $\mathbf{M}$ in the xz-plane $\Rightarrow$ $\mathbf{M} = M \sin\theta\,\hat{\mathbf{x}} + M\cos\theta\,\hat{\mathbf{z}}$

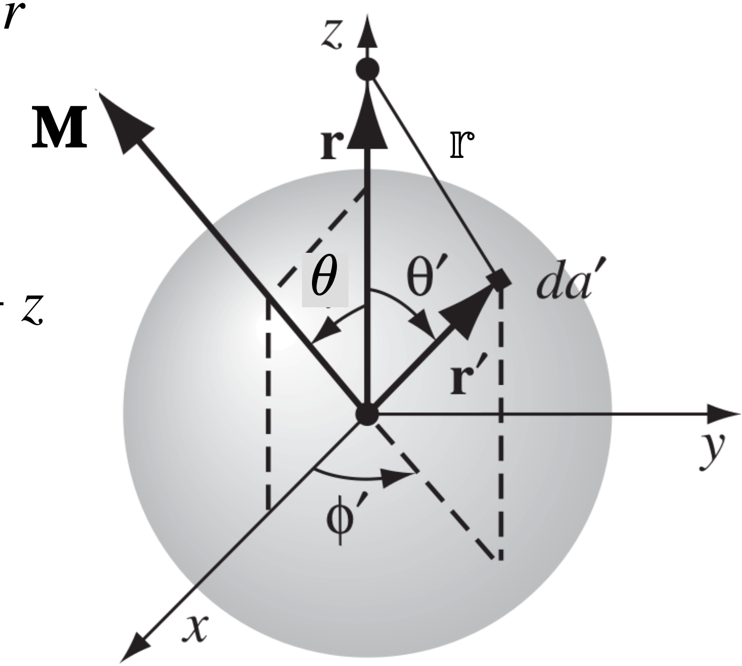
And $\hat{\mathbf{r}}' = \sin\theta' \cos\phi'\,\hat{\mathbf{x}} + \sin\theta' \sin\phi'\,\hat{\mathbf{y}} + \cos\theta'\,\hat{\mathbf{z}}$

$$\Rightarrow \quad \sigma_M = \mathbf{M}\cdot\hat{\mathbf{r}}' = M\left(\sin\theta \sin\theta' \cos\phi' + \cos\theta\cos\theta'\right)$$

$$\Rightarrow \quad \oint \frac{\sigma_M}{\mathscr{r}}\, d\, a' = M \oint \frac{\sin\theta\sin\theta'\cos\phi' + \cos\theta\cos\theta'}{\sqrt{r^2 + R^2 - 2\,r\,R\cos\theta'}} R^2 \sin\theta'\, d\,\theta'\, d\,\phi'$$

$$= 2\pi M R^2 \cos\theta \int_0^\pi \frac{\cos\theta' \sin\theta'\, d\,\theta'}{\sqrt{r^2+R^2-2\,r\,R\cos\theta'}} = \frac{2\pi M}{3 r^2}\left(R^3 + r^3 - \left|R^3 - r^3\right|\right)\cos\theta$$

$$= \frac{4\pi M}{3}\frac{r_<^3}{r^2}\cos\theta \quad \Leftarrow \quad r_< = \min(r, R)$$



$$\Rightarrow \quad \Phi = \frac{M}{3}\frac{r_<^3}{r^2}\cos\theta \quad \Rightarrow \quad \begin{cases} \Phi_{in}(r,\theta) = \dfrac{M}{3} r\cos\theta = \dfrac{M}{3} z \\ \Phi_{out}(r,\theta) = \dfrac{M}{3}\dfrac{R^3}{r^2}\cos\theta \end{cases}$$

$$\Rightarrow \quad \mathbf{H} = -\nabla\Phi, \quad \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$$

Example 6.1: Using a scalar potential as the alternative **3**:

$$\mathbf{H} = -\nabla\Phi \quad \Leftarrow \quad \nabla\times\mathbf{H} = \boldsymbol{J}_f = 0 \quad \Leftarrow \quad \text{no free current}$$

$$\nabla^2\Phi = 0 \quad \Leftarrow \quad \nabla\cdot\mathbf{H} = \nabla\cdot\mathbf{M} = 0 \quad \Leftarrow \quad \mathbf{M} = M\,\hat{\mathbf{z}}$$

$$\Rightarrow \quad \Phi_{\text{in}}(r,\theta) = \sum_{\ell=0} C_\ell\, r^\ell\, P_\ell(\cos\theta), \quad \Phi_{\text{out}}(r,\theta) = \sum_{\ell=0} \frac{D_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

Boundary conditions: $(1)\ \Phi_{\text{in}}(R) = \Phi_{\text{out}}(R), \quad (2)\ B_{\text{in},r}(R) = B_{\text{out},r}(R) \quad \Leftarrow \quad \mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$

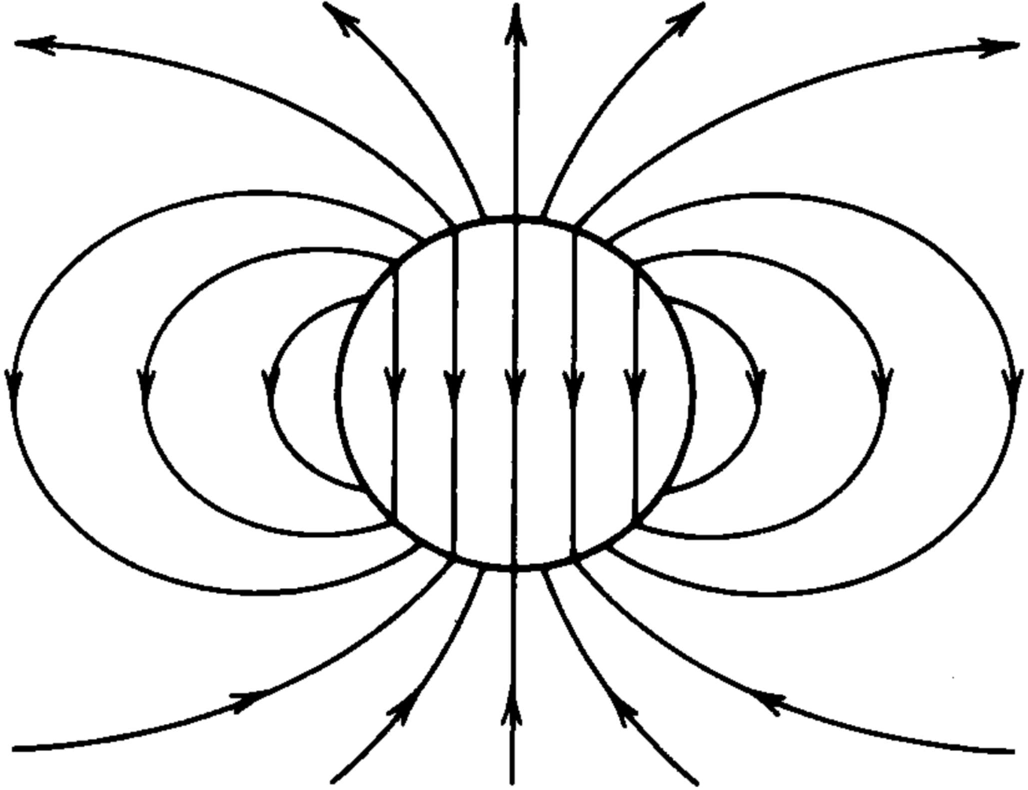
$$\Rightarrow (1)\ C_\ell = \frac{D_\ell}{R^{2\ell+1}} \quad (2)\ \mu_0(C_1 + M) = -2\mu_0\frac{D_1}{R^3}, \quad \mu_0\,\ell\,C_\ell = -\mu_0\frac{\ell+1}{R^{2\ell+1}}D_\ell \ \text{ for } \ \ell\neq 1$$

$$\Rightarrow \quad C_1 = \frac{M}{3}, \quad D_1 = \frac{M}{3}R^3, \quad C_\ell = D_\ell = 0 \ \text{ for } \ \ell\neq 1$$

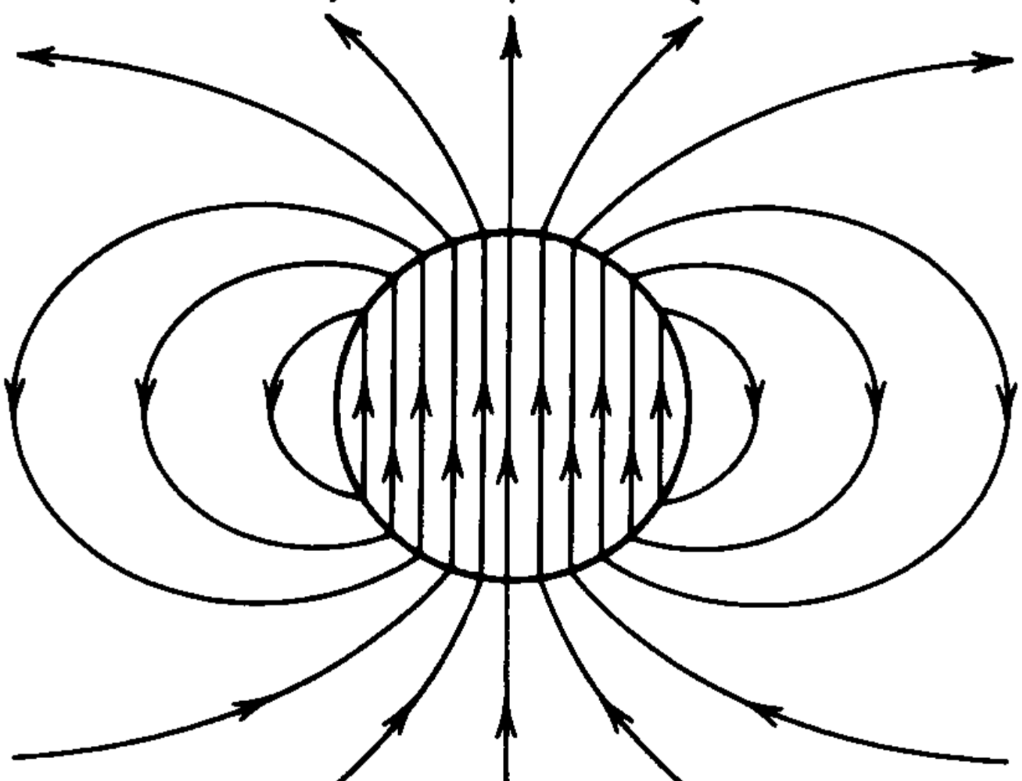
$$\Rightarrow \quad \Phi_{\text{in}}(r,\theta) = \frac{M}{3}z, \quad \Phi_{\text{out}}(r,\theta) = \frac{M}{3}\frac{R^3}{r^2}\cos\theta$$

$$\Rightarrow \quad \mathbf{H}_{\text{in}} = -\frac{1}{3}\mathbf{M}, \quad \mathbf{H}_{\text{out}} = \frac{1}{3}\frac{R^3}{r^3}\left[3(\hat{\mathbf{r}}\cdot\mathbf{M})\hat{\mathbf{r}} - \mathbf{M}\right]$$

$$\Rightarrow \quad \mathbf{B}_{\text{in}} = \frac{2}{3}\mu_0\mathbf{M}, \quad \mathbf{B}_{\text{out}} = \frac{\mu_0}{3}\frac{R^3}{r^3}\left[3(\hat{\mathbf{r}}\cdot\mathbf{M})\hat{\mathbf{r}} - \mathbf{M}\right]$$

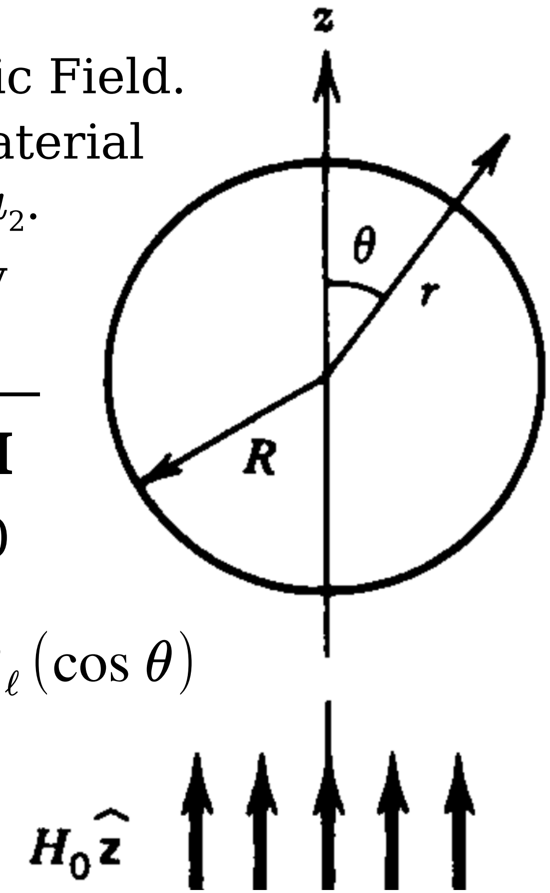


H



B

Example: A Magnetic Sphere in a Uniform External Magnetic Field. Consider a sphere of radius R, made of a linear magnetic material of permeability μ_1, embedded in a medium of permeability μ_2. The sphere is placed in a magnetic field $\mathbf{H}_0$ which is initially uniform and pointing along the z direction.



$$\text{Current} = 0 \;\Rightarrow\; \mathbf{H} = -\nabla\Phi \;\Leftarrow\; \nabla\times\mathbf{H} = \boldsymbol{J}_f = 0\,, \quad \mathbf{B} = \mu\,\mathbf{H}$$

$$\Rightarrow\; \Phi(r\to\infty) = -H_0\,z = -H_0\,r\cos\theta\,, \quad \text{choose } \Phi(r=0) = 0$$

$$\Rightarrow\; \Phi_{\text{in}} = \sum C_\ell\, r^\ell P_\ell(\cos\theta)\,, \quad \Phi_{\text{out}} = -H_0\,r\cos\theta + \sum_{\ell=0} \frac{D_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

Boundary conditions: $(1)\;\Phi_{\text{in}}(R) = \Phi_{\text{out}}(R)\,,\;(2)\;B_{\text{in},r}(R) = B_{\text{out},r}(R)$

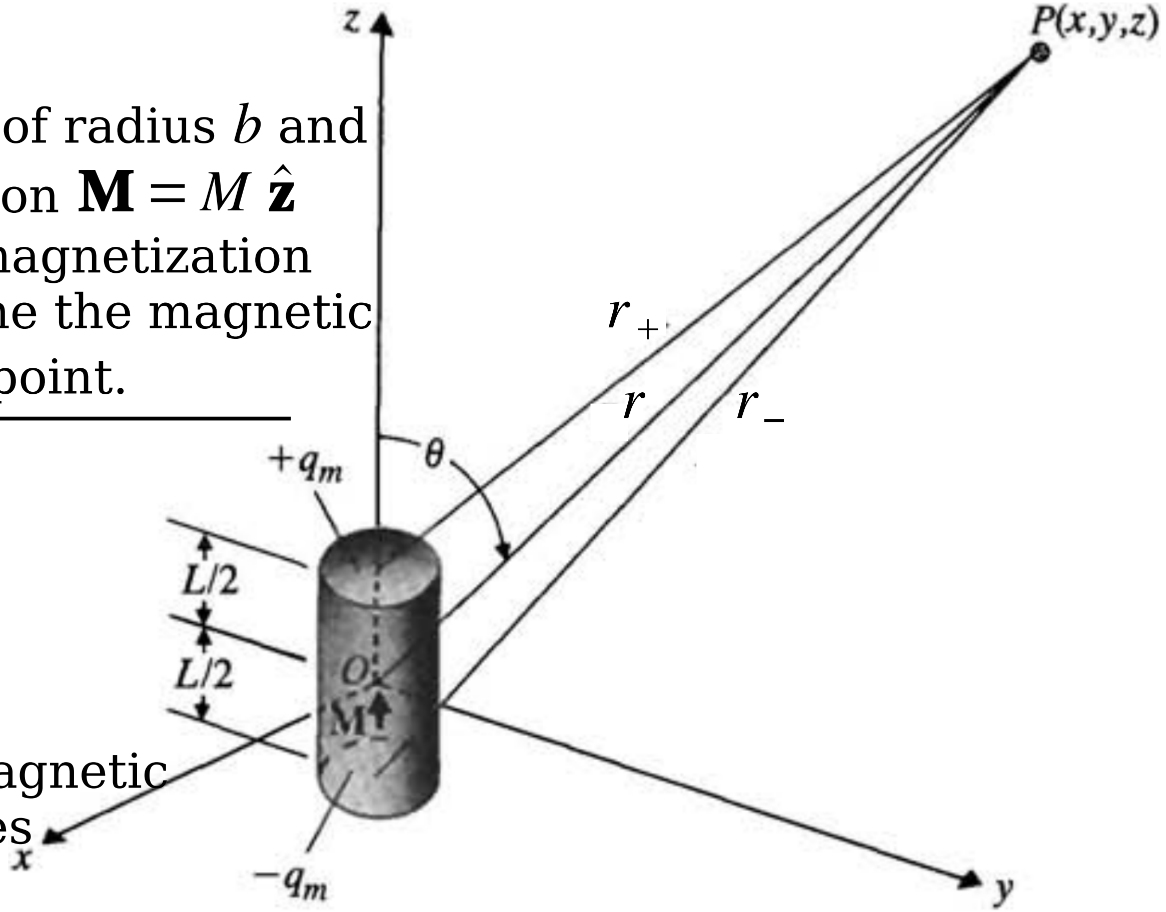
$$\Rightarrow\; C_1 = \frac{D_1}{R^3} - H_0\,, \qquad C_\ell = \frac{D_\ell}{R^{2\ell+1}} \;\text{for}\; \ell\neq 1 \;\Leftarrow\; (1)$$

$$\mu_1 C_1 = -\mu_2\left(2\frac{D_1}{R^3} + H_0\right), \quad \mu_1\,\ell\,C_\ell = -\mu_2(\ell+1)\frac{D_\ell}{R^{2\ell+1}} \;\text{for}\; \ell\neq 1 \;\Leftarrow\; (2)$$

$$\Rightarrow\; C_1 = -\frac{3\mu_2}{\mu_1+2\mu_2}H_0\,, \quad D_1 = \frac{\mu_1-\mu_2}{\mu_1+2\mu_2}H_0 R^3\,, \quad C_\ell = D_\ell = 0 \;\text{for}\; \ell\neq 1$$

$$\Rightarrow\; \mathbf{B}_{\text{in}} = \frac{3\mu_1\mu_2}{\mu_1+2\mu_2}\mathbf{H}_0\,, \quad \mathbf{B}_{\text{out}} = \mu_2\left(\mathbf{H}_0 + \frac{\mu_1-\mu_2}{\mu_1+2\mu_2}\frac{R^3}{r^3}\left[3(\hat{\mathbf{r}}\cdot\mathbf{H}_0)\hat{\mathbf{r}} - \mathbf{H}_0\right]\right)$$

Example: A cylindrical bar magnet of radius b and length L has a uniform magnetization $\mathbf{M} = M\,\hat{\mathbf{z}}$ along its axis. Use the equivalent magnetization charge density concept to determine the magnetic flux density at an arbitrary *distant* point.



$$\rho_M = -\nabla\cdot\mathbf{M} = 0$$

$$\sigma_M = \begin{cases} \pm M & \text{on top/bottom face} \\ 0 & \text{on side wall} \end{cases}$$

At a distant point the equivalent magnetic charges on the top and bottom faces appear as point charges:

$$\pm q_M = \pi\, b^2\,\sigma_M = \pm\pi\, b^2 M$$

$$\Rightarrow \quad \Phi = \frac{1}{4\pi}\oint \frac{\sigma_M}{\mathfrak{r}}\,\mathrm{d}\,a' \approx \frac{q_M}{4\pi}\left(\frac{1}{r_+} - \frac{1}{r_-}\right) \quad \Leftarrow \quad r_\pm = \sqrt{r^2 \mp r\,L\cos\theta + \frac{L^2}{4}}$$

$$= \frac{q_M}{4\pi r}\left(1 + \frac{L}{2r}\cos\theta + \cdots - 1 + \frac{L}{2r}\cos\theta + \cdots\right) \approx \frac{q_M L}{4\pi r^2}\cos\theta$$

$$\Rightarrow \quad \mathbf{B} = -\mu_0\nabla\Phi = \frac{\mu_0}{4\pi}\frac{(3\,\hat{\mathbf{r}}\cdot\mathbf{m})\,\hat{\mathbf{r}} - \mathbf{m}}{r^3} \quad \Leftarrow \quad \mathbf{m} = m\,\hat{\mathbf{z}} = q_M L\,\hat{\mathbf{z}} = \pi\, b^2 L\, M\,\hat{\mathbf{z}}$$

total magnetic dipole moment

Ferromagnetism

- In a linear medium, the alignment of atomic dipoles is maintained by a magnetic field imposed from the outside.

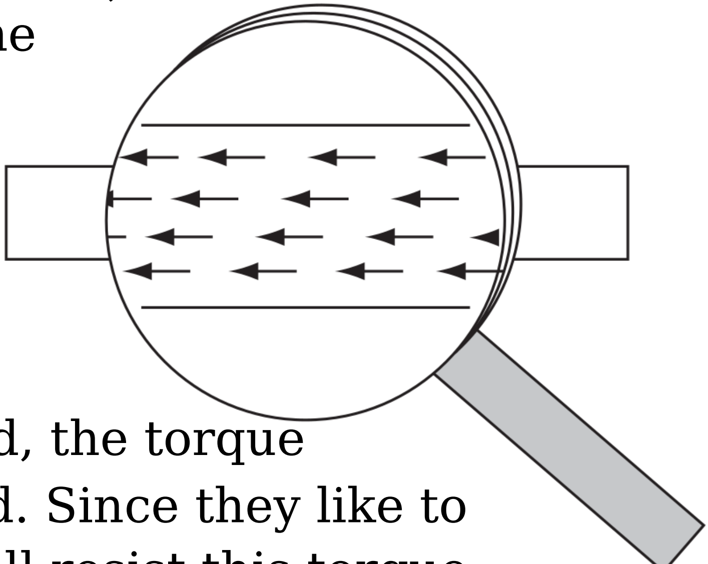
- Ferromagnets—which are *not* linear—no external fields needed to sustain the magnetization; the alignment is "frozen in."

- Like paramagnetism, ferromagnetism involves the magnetic dipoles associated with the spins of unpaired electrons. (An iron atom ^{26}Fe has 2 lone electrons.)

- The new feature is the interaction between nearby dipoles: In a ferromagnet, *each dipole "likes" to point in the same direction as its neighbors*.

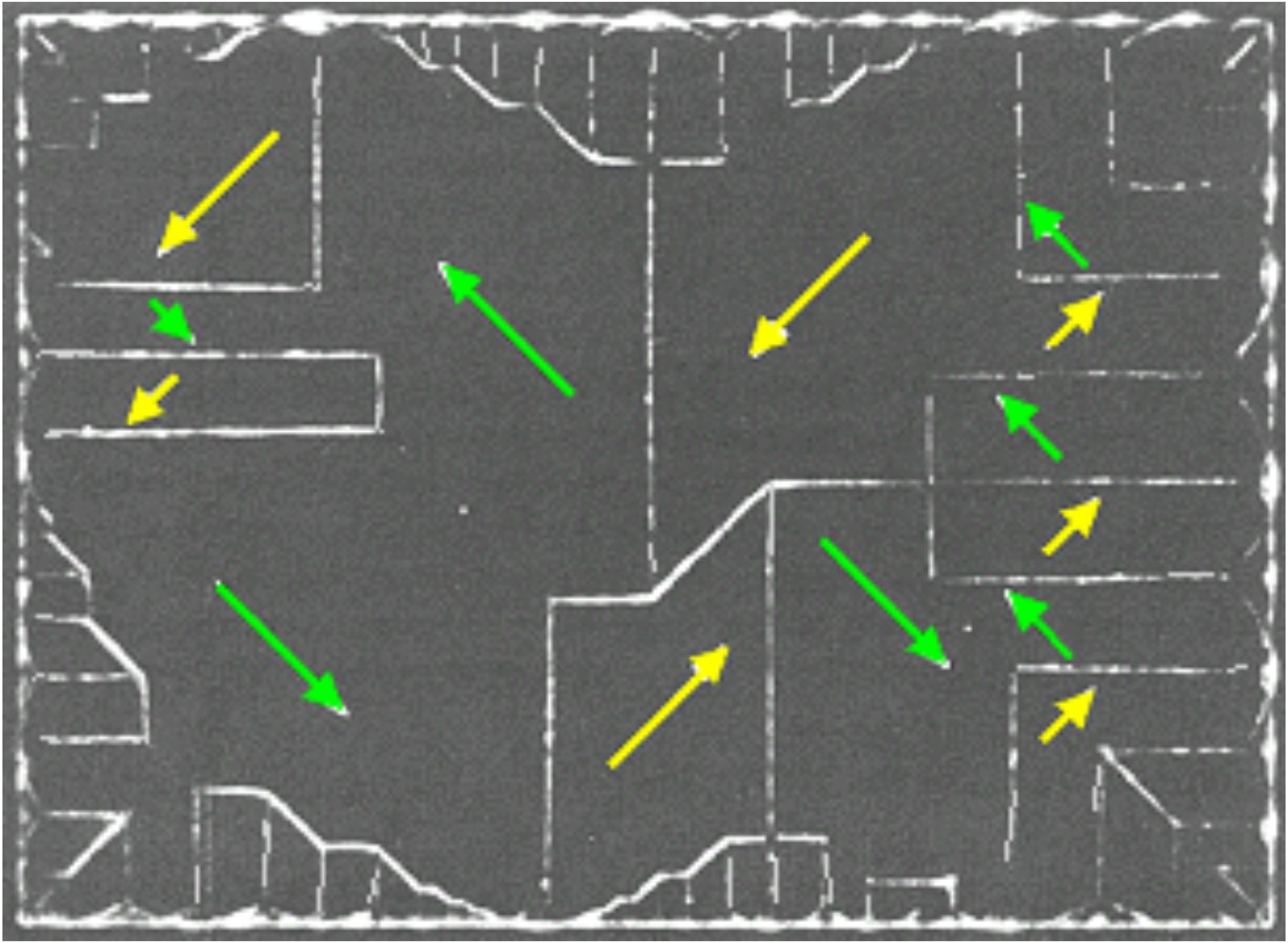
- The reason for this preference is essentially quantum mechanical. The correlation is so strong as to align virtually 100% of the unpaired electron spins.

- This kind of alignment occurs in relatively small patches, called **domains**. Each domain contains billions of dipoles, all lined up, but the domains *themselves* are randomly oriented.



- The household wrench contains an great number of domains, and their magnetic fields cancel, so the wrench as a whole is not magnetized.

- If you put a piece of iron into a strong magnetic field, the torque $\mathbf{N}=\mathbf{m}\times\mathbf{B}$ tends to align the dipoles parallel to the field. Since they like to stay parallel to their neighbors, most of the dipoles will resist this torque.



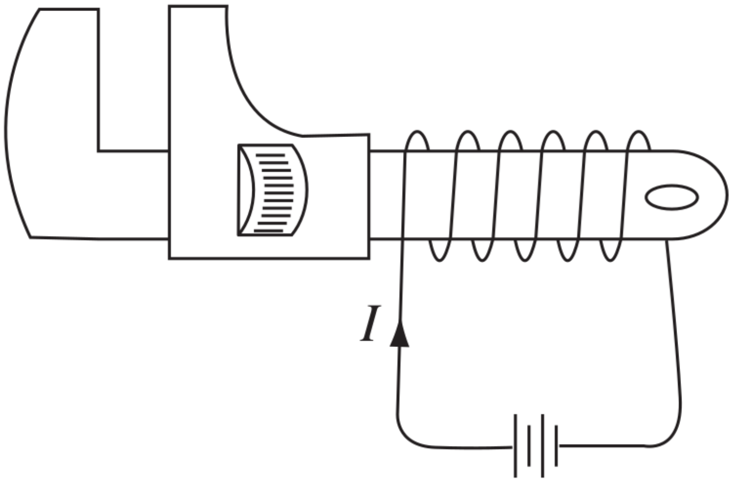
- At the *boundary* between 2 domains, there are *competing* neighbors, and the torque will throw its weight on the side of the domain most nearly parallel to the field; this domain will win some converts, at the expense of the less favorably oriented one.

- The net effect of the magnetic field is to *move the domain boundaries*. Domains parallel to the field grow, and the others shrink.

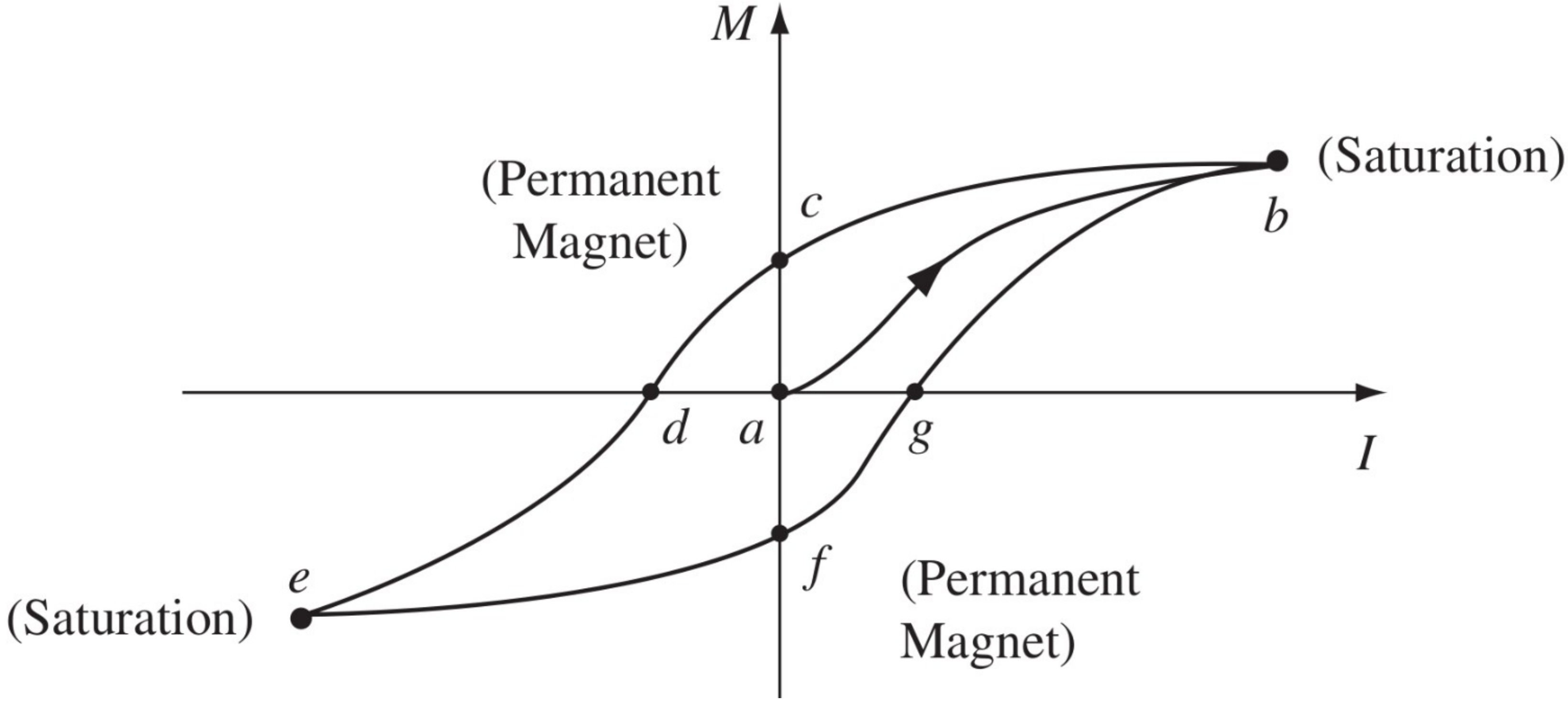
- If the field is strong enough, one domain takes over entirely, and the iron is said to be **saturated**.

- This process is not entirely reversible: when the field is switched off, there are some return to randomly oriented domains, but there remains a preponderance of domains in the original direction—**permanent magnet**.

- A simple way to accomplish this is to wrap a coil of wire around the object to be magnetized.



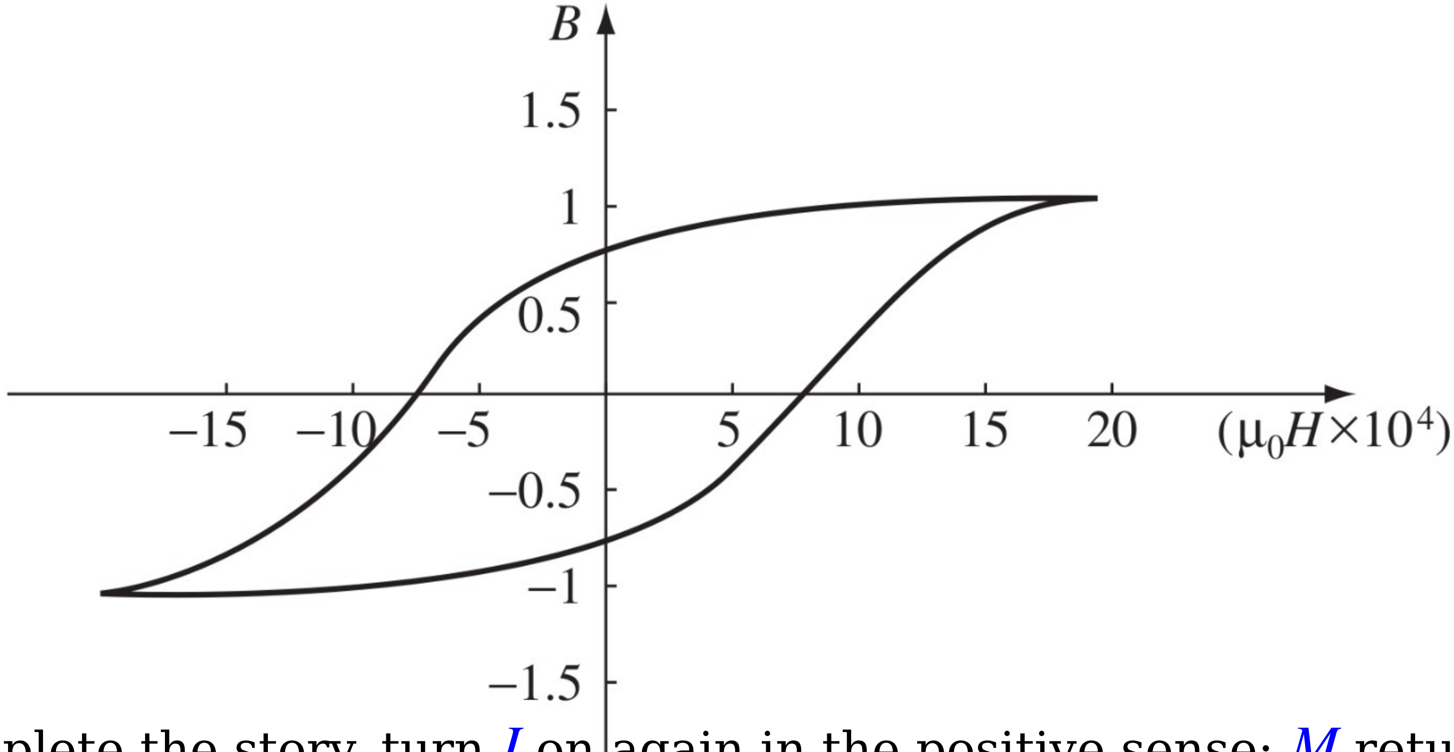
- As you increase the current, the field increases, the domain boundaries move, and the magnetization grows. Eventually, you reach the saturation point, with all the dipoles aligned, and a further increase in current has no effect on $\mathbf{M}$ (point *b*).



- Suppose you *reduce* the current. Instead of retracing the path back to $M=0$, there is only a *partial* return to randomly oriented domains; M decreases, but even with the current off there is some residual magnetization (point *c*).

- If you want to eliminate the remaining magnetization, you have to run a current backwards through the coil (a negative I). Now the external field points to the right, and as you increase I (negatively), M drops down to 0 (point *d*).

- If you turn I still higher, you soon reach saturation in the other direction—all the dipoles pointing to the right (point *e*). At this stage, switching off the current will leave the wrench with a permanent magnetization to the right (point *f*).



- To complete the story, turn I on again in the positive sense: M returns to 0 (point g), and eventually to the forward saturation (point b).

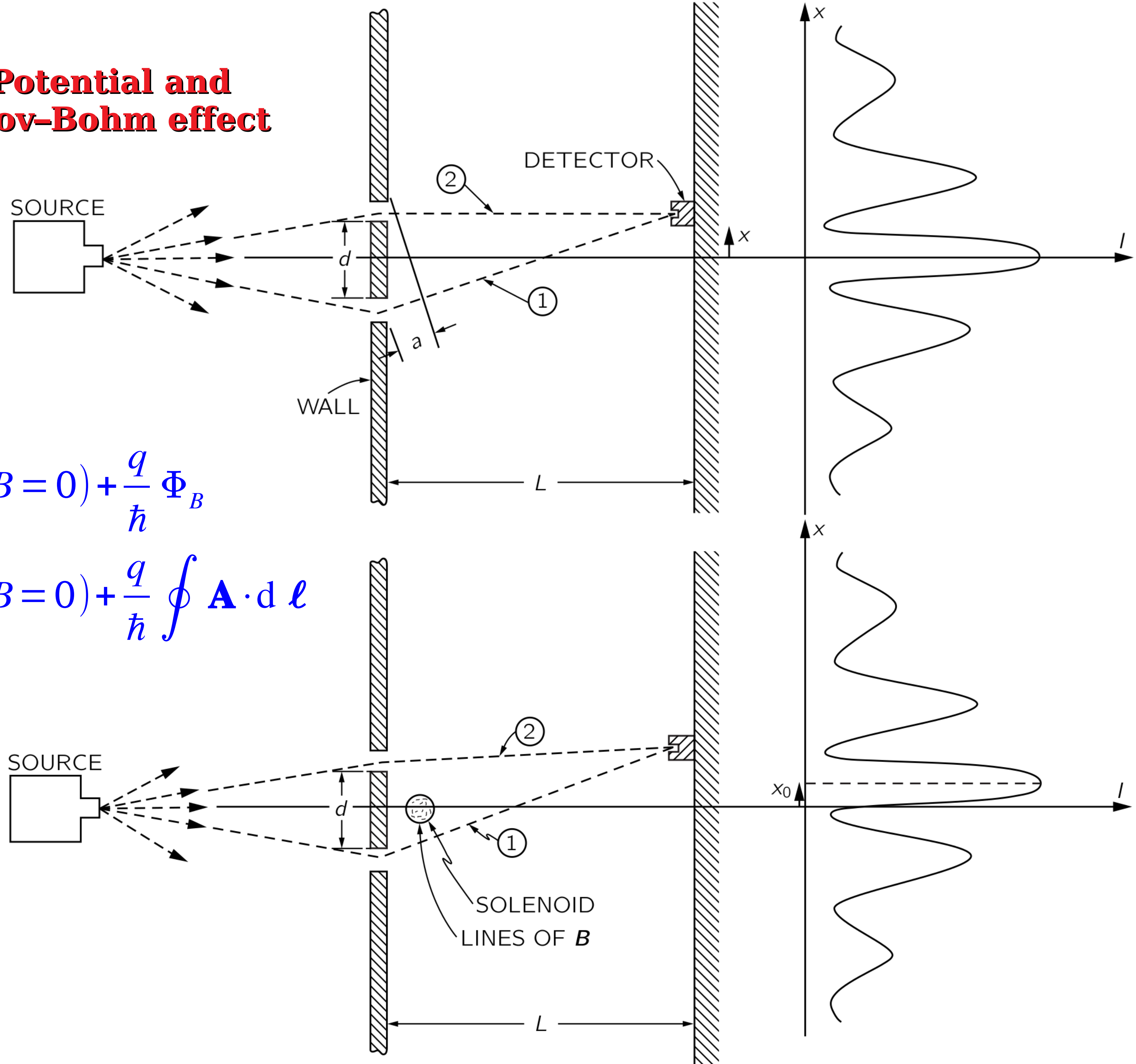
- The path we traced out is called a **hysteresis loop**. So the magnetization of material depends not only on the applied field (ie, on I), but also on its previous magnetic "history."

- It usually draws hysteresis loops as plots of B vs H, rather than M vs I, for $H=n\,I$, $\mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M})$., and usually $M \gg H$.

- A little current is enough when you have ferromagnetic materials. That's why anyone who wants to make a powerful electromagnet wraps the coil around an iron. It doesn't take much of an external field to move the domain boundaries, and when you do that, you have all the dipoles in the iron working with you.

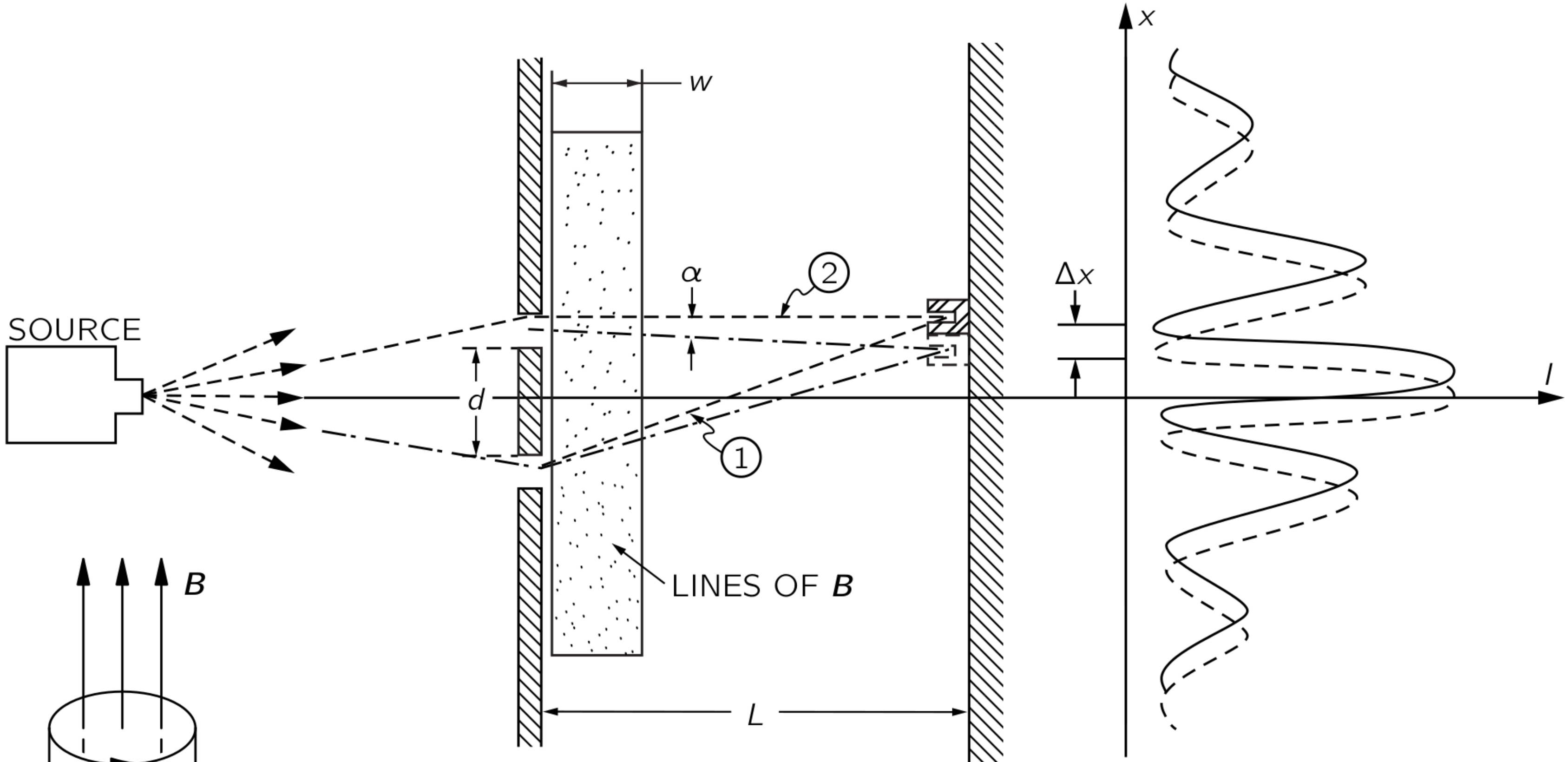
- Dipoles within a domain in ferromagnetism line up parallel to one another.
- Random thermal motions compete with this ordering, but as long as the temperature doesn't get too high, they cannot budge the dipoles out of line.
- *Very* high temperatures do destroy the alignment. This occurs at a precise temperature of 770° C for iron.
- Below this temperature (called the **Curie point**), iron is ferromagnetic; above, it is paramagnetic.
- The Curie point is rather like the boiling/freezing point in that there is no *gradual* transition from ferro- to para-magnetic behavior, any more than there is between water and ice.
- These abrupt changes in the properties of a substance, occurring at sharply defined temperatures, are known in statistical mechanics as **phase transitions**.

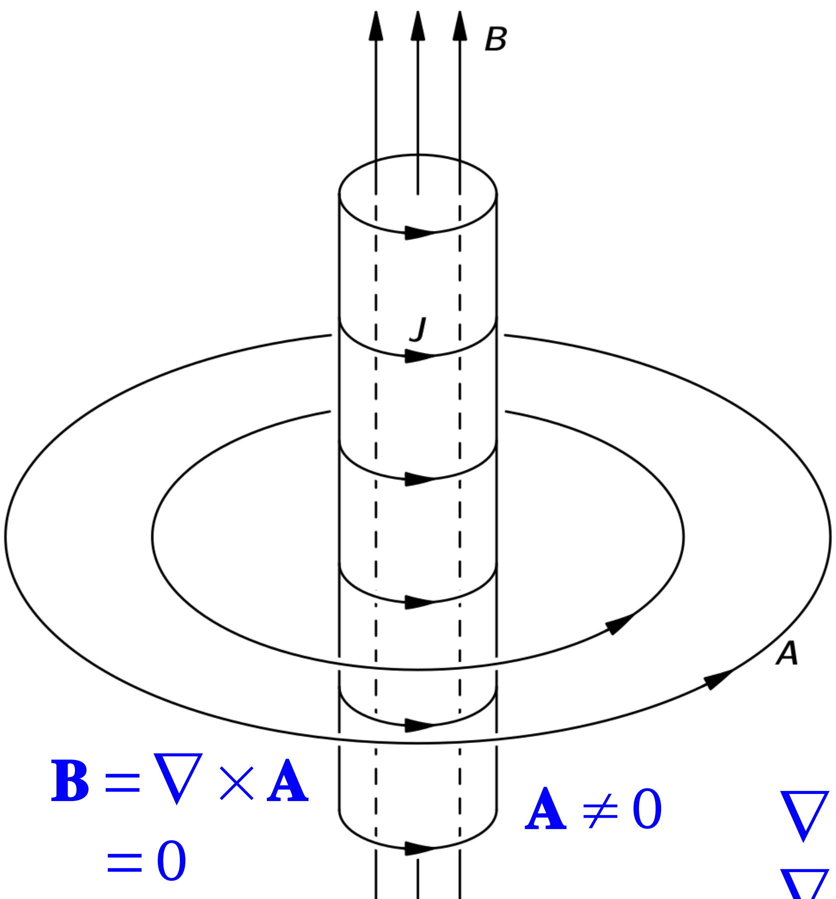
Vector Potential and Aharonov–Bohm effect



$$\delta = \delta\left(B=0\right) + \frac{q}{\hbar}\,\Phi_B$$

$$= \delta\left(B=0\right) + \frac{q}{\hbar}\oint \mathbf{A}\cdot \mathrm{d}\,\boldsymbol{\ell}$$





- In the solenoid case, the electrons don't feel the magnetic field inside but only the magnetic vector potential outside, and the phase difference change still happen.

- Quantum-mechanically, the magnetic vector potential $\mathbf{A}$ is as real physically as the magnetic field $\mathbf{B}$.

$$\mathbf{B} = \nabla \times \mathbf{A} = 0$$

$$\mathbf{A} \neq 0$$

$$\begin{array}{l} \nabla \times \mathbf{A} = 0 \\ \nabla \cdot \mathbf{A} = 0 \end{array} \Rightarrow \begin{array}{l} \mathbf{A} = -\nabla \Phi \\ \nabla^2 \Phi = 0 \end{array} + \begin{array}{l} \text{boundary} \\ \text{conditions} \end{array} \Leftarrow \begin{array}{l} \text{classical} \\ \text{treatment} \end{array}$$