Chapter 5 Magnetostatics

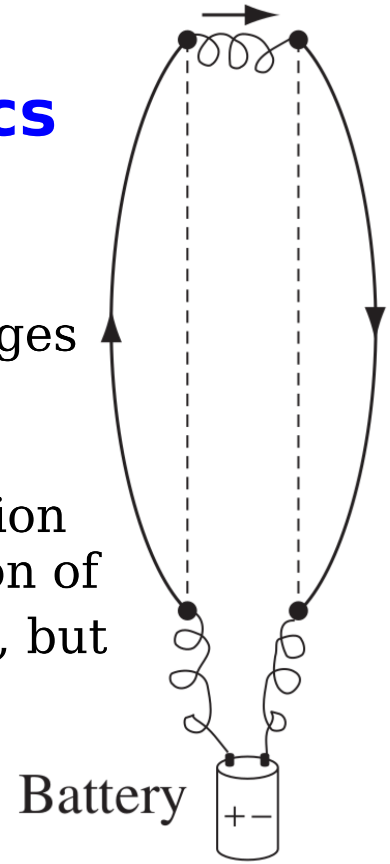
The Lorentz Force Law

Magnetic Fields

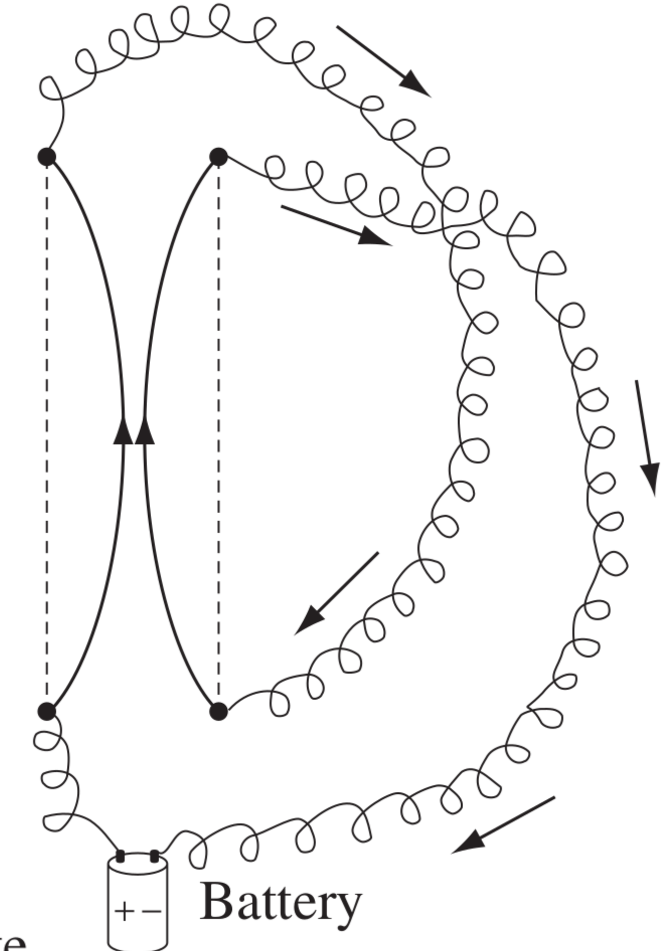
- Consider the forces between charges *in motion*.

- The force accounts for the attraction of parallel currents and the repulsion of antiparallel ones is *not* electrostatic, but a *magnetic* force.

- Whereas a *stationary* charge produces only an electric field **E** in the space around it, a moving charge generates, in addition, a magnetic field **B**.



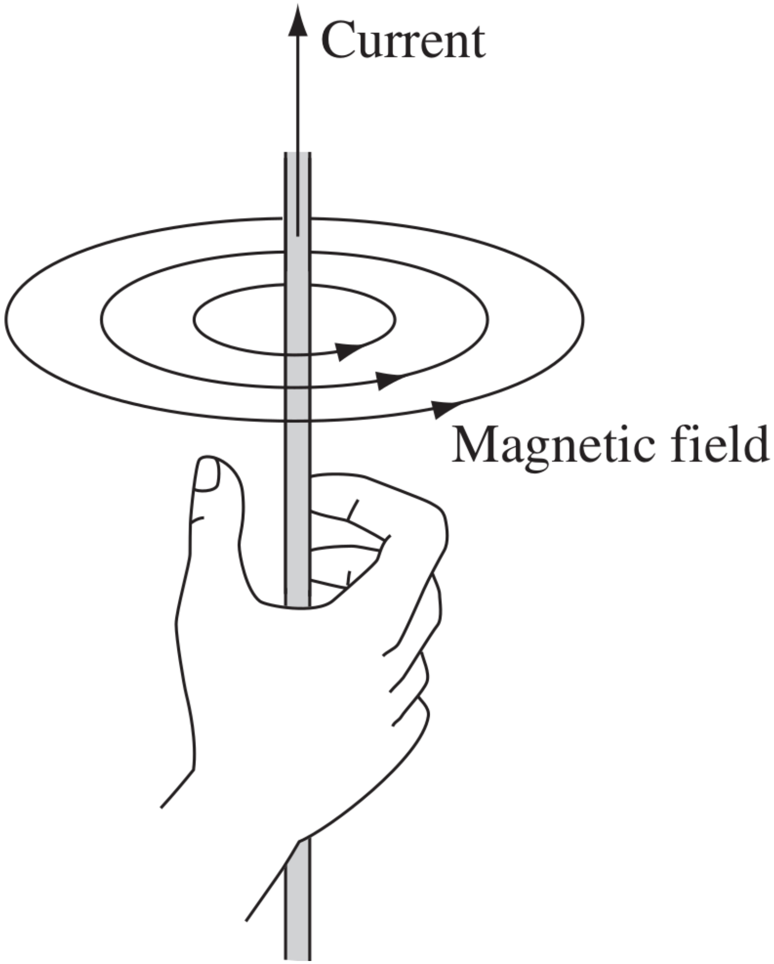
Currents in opposite directions repel.

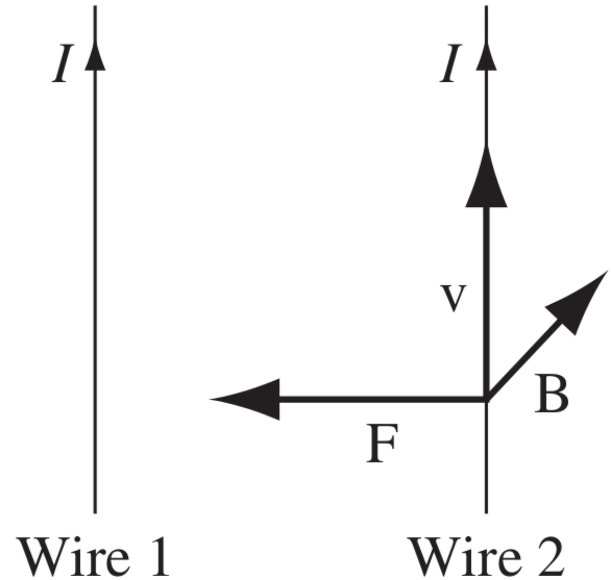


Currents in same directions attract.

- Magnetic fields can be easily detected with a compass. And its needle points in the direction of the local magnetic field.

- For the earth's magnetic field, the pointing means *north*, but in a lab, it is the direction of whatever magnetic field is present.

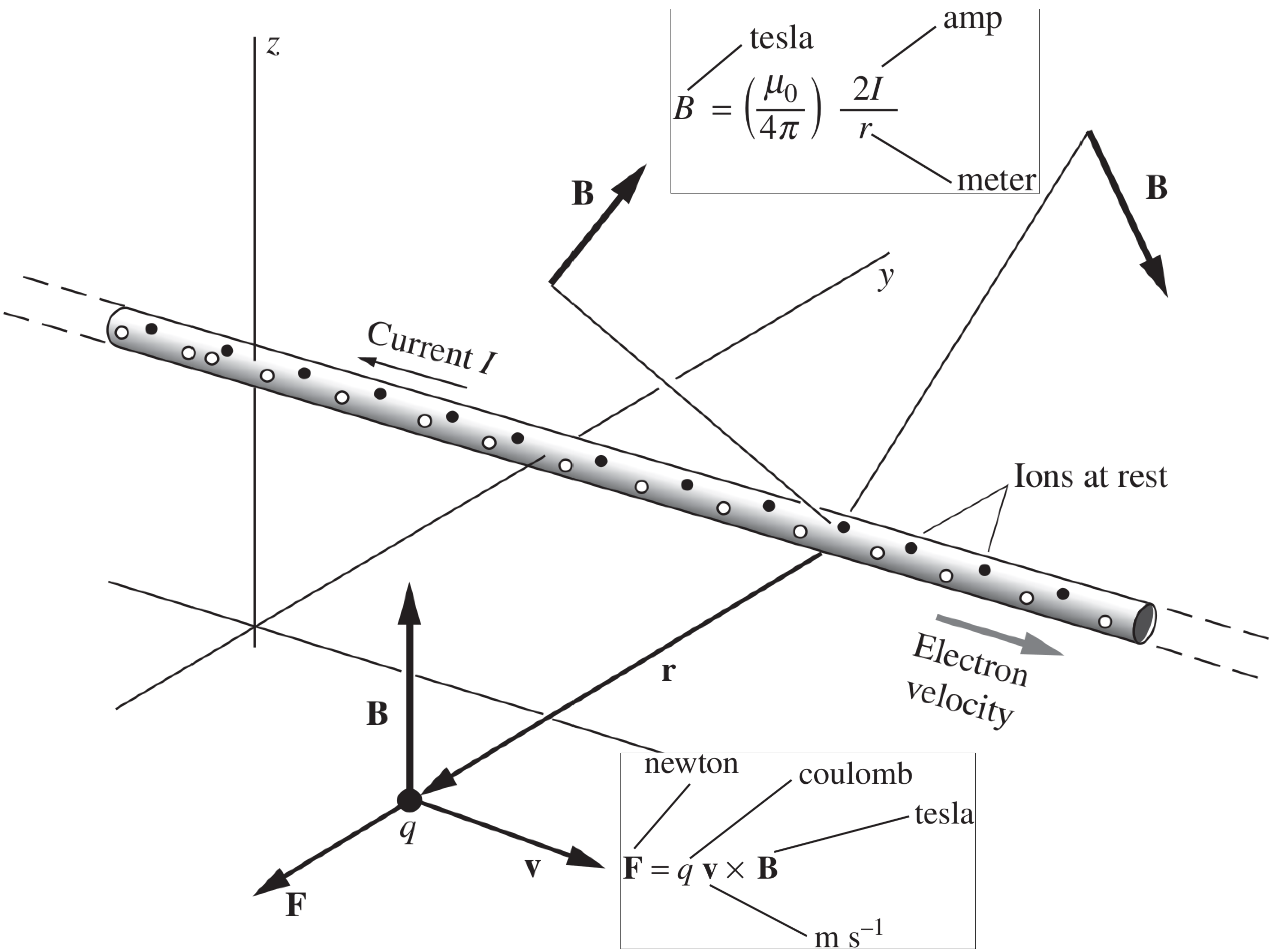




- The magnetic field does not point *toward* the wire, nor *away* from it, but rather it *circles around the wire*.

- If you grab the wire with your right hand—thumb in the direction of the current—your fingers curl around in the direction of the magnetic field.

- At the 2nd wire, the magnetic field points *into the page*, the current is *upward*, and yet the resulting force is *to the left*!



tesla
amp
$B = \left(\frac{\mu_0}{4\pi}\right)\frac{2I}{r}$
meter
z
B
B
y
Current I
Ions at rest
r
B
Electron velocity
newton
coulomb
tesla
q
v
$\mathbf{F} = q\,\mathbf{v} \times \mathbf{B}$
F
m s^{-1}

Magnetic Forces

- The magnetic force on a charge Q, moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$,

$$\mathbf{F}_{\text{mag}} = Q\,\mathbf{v} \times \mathbf{B}$$

- In the presence of both electric *and* magnetic fields, the net force on Q

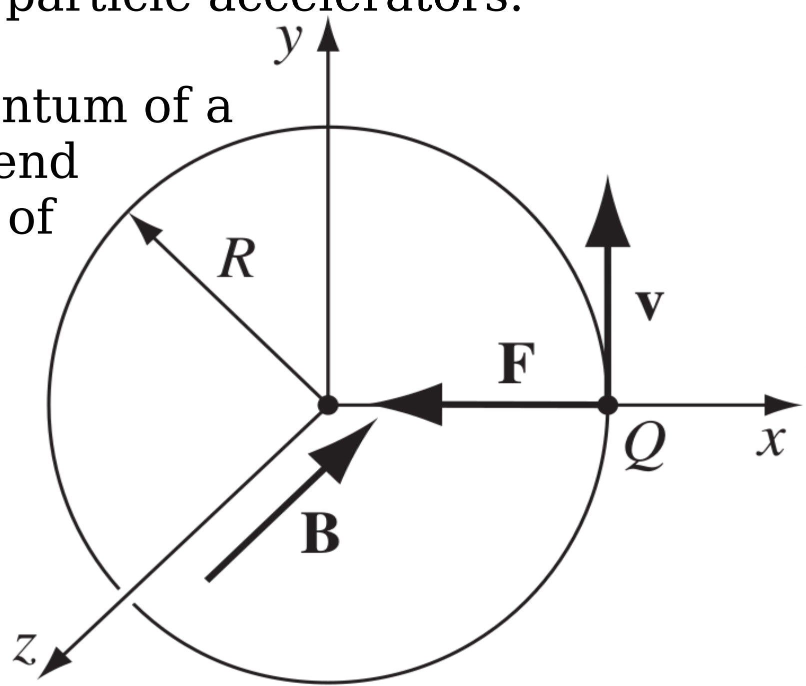
$$\mathbf{F} = Q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \quad \Leftarrow \quad \text{Lorentz force law}$$

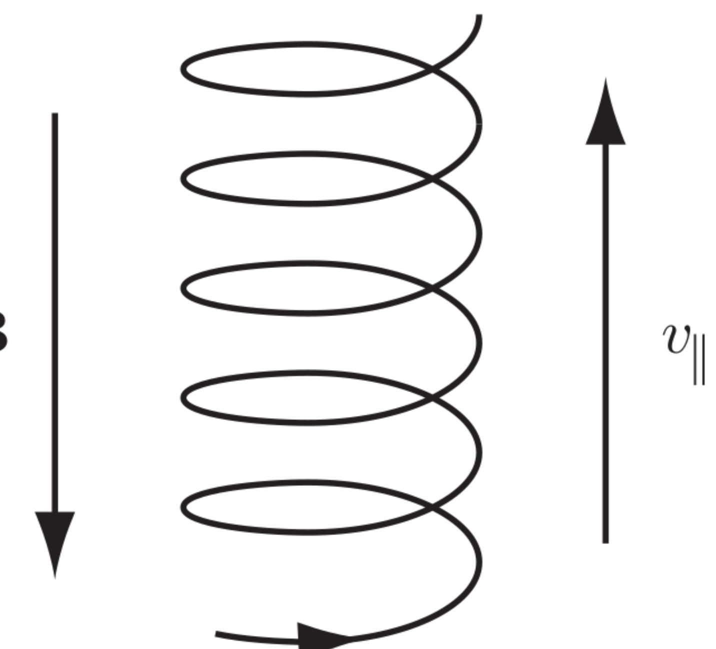
- **Cyclotron motion**: The archetypal motion of a charged particle in a magnetic field is circular, with the magnetic force providing the centripetal acceleration.

$$F = m\,a = m\,\frac{v^2}{R} = Q\,v\,B \quad \Rightarrow \quad p = Q\,B\,R \quad \Leftarrow \quad p = m\,v$$

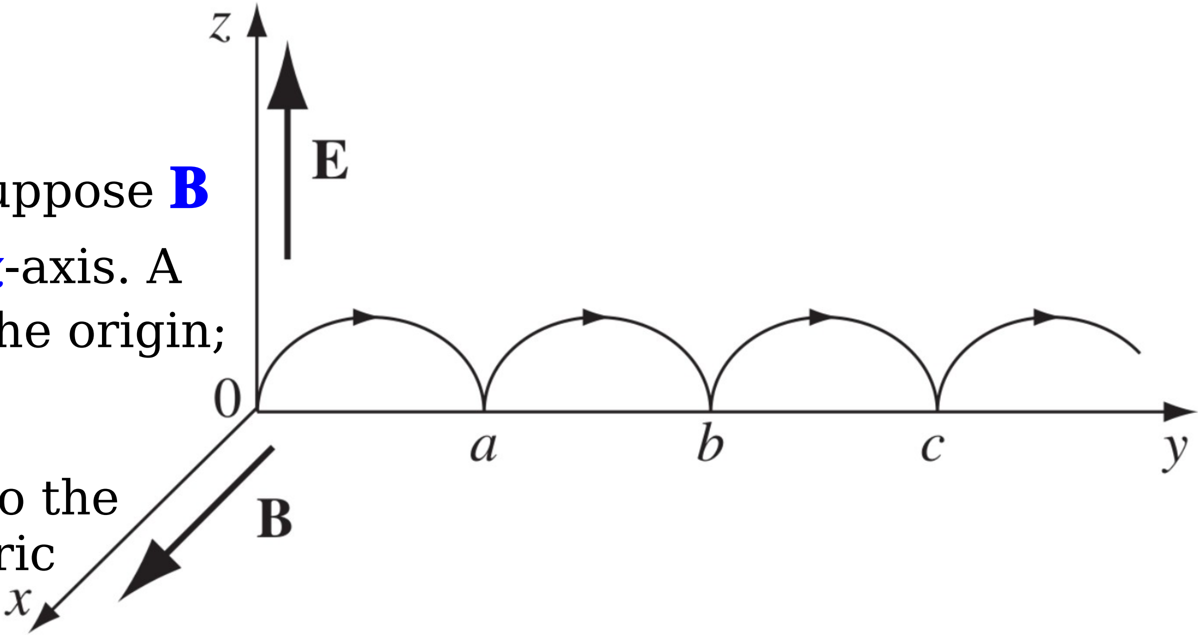
- The **cyclotron formula** describes the motion of a particle in a cyclotron—the first of the modern particle accelerators.

- To find the momentum of a charged particle: send it through a region of known magnetic field, and measure the radius of its trajectory.





- If it starts out with some additional speed $v_{\|}$ *parallel* to **B**, this component of the motion is unaffected by the magnetic field, and the particle moves in a helix.
- The radius is $m\frac{v_{\perp}^{2}}{R}=Q\,v_{\perp}\,B$



<u>Example 5.2</u>: **Cycloid Motion**: Suppose **B** points in the x-axis, and **E** in the z-axis. A positive charge is released from the origin; what path will it follow?

- Initially, the particle is at rest, so the magnetic force is 0, and the electric field accelerates the charge in the z-direction.

- As it picks up speed, F_{mag} develops which pulls the charge around to the right.

- The faster it goes, the stronger F_{mag} becomes; eventually, it curves the particle back around towards the y axis.

- At this point the charge is moving against the electrical force, so it begins to slow down—the magnetic force then decreases, and the electrical force takes over, bringing the particle to rest at point a, and so on.

- No force in the x-direction, the position of the particle can be described by the vector $(0, y(t), z(t))$; the velocity is $\mathbf{v}=(0\,,\,\dot{y}\,,\,\dot{z})$

$$\mathbf{v}\times\mathbf{B}=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ 0 & \dot{y} & \dot{z}\\ B & 0 & 0\end{vmatrix}=B\left(\dot{z}\,\hat{\mathbf{y}}-\dot{y}\,\hat{\mathbf{z}}\right)\quad\Rightarrow\quad \begin{aligned}\mathbf{F}&=Q\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)=Q\left[E\,\hat{\mathbf{z}}+B\left(\dot{z}\,\hat{\mathbf{y}}-\dot{y}\,\hat{\mathbf{z}}\right)\right]\\ &=m\,\mathbf{a}=m\left(\ddot{y}\,\hat{\mathbf{y}}+\ddot{z}\,\hat{\mathbf{z}}\right)\end{aligned}$$

$$\Rightarrow\quad \begin{aligned}m\,\ddot{y}&=Q\,B\,\dot{z}\\ m\,\ddot{z}&=Q\left(E-B\,\dot{y}\right)\end{aligned}\quad\Rightarrow\quad \begin{aligned}\ddot{y}&=\omega\,\dot{z}\\ \ddot{z}&=\omega\left(\frac{E}{B}-\dot{y}\right)\end{aligned}\quad\Leftarrow\quad \omega\equiv\frac{Q\,B}{m}\quad \text{cyclotron frequency}$$

$$\Rightarrow\quad \begin{aligned}y(t)&=C_1\cos\omega t+C_2\sin\omega t+\frac{E}{B}t+C_3=\frac{E}{\omega B}\left(\omega t-\sin\omega t\right)\\ z(t)&=C_2\cos\omega t-C_1\sin\omega t+C_4=\frac{E}{\omega B}\left(1-\cos\omega t\right)\end{aligned}\quad\Leftarrow\quad \begin{aligned}y(0)&=z(0)=0\\ \dot{y}(0)&=\dot{z}(0)=0\end{aligned}$$

$R\equiv\dfrac{E}{\omega B}\quad\Rightarrow\quad\left(y-R\,\omega\,t\right)^2+\left(z-R\right)^2=R^2$ a *circle*, of radius R, whose center $(0, R\,\omega\,t, R)$ travels in the y-direction at a constant speed $u=\omega\,R=\dfrac{E}{B}$

- The particle moves as though it were a spot on the rim of a wheel rolling along the y axis. The curve is called a **cycloid**.

- The overall motion is not in the direction of $\mathbf{E}$, but perpendicular to it.

- One implication of the Lorentz force law deserves special attention:

Magnetic forces do no work.

- If Q moves an amount $\mathrm{d}\boldsymbol{\ell} = \mathbf{v}\,\mathrm{d}t$

$$\Rightarrow \quad \text{work } \mathrm{d}W_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot \mathrm{d}\boldsymbol{\ell} = Q\left(\mathbf{v} \times \mathbf{B}\right) \cdot \mathrm{d}\boldsymbol{\ell} = Q\,\mathbf{v} \times \mathbf{B} \cdot \mathbf{v}\,\mathrm{d}t = 0$$

- Magnetic forces may alter the *direction* in which a particle moves, but they cannot speed it up or slow it down.

Currents

- The **current** in a wire is the *charge per unit time* passing a given point: $I = \frac{\mathrm{d}\,Q}{\mathrm{d}\,t}$

- $-Q$ moving to the left count the same as $+Q$ to the right.

- Almost all phenomena involving moving charges depend on the product of charge & velocity—if you reverse the signs of q & $\mathbf{v}$, you get the same answer, so it doesn't matter which you have — CT invariance.

- The Lorentz force law is a case in point; the Hall effect is a notorious exception.

- It is ordinarily the negatively charged electrons that do the moving—in the direction opposite to the electric current.

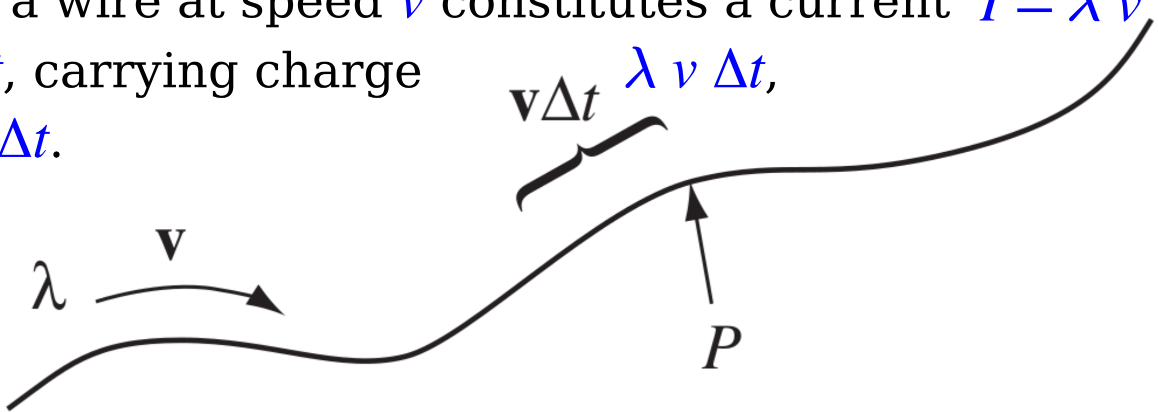
- To avoid the petty complications, pretend it's the positive charges that move.

- Current is measured in coulombs-per-second, or **amperes** (A): $1\ \mathrm{A} = 1\ \frac{\text{Coulomb}}{\text{second}}$

- A line charge λ traveling down a wire at speed v constitutes a current $I = \lambda\, v$ because a segment of length $v\,\Delta t$, carrying charge $\lambda\, v\,\Delta t$, passes point P in a time interval Δt.

- Current is actually a *vector*:

$$\boldsymbol{I} = \lambda\,\mathbf{v}$$



- Because the path of the flow is dictated by the shape of the wire, one doesn't ordinarily bother to display the direction of $\boldsymbol{I}$ explicitly.

- A neutral wire contains as many stationary positive charges as mobile negative ones. The former do not contribute to the current—the charge density λ refers only to the *moving* charges.

- In the unusual situation where both types move, $\boldsymbol{I} = \lambda_+ \mathbf{v}_+ + \lambda_- \mathbf{v}_-$

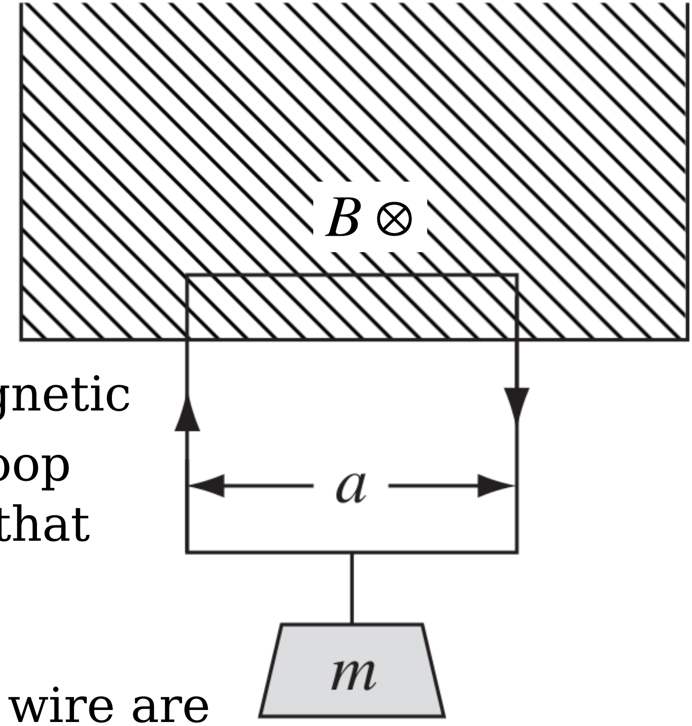
- The magnetic force on a segment of current-carrying wire is

$$\mathbf{F}_{\text{mag}} = \int \mathbf{v} \times \mathbf{B} \, \mathrm{d} q = \int \mathbf{v} \times \mathbf{B} \, \lambda \, \mathrm{d} \ell = \int \boldsymbol{I} \times \mathbf{B} \, \mathrm{d} \ell \quad ⬃ \quad \mathrm{d} \mathbf{F} = I \, \mathrm{d} \boldsymbol{\ell} \times \mathbf{B}$$

$$\boldsymbol{I} \parallel \mathrm{d} \boldsymbol{\ell} \;\Rightarrow\; \boxed{\mathbf{F}_{\text{mag}} = \int I \, \mathrm{d} \boldsymbol{\ell} \times \mathbf{B}} = I \int \mathrm{d} \boldsymbol{\ell} \times \mathbf{B} \;\Leftarrow\; I = \text{const}$$

- Example 5.3: The magnetic force upward exactly balance the gravitational force downward

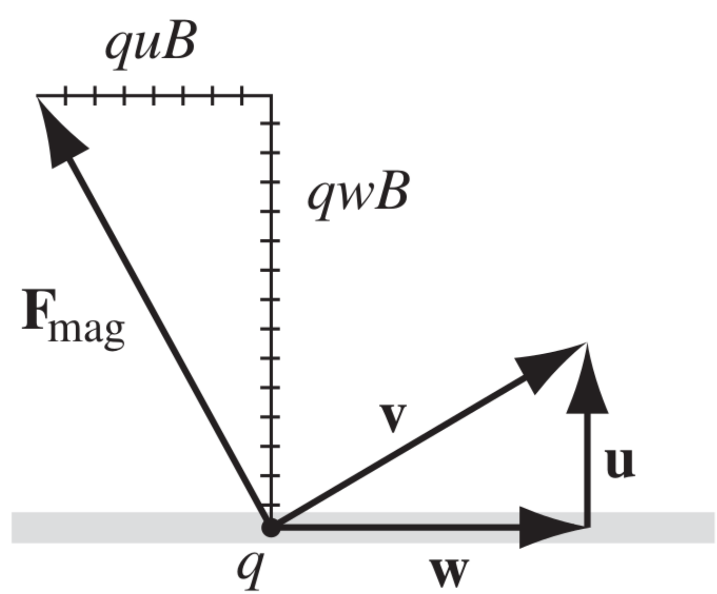
$$F_{\text{mag}} - F_{\text{grav}} = I\,B\,a - m\,g = 0 \quad \Rightarrow \quad I = \frac{m\,g}{B\,a}$$



- If we now *increase* the current, then the upward magnetic force *exceeds* the downward force of gravity, and the loop rises, lifting the weight and doing work. But we know that magnetic forces *never* do work. What's going on here?

- Well, when the loop starts to rise, the charges in the wire are no longer moving horizontally—their velocity now acquires an upward component u, in addition to the horizontal component w associated with the current $I=\lambda w$.

- $\mathbf{F}_{\text{mag}}$, always ⊥ the velocity, no longer points straight up, but tilts back. $\mathbf{F}_{\text{mag}}$ ⊥ the *net* displacement of the charge (in the direction of $\mathbf{v}$), and therefore it *does no work on* q.



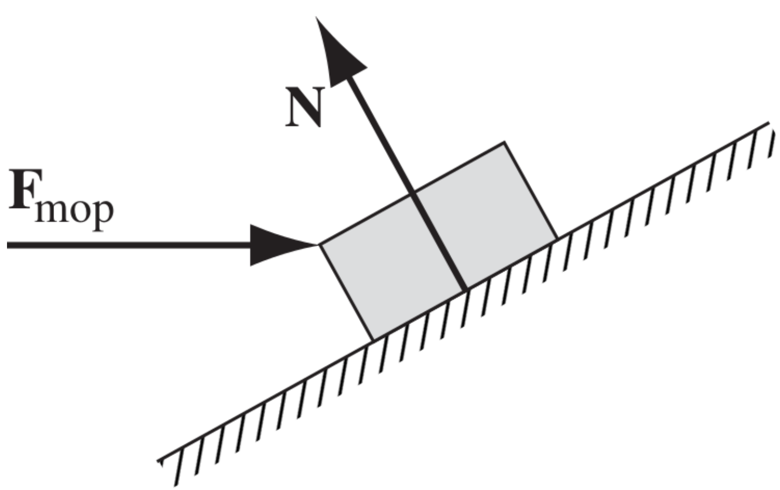
- It does have a vertical *component* (qwB); the net vertical force on all the charge (λa) in the upper segment is $F_{\text{vert}} = \lambda\,a\,w\,B = I\,B\,a$ as before.

- Now it also has a *horizontal* component ($q\,u\,B$), which opposes the current flow. $= \lambda\,a\,u\,B$

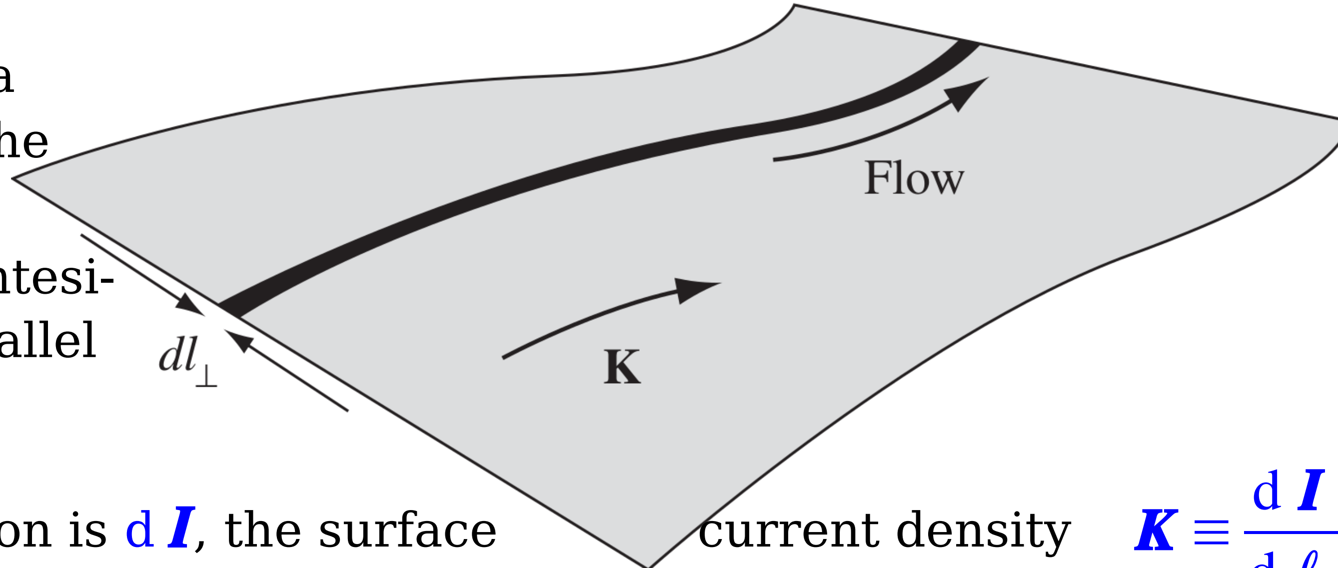
- The total horizontal force on the top segment is $F_{\text{horiz}} = \lambda\, a\, u\, B$
- In $\mathrm{d}t$, the charges move a horizontal distance $\mathrm{d}\ell = w\, \mathrm{d}t$, so the work done by this agency (battery/generator) is $W_{\text{battery}} = \int F_{\text{horiz}}\, \mathrm{d}\,\ell = \lambda\, a\, B \int u\, w\, \mathrm{d}\, t = I\, B\, a\, h$
- The work is done by the battery! The magnetic force *redirects* the horizontal force of the battery into the *vertical* motion of the loop and the weight.

- Slide a trunk up a frictionless ramp by pushing on it horizontally. The normal force **N** does no work, because it ⊥ the displacement. But it does have a vertical component, and a (backward) horizontal component.

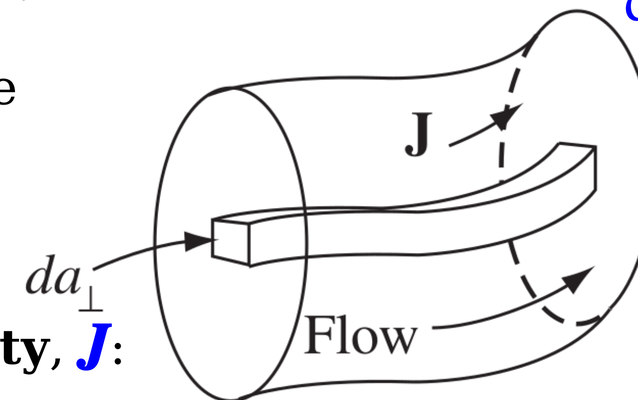
- You do the work but your force does not (directly) lift the box. **N** plays the same passive role as the magnetic force, doing no work itself, but *redirects* the efforts of the active agent.



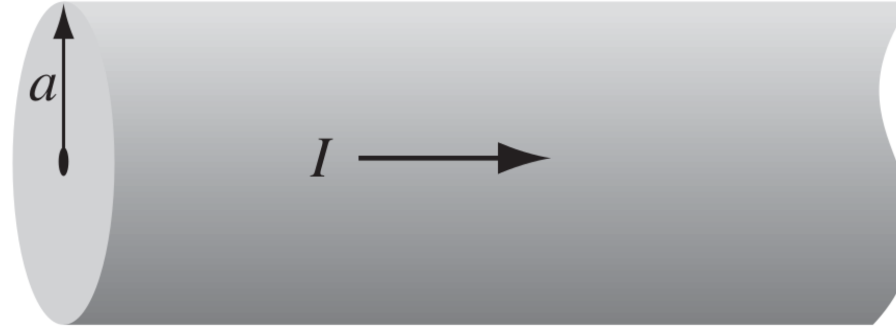
- When charge flows over a *surface*, we describe it by the **surface current density**, $\boldsymbol{K}$: Consider a "ribbon" of infintesimal width $d\ell_\perp$, running parallel to the flow.



- If the current in this ribbon is $d\boldsymbol{I}$, the surface current density $\boldsymbol{K} \equiv \frac{d\boldsymbol{I}}{d\ell_\perp}$
- K is the *current per unit width*: If the surface charge density is σ and its velocity is $\mathbf{v}$, then $\boldsymbol{K} = \sigma\,\mathbf{v}$

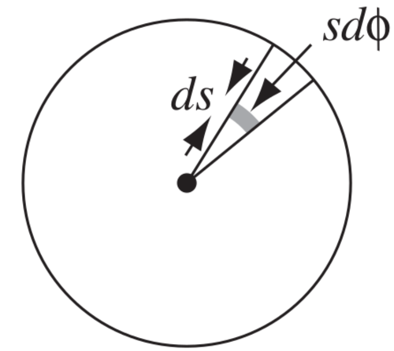


- If the flow of charge is distributed throughout a 3d region, we describe it by the **volume current density**, $\boldsymbol{J}$: Consider a "tube" of infinitesimal cross section $d\,a_\perp$, running parallel to the flow.
- If the current in this tube is $d\boldsymbol{I}$, the volume current density is $\boldsymbol{J} \equiv \frac{d\boldsymbol{I}}{d\,a_\perp}$
- J is the *current per unit area*: $I = \int_{\mathcal{S}} \boldsymbol{J}\cdot d\boldsymbol{a}$
- If the (mobile) volume charge density is ρ and the velocity is $\mathbf{v}$, then $\boldsymbol{J} = \rho\,\mathbf{v}$
- The magnetic force on a volume current $\mathbf{F}_{\mathrm{mag}} = \int \mathbf{v}\times\mathbf{B}\,\rho\,d\tau = \int \boldsymbol{J}\times\mathbf{B}\,d\tau$



- A current I is uniformly distributed over a wire of circular cross section with radius a.

The area is πa^2, so $J = \dfrac{I}{\pi a^2}$



- If the current density in the wire is proportional to the distance from the axis, $J = k\, s \;\Leftarrow\; k = \text{const},\; s$ is radius $\Rightarrow\; \mathrm{d}I = J\,\mathrm{d}a_{\perp}$

$$\Rightarrow\quad I = \int \mathrm{d}I = \int (ks)(s\,\mathrm{d}\phi\,\mathrm{d}s) = 2\pi k \int_0^a s^2\,\mathrm{d}s = \frac{2\pi}{3} k a^3$$

- The total current crossing a surface $\mathcal{S}$ can be $I = \int_{\mathcal{S}} J\,\mathrm{d}a_{\perp} = \int_{\mathcal{S}} \boldsymbol{J}\cdot\mathrm{d}\boldsymbol{a}$

- The charge per unit time leaving a volume $\mathcal{V}$ is $\oint_{\mathcal{S}} \boldsymbol{J}\cdot\mathrm{d}\boldsymbol{a} = \int_{\mathcal{V}} \nabla\cdot\boldsymbol{J}\,\mathrm{d}\tau$

- *Because charge is conserved,* whatever flows out through the surface must come at the expense of what remains inside:

$$\oint_{\mathcal{S}} \boldsymbol{J}\cdot\mathrm{d}\boldsymbol{a} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \rho\,\mathrm{d}\tau = -\int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\,\mathrm{d}\tau \quad\Rightarrow\quad \frac{\partial \rho}{\partial t} + \nabla\cdot\boldsymbol{J} = 0$$

continuity equation

- Summary for translating equations:

$$\sum_{i=1}^{n} (\;)\, q_i \mathbf{v}_i \sim \int_{\text{line}} (\;)\, \boldsymbol{I}\,\mathrm{d}\ell \sim \int_{\text{surface}} (\;)\, \boldsymbol{K}\,\mathrm{d}a \sim \int_{\text{volume}} \boldsymbol{J}\,\mathrm{d}\tau$$

vs charge $q \sim \lambda\,\mathrm{d}\ell \sim \sigma\,\mathrm{d}a \sim \rho\,\mathrm{d}\tau$

The Biot-Savart Law

Steady Currents

- *Stationary charges* produce electric fields that are constant in time; hence the term **electrostatics**.

- *Steady currents* produce magnetic fields that are constant in time; the theory of steady currents is called **magnetostatics**.

$$\text{Stationary charges} \quad \Rightarrow \quad \text{constant electric fields: electrostatics.}$$
$$\text{Steady currents} \quad \Rightarrow \quad \text{constant magnetic fields: magnetostatics.}$$

- By **steady current** we mean a continuous flow that has been going on forever, without change and without charge piling up anywhere.

- Electro/magnetostatics mean $\dfrac{\partial \rho}{\partial t} = 0\,,\ \dfrac{\partial \boldsymbol{J}}{\partial t} = 0$ at all places and all times.

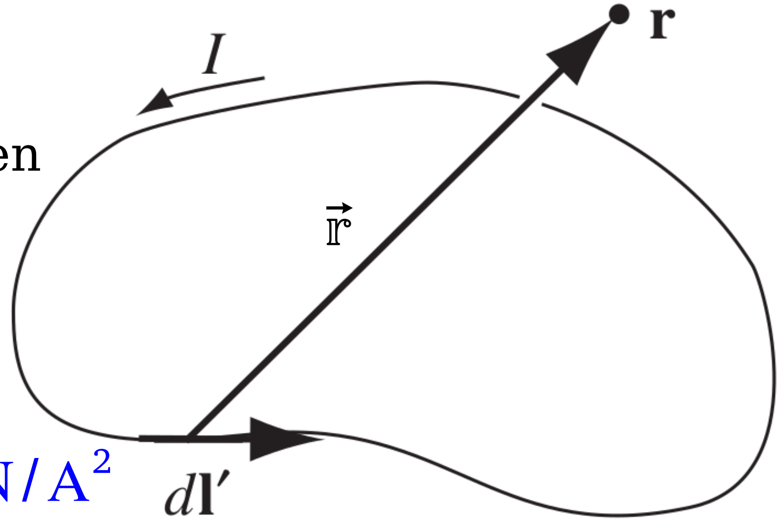
- Electrostatics and magnetostatics represent suitable *approximations* as long as the actual fluctuations are remote, or gradual.

- A moving point charge cannot possibly constitute a steady current. We are forced to deal with extended current distributions right from the start.

- When a steady current flows in a wire, its magnitude I must be the same all along the line

$$\frac{\partial \rho}{\partial t} = 0 \ \text{in magnetostatics} \quad \Rightarrow \quad \nabla \cdot \boldsymbol{J} = 0 \quad \Leftarrow \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$$

The Magnetic Field of a Steady Current



- The magnetic field of a steady line current is given by the **Biot-Savart law**:

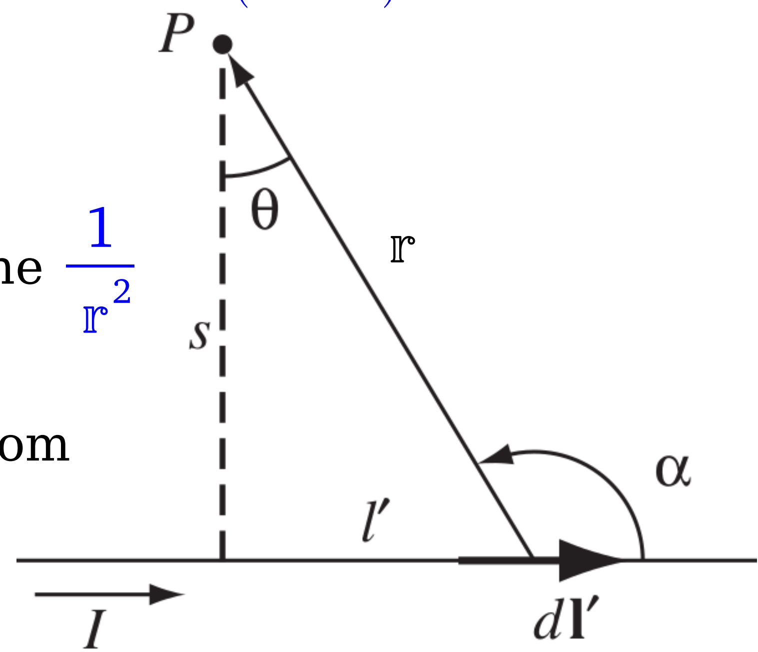
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{I}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\ell' = \frac{\mu_0 I}{4\pi}\int \frac{d\boldsymbol{\ell}'\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

μ_0: **permeability of free space**: $\mu_0 = 4\pi\times 10^{-7}\ \mathrm{N/A^2}$

- **B**'s is newtons per ampere-meter, or **teslas** (T): $1\ \mathrm{T} = 1\ \mathrm{N/(A\cdot m)}$

- 1 tesla = 10^4 gauss. $\mu_0 = \dfrac{1}{\epsilon_0 c^2} \iff c^2 = \dfrac{1}{\epsilon_0\mu_0}$

- For magnetostatics, the Biot-Savart law plays a role analogous to Coulomb's law in electrostatics. the $\dfrac{1}{\mathscr{r}^2}$ dependence is common to both laws.

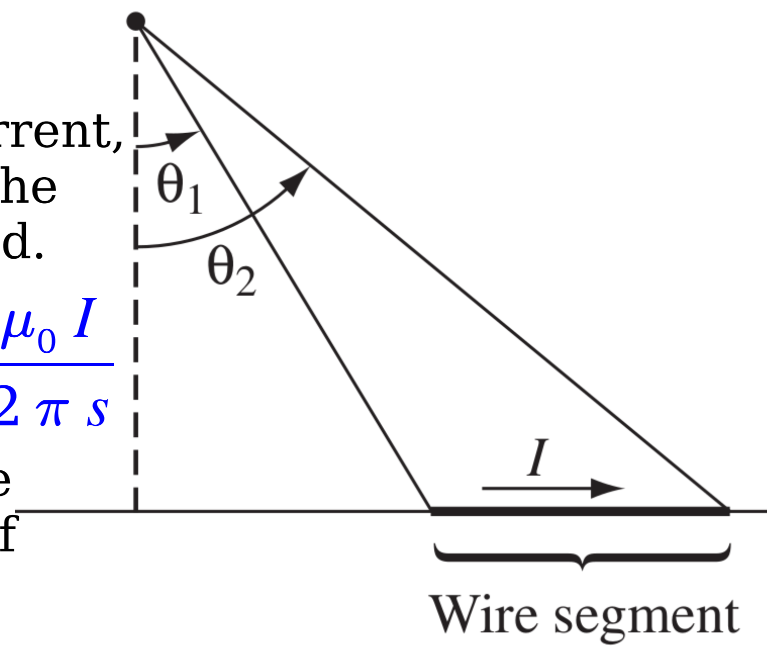


Example 5.5: Find the magnetic field a distance s from a long straight wire carrying a steady current I.

$$d\boldsymbol{\ell}'\times\hat{\mathscr{r}} \to \odot, \quad \sin\alpha\, d\ell' = \cos\theta\, d\ell' \quad \Leftarrow \quad \alpha = \theta + \frac{\pi}{2}$$

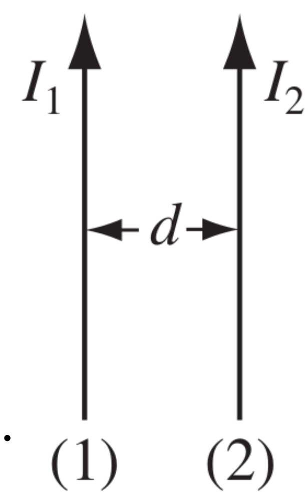
$$\ell' = s\tan\theta, \quad \mathscr{r} = s\sec\theta \quad \Rightarrow \quad d\ell' = s\sec^2\theta\, d\theta, \quad \mathscr{r}^2 = s^2\sec^2\theta$$

$$\Rightarrow \quad B = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\frac{s\sec^2\theta}{s^2\sec^2\theta}\cos\theta\, d\theta = \frac{\mu_0 I}{4\pi s}\int_{\theta_1}^{\theta_2}\cos\theta\, d\theta = \frac{\mu_0 I}{4\pi s}\left(\sin\theta_2 - \sin\theta_1\right)$$



- A finite segment could never support a steady current, but it might be a piece of some closed circuit, and the equation represents its contribution to the total field.
- For an *infinite* wire, $\theta_1 = -\dfrac{\pi}{2}$ and $\theta_2 = \dfrac{\pi}{2}$, so $B = \dfrac{\mu_0 I}{2\pi s}$
- The magnetic field is inversely proportional to the distance from the wire—just like the electric field of an infinite line charge.
- In general, $\mathbf{B}$ "circles around" the wire, with the right-hand rule: $\mathbf{B} = \dfrac{\mu_0 I}{2\pi s}\hat{\phi}$
- Find the force of attraction between 2 long, parallel wires a distance d apart,

carrying currents I_1 and I_2. The field at (2) due to (1) is $B = \dfrac{\mu_0 I_1}{2\pi d} \otimes$

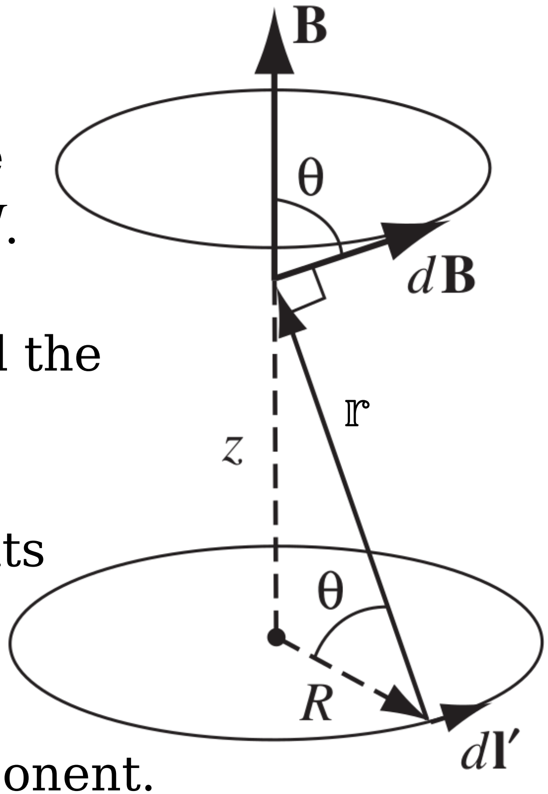


- The Lorentz force law predicts a force directed towards (1), and

$$F = I_2 \frac{\mu_0 I_1}{2\pi d}\int d\ell \quad \Rightarrow \quad f = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d} \quad \text{force per unit length}$$

- If the currents are antiparallel (up and down), the force is repulsive.

Example 5.6: Find the magnetic field a distance z above the center of a circular loop of radius R, with a steady current I.



- $\mathrm{d}\mathbf{B}$ attributable to the segment $\mathrm{d}\boldsymbol{\ell}'$, integrate $\mathrm{d}\boldsymbol{\ell}'$ around the loop, $\mathrm{d}\mathbf{B}$ sweeps out a cone.

- The horizontal components cancel, the vertical components combine, to give $B(z) = \dfrac{\mu_0 I}{4\pi} \displaystyle\int \dfrac{\cos\theta \, \mathrm{d}\ell'}{\mathfrak{r}^2}$

- $\mathrm{d}\boldsymbol{\ell}' \perp \vec{\mathfrak{r}}$; the factor of $\cos\theta$ projects out the vertical component.

- $\cos\theta$ and $\mathfrak{r}^2$ are constants, $\displaystyle\int \mathrm{d}\ell' = 2\pi R$

$$\Rightarrow \quad B(z) = \frac{\mu_0 I}{4\pi} \frac{\cos\theta}{\mathfrak{r}^2} 2\pi R = \frac{\mu_0 I}{2} \frac{R^2}{\left(R^2 + z^2\right)^{3/2}} \quad \Rightarrow \quad B(0) = \frac{\mu_0 I}{2R}$$

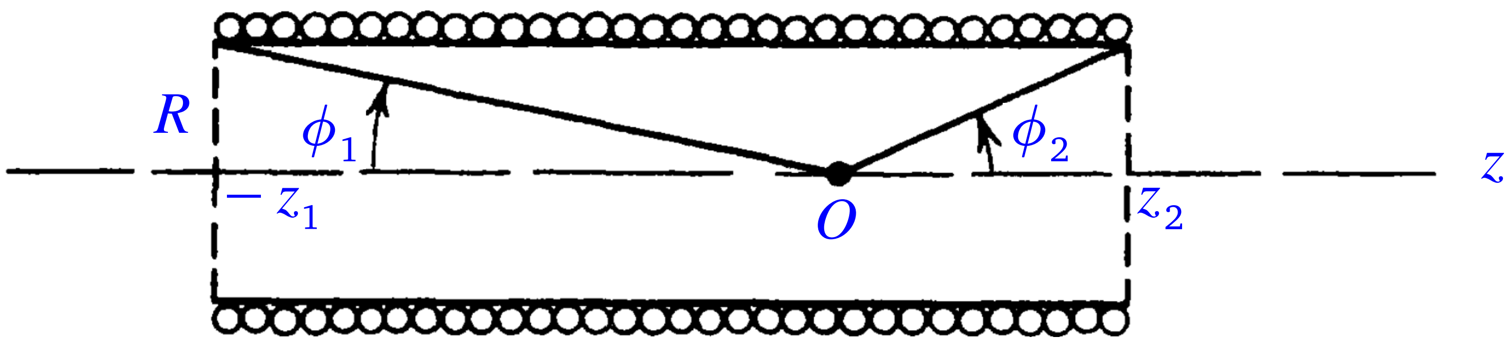
- For surface and volume currents, the Biot-Savart law becomes

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{K}(\mathbf{r}') \times \hat{\mathbb{r}}}{\mathbb{r}^2}\, \mathrm{d}a', \quad \& \quad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\mathbf{r}') \times \hat{\mathbb{r}}}{\mathbb{r}^2}\, \mathrm{d}\tau'$$

- It's *wrong* to write down the formula for a moving charge $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{q\,\mathbf{v} \times \hat{\mathbb{r}}}{\mathbb{r}^2}$

- A point charge does not constitute a steady current, and the Biot-Savart law, which only holds for steady currents, does *not* determine its field.

- The superposition principle also applies to magnetic fields: For a *collection* of source currents, the net field is the sum of the fields from each of them separately.



$$dB_z(z) = \frac{\mu_0\, dI}{2}\frac{R^2}{\left(R^2+z^2\right)^{3/2}} = \frac{\mu_0\, n\, I}{2}\frac{R^2}{\left(R^2+z^2\right)^{3/2}}\, dz \quad \Leftarrow \quad dI = n\, I\, dz$$

$$= \frac{\mu_0\, n\, I}{2}\sin^3\phi\,(-d\cot\phi) \quad \Leftarrow \quad \sin\phi = \frac{R}{\sqrt{R^2+z^2}}, \quad \cot\phi = -\frac{z}{R}$$

$$= \frac{\mu_0\, n\, I}{2}\sin\phi\, d\phi \quad \Leftarrow \quad d\cot\phi = -\csc^2\phi\, d\phi$$

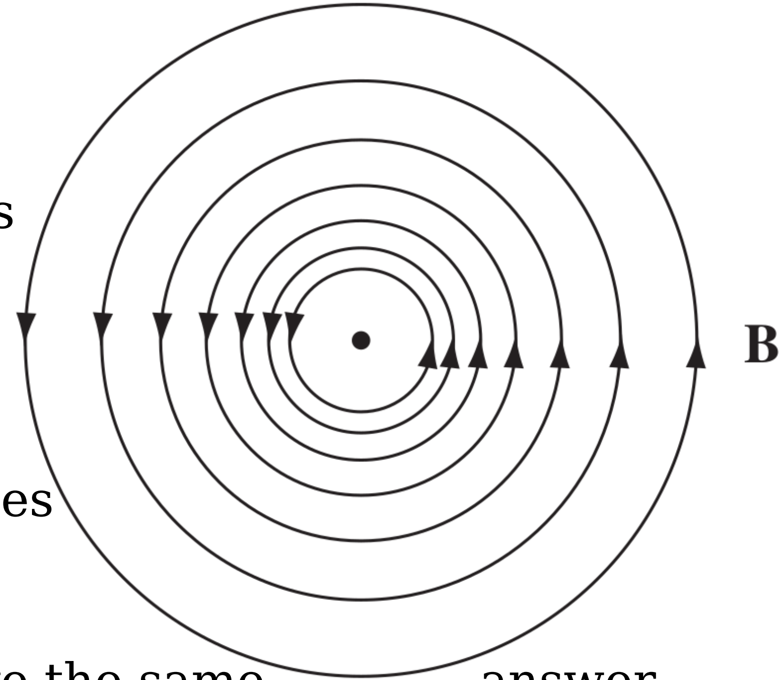
$$\Rightarrow \quad B_z = \int_{-z_1}^{z_2} dB_z = \frac{\mu_0\, n\, I}{2}\int_{\phi_1}^{\pi-\phi_2}\sin\phi\, d\phi = -\frac{\mu_0\, n\, I}{2}\cos\phi\Big|_{\phi_1}^{\pi-\phi_2}$$

$$= -\frac{\mu_0\, n\, I}{2}\left[\cos(\pi-\phi_2) - \cos\phi_1\right] = \frac{\mu_0\, n\, I}{2}\left(\cos\phi_1 + \cos\phi_2\right)$$

For an infinite solinoid $\phi_1 = \phi_2 = 0 \quad \Rightarrow \quad B_z = \mu_0\, n\, I$

The Divergence and Curl of B

Straight-Line Currents



- The magnetic field of an infinite straight wire has a nonzero curl:

$$\oint \mathbf{B}\cdot d\boldsymbol{\ell} = \oint \frac{\mu_0 I}{2\pi s} d\ell = \frac{\mu_0 I}{2\pi s}\oint d\ell = \mu_0 I$$

- The result is independent of s because B decreases at the same rate as the circumference increases.

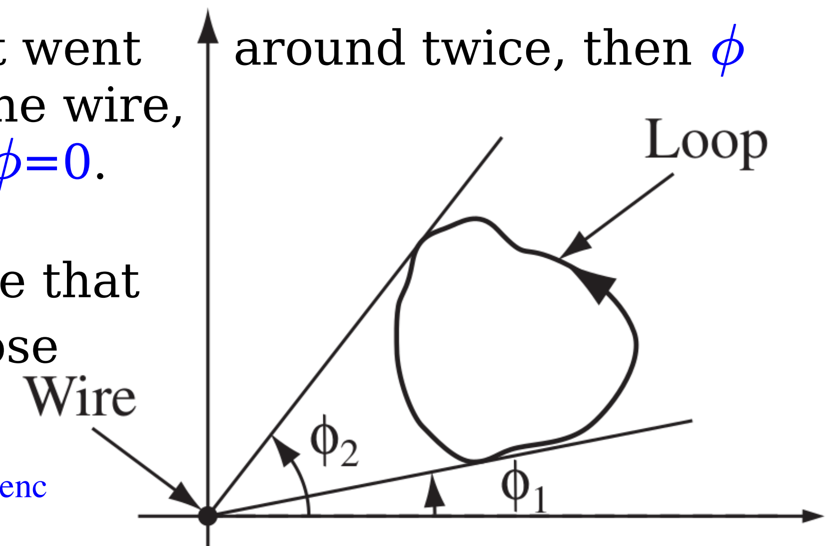
- In fact, *any* loop that encloses the wire would give the same answer.

- Use cylindrical coordinates (s, ϕ, z), with the current flowing along the z axis

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}} \qquad \Rightarrow \qquad \oint \mathbf{B}\cdot d\boldsymbol{\ell} = \oint \frac{\mu_0 I}{2\pi s} s\, d\phi = \frac{\mu_0 I}{2\pi}\oint d\phi = \mu_0 I$$

$$d\boldsymbol{\ell} = ds\,\hat{\mathbf{s}} + s\, d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$$

- Here the loop encircles the wire only once; if it went around twice, then ϕ would run from 0 to 4π, and if it didn't enclose the wire, ϕ would go from ϕ_1 to ϕ_2 and back again, with $\Delta\phi=0$.



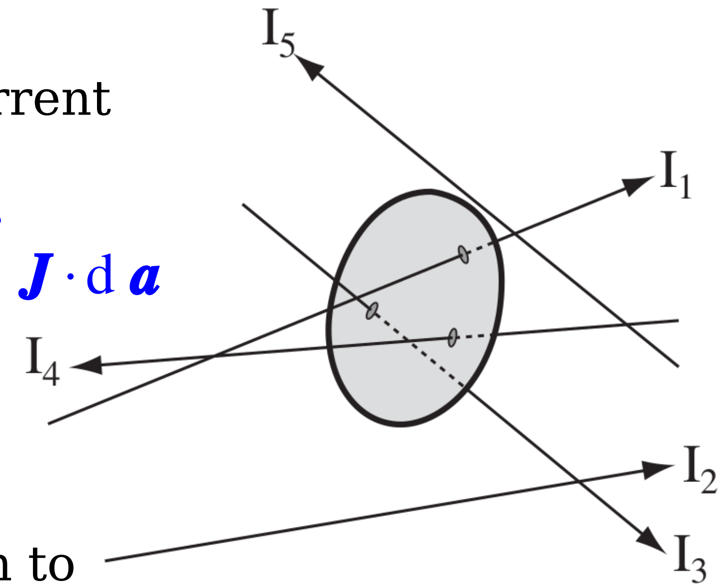
- If we have a bundle of straight wires. Each wire that passes through our loop contributes $\mu_0 I$, and those outside contribute nothing. Then $\oint \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 I_{enc}$

- If the flow of charge is represented by a volume current density $\boldsymbol{J}$,

$$I_{\text{enc}} = \int \boldsymbol{J} \cdot d\boldsymbol{a} \quad \Rightarrow \quad \oint \mathbf{B} \cdot d\boldsymbol{\ell} = \int \nabla \times \mathbf{B} \cdot d\boldsymbol{a} = \mu_0 \int \boldsymbol{J} \cdot d\boldsymbol{a}$$

Stokes' theorem

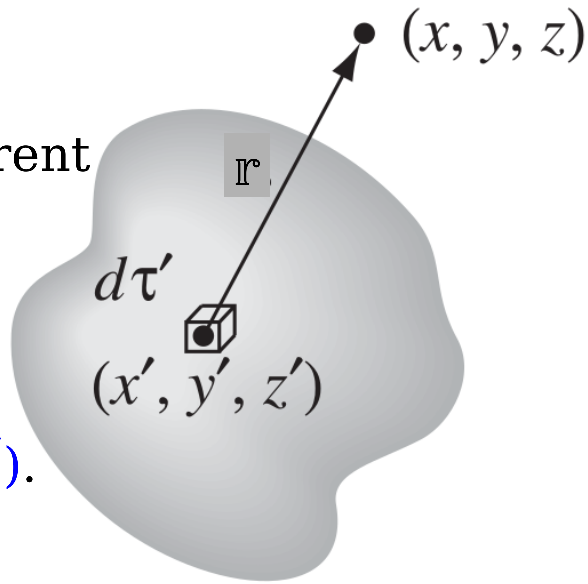
$$\Rightarrow \quad \nabla \times \mathbf{B} = \mu_0 \boldsymbol{J}$$ formula for the curl of $\mathbf{B}$



- This derivation is seriously flawed by the restriction to infinite straight line currents (and combinations thereof).

- Do it right in the next section.

The Divergence and Curl of B



- The Biot-Savart law for the general case of a volume current

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\mathbf{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau' \qquad (\$)$$

- This formula gives the magnetic field at $\mathbf{r}=(x, y, z)$ in terms of an integral over the current distribution $\boldsymbol{J}(x', y', z')$.

- $\mathbf{B}$ is a function of (x, y, z), $\quad d\tau' = dx'\,dy'\,dz'$

  $\boldsymbol{J}$ is a function of (x', y', z'), $\quad \vec{\mathscr{r}} = (x-x')\,\hat{\mathbf{x}} + (y-y')\,\hat{\mathbf{y}} + (z-z')\,\hat{\mathbf{z}}$,

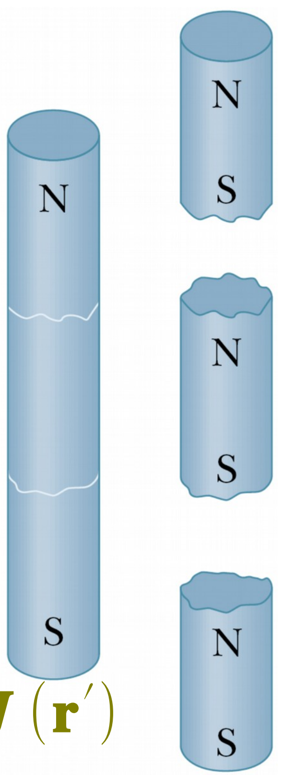
- $(\$) \;\Rightarrow\; \nabla\cdot\mathbf{B} = \dfrac{\mu_0}{4\pi}\displaystyle\int \nabla\cdot\left(\boldsymbol{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) d\tau'$

$$\nabla\cdot\left(\boldsymbol{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot(\nabla\times\boldsymbol{J}) - \boldsymbol{J}\cdot\left(\nabla\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 0 \;\Leftarrow\; \begin{aligned} \nabla\times\boldsymbol{J}(\mathbf{r}') &= 0 \\ \nabla\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} &= 0 \end{aligned}$$

$\Rightarrow$ $\nabla\cdot\mathbf{B} = 0$ the *divergence* of the magnetic field is 0

- $(\$) \;\Rightarrow\; \nabla\times\mathbf{B} = \dfrac{\mu_0}{4\pi}\displaystyle\int \nabla\times\left(\boldsymbol{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) d\tau'$

$$\nabla\times\left(\boldsymbol{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\nabla\right)\boldsymbol{J}(\mathbf{r}') - (\boldsymbol{J}\cdot\nabla)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} + \boldsymbol{J}\,\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} - \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\,\nabla\cdot\boldsymbol{J}(\mathbf{r}')$$



$$-(\boldsymbol{J}\cdot\nabla)\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}=(\boldsymbol{J}\cdot\nabla')\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}=\sum_j \partial'_j\left(J^j\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\right)-\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\cancel{\nabla'\cdot\boldsymbol{J}} \quad \Leftarrow \quad \nabla'\cdot\boldsymbol{J}=0$$

for steady current

$$\sum_j\int_{\mathcal{V}}\partial'_j\left(J^j\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\right)\mathrm{d}\tau'=\oint_{\mathcal{S}}\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}(\boldsymbol{J}\cdot\mathrm{d}\boldsymbol{a}')=0 \quad \Leftarrow$$

on a large enough boundary, the current is 0

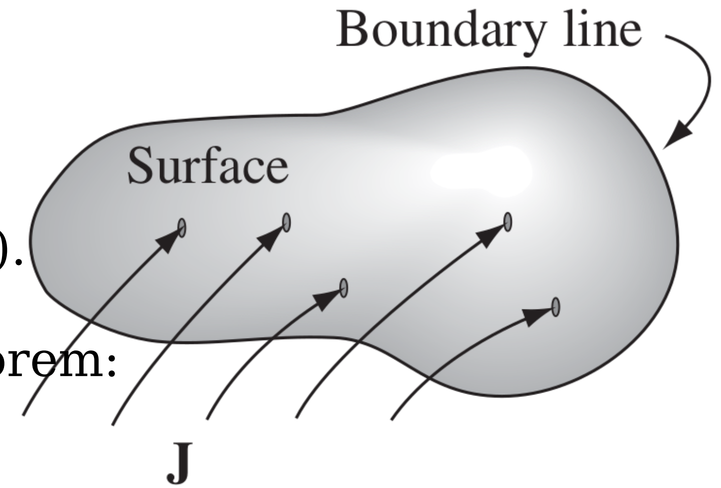
$$\nabla\cdot\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}=4\pi\,\delta^3(\vec{\mathfrak{r}}) \quad \Rightarrow \quad \nabla\times\mathbf{B}=\frac{\mu_0}{4\pi}\int\boldsymbol{J}(\mathbf{r}')\,4\pi\,\delta^3(\mathbf{r}-\mathbf{r}')\,\mathrm{d}\tau'=\mu_0\boldsymbol{J}(\mathbf{r})$$

$\Rightarrow$ $\nabla\times\mathbf{B}=\mu_0\boldsymbol{J}$ holds generally in magnetostatics

$$\Rightarrow \quad 0\Leftarrow\nabla\cdot(\nabla\times\mathbf{B})=\mu_0\nabla\cdot\boldsymbol{J} \quad \Rightarrow \quad \nabla\cdot\boldsymbol{J}=0 \quad \text{check}$$

Ampère's Law

- $\nabla \times \mathbf{B} = \mu_0 \boldsymbol{J}$ is **Ampère's law** (in differential form).



- It can be converted to integral form by Stokes' theorem:

$$\int \nabla \times \mathbf{B} \cdot \mathrm{d}\boldsymbol{a} = \oint \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \int \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{a}$$

$$\Rightarrow \quad \oint \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 I_{enc} \quad \Leftarrow \quad \text{the current enclosed by the Amperian loop}$$

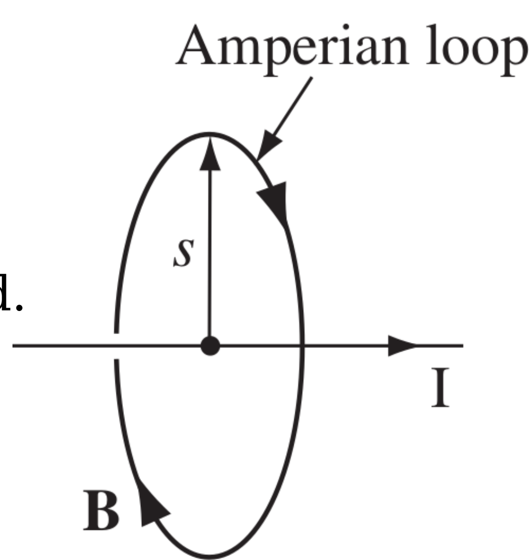
- This is the integral version of Ampère's law; it is generalized into *arbitrary* steady currents.

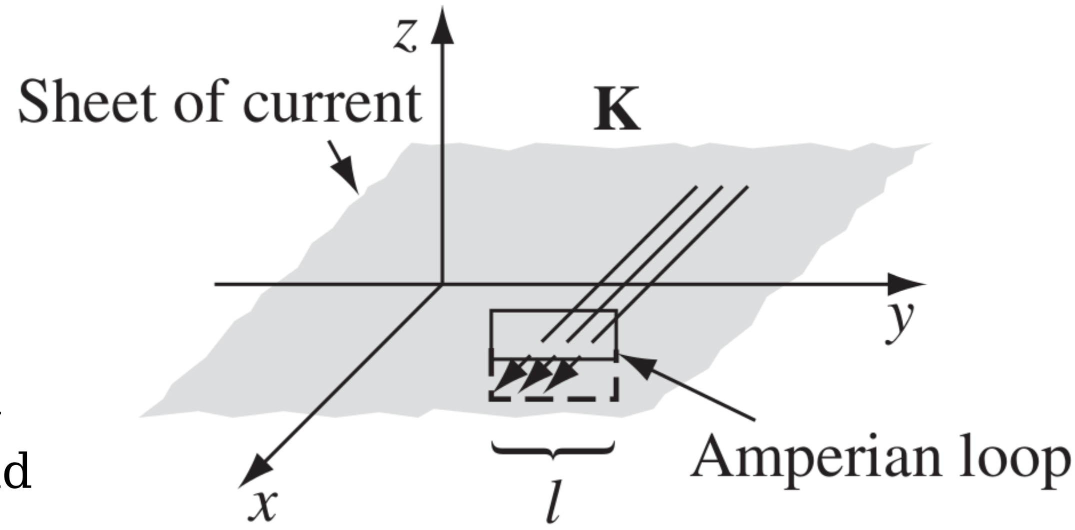
- Use the right-hand rule to decide the direction: If the fingers of your right hand indicate the direction of integration around the boundary, your thumb defines the direction of a positive current.

- Electrostatics : Coulomb → Gauss
  Magnetostatics : Biot–Savart → Ampère

- For currents with appropriate symmetry, Ampère's law in integral form is quite useful in calculating the magnetic field.

Example 5.7





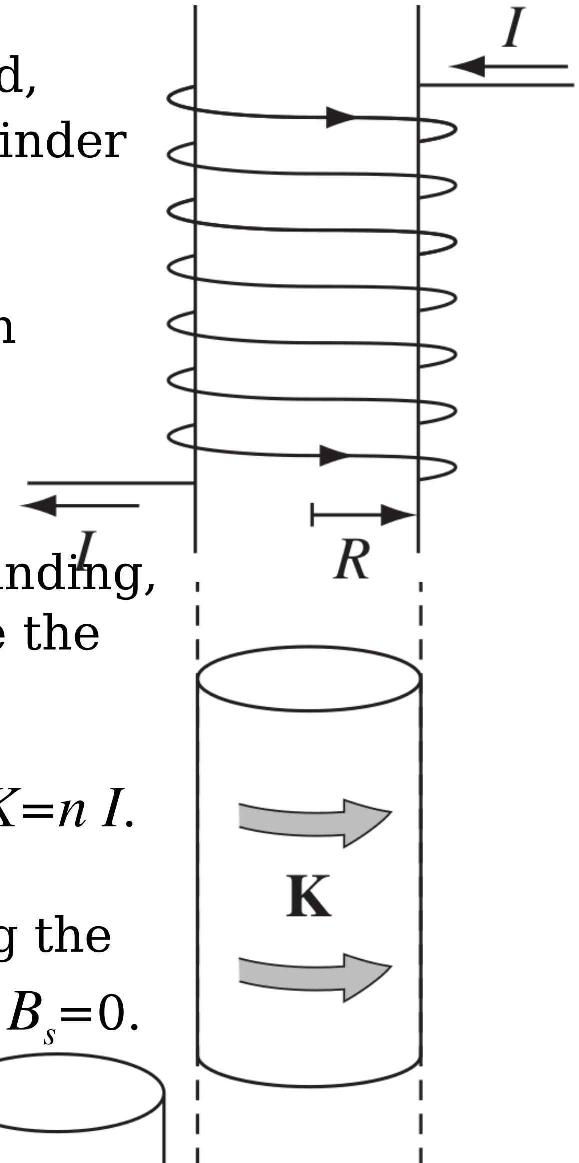
Example 5.8: Find the magnetic field of an infinite uniform surface current $\boldsymbol{K} = K\,\hat{\mathbf{x}}$, flowing over the *xy* plane.

- $\mathbf{B}$ can only have a *y* component, and it points to the *left* above the plane and to the *right* below it.

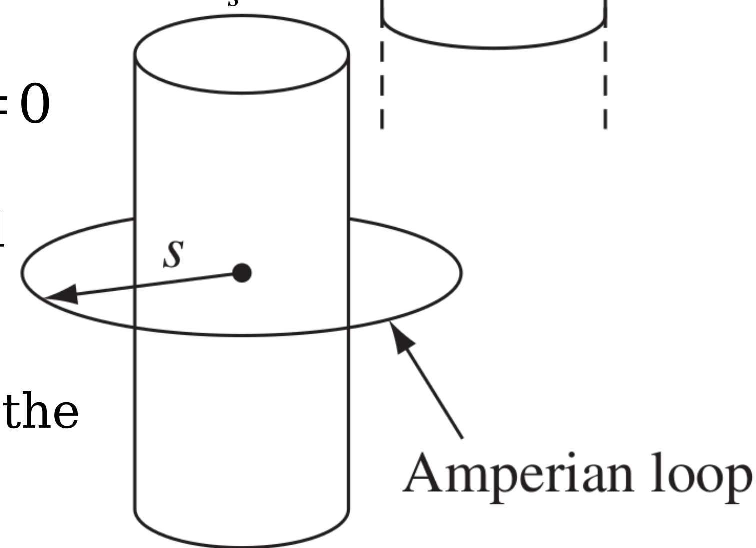
- $\oint \mathbf{B}\cdot d\boldsymbol{\ell} = 2B\ell = \mu_0 I_{\text{enc}} = \mu_0 K \ell$ Ampere's law

$$\Rightarrow \quad B = \frac{\mu_0}{2}K \quad \Rightarrow \quad \mathbf{B} = \pm\frac{\mu_0}{2}K\,\hat{\mathbf{y}} \quad \text{for} \quad z \lessgtr 0 \quad \text{const} \quad \text{for all } z$$

Example 5.9: Find the magnetic field of a very long solenoid, consisting of n closely wound turns per unit length on a cylinder of radius R, each carrying a steady current I.

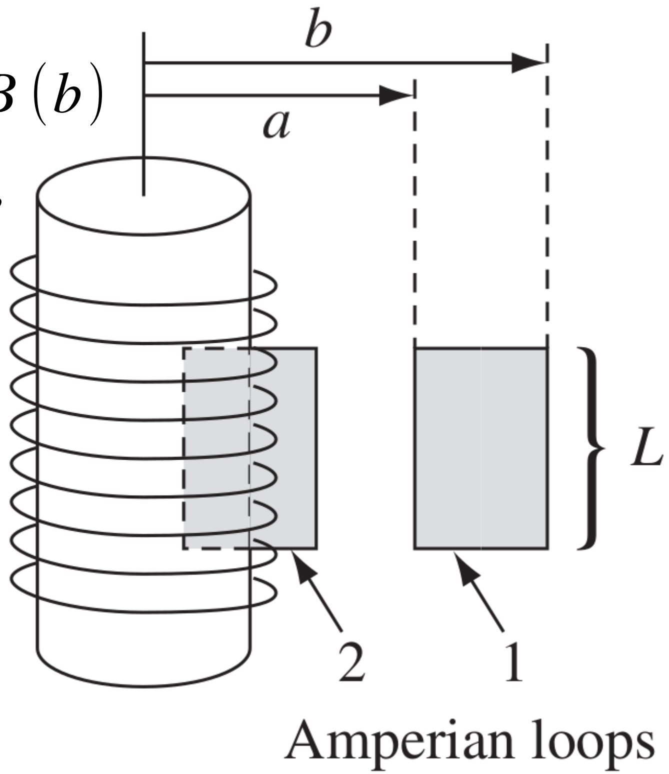


- The point of making the windings so close is that one can then pretend each turn is circular.
- There is a net current I in the direction of the solenoid's axis, no matter *how* tight the winding. Or make a double winding, going up to one end and then going back down to eliminate the net longitudinal current—unnecessary!
- Imagine a sheet of foil with the uniform surface current $K=nI$.
- If $I \to B_s \Rightarrow -I \to -B_s$. But switching I is equivalent to turning the solenoid upside down, and no change for the radial field $\Rightarrow B_s=0$.
- For B_ϕ, $\oint \mathbf{B}\cdot d\boldsymbol{\ell} = B_\phi(2\pi s) = \mu_0 I_{\text{enc}} = 0 \Rightarrow B_\phi = 0$
- So the magnetic field of an infinite, closely wound solenoid runs *parallel to the axis*.
- From the right-hand rule, it points upward inside the solenoid and downward outside.



- $\oint_1 \mathbf{B}\cdot d\boldsymbol{\ell} = [B(a) - B(b)]L = \mu_0 I_{enc} = 0 \;\Rightarrow\; B(a) = B(b)$

the *field outside does not depend on the distance from the axis*.



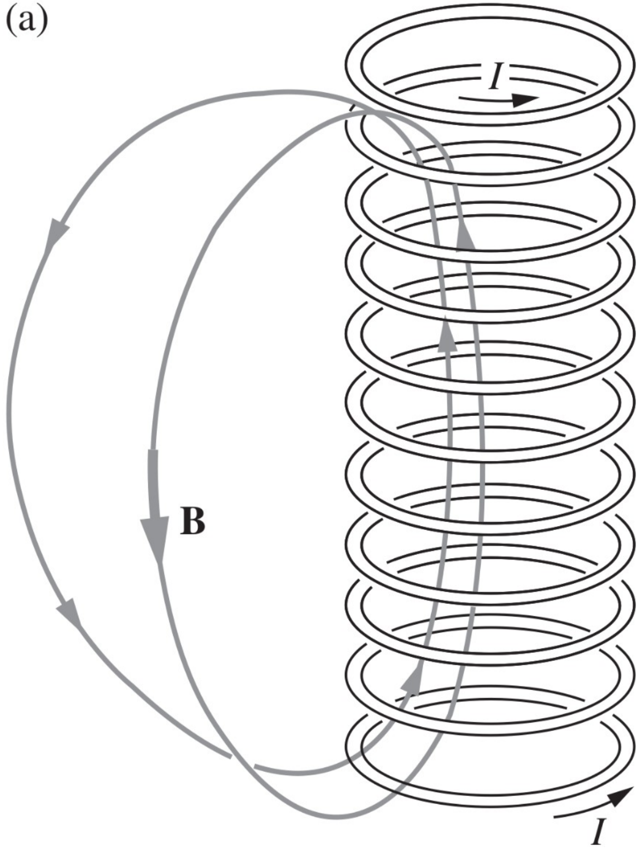
- But it should go to 0 for large *s*. It must therefore be 0 *everywhere*!

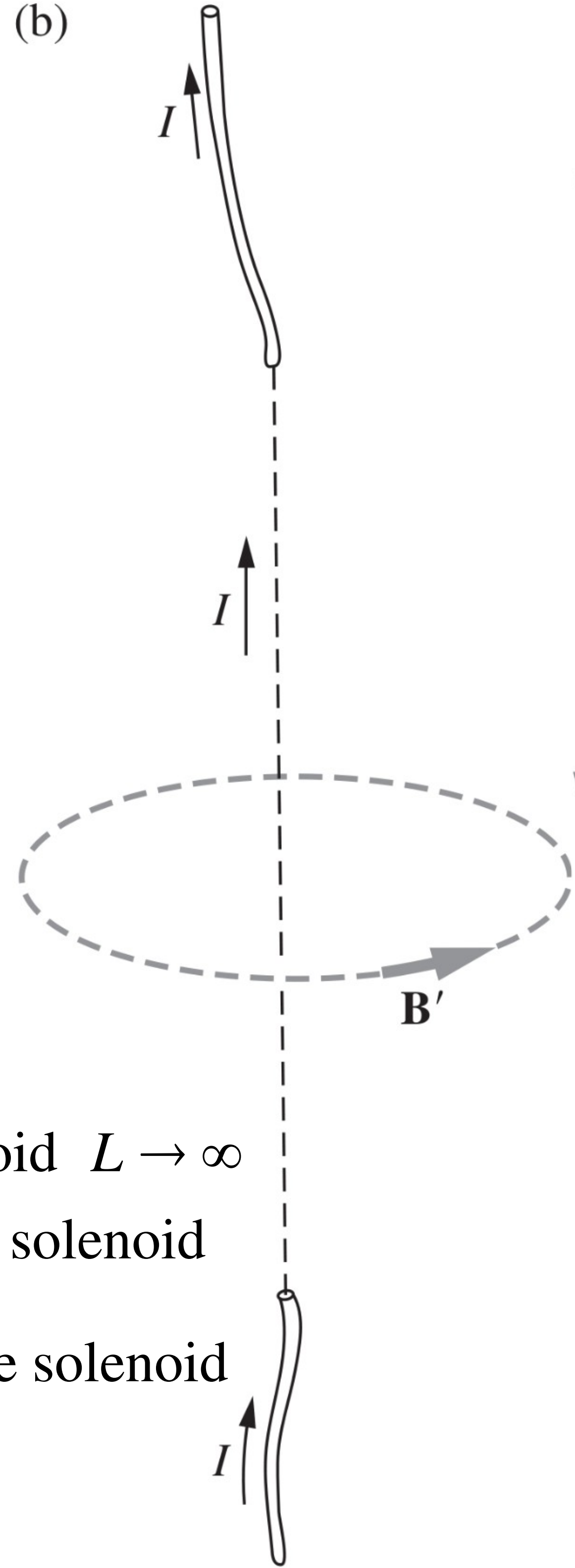
- $\oint_2 \mathbf{B}\cdot d\boldsymbol{\ell} = BL = \mu_0 I_{enc} = \mu_0 n I L \;\Leftarrow\; B_{outside} = 0$

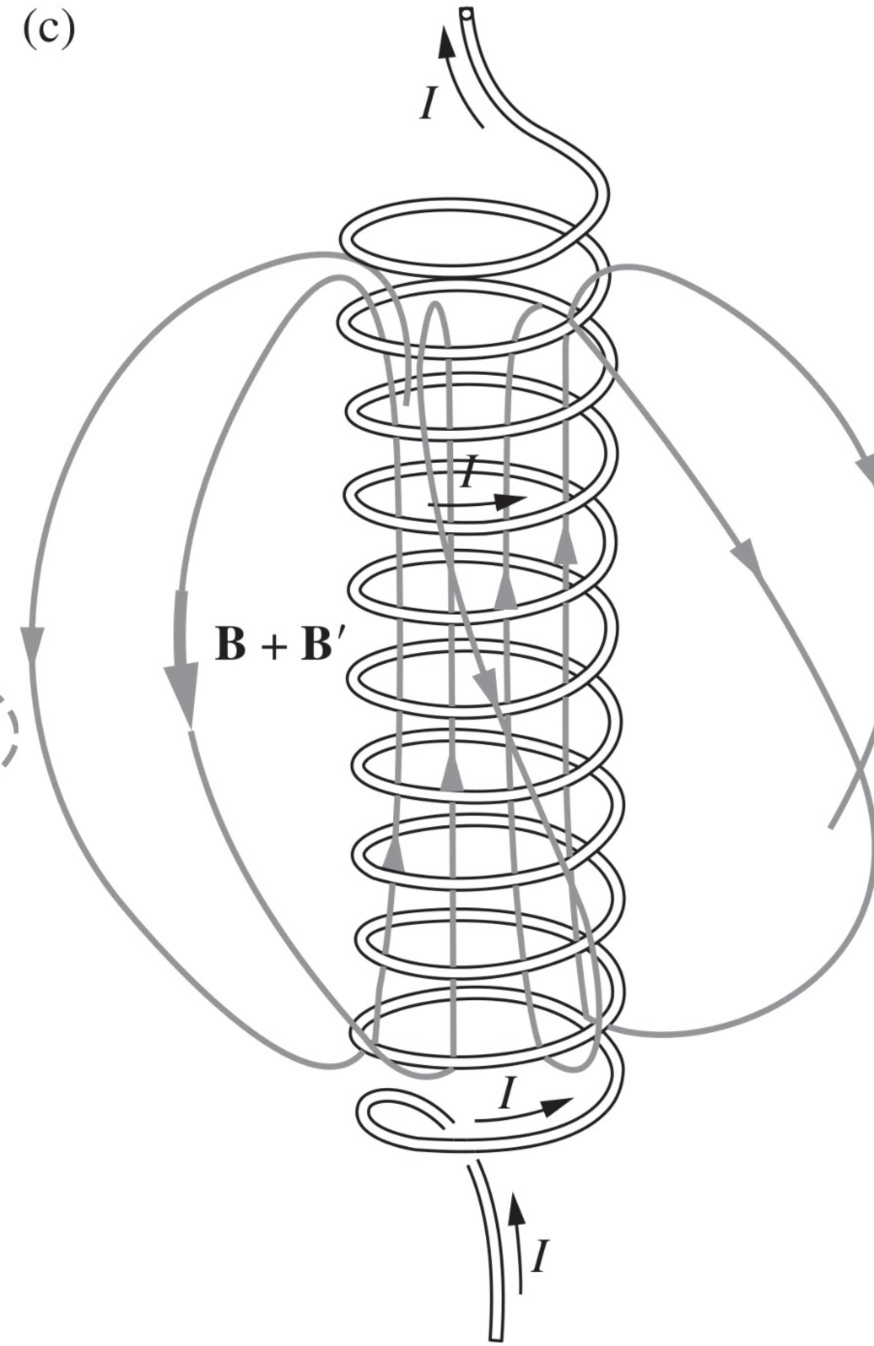
$$\Rightarrow \mathbf{B} = \begin{cases} \mu_0 n I\,\hat{\mathbf{z}}, & \text{inside the solenoid} \\ 0, & \text{outside the solenoid} \end{cases}$$

- The field inside is *uniform*—it doesn’t depend on the distance from the axis.

- The solenoid to magnetostatics is the parallel-plate capacitor to electrostatics: a simple device for producing strong uniform fields.







As the length of the solenoid $L \to \infty$

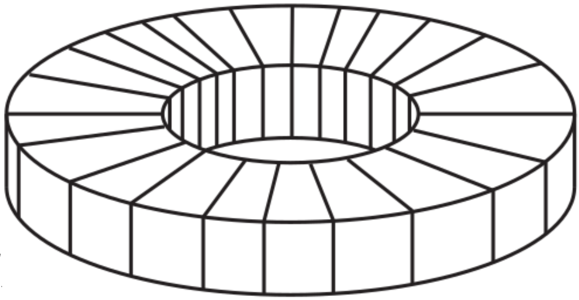
$$\mathbf{B} = \begin{cases} \mu_0 \, n \, I \, \hat{\mathbf{z}} & \text{inside the solenoid} \\ \dfrac{\mu_0 \, I}{2 \pi \rho} \hat{\boldsymbol{\phi}} & \text{outside the solenoid} \end{cases}$$

- Ampère's law is always *true* (for steady currents), but it is not always *useful*.

- When it *does* work, it's the fastest method; when it doesn't, you have to fall back on the Biot-Savart law.

- The current configurations that can be handled by Ampère's law are

  1. Infinite straight lines 2. Infinite planes
  3. Infinite solenoids 4. Toroids



Example 5.10: A toroidal coil consists of a circular ring, or "donut," around which a long wire is wrapped. In a toroid, the *magnetic field of the toroid is circumferential at all points both inside and outside the coil*.

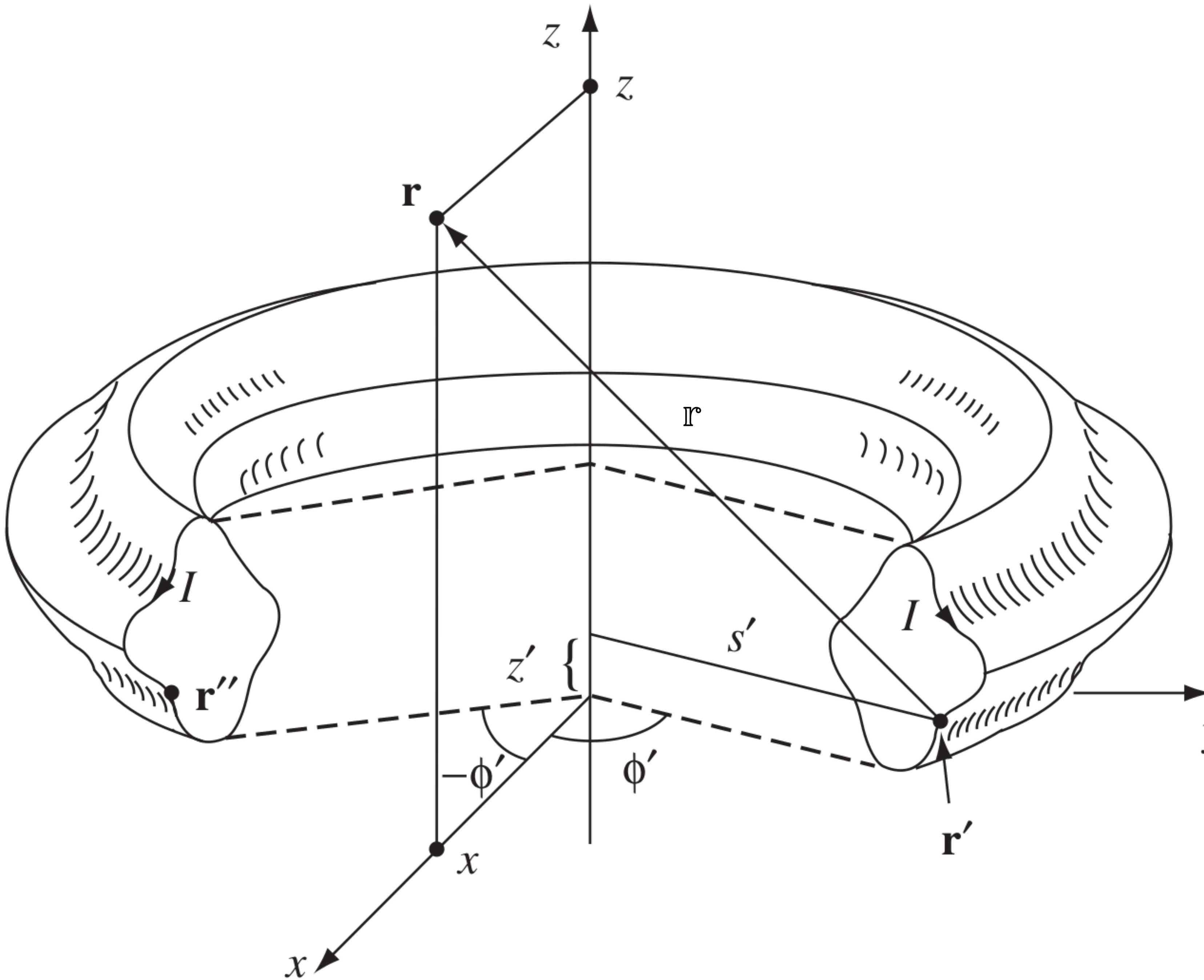
Proof: According to the Biot-Savart law $\mathrm{d}\,\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\dfrac{\boldsymbol{I}\times\vec{\mathfrak{r}}}{\mathfrak{r}^3}\,\mathrm{d}\,\ell'$

- Put $\mathbf{r}$ in the xz plane $\mathbf{r}=(x,0,z)$, and the source coordinates $\mathbf{r}'=(s'\cos\phi', s'\sin\phi', z')$

$$\Rightarrow\quad \vec{\mathfrak{r}} = \mathbf{r} - \mathbf{r}' = (x - s'\cos\phi',\, -s'\sin\phi',\, z - z')$$

- The current has no ϕ component: $\boldsymbol{I} = I_s\,\hat{\mathbf{s}} + I_z\,\hat{\mathbf{z}} = (I_s\cos\phi',\, I_s\sin\phi',\, I_z)$

$$\Rightarrow\ \boldsymbol{I}\times\vec{\mathfrak{r}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ I_s\cos\phi' & I_s\sin\phi' & I_z \\ x - s'\cos\phi' & -s'\sin\phi' & z - z' \end{vmatrix} = \begin{aligned} &[I_s(z-z') + I_z s']\sin\phi'\,\hat{\mathbf{x}} \\ &+[I_z x - (I_z s' + I_s z - I_s z')\cos\phi']\,\hat{\mathbf{y}} \\ &- I_s x\sin\phi'\,\hat{\mathbf{z}} \end{aligned}$$



z
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- But there is a symmetrically situated current element at $\mathbf{r}''$, with the same s', $\mathscr{r}$, $d\ell'$, I_s, I_z, *but negative* ϕ'.

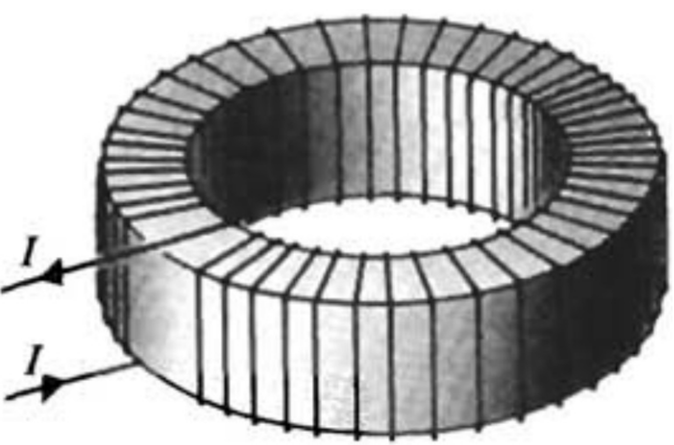
- Because $\sin\phi'$ changes sign, the $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ contributions from $\mathbf{r}'$ and $\mathbf{r}''$ cancel, leaving only a $\hat{\mathbf{y}}$ term, in general the field points in the $\hat{\boldsymbol{\phi}}$ direction.

- To determine the magnitude, apply Ampère's law to a circle of radius s about the axis of the toroid:

$$B\,2\pi s = \mu_0 I_{\text{enc}} \quad \Rightarrow \quad \mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 N I}{2\pi s}\hat{\boldsymbol{\phi}}, & \text{for points inside the coil} \\ 0 & \text{for points outside the coil} \end{cases}$$

N is the total number of turns.

- True toroid = Ideal toroid + Single coil



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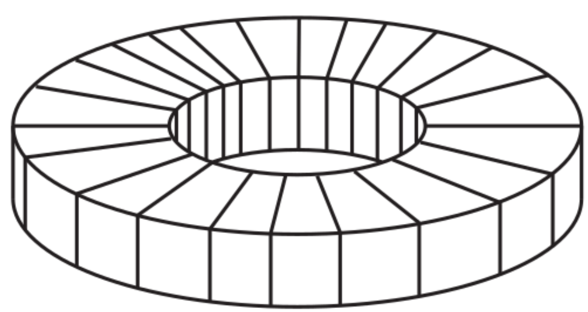

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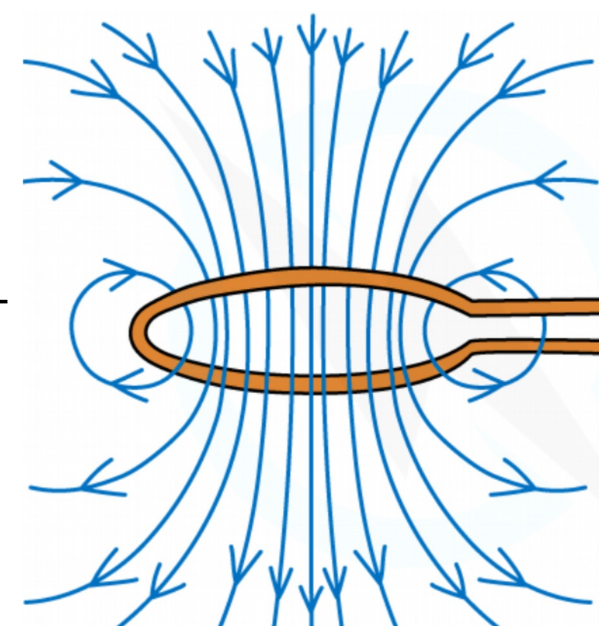


Comparison of Magnetostatics and Electrostatics

- The divergence and curl of the *electrostatic* field

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0} & \text{Gauss's law} \\ \nabla \times \mathbf{E} = 0 & \text{Faraday's law} \end{cases}$$

- They are **Maxwell's eqns** for electrostatics. With boundary condition (eg, $\mathbf{E} \to 0$ far from charges), Maxwell's eqns determine the field, if the source ρ is given.

- They contain essentially the same information as Coulomb's law plus the principle of superposition.

- The divergence and curl of the magnetostatic field

$$\begin{cases} \nabla \cdot \mathbf{B} = 0 & \text{no magnetic monopole} \\ \nabla \times \mathbf{B} = \mu_0 \boldsymbol{J} & \text{Ampere's law} \end{cases}$$

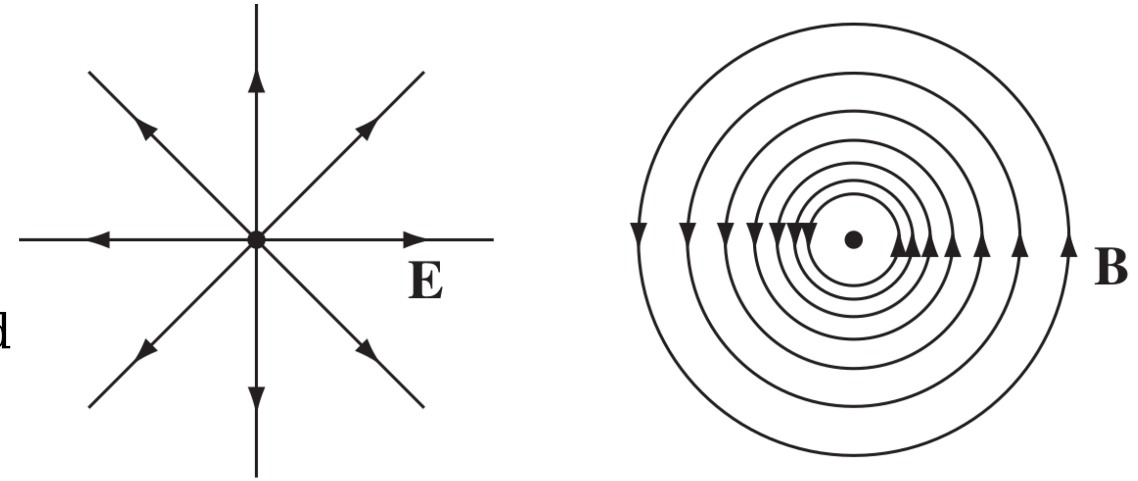
- They are Maxwell's eqns for magnetostatics. With boundary condition (eg, $\mathbf{B} \to 0$ far from all currents), Maxwell's eqns determine the magnetic field, with given $\boldsymbol{J}$.

- They are equivalent to the Biot-Savart law (plus superposition).

- Maxwell's eqns and the Lorentz force law $\mathbf{F} = Q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$ constitute the most elegant formulation of electrostatics and magnetostatics.

- The electric field *diverges away from* a (positive) charge; the magnetic field line *curls around* a current.

- Electric field lines originate on positive charges and terminate on negative ones; magnetic field lines do not begin or end anywhere. They typically form closed loops or extend out to infinity.



(a) Electrostatic field of a point charge

(b) Magnetostatic field of a long wire

- *There are no point sources for* $\mathbf{B}$, the physical content of $\nabla \cdot \mathbf{B} = 0$.

- Ampère was the first who speculated that all magnetic effects are attributable to *electric* charges *in motion* (currents).

- $\mathbf{B}$ is divergenceless and no magnetic monopoles. It takes a *moving* electric charge to *produce* a magnetic field, and another moving electric charge to "feel" the magnetic field.

- Typically, electric forces are enormously larger than magnetic ones. It is usually with the sizes of the fundamental constants ϵ_0 and μ_0.

- In general, it is only when both the source charges and the test charge are moving at velocities comparable to the speed of light that the magnetic force approaches the electric force in strength.

- If we arrange to keep the wire *neutral*, the magnetic field can easily stand out.

Magnetic Vector Potential

The Vector Potential

- $\nabla\times\mathbf{E}=0$ introduces a scalar potential (Φ) in electrostatics, $\mathbf{E} = -\nabla\Phi$.

- The electric potential had a built-in ambiguity: you can add to Φ any function whose gradient is 0 (ie, any constant), without altering the physical quantity $\mathbf{E}$.

- $\nabla\cdot\mathbf{B}=0$ introduces a *vector* potential $\mathbf{A}$ in magnetostatics: $\mathbf{B} = \nabla\times\mathbf{A}$

$$\Rightarrow \quad \nabla\times\mathbf{B}=\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}=\mu_0\boldsymbol{J}$$

- You can add to $\mathbf{A}$ any function whose *curl* vanishes (ie, the *gradient of any scalar*), with no effect on $\mathbf{B}$.

Coulomb gauge

- We can exploit this freedom to eliminate the divergence of $\mathbf{A}$: $\nabla\cdot\mathbf{A} = 0$

Proof: Let the original potential, $\mathbf{A}_0$, is *not* divergenceless. Add to it the gradient of λ $\Rightarrow \quad \mathbf{A}=\mathbf{A}_0+\nabla\lambda \quad \Rightarrow \quad \nabla\cdot\mathbf{A}=\nabla\cdot\mathbf{A}_0+\nabla^2\lambda=0 \quad \Rightarrow \quad \nabla^2\lambda=-\nabla\cdot\mathbf{A}_0$

This is mathematically identical to Poisson's equation $\nabla^2\Phi=-\dfrac{\rho}{\epsilon_0}$ with $\nabla\cdot\mathbf{A}_0$ in place of ρ/ϵ_0 as the "source."

If $\nabla\cdot\mathbf{A}_0$ goes to 0 at ∞, $\lambda=\dfrac{1}{4\pi}\displaystyle\int\frac{\nabla\cdot\mathbf{A}_0}{\mathfrak{r}}\,d\tau'$

If $\nabla \cdot \mathbf{A}_0$ does *not* go to 0 at ∞, we use other means to discover the appropriate λ.

So *it is always possible to make the vector potential divergenceless*.

- $\mathbf{B}=\nabla\times\mathbf{A}$ specifies the *curl* of $\mathbf{A}$, but it never say anything about the divergence —we are at will to pick that as we see fit, and 0 is ordinarily the simplest choice.
- With $\nabla\cdot\mathbf{A}=0$, Ampère's law becomes $\nabla^2\mathbf{A}=-\mu_0\boldsymbol{J}$
- This *again* is nothing but Poisson's eqn, but it is 3 Poisson's eqns, one for each spatial dimension.
- Assuming $\boldsymbol{J}$ goes to 0 at ∞, $\Rightarrow$ $\mathbf{A}(\mathbf{r})=\dfrac{\mu_0}{4\pi}\displaystyle\int\frac{\boldsymbol{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau'$
- For line and surface currents,

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\boldsymbol{I}}{\mathscr{r}}\,d\ell'=\frac{\mu_0 I}{4\pi}\int\frac{d\boldsymbol{\ell}'}{\mathscr{r}},\quad \mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\boldsymbol{K}}{\mathscr{r}}\,da'$$

- $\mathbf{A}$ is not as *useful* as Φ because it's still a *vector*.
- There is only quite limited usage (when $\nabla\times\mathbf{B}=0$) with a scalar potential $\mathbf{B}=-\nabla\Psi$ because it is incompatible with Ampère's law, since the curl of a gradient is always 0. See Chapter 6 for further discussions.

The expression of the Vector Potential

$$\nabla^2 \mathbf{A} = -\mu_0 \boldsymbol{J} \quad \Rightarrow \quad \nabla^2 A_x = -\mu_0 J_x , \quad \nabla^2 A_y = -\mu_0 J_y , \quad \nabla^2 A_z = -\mu_0 J_z$$

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \quad \Rightarrow \quad \Phi = \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{\mathscr{r}}\, \mathrm{d}\tau'$$

$$\Rightarrow \quad A_x = \frac{\mu_0}{4\pi}\int \frac{J_x}{\mathscr{r}}\, \mathrm{d}\tau' , \quad A_y = \frac{\mu_0}{4\pi}\int \frac{J_y}{\mathscr{r}}\, \mathrm{d}\tau' , \quad A_z = \frac{\mu_0}{4\pi}\int \frac{J_z}{\mathscr{r}}\, \mathrm{d}\tau'$$

$$\Rightarrow \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\mathbf{r}')}{\mathscr{r}}\, \mathrm{d}\tau' \quad \Leftarrow \quad \vec{\mathscr{r}} = \mathbf{r} - \mathbf{r}' , \quad \mathscr{r} = |\vec{\mathscr{r}}|$$

Derivation of the Biot-Savart Law from the Vector Potential

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi}\nabla \times \int \frac{\boldsymbol{J}(\mathbf{r}')}{\mathscr{r}}\, \mathrm{d}\tau'$$

$$= \frac{\mu_0}{4\pi}\oint \nabla \times \frac{I\, \mathrm{d}\boldsymbol{\ell}'}{\mathscr{r}} = \frac{\mu_0 I}{4\pi}\oint \nabla \times \frac{\mathrm{d}\boldsymbol{\ell}'}{\mathscr{r}} \quad \Leftarrow \quad \boldsymbol{J}\, \mathrm{d}\tau' = I\, \mathrm{d}\boldsymbol{\ell}'$$

$$= \frac{\mu_0 I}{4\pi}\oint \left(\cancel{\frac{\nabla \times \mathrm{d}\boldsymbol{\ell}'}{\mathscr{r}}} + \nabla \frac{1}{\mathscr{r}} \times \mathrm{d}\boldsymbol{\ell}' \right) = \frac{\mu_0 I}{4\pi}\oint \left(-\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \times \mathrm{d}\boldsymbol{\ell}' \right) \quad \Leftarrow \quad \nabla \frac{1}{\mathscr{r}} = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$= \frac{\mu_0 I}{4\pi}\oint \frac{\mathrm{d}\boldsymbol{\ell}' \times \hat{\mathscr{r}}}{\mathscr{r}^2} \quad \Rightarrow \quad \mathrm{d}\mathbf{B} = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}\boldsymbol{\ell}' \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}\boldsymbol{\ell}' \times \vec{\mathscr{r}}}{\mathscr{r}^3} \quad \Leftarrow \quad \mathbf{B} = \oint \mathrm{d}\mathbf{B}$$

Example :

$$\mathbf{A}=\frac{\mu_0 I}{4\pi}\int\frac{\mathrm{d}\,\boldsymbol{\ell}'}{\mathscr{r}}=\frac{\mu_0 I}{4\pi}\,\hat{\mathbf{z}}\int_{-L}^{L}\frac{\mathrm{d}\,z'}{\sqrt{s^2+z'^2}}=\frac{\mu_0 I}{2\pi}\,\hat{\mathbf{z}}\int_{0}^{L}\frac{\mathrm{d}\,z'}{\sqrt{s^2+z'^2}}\qquad ⇦\quad \tan\theta_L\equiv\frac{L}{s}$$

$$=\frac{\mu_0 I}{2\pi}\,\hat{\mathbf{z}}\int_0^{\theta_L}\cos\theta\,\mathrm{d}\tan\theta=\frac{\mu_0 I}{2\pi}\,\hat{\mathbf{z}}\int_0^{\theta_L}\sec\theta\,\mathrm{d}\,\theta\quad\Leftarrow\quad\cos\theta=\frac{s}{\sqrt{s^2+z'^2}}\,,\quad\tan\theta=\frac{z'}{s}$$

$$=\frac{\mu_0 I}{2\pi}\,\hat{\mathbf{z}}\ln\left(\sec\theta+\tan\theta\right)\Big|_0^{\theta_L}=\frac{\mu_0 I}{2\pi}\ln\frac{L+\sqrt{s^2+L^2}}{s}\,\hat{\mathbf{z}}\quad\Rightarrow\quad A_z=\frac{\mu_0 I}{2\pi}\ln\frac{L+\sqrt{s^2+L^2}}{s}$$

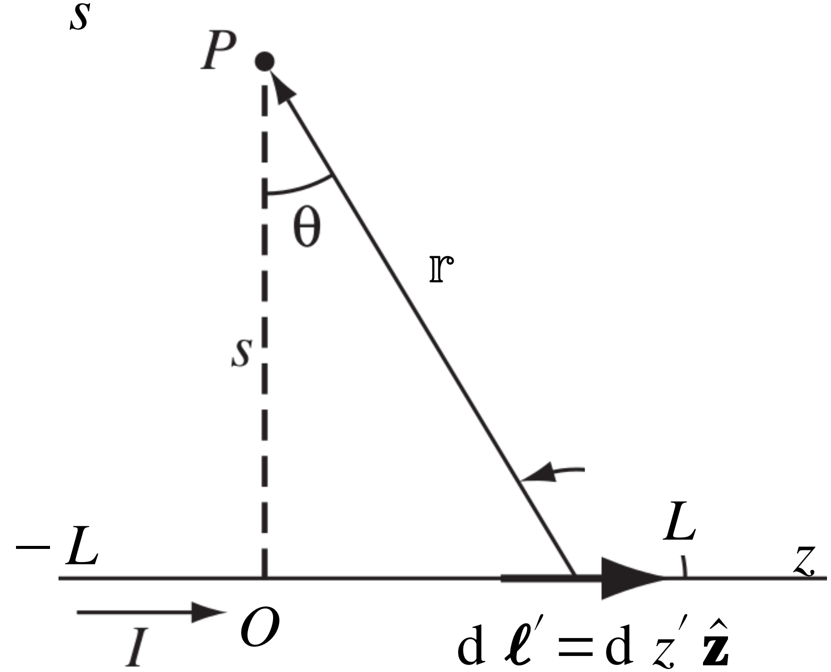
For $s\ll L\quad\Rightarrow\quad\mathbf{A}\simeq\frac{\mu_0 I}{2\pi}\ln\frac{2L}{s}\,\hat{\mathbf{z}}\quad\Rightarrow\quad A_z\to\infty$ as $\frac{L}{s}\to\infty$

$$\Rightarrow\quad\mathbf{A}=A_z\,\hat{\mathbf{z}}\quad\Leftarrow\quad A_z\simeq-\frac{\mu_0 I}{2\pi}\ln s\;+\;\text{constant}$$

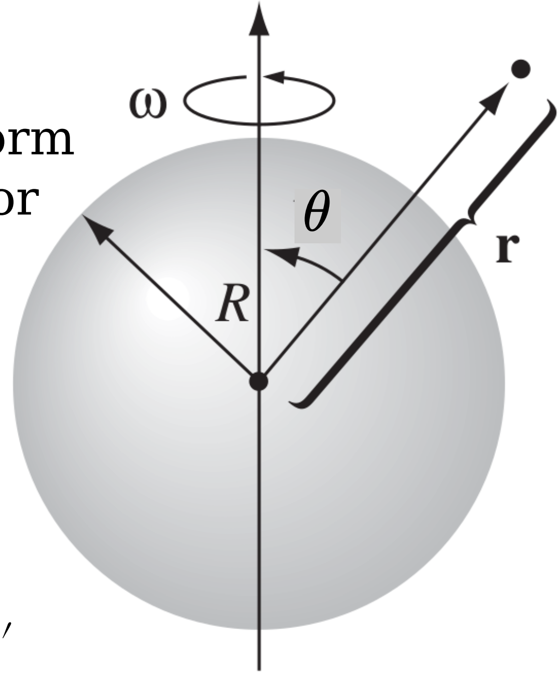
$$\mathbf{B}=\nabla\times\mathbf{A}=-\frac{\partial A_z}{\partial s}\,\hat{\boldsymbol{\phi}}=\frac{\mu_0 I}{2\pi s}\frac{L}{\sqrt{L^2+s^2}}\,\hat{\boldsymbol{\phi}}$$

$$\Rightarrow\quad\mathbf{B}=\frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}\quad\text{as}\quad L\to\infty$$

$$\Rightarrow\quad\nabla\cdot\mathbf{A}=0\,?$$



Example 5.11: A spherical shell of radius R, carrying a uniform surface charge σ, spins at angular velocity $\boldsymbol{\omega}$. Find the vector potential it produces at $\mathbf{r}$.



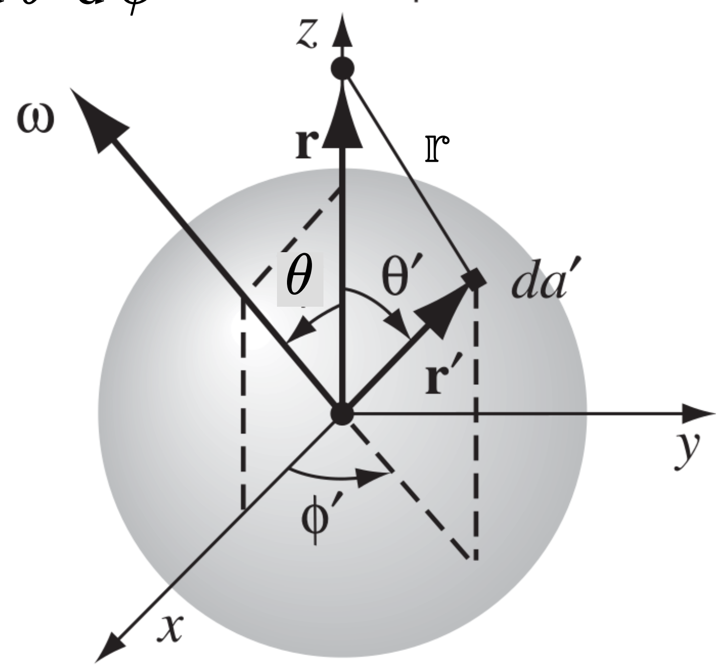
- The integration is easier if we let $\mathbf{r}$ lie on the z axis, so that $\boldsymbol{\omega}$ is tilted at an angle θ. We orient the x axis so that $\boldsymbol{\omega}$ lies in the xz plane.

- For $\mathbf{A}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\boldsymbol{K}(\mathbf{r}')}{\mathscr{r}}\, \mathrm{d}a' \quad \Leftarrow \quad \mathrm{d}a' = R^2 \sin\theta' \, \mathrm{d}\theta' \, \mathrm{d}\phi'$

$$\boldsymbol{K} = \sigma\,\mathbf{v}\,, \quad \mathscr{r} = \sqrt{R^2 + r^2 - 2\,r\,R\cos\theta'} \quad \Leftarrow \quad \mathbf{r} = r\,\hat{\mathbf{z}}$$

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}' = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega\sin\theta & 0 & \omega\cos\theta \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$= R\,\omega\,[\,\sin\theta\sin\theta'\sin\phi'\,\hat{\mathbf{z}} - \cos\theta\sin\theta'\sin\phi'\,\hat{\mathbf{x}} + (\cos\theta\sin\theta'\cos\phi' - \sin\theta\cos\theta')\,\hat{\mathbf{y}}\,]$$



- $\displaystyle\int_0^{2\pi} \sin\phi' \, \mathrm{d}\phi' = \int_0^{2\pi} \cos\phi' \, \mathrm{d}\phi' = 0$

$$\Rightarrow \quad \mathbf{A}(\mathbf{r}) = -\frac{\mu_0\,\sigma\,\omega\,R^3\sin\theta}{2} \int_0^{\pi} \frac{\cos\theta'\sin\theta'\,\mathrm{d}\theta'}{\sqrt{R^2 + r^2 - 2\,r\,R\cos\theta'}}\,\hat{\mathbf{y}}$$

- $$\int_{-1}^{1} \frac{u\,\mathrm{d}\,u}{\sqrt{R^2+r^2-2\,r\,R\,u}} = -\frac{2}{2\,r\,R}\int_{-1}^{1} u\,\mathrm{d}\sqrt{R^2+r^2-2\,r\,R\,u} \quad \Leftarrow \quad u=\cos\theta'$$

$$= -\left.\frac{u\sqrt{R^2+r^2-2\,r\,R\,u}}{r\,R}\right|_{-1}^{1} + \frac{1}{r\,R}\int_{-1}^{1}\sqrt{R^2+r^2-2\,r\,R\,u}\,\mathrm{d}\,u$$

$$= -\frac{|R-r|+R+r}{r\,R} - \frac{1}{2\,r^2\,R^2}\,\frac{2}{3}\left(R^2+r^2-2\,r\,R\,u\right)^{3/2}\Big|_{-1}^{1} \quad ⇦ \quad r_{\lessgtr} = \begin{matrix}\min\\ \max\end{matrix}\,(r\,,\,R)$$

$$= \frac{(R+r)^3-|R-r|^3}{3\,r^2\,R^2} - \frac{|R-r|+R+r}{r\,R} = \frac{R^3+r^3-|R^3-r^3|}{3\,r^2\,R^2} = \frac{2}{3}\,\frac{r_<}{r_>^2}$$

$$\Rightarrow \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0\,\sigma\,R}{3}\,\frac{r_<^3}{r^3}\,\boldsymbol{\omega}\times\mathbf{r} \quad \Leftarrow \quad \boldsymbol{\omega}\times\mathbf{r} = -r\,\omega\sin\theta\,\hat{\mathbf{y}}$$

$$= \frac{\mu_0\,\sigma\,R\,\omega}{3}\,\frac{r_<^3}{r^2}\sin\theta\,\hat{\boldsymbol{\phi}} \quad \Leftarrow \quad \text{revert the coordinates} \;\; \boldsymbol{\omega}\,\|\,\hat{\mathbf{z}}\,, \quad \mathbf{r} = (r\,,\,\theta\,,\,\phi)$$

$$\Rightarrow \quad \mathbf{B} = \nabla\times\mathbf{A} = \begin{cases} \dfrac{2}{3}\,\mu_0\,\sigma\,R\,\boldsymbol{\omega} & \textit{uniform} \text{ inside the spherical shell} \\[2ex] \dfrac{1}{3}\,\mu_0\,\sigma\,R^4\,\dfrac{3\,(\hat{\mathbf{r}}\cdot\boldsymbol{\omega})\,\hat{\mathbf{r}}-\boldsymbol{\omega}}{r^3} & \textit{dipole} \text{ outside the spherical shell} \end{cases}$$

Example 5.12: Find the vector potential of an infinite solenoid with n turns per unit length, radius R, and current I.

- $\Phi_B = \int \mathbf{B} \cdot d\boldsymbol{a} = \int \nabla \times \mathbf{A} \cdot d\boldsymbol{a} = \oint \mathbf{A} \cdot d\boldsymbol{\ell} \quad \Leftarrow \quad \Phi_B :$ magnetic flux through the loop

- Similar with the Ampere's law $\oint \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{enc}$ with $\mathbf{B} \rightarrow \mathbf{A}$ and $\mu_0 I_{enc} \rightarrow \Phi_B$.

- If symmetry permits, we can determine $\mathbf{A}$ from Φ_B in the same way $\mathbf{B}$ from I_{enc}.

- Use a circular "Amperian loop" at radius s *inside* the solenoid,

$$\oint \mathbf{A} \cdot d\boldsymbol{\ell} = A(2\pi s) = \int \mathbf{B} \cdot d\boldsymbol{a} = \mu_0 n I (\pi s^2) \quad \Rightarrow \quad \mathbf{A} = \frac{\mu_0 n I}{2} s \, \hat{\boldsymbol{\phi}} \quad \text{for } s < R$$

- For an Amperian loop *outside* the solenoid,

$$\Phi_B = \int \mathbf{B} \cdot d\boldsymbol{a} = \mu_0 n I (\pi R^2) \quad \Rightarrow \quad \mathbf{A} = \frac{\mu_0 n I}{2} \frac{R^2}{s} \hat{\boldsymbol{\phi}} \quad \text{for } s > R$$

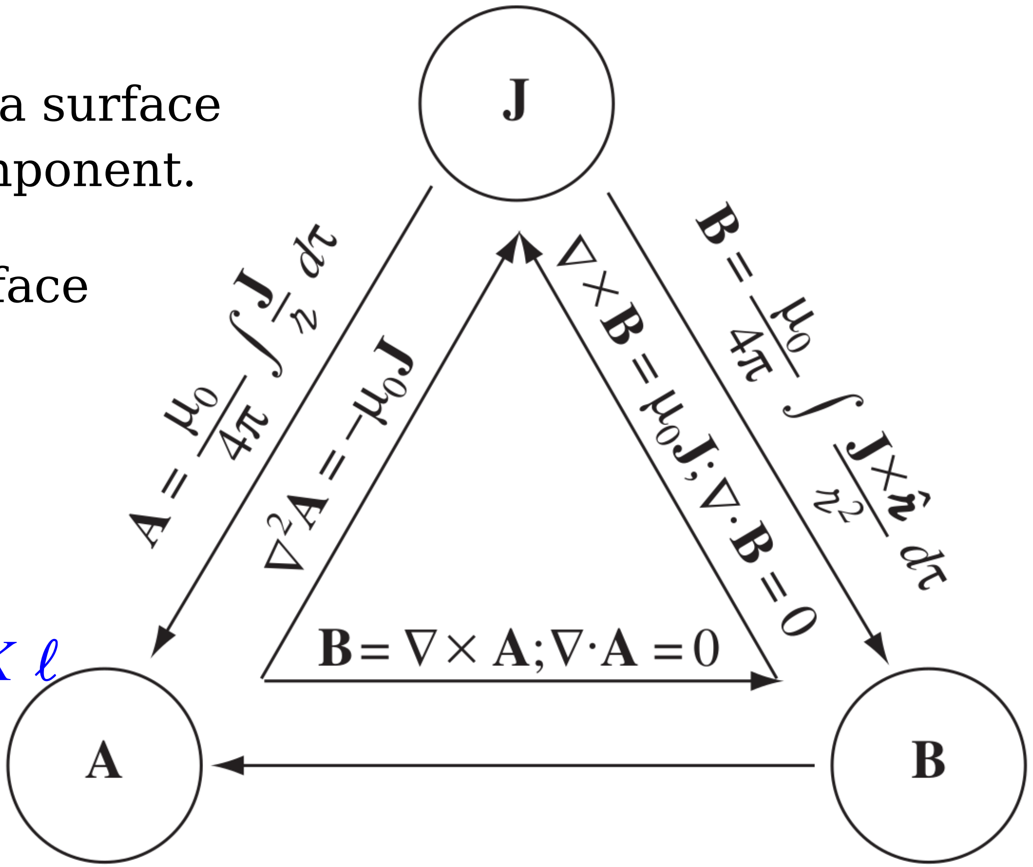
- Check if $\nabla \times \mathbf{A} = \mathbf{B}$? $\nabla \cdot \mathbf{A} = 0$?

- *Ordinarily*, the direction of $\mathbf{A}$ mimics the direction of the current. If all the current flows in one direction, $\mathbf{A}$ *must* point that way, too.

- You can always add an arbitrary constant vector to $\mathbf{A}$ —analogous to changing the reference point for Φ, and it won't affect the divergence or curl of $\mathbf{A}$, which is all that matters.

Boundary Conditions

- The magnetic field is discontinuous at a surface *current*, but it is about the *tangential* component.



- A wafer-thin pillbox straddling the surface

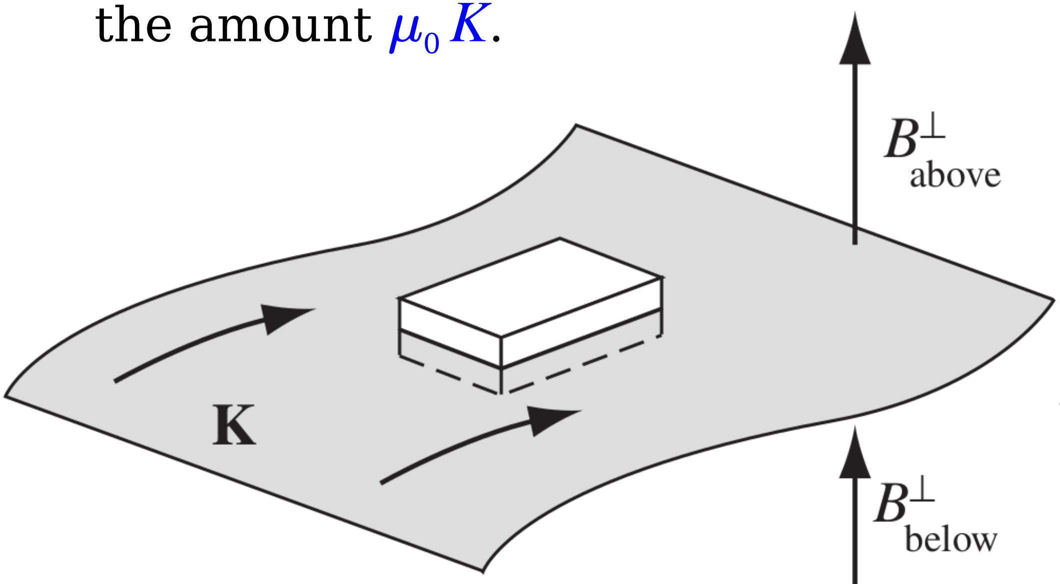
$$\oint \mathbf{B}\cdot d\boldsymbol{a} = 0 \quad\Rightarrow\quad B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}$$

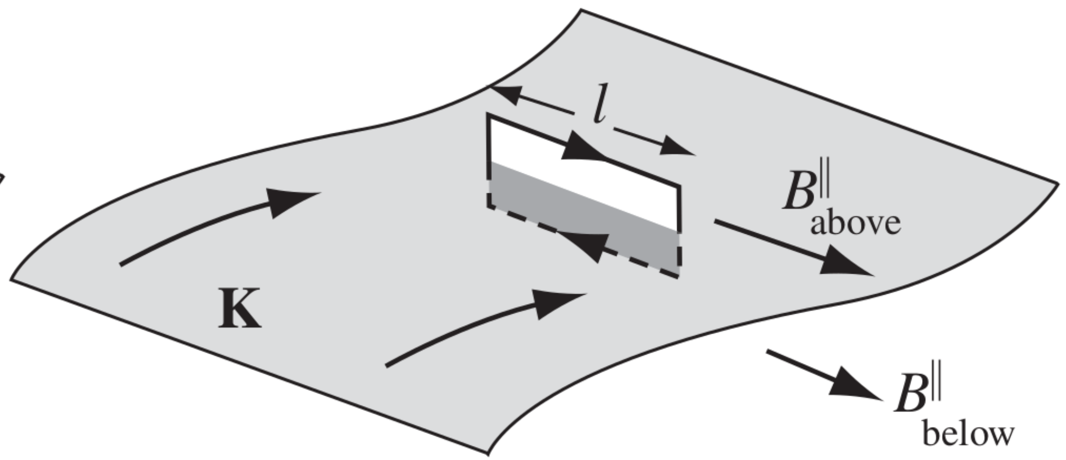
- For the tangential components, an Amperian loop running ⊥ the current,

$$\oint \mathbf{B}\cdot d\boldsymbol{\ell} = \left(B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}}\right)\ell = \mu_0 I_{\text{enc}} = \mu_0 K \ell$$

$$\Rightarrow\quad B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K$$

- The component of $\mathbf{B}$ that is $\parallel$ the surface but ⊥ the current is discontinuous in the amount $\mu_0 K$.





- A similar Amperian loop $\parallel$ the current reveals that the *parallel* component is *continuous*.

- $\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 \boldsymbol{K} \times \hat{\mathbf{n}} \quad \Leftarrow \quad \hat{\mathbf{n}}:$ unit vector $\perp$ the surface, pointing *upward* .

- The vector potential is continuous across any boundary: $\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$, for $\nabla \cdot \mathbf{A} = 0$ guarantees that the *normal* component is continuous.

- $\nabla \times \mathbf{A} = \mathbf{B} \quad \Rightarrow \quad \oint \mathbf{A} \cdot \mathrm{d}\boldsymbol{\ell} = \int \mathbf{B} \cdot \mathrm{d}\boldsymbol{a} = \Phi_B$ means that the tangential components are continuous (the flux through an Amperian loop of vanishing thickness is 0).

- $\mathbf{A}$'s *derivative* inherits the discontinuity of $\mathbf{B}$: $\dfrac{\partial}{\partial n} \mathbf{A}_{\text{above}} - \dfrac{\partial}{\partial n} \mathbf{A}_{\text{below}} = -\mu_0 \boldsymbol{K}$

$$\mathbf{B} = \nabla \times \mathbf{A} = \hat{\mathbf{n}} \frac{\partial A_k}{\partial \ell} - \hat{\boldsymbol{\ell}} \frac{\partial A_k}{\partial n} = \hat{\mathbf{n}} \frac{\partial A_k}{\partial \ell} - \frac{\partial \mathbf{A}}{\partial n} \times \hat{\mathbf{n}} \quad \Leftarrow \quad \mathbf{A} = A_k \hat{\mathbf{k}} \quad \Leftarrow \quad \begin{array}{l} \boldsymbol{K} \equiv K \hat{\mathbf{k}} \\ \hat{\boldsymbol{\ell}} = \hat{\mathbf{k}} \times \hat{\mathbf{n}} \end{array}$$

$$\begin{array}{l} B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}} \\ \nabla \cdot \mathbf{A} = 0 \end{array} \quad \Rightarrow \quad \frac{\partial}{\partial \ell} \mathbf{A}_{\text{above}} = \frac{\partial}{\partial \ell} \mathbf{A}_{\text{below}}, \quad \frac{\partial}{\partial k} \mathbf{A}_{\text{above}} = \frac{\partial}{\partial k} \mathbf{A}_{\text{below}} = 0$$

$$\Rightarrow \quad \mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = -\left(\frac{\partial}{\partial n} \mathbf{A}_{\text{above}} - \frac{\partial}{\partial n} \mathbf{A}_{\text{below}} \right) \times \hat{\mathbf{n}} = \mu_0 \boldsymbol{K} \times \hat{\mathbf{n}}$$

$$\Rightarrow \quad \frac{\partial}{\partial n} \mathbf{A}_{\text{above}} - \frac{\partial}{\partial n} \mathbf{A}_{\text{below}} = -\mu_0 \boldsymbol{K}$$

<u>Example</u>: Consider an infinitely long cylindrical conductor of radius *a*, with a constant current *I* flowing in. Find the magnetic vector potential.

Take the *z*-axis along the axis of the conductor,

$$\Rightarrow \quad \nabla^2 A_z = -\mu_0 J_z \quad \Leftarrow \quad \boldsymbol{J} = J_z \hat{\mathbf{z}} = J \hat{\mathbf{z}}, \quad \mathbf{A} = A_z \hat{\mathbf{z}} \quad \Leftarrow \quad J = \frac{I}{\pi a^2} \Theta(a-s)$$

symmetry in z & ϕ $\Rightarrow \frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}s}\left(s\frac{\mathrm{d}A_z}{\mathrm{d}s}\right) = -\mu_0 J \quad \Rightarrow \quad \mathrm{d}\left(s'\frac{\mathrm{d}A_z}{\mathrm{d}s'}\right) = -\mu_0 J s' \mathrm{d}s'$

For $s<a$: $\quad s\frac{\mathrm{d}A_z}{\mathrm{d}s} = -\frac{\mu_0 J}{2}s^2 \quad \Leftarrow \quad \int_0^s \mathrm{d}\left(s'\frac{\mathrm{d}A_z}{\mathrm{d}s'}\right) = -\mu_0 \int_0^s J(s') s' \mathrm{d}s'$

$$\Rightarrow \quad A_z(s) - A_z(0) = \int_0^s -\frac{\mu_0 J}{2} s' \mathrm{d}s' = -\frac{\mu_0 I}{4\pi}\frac{s^2}{a^2} \quad \Rightarrow \quad A_z(s) = -\frac{\mu_0 I}{4\pi}\frac{s^2}{a^2} + A_z(0)$$

For $s \geq a$: $\quad s\frac{\mathrm{d}A_z}{\mathrm{d}s} = a\frac{\mathrm{d}A_z}{\mathrm{d}s}(a) \Leftarrow \int_a^s \mathrm{d}\left(s'\frac{\mathrm{d}A_z}{\mathrm{d}s'}\right) = -\mu_0 \int_a^s J(s') s' \mathrm{d}s' = 0$

surface current density vanishes $\quad \Rightarrow \quad a\frac{\mathrm{d}A_z}{\mathrm{d}s}(a^+) = a\frac{\mathrm{d}A_z}{\mathrm{d}s}(a^-) = -\frac{\mu_0 I}{2\pi}$

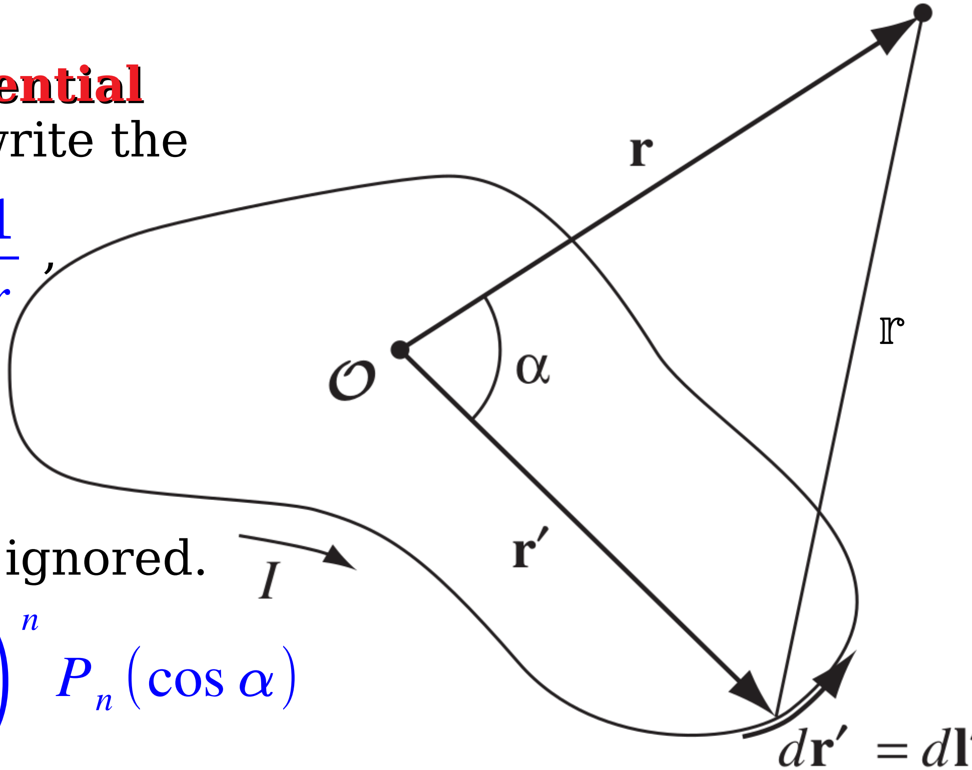
$$\Rightarrow \; A_z(s) - A_z(a^+) = -\frac{\mu_0 I}{2\pi} \int_a^s \frac{d\,s'}{s'} = -\frac{\mu_0 I}{2\pi} \ln \frac{s}{a}$$

$$\Rightarrow \; A_z(s) = -\frac{\mu_0 I}{2\pi} \ln \frac{s}{a} - \frac{\mu_0 I}{4\pi} + A_z(0) \quad \Leftarrow \quad A_z(a^-) = A_z(a^+)$$

$$\Rightarrow \quad A_z = -\frac{\mu_0 I}{4\pi} \begin{bmatrix} \dfrac{s^2}{a^2} \\ \ln \dfrac{s^2}{a^2} + 1 \end{bmatrix} + A_z(0) , \quad \mathbf{B} = \frac{\mu_0 I}{2\pi} \begin{bmatrix} \dfrac{s}{a^2} \\ \dfrac{1}{s} \end{bmatrix} \hat{\phi} , \quad \text{for} \quad \begin{bmatrix} s \le a \\ s > a \end{bmatrix}$$

$$\Rightarrow \quad A_z = -\frac{\mu_0 I}{4\pi} \left(\frac{s^2}{s_>^2} + 2 \ln \frac{s_>}{a} \right) + A_z(0) , \quad \mathbf{B} = \frac{\mu_0 I}{2\pi} \frac{s}{s_>^2} \hat{\phi} \quad \Leftarrow \quad s_> = \max(s, a)$$

Multipole Expansion of the Vector Potential



- The idea of a multipole expansion is to write the potential in the form of a power series in $\dfrac{1}{r}$, where r is the distance to the point.

- If r is sufficiently large, the series will be dominated by the lowest nonvanishing contribution, and the higher terms can be ignored.

- $$\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{r^2 + r'^2 - 2\, r\, r' \cos\alpha}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

$$\Rightarrow \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{d\boldsymbol{\ell}'}{\mathscr{r}} = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint r'^n P_n(\cos\alpha)\, d\boldsymbol{\ell}'$$

$$= \frac{\mu_0 I}{4\pi}\left(\frac{1}{r}\oint d\boldsymbol{\ell}' + \frac{1}{r^2}\oint r'\cos\alpha\, d\boldsymbol{\ell}' + \frac{1}{r^3}\oint r'^2\,\frac{3\cos^2\alpha - 1}{2}\, d\boldsymbol{\ell}' + \cdots\right)$$

- We call the 1st term (with $\dfrac{1}{r}$) the **monopole** term, the 2nd (with $\dfrac{1}{r^2}$) **dipole**, the 3rd **quadrupole**, and so on.

- The *magnetic monopole term is always* 0, for the integral is just the total vector displacement around a closed loop: $\oint d\boldsymbol{\ell}' = 0$

- This reflects the fact that there are no magnetic monopoles in nature $\nabla \cdot \mathbf{B} = 0$

- So the dominant term is the dipole:

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\alpha \, d\boldsymbol{\ell}' = \frac{\mu_0 I}{4\pi r^2} \oint \hat{\mathbf{r}} \cdot \mathbf{r}' \, d\boldsymbol{\ell}'$$

$$\oint \hat{\mathbf{r}} \cdot \mathbf{r}' \, d\boldsymbol{\ell}' = -\int \nabla' (\hat{\mathbf{r}} \cdot \mathbf{r}') \times d\boldsymbol{a}' = -\hat{\mathbf{r}} \times \int d\boldsymbol{a}' = \int d\boldsymbol{a}' \times \hat{\mathbf{r}}$$

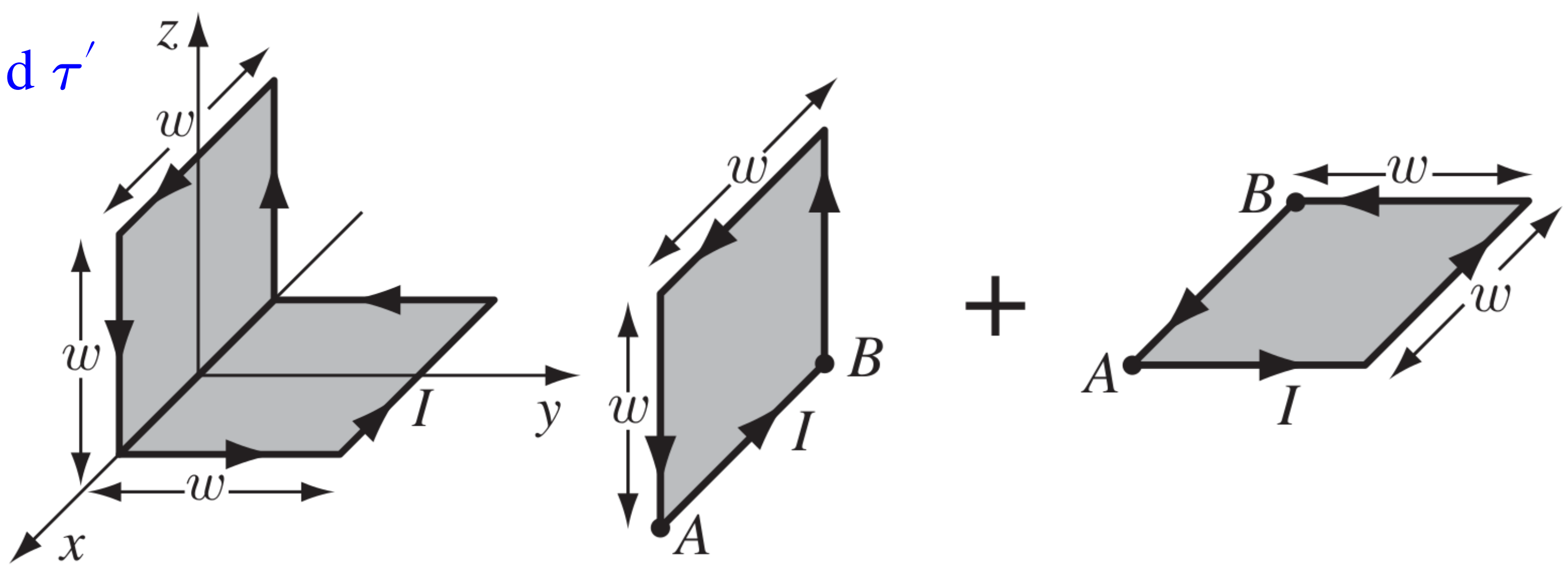
$$\Rightarrow \mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2} = -\frac{\mu_0}{4\pi} \mathbf{m} \times \nabla \frac{1}{r} \quad \Leftarrow \quad \mathbf{m} \equiv I \int d\boldsymbol{a} = I\boldsymbol{a} : \text{ magnetic dipole moment}$$

here $\boldsymbol{a}$ is the "vector area" of the loop; if the loop is *flat*, $\boldsymbol{a}$ is the ordinary area enclosed, with the direction by the usual right-hand rule.

- $d\boldsymbol{a}' = \dfrac{\mathbf{r}' \times d\boldsymbol{\ell}'}{2} \quad \Rightarrow \quad \mathbf{m} = I \displaystyle\int \frac{\mathbf{r}' \times d\boldsymbol{\ell}'}{2} = \frac{1}{2} \int \mathbf{r}' \times (I \, d\boldsymbol{\ell}') = \frac{1}{2} \int \mathbf{r}' \times \boldsymbol{J} \, d\tau'$

$$\Rightarrow \quad d\mathbf{m} = \frac{1}{2} \mathbf{r}' \times \boldsymbol{J} \, d\tau'$$

Example 5.13:



$$\int (\nabla \times \mathbf{v}) \cdot \mathrm{d}\,\boldsymbol{a} = \oint \mathbf{v} \cdot \mathrm{d}\,\boldsymbol{\ell} \quad \Leftarrow \quad \text{Stokes's theorem}$$

Let $\mathbf{v} = T\,\mathbf{c}$ where $\mathbf{c}$ is constant $\Rightarrow \nabla \times \mathbf{c} = 0 \Rightarrow \nabla \times (T\,\mathbf{c}) = \nabla T \times \mathbf{c}$

$$\Rightarrow \int [\nabla \times (T\,\mathbf{c})] \cdot \mathrm{d}\,\boldsymbol{a} = \int (\nabla T \times \mathbf{c}) \cdot \mathrm{d}\,\boldsymbol{a} = \mathbf{c} \cdot \int \mathrm{d}\,\boldsymbol{a} \times \nabla T$$

$$\oint T\,\mathbf{c} \cdot \mathrm{d}\,\boldsymbol{\ell} \quad = \quad \mathbf{c} \cdot \oint T\,\mathrm{d}\,\boldsymbol{\ell}$$

$$\Rightarrow \int_S \mathrm{d}\,\boldsymbol{a} \times \nabla T = -\int_S \nabla T \times \mathrm{d}\,\boldsymbol{a} = \oint_C T\,\mathrm{d}\,\boldsymbol{\ell} \quad \Leftarrow \quad \mathbf{c} \text{ can be any constant.}$$

Check Problem 1.61(e).

$$(\mathbf{r}' \times \mathrm{d}\,\mathbf{r}') \times \mathbf{r} = -(\mathbf{r} \cdot \mathrm{d}\,\mathbf{r}')\,\mathbf{r}' + (\mathbf{r} \cdot \mathbf{r}')\,\mathrm{d}\,\mathbf{r}' \quad \Leftarrow \quad \mathrm{d}\,\boldsymbol{\ell}' \to \mathrm{d}\,\mathbf{r}' \text{ in general}$$

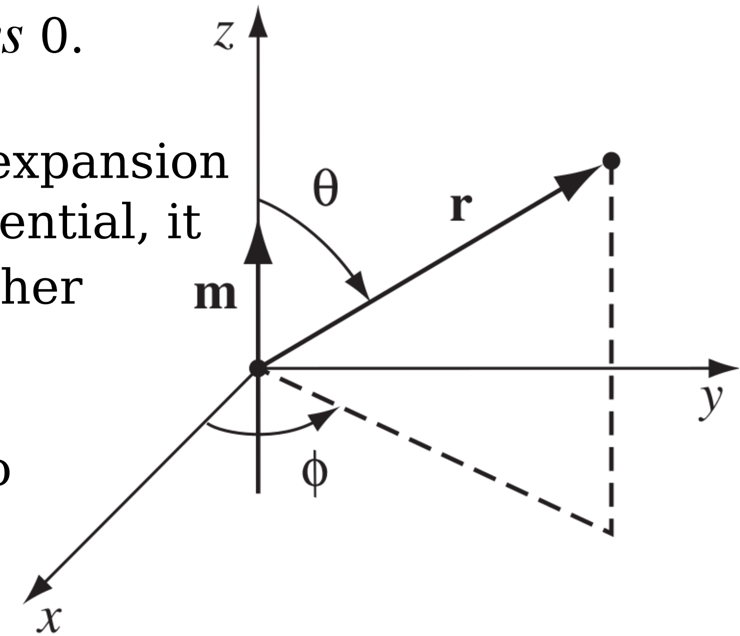
$$\mathrm{d}\,[(\mathbf{r} \cdot \mathbf{r}')\,\mathbf{r}'] \;=\; (\mathbf{r} \cdot \mathrm{d}\,\mathbf{r}')\,\mathbf{r}' + (\mathbf{r} \cdot \mathbf{r}')\,\mathrm{d}\,\mathbf{r}' \quad \text{due to small change } \mathrm{d}\,\mathbf{r}' \text{ in } \mathbf{r}'$$

$$\Rightarrow (\mathbf{r} \cdot \mathbf{r}')\,\mathrm{d}\,\mathbf{r}' = \frac{\mathbf{r}' \times \mathrm{d}\,\mathbf{r}'}{2} \times \mathbf{r} + \frac{1}{2}\,\mathrm{d}\,[\mathbf{r}'(\mathbf{r} \cdot \mathbf{r}')] \quad \Rightarrow$$

total derivative in a closed path for an integral

$$\Rightarrow (\hat{\mathbf{r}} \cdot \mathbf{r}')\,\mathrm{d}\,\boldsymbol{\ell}' = \frac{\mathbf{r}' \times \mathrm{d}\,\boldsymbol{\ell}'}{2} \times \hat{\mathbf{r}} = \mathrm{d}\,\boldsymbol{a}' \times \hat{\mathbf{r}}$$

- It is clear that the magnetic dipole moment is independent of the choice of origin since the *magnetic* monopole moment is *always* 0.



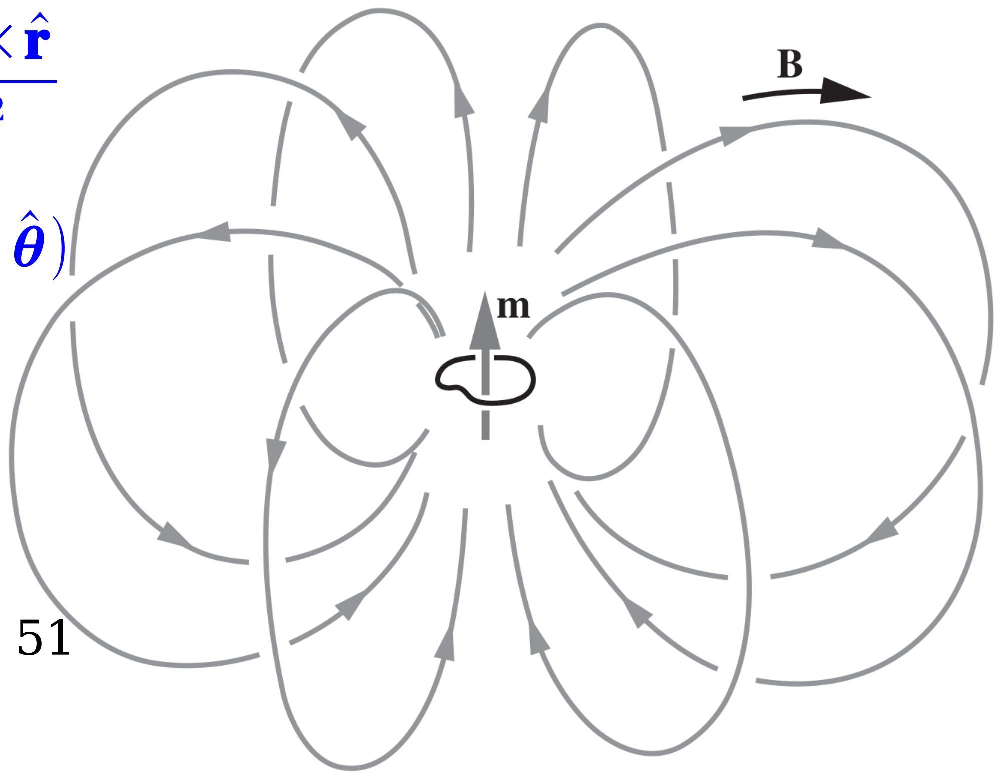
- Although the dipole term *dominates* the multipole expansion and thus offers a good approximation to the true potential, it is not ordinarily the *exact* potential; there will be higher multipoles' influence.

- The magnetic *field* of a (perfect) dipole is easiest to calculate if we put $\mathbf{m}$ at the origin and let it point in the z-direction,

$$\Rightarrow \quad \mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}} \quad \Leftarrow \quad \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$$

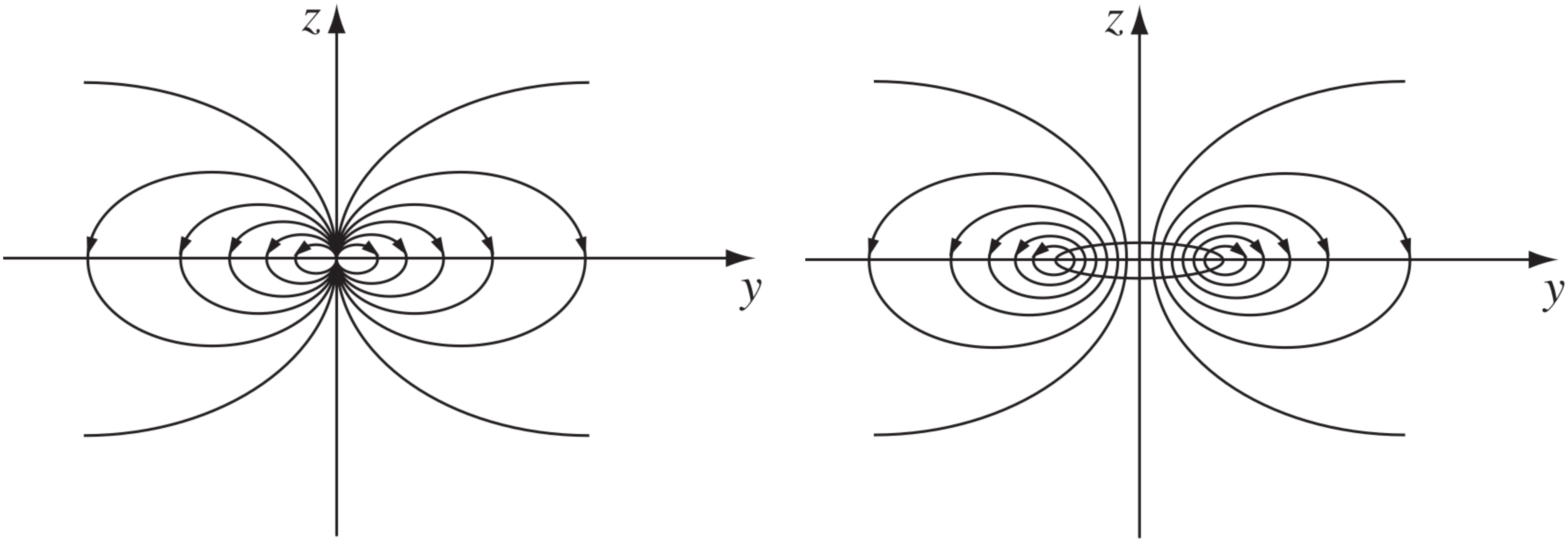
$$\Rightarrow \quad \mathbf{B}_{\text{dip}} = \nabla\times\mathbf{A} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

$$= \frac{\mu_0}{4\pi}\frac{3(\hat{\mathbf{r}}\cdot\mathbf{m})\hat{\mathbf{r}} - \mathbf{m}}{r^3}$$



Selected problems: 11, 13, 26, 30, 44, 48, 51

- This is *identical* in structure to the field of an *electric* dipole.



(a) Field of a "pure" dipole

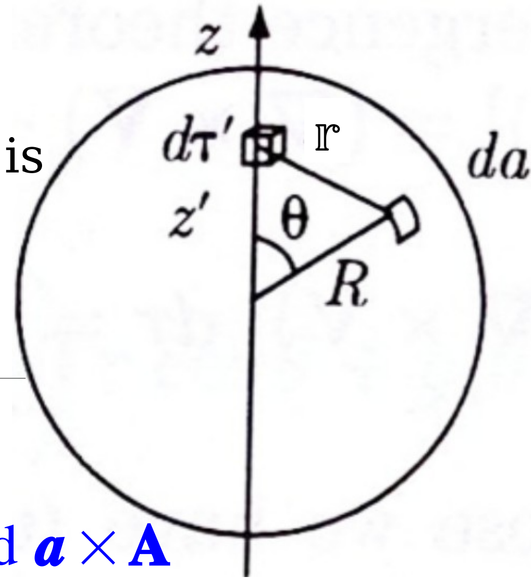
(b) Field of a "physical" dipole

Postulates of Magnetostatics in Free Space

Differential Form	Integral Form
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\boldsymbol{a} = 0$
$\nabla \times \mathbf{B} = \mu_0 \boldsymbol{J}$	$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I$

Problem 5.59: Prove that the average magnetic field, over a sphere of radius R, due to steady currents inside the sphere, is

$$\mathbf{B}_{\text{ave}} = \frac{\mu_0}{2\pi}\frac{\mathbf{m}}{R^3}$$, $\mathbf{m}$ is the total dipole moment of the sphere.



The average field $\mathbf{B}$ due to the current density $\boldsymbol{J}$ at $\mathbf{r}'$ is,

$$\mathbf{B}_{\text{ave}} = \frac{1}{4\pi R^3/3}\int \mathbf{B}\,\mathrm{d}\tau = \frac{3}{4\pi R^3}\int \nabla\times\mathbf{A}\,\mathrm{d}\tau = \frac{3}{4\pi R^3}\oint \mathrm{d}\boldsymbol{a}\times\mathbf{A}$$

$$= \frac{3}{4\pi R^3}\frac{\mu_0}{4\pi}\oint \mathrm{d}\boldsymbol{a}\times\int\frac{\boldsymbol{J}}{\mathscr{r}}\,\mathrm{d}\tau' = \frac{3\mu_0}{16\pi^2 R^3}\int\left(\oint\frac{\mathrm{d}\boldsymbol{a}}{\mathscr{r}}\right)\times\boldsymbol{J}\,\mathrm{d}\tau'$$

$$\oint\frac{\mathrm{d}\boldsymbol{a}}{\mathscr{r}} = \hat{\mathbf{z}}\int\frac{\cos\theta\,R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}{\sqrt{r'^2+R^2-2r'R\cos\theta}} \Leftarrow \text{let } \mathbf{r}' = r'\hat{\mathbf{z}},\quad \mathrm{d}\boldsymbol{a} = R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{\mathbf{r}}$$

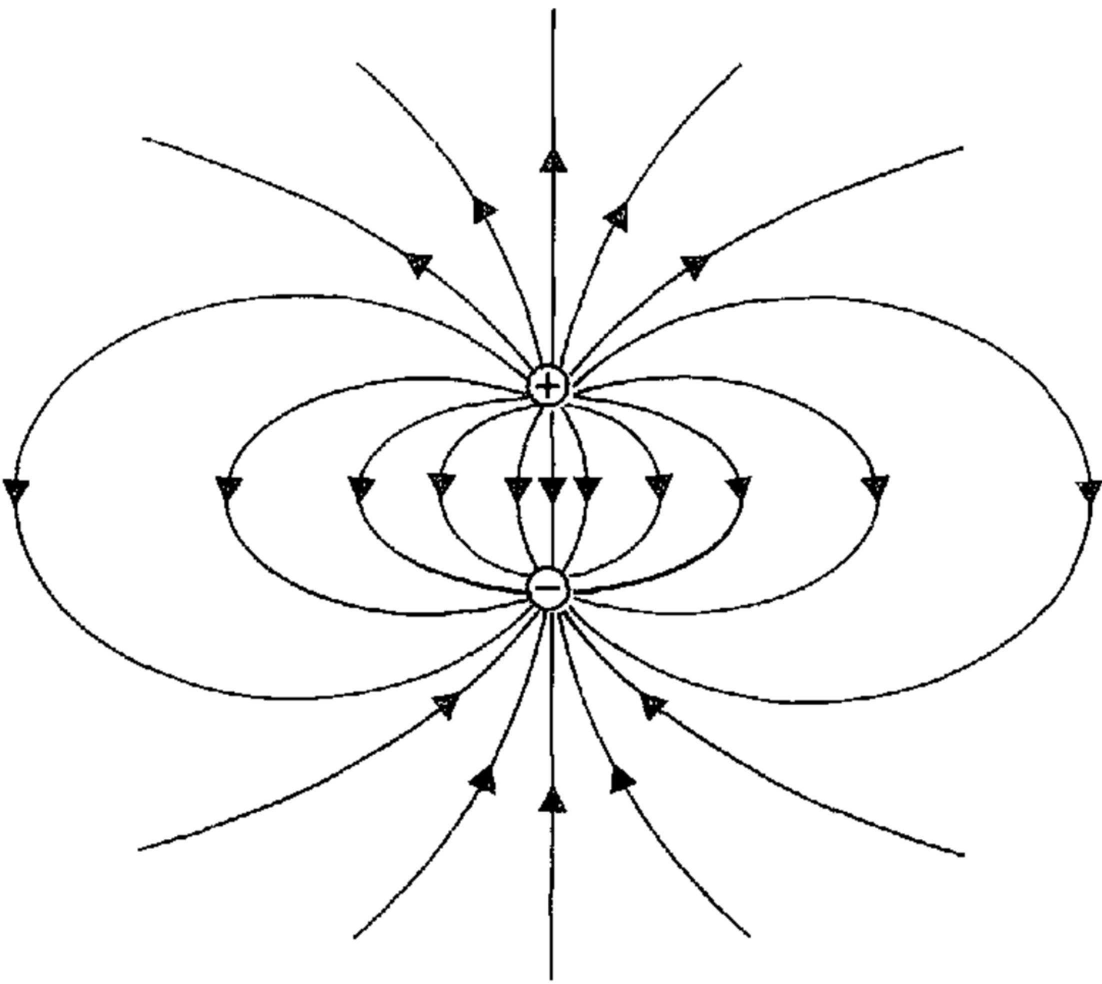
$+$ Ex. 5.11 experience

$$= -2\pi R^2\hat{\mathbf{z}}\int_0^{\pi}\frac{\cos\theta\,\mathrm{d}\cos\theta}{\sqrt{r'^2+R^2-2r'R\cos\theta}} = \frac{4\pi}{3}r'\hat{\mathbf{z}} = \frac{4\pi}{3}\mathbf{r}' \quad\Leftarrow\quad R > r'$$

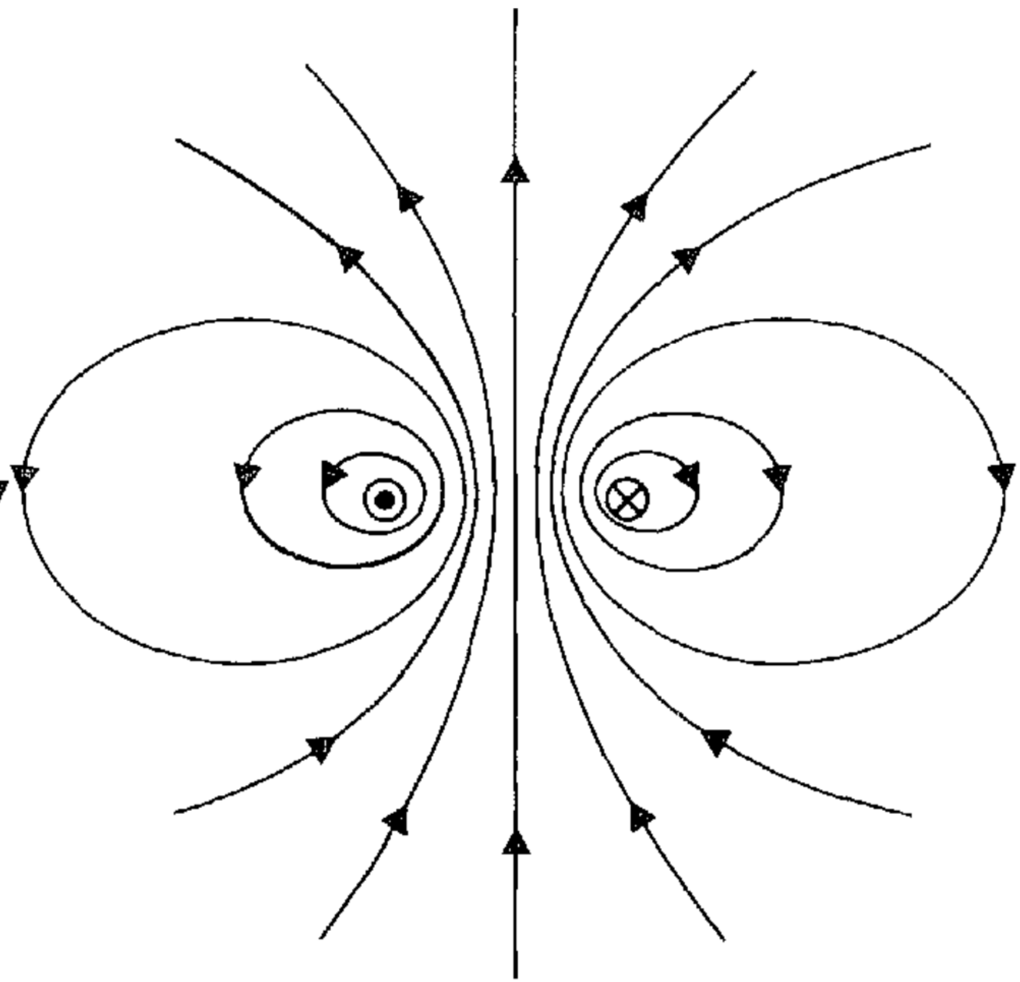
$$\Rightarrow\quad \mathbf{B}_{\text{ave}} = \frac{3\mu_0}{16\pi^2 R^3}\int\frac{4\pi}{3}\mathbf{r}'\times\boldsymbol{J}\,\mathrm{d}\tau' = \frac{\mu_0}{2\pi}\frac{\mathbf{m}}{R^3} \Rightarrow \frac{1}{4\pi R^3/3}\int\frac{2\mu_0}{3}\mathbf{m}\,\delta^3(\mathbf{r})\,\mathrm{d}\tau$$

by turning into an infinitestimal sphere centered at a pure dipole $\mathbf{m}$

$$\Rightarrow\quad \mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3(\hat{\mathbf{r}}\cdot\mathbf{m})\hat{\mathbf{r}}-\mathbf{m}}{r^3} + \frac{2\mu_0}{3}\mathbf{m}\,\delta^3(\mathbf{r}) \quad\Leftarrow\quad \text{Problem 5.61}$$



(a) Electric dipole.



(b) Magnetic dipole.

$$\int \nabla \cdot \mathbf{v}\, d\tau = \oint \mathbf{v} \cdot d\boldsymbol{a} \quad \Leftarrow \quad \text{divergence theorem}$$

Let $\mathbf{v} \to \mathbf{v} \times \mathbf{c}$ where $\mathbf{c}$ is a constant vector $\Rightarrow \nabla \times \mathbf{c} = 0$

$$\nabla \cdot (\mathbf{u} \times \mathbf{w}) = \mathbf{w} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{w})$$

$$\Rightarrow \int \nabla \cdot (\mathbf{v} \times \mathbf{c})\, d\tau = \int [\mathbf{c} \cdot (\nabla \times \mathbf{v}) - \mathbf{v} \cdot (\nabla \times \mathbf{c})]\, d\tau = \mathbf{c} \cdot \int \nabla \times \mathbf{v}\, d\tau$$

$$\oint (\mathbf{v} \times \mathbf{c}) \cdot d\boldsymbol{a} = \qquad \mathbf{c} \cdot \oint d\boldsymbol{a} \times \mathbf{v}$$

$$\Rightarrow \int_{\mathcal{V}} \nabla \times \mathbf{v}\, d\tau = \oint_{\mathcal{S}} d\boldsymbol{a} \times \mathbf{v} = -\oint_{\mathcal{S}} \mathbf{v} \times d\boldsymbol{a} \quad \Leftarrow \quad \mathbf{c} \text{ can be any constant.}$$

Check Problem 1.61(b).