# Chapter 4 Electric Fields in Matter
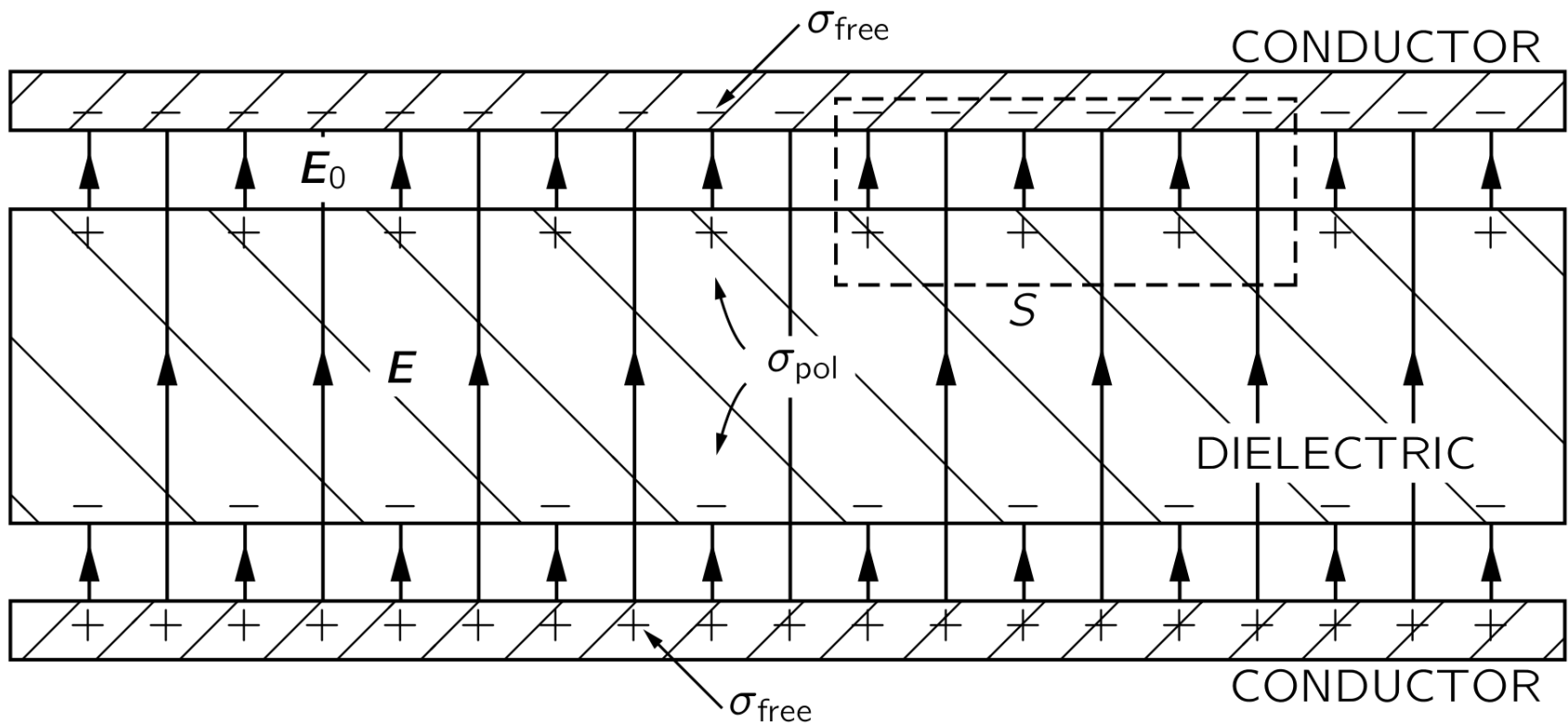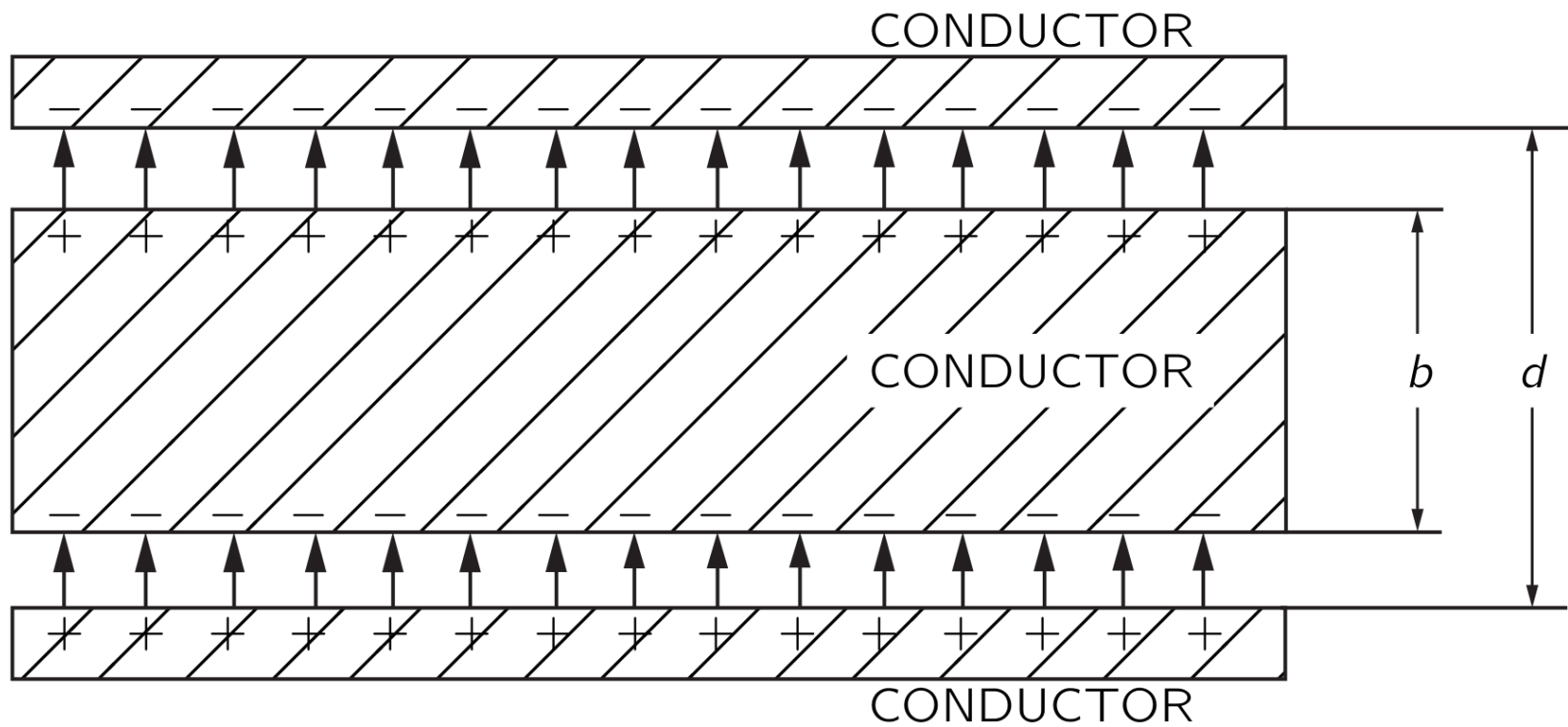
## Polarization

### Dielectrics

- Most everyday objects belong to one of 2 large classes: **conductors** and **insulators** (or **dielectrics**).

- Conductors are substances that contain an "unlimited" supply of charges that are free to move about through the material.

- It means that many of the electrons (1 or 2 per atom, in a typical metal) are not associated with any particular nucleus, but roam around at will.

- In dielectrics, *all charges are attached to specific atoms or molecules*—they're on a tight leash, and all they can do is move a bit *within* the atom or molecule.

- Such microscopic displacements are not as dramatic as the wholesale rearrangement of charge in a conductor, but their cumulative effects account for the characteristic behavior of dielectric materials.

- There are actually 2 principal mechanisms by which electric fields can distort the charge distribution of a dielectric atom or molecule: *stretching* and *rotating*.

| H | He | Li | Be | C | Ne | Na | Ar | K | Cs |
|------|-------|------|------|------|-------|------|------|------|------|
| 0.667 | 0.205 | 24.3 | 5.60 | 1.67 | 0.396 | 24.1 | 1.64 | 43.4 | 59.4 |

$$\frac{\alpha}{4\pi\epsilon_0}$$

$$(10^{-30} \, \mathrm{m^3})$$

CONDUCTOR

CONDUCTOR

CONDUCTOR

$b$

$d$

$\sigma_{\text{free}}$

CONDUCTOR

$E_0$

$E$

$\sigma_{\text{pol}}$

$S$

DIELECTRIC

CONDUCTOR

$\sigma_{\text{free}}$

## Induced Dipoles

● Although the atom as a whole is electrically neutral, there is a positively charged core (nucleus) and a negatively charged electron cloud surrounding it.

● When a neutral atom is placed in an electric field $\mathbf{E}$, the nucleus is pushed in the direction of the field, and the electrons the opposite way.

● If $\mathbf{E}$ is large enough, it can pull the atom apart completely, "ionizing" it (the substance then becomes a conductor).

● With less extreme fields, the 2 opposing forces—$\mathbf{E}$ pulling the electrons and nucleus apart, their mutual attraction drawing them back together—reach a balance, leaving the atom **polarized**, with plus charge shifted slightly one way, and minus the other.

● The atom now has a tiny dipole moment $\mathbf{p}$, which points in the *same direction as* $\mathbf{E}$. This induced dipole moment is approximately proportional to $\mathbf{E}$:  $\mathbf{p} = \alpha\, \mathbf{E}$

$\alpha$ is called **atomic polarizability**.

Example 4.1
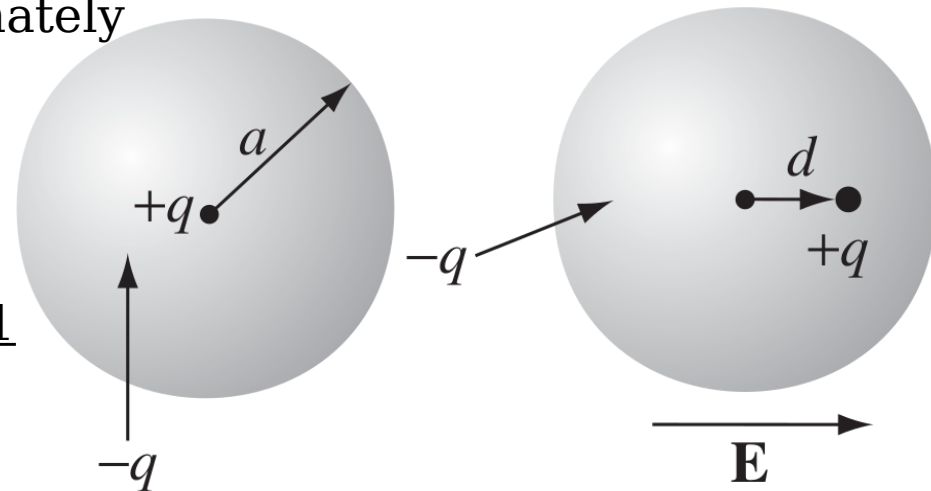
● For molecules the situation is not so simple, because they polarize more readily in some directions than in others.
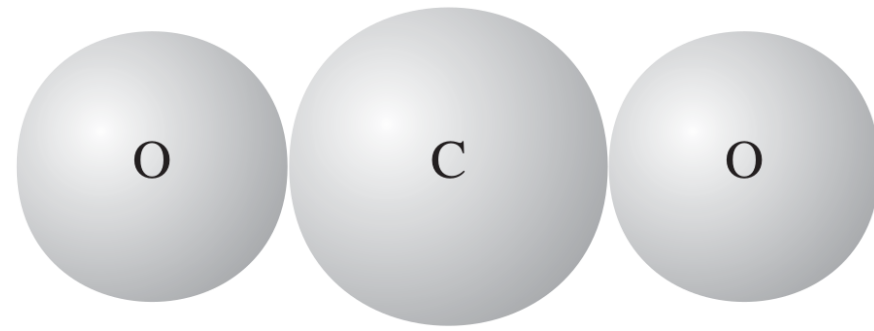


● Carbon dioxide has a polarizability of $4.5{\times}10^{-40}$ C$^2{\cdot}$m/N when applying the field along the axis of the molecule, but only $2{\times}10^{-40}$ for fields $\perp$ this direction.

● When the field is at some *angle* to the axis, you must resolve it into parallel and perpendicular components, and multiply each by the pertinent polarizability:

$$\mathbf{p} = \alpha_{\|}\,\mathbf{E}_{\|} + \alpha_{\perp}\,\mathbf{E}_{\perp}$$

● **p** may not even be in the same *direction* as **E**:

● For a completely asymmetrical molecule, the relation between **E** and **p** is

$$\mathbf{p} = \boldsymbol{\alpha}\cdot\mathbf{E} \ \Rightarrow\ \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \ \Leftarrow\ \boldsymbol{\alpha}: \text{polarizability tensor}$$

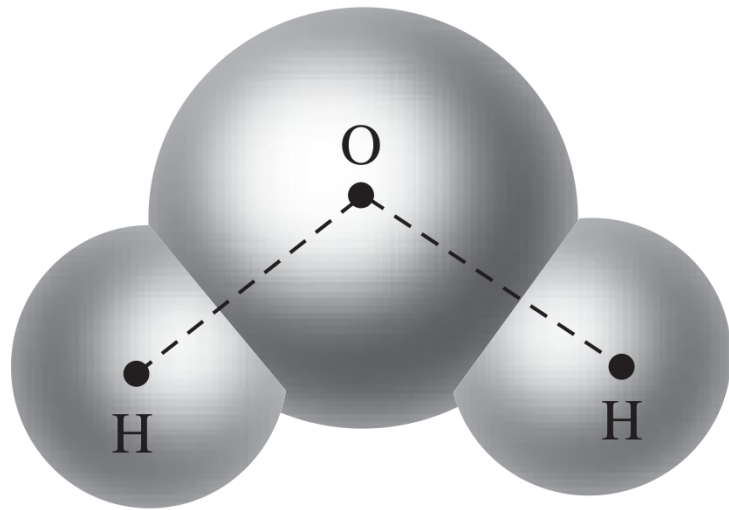$$vs\quad \mathbf{L} = \mathbf{I}\cdot\boldsymbol{\omega}$$

● Th values of $\boldsymbol{\alpha}$ depend on the orientation of the axes you use, though it is always possible to choose "principal" axes such that only the diagonal terms are nonvanishing.

# Permanent Dipole Moments

- Some molecules are so constructed that they have electric dipole moments in the absence of an electric field. They are unsymmetrical in their normal state.

- By contrast, the hydrogen atom in a field of 1 megavolt per meter, with its polarizability, acquires an induced moment $< 10^{-34}$ coulomb-meter.

- Permanent dipole moments, when they exist, are enormously larger than any moment that can be induced by ordinary laboratory electric fields.

- Because of this, the distinction between *polar* molecules, as molecules with "built-in" dipole moments are called, and *nonpolar* molecules is very sharp.

- The behavior of a polar substance as a dielectric is strikingly different from that of material composed of nonpolar molecules. The dielectric constant of water is about 80, that of methyl alcohol 33, while a typical nonpolar liquid might have a dielectric constant around 2.
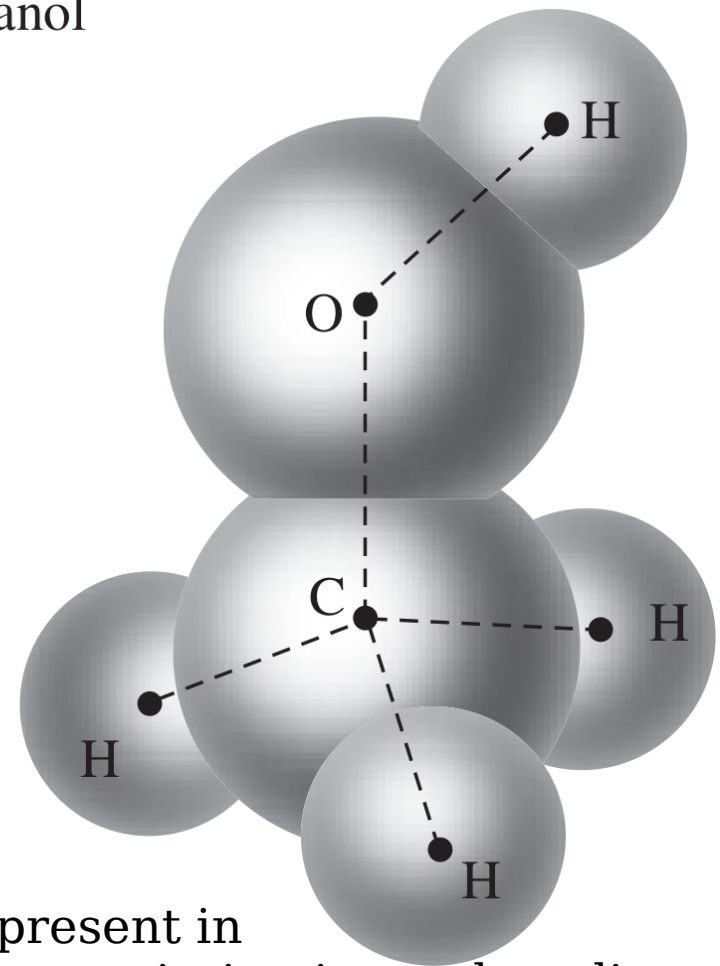
Carbon monoxide

Ammonia

Hydrogen chloride

H

Cl

$p = 3.43$

N

H

H

H

$p = 4.76$

O

C

?

$p = 0.33$
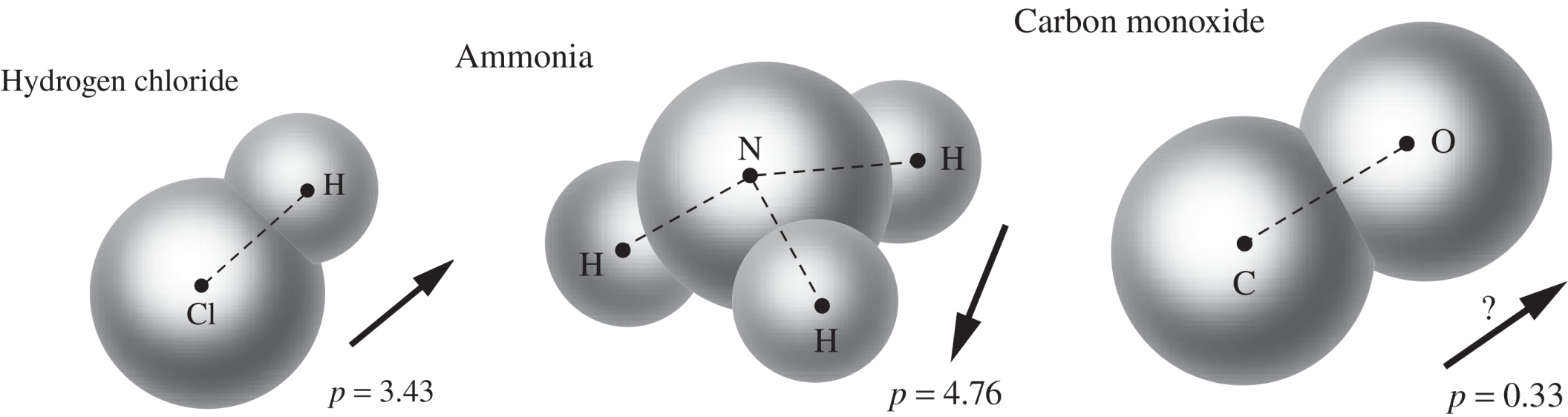
Water

Methanol



$p = 6.13$

$p = 5.66$

?

- In a nonpolar substance the application of an electric field induces a slight dipole moment in each molecule.

- In the polar substance dipoles are already present in great strength but, in the absence of a field, are pointing in random directions so that they have no large-scale effect.

- An applied electric field merely *aligns* them to a certain degree.

- In either process, the macroscopic effects will be determined by the net amount of polarization per unit volume.
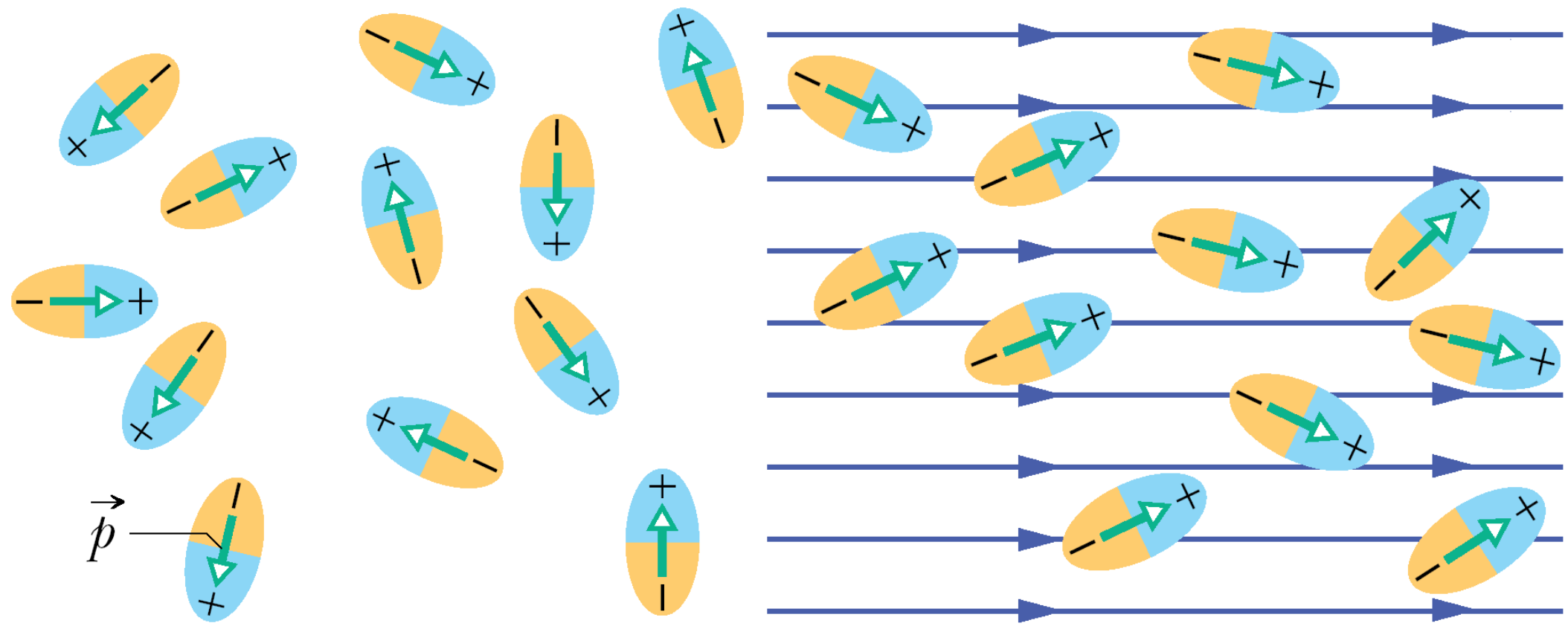
## Alignment of Polar Molecules

● A neutral atom had no dipole moment—**p** was *induced* by the applied field. Some molecules have built-in, permanent dipole moments.

● In the water molecule, the electrons tend to cluster around the oxygen atom, and since the molecule is bent at $105°$, this leaves a negative charge at the vertex and a net positive charge on the opposite side. ($\mathbf{p}_{water}=6.1\times10^{-30}$ C·m, quite large)
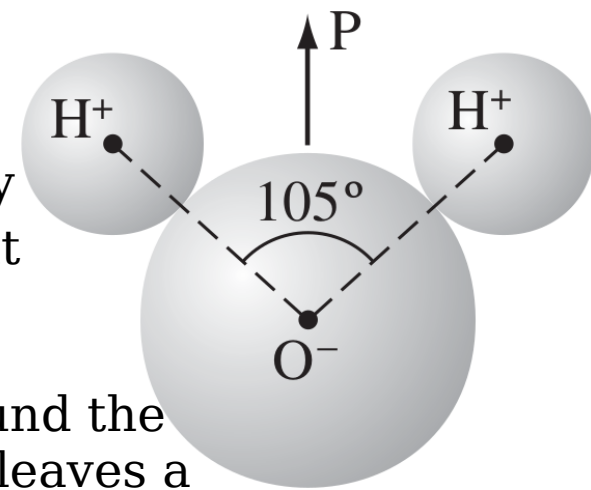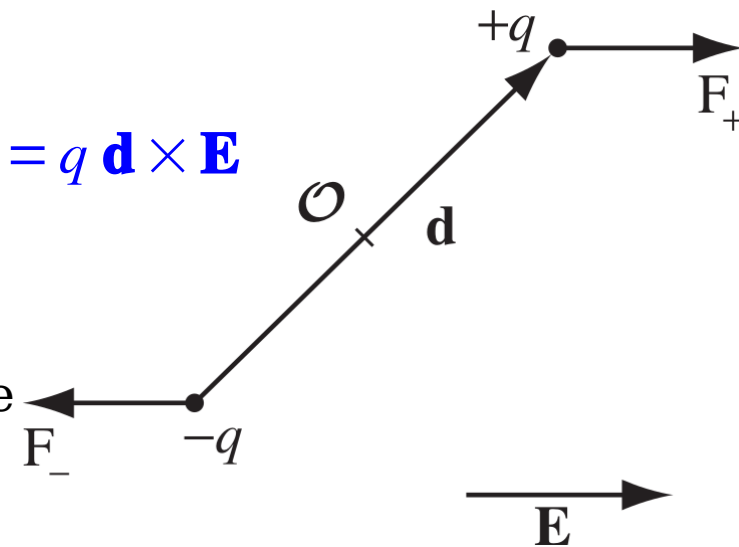
● When such **polar molecules** are placed in a uniform electric field, the *force* on the positive end $\mathbf{F}_{+}=q\,\mathbf{E}$, exactly cancels the force on the negative end $\mathbf{F}_{-}=-q\,\mathbf{E}$. However, there will be a *torque*:

$$\mathbf{N}=\mathbf{r}_{+}\times\mathbf{F}_{+}+\mathbf{r}_{-}\times\mathbf{F}_{-}=\frac{\mathbf{d}}{2}\times q\,\mathbf{E}+\left(-\frac{\mathbf{d}}{2}\right)\times(-q\,\mathbf{E})=q\,\mathbf{d}\times\mathbf{E}$$

$\mathbf{E}$ uniform $\Rightarrow$ $\boxed{\mathbf{N}=\mathbf{p}\times\mathbf{E}}$ ($) $\Leftarrow$ $\mathbf{p}=q\,\mathbf{d}$

● $\mathbf{N}$ tends to line $\mathbf{p}$ up *parallel* to $\mathbf{E}$; a polar molecule free to rotate will swing around until it points in the direction of the applied field.

● If $\mathbf{E}$ is *nonuniform*, so that $\mathbf{F}_{+}$ does not exactly balance $\mathbf{F}_{-}$, there will be a net *force* on the dipole, in addition to the torque.

- But $\mathbf{E}$ must change abruptly to be significant variation within 1 molecule, so this is not ordinarily a major consideration for the behavior of dielectrics.

- The force on a dipole in a nonuniform field: $\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = q\left(\mathbf{E}_+ - \mathbf{E}_-\right) = q\,\Delta\mathbf{E}$

$$\Delta E_x \simeq \nabla E_x \cdot \mathbf{d} \quad \Rightarrow \quad \Delta\mathbf{E} \simeq (\mathbf{d}\cdot\nabla)\mathbf{E} \quad \Rightarrow \quad \boxed{\mathbf{F} \simeq (\mathbf{p}\cdot\nabla)\mathbf{E}} \quad \Leftarrow q\,(\mathbf{d}\cdot\nabla)\mathbf{E}$$
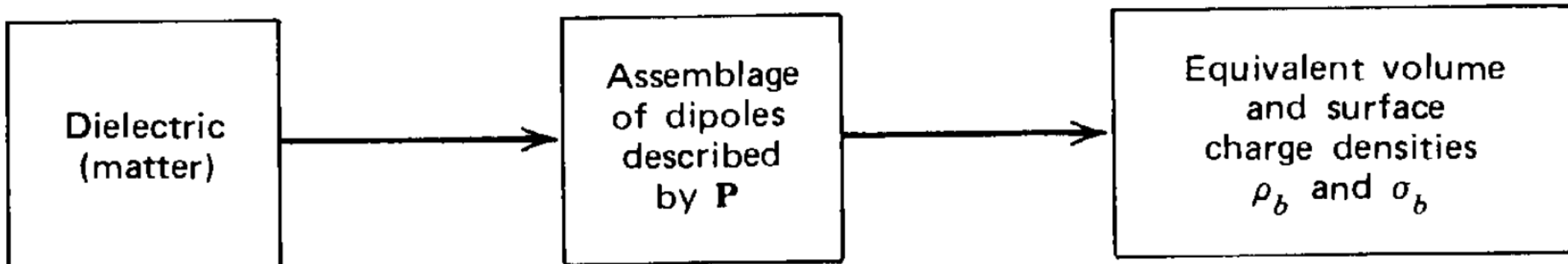
- For a "perfect" dipole of infinitesimal length, ($\$$) gives the torque *about the center of the dipole* even in a *nonuniform* field; about any *other* point

$$\mathbf{N} = \mathbf{p}\times\mathbf{E} + \mathbf{r}\times\mathbf{F} = \mathbf{p}\times\mathbf{E} + \mathbf{r}\times(\mathbf{p}\cdot\nabla)\mathbf{E}$$

# Polarization

- When a piece of dielectric material is placed in an electric field, if it consists of neutral atoms (or nonpolar molecules), the field induces in each a tiny dipole moment, pointing in the same direction as the field.

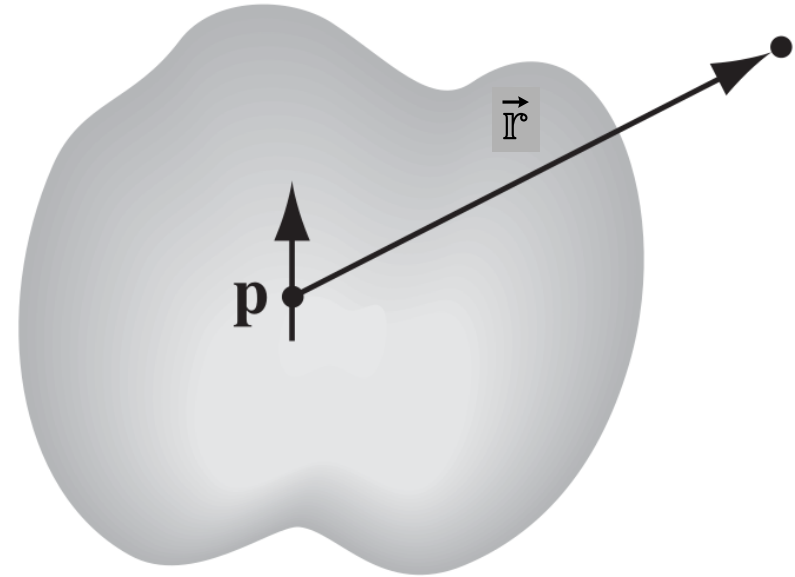- If the material is made up of polar molecules, each permanent dipole will experience a torque, tending to line it up along the field direction.

- Thermal motions compete with the process, so the alignment is never complete, especially at higher temperatures, and disappears when the field is removed.

- These 2 mechanisms produce *a lot of little dipoles pointing along the direction of the field*—the material becomes **polarized**:

$$\text{polarization } \mathbf{P} \equiv \frac{\text{dipole moment}}{\text{volume}} = \lim_{\Delta\tau \to 0} \frac{\sum \mathbf{p}_k}{\Delta\tau}$$

- Even in polar molecules there will be some polarization by displacement, but the rotation mechanism dominates.

- Some materials can "freeze in" polarization, so that it persists after the field is removed. But this will not be discussed here.

| Dielectric (matter) | → | Assemblage of dipoles described by **P** | → | Equivalent volume and surface charge densities $\rho_b$ and $\sigma_b$ |
|---|---|---|---|---|

# The Field of a Polarized Object

## Bound Charges

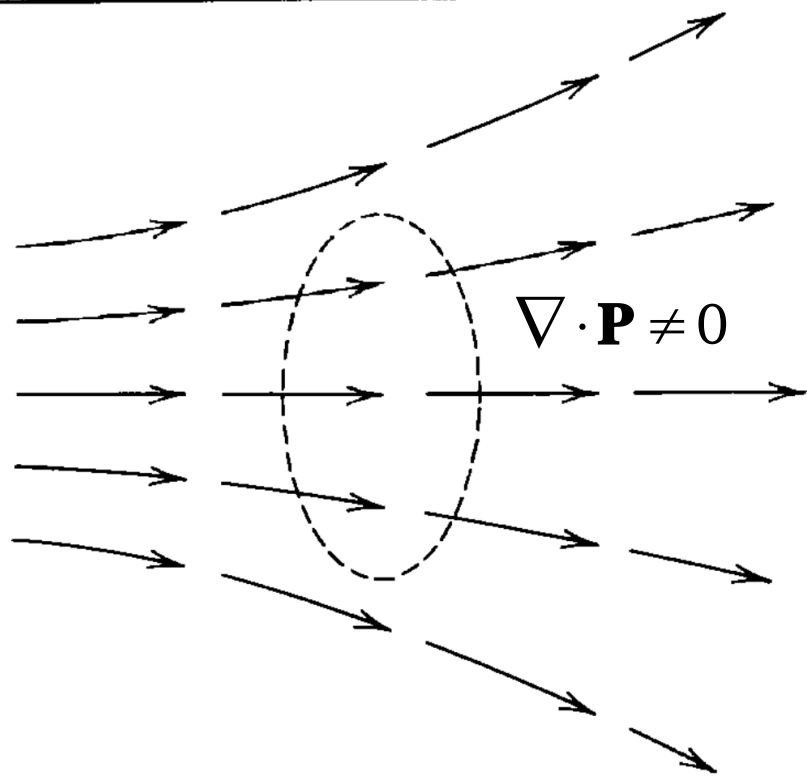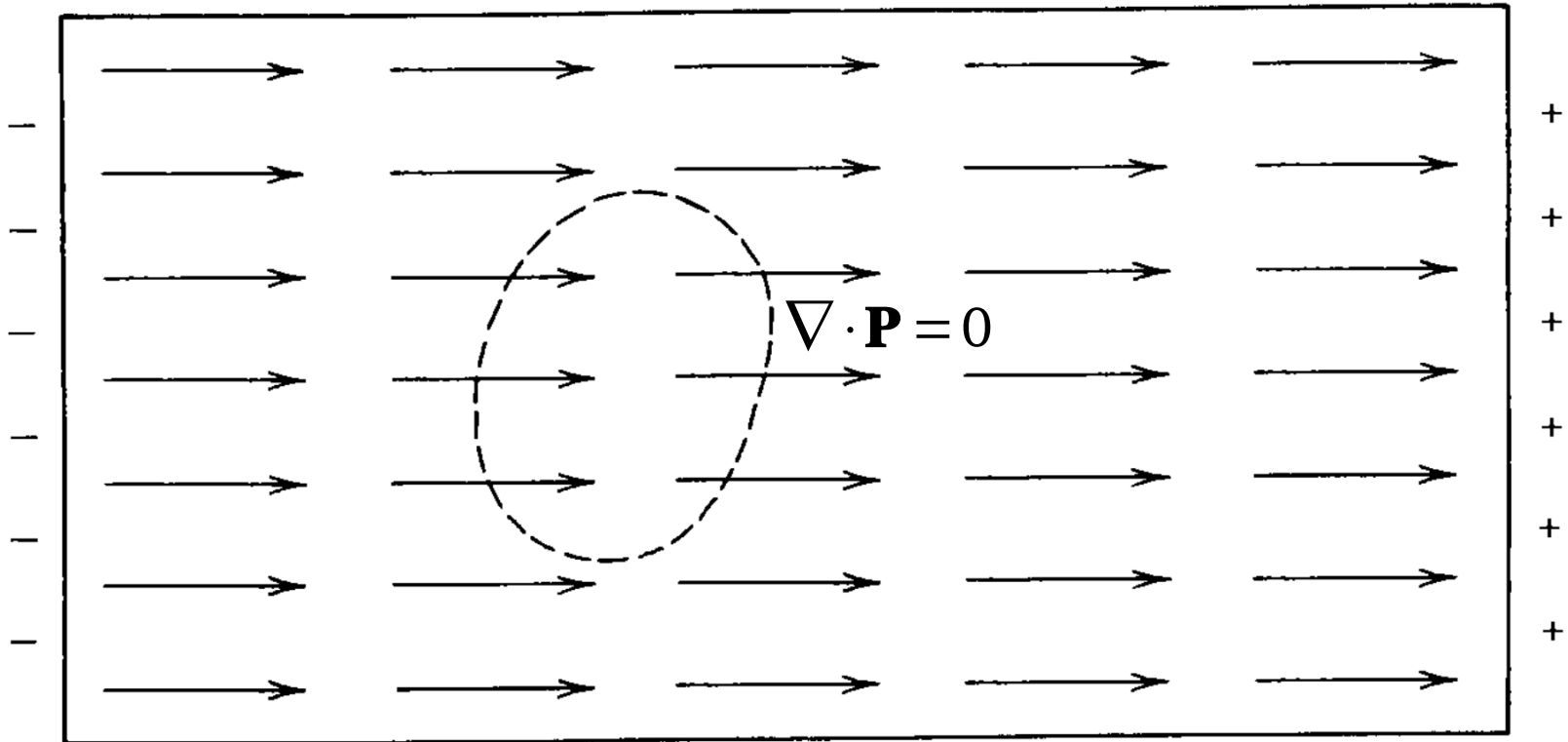- For polarized material with polarization $\mathbf{P}$, we would like to know what the field is produced by this object.

- For a single dipole $\mathbf{p}$: $\Phi(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0} \dfrac{\mathbf{p} \cdot \hat{\mathbb{r}}}{\mathbb{r}^2}$

- For dipole moment $\mathrm{d}\mathbf{p} = \mathbf{P}\,\mathrm{d}\tau'$ in each volume element $\mathrm{d}\tau'$, the total potential
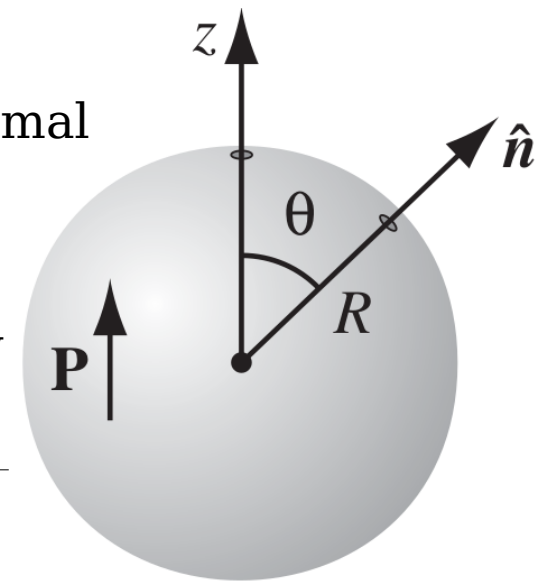
$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{\mathbb{r}} \cdot \mathrm{d}\mathbf{p}}{\mathbb{r}^2} = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\hat{\mathbb{r}} \cdot \mathbf{P}(\mathbf{r}')}{\mathbb{r}^2}\,\mathrm{d}\tau' \quad \Leftarrow \vec{\mathbb{r}} = \mathbf{r} - \mathbf{r}' \quad \Leftarrow \mathbf{r}': \begin{array}{l} \text{source} \\ \text{coordinates} \end{array}$$

$$= \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \mathbf{P} \cdot \nabla' \frac{1}{\mathbb{r}}\,\mathrm{d}\tau' \quad \Leftarrow \quad \frac{\hat{\mathbb{r}}}{\mathbb{r}^2} = \nabla' \frac{1}{\mathbb{r}} = -\nabla \frac{1}{\mathbb{r}} \qquad \Downarrow \quad \mathrm{d}\mathbf{a}' = \hat{\mathbf{n}}\,\mathrm{d}a'$$

$$= \frac{1}{4\pi\epsilon_0} \left( \int_{\mathcal{V}} \nabla' \cdot \frac{\mathbf{P}}{\mathbb{r}}\,\mathrm{d}\tau' - \int_{\mathcal{V}} \frac{\nabla' \cdot \mathbf{P}}{\mathbb{r}}\,\mathrm{d}\tau' \right) = \frac{1}{4\pi\epsilon_0} \left( \oint_{\mathcal{S}} \frac{\mathbf{P}}{\mathbb{r}} \cdot \mathrm{d}\mathbf{a}' - \int_{\mathcal{V}} \frac{\nabla' \cdot \mathbf{P}}{\mathbb{r}}\,\mathrm{d}\tau' \right)$$

$$= \frac{1}{4\pi\epsilon_0} \oint_{\mathcal{S}} \frac{\sigma_b}{\mathbb{r}}\,\mathrm{d}a' + \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho_b}{\mathbb{r}}\,\mathrm{d}\tau' \quad \Leftarrow \quad \boxed{\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}}, \quad \rho_b \equiv -\nabla \cdot \mathbf{P}}$$

- The potential (and hence also the field) of a polarized object is the same as that produced by a volume charge density $\rho_b = -\nabla \cdot \mathbf{P}$ plus a surface charge density $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$

$$\nabla \cdot \mathbf{P} = 0$$

$$\nabla \cdot \mathbf{P} \neq 0$$

● Instead of integrating the contributions of all the infinitesimal dipoles, we could first find those **bound charges**, and then calculate the fields they produce.

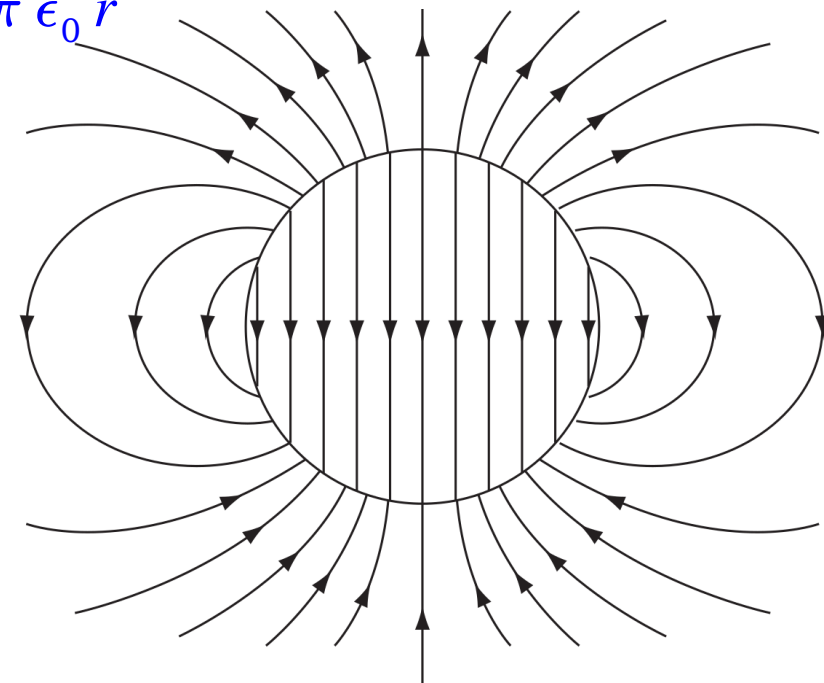Example 4.2: Find the electric field produced by a uniformly polarized sphere of radius $R$.

$$\mathbf{P} = \text{const} \implies \rho_b = -\nabla \cdot \mathbf{P} = 0, \quad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P \cos\theta$$

$$\Phi(r, \theta) = \frac{1}{4\pi\epsilon_0} \oint \frac{\sigma_b}{\imath} \, da' = \begin{cases} \dfrac{P}{3\epsilon_0} r \cos\theta = \dfrac{Pz}{3\epsilon_0}, & \text{for } r \leq R \\[2ex] \dfrac{P}{3\epsilon_0} \dfrac{R^3}{r^2} \cos\theta = \dfrac{\mathbf{p} \cdot \hat{\mathbf{r}}}{4\pi\epsilon_0 r^2}, & \text{for } r \geq R \end{cases} \quad \Leftarrow \text{Ex. 3.9}$$

$$\implies \mathbf{E} = -\nabla\Phi = \begin{cases} -\dfrac{P}{3\epsilon_0}\hat{\mathbf{z}} = -\dfrac{\mathbf{P}}{3\epsilon_0}, & r \leq R \\[2ex] p\dfrac{2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}}{4\pi\epsilon_0 r^3}, & r \geq R \end{cases}$$

where $p = |\mathbf{p}| \iff \mathbf{p} = \mathbf{P} \times \text{Volume} = \dfrac{4\pi}{3}R^3\mathbf{P}$

- Calculate the potential directly:

$$\Phi = \frac{1}{4\pi\epsilon_0} \oint_{\mathcal{S}} \frac{\sigma_b}{\mathit{r}} \, d\,a' \quad \text{(compare it with Chapter 6)}$$



Choose $\mathbf{r} = r\,\hat{\mathbf{z}}$, put $\mathbf{P}$ in the $x\,z$-plane

$$\Rightarrow \quad \mathbf{P} = P \sin\theta\,\hat{\mathbf{x}} + P\cos\theta\,\hat{\mathbf{z}}$$

And $\hat{\mathbf{n}} = \sin\theta'\cos\phi'\,\hat{\mathbf{x}} + \sin\theta'\sin\phi'\,\hat{\mathbf{y}} + \cos\theta'\,\hat{\mathbf{z}}$
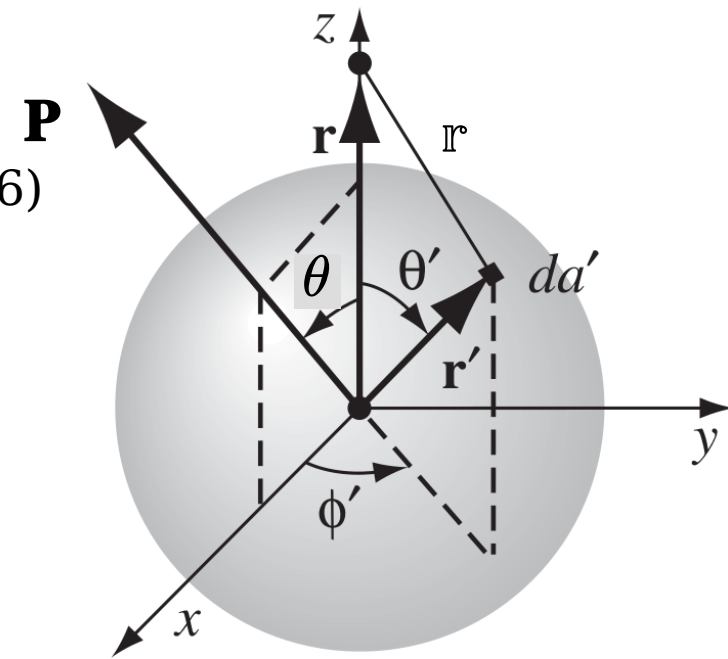
$$\Rightarrow \quad \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = P\left(\sin\theta \sin\theta' \cos\phi' + \cos\theta \cos\theta'\right)$$

$$\Rightarrow \quad \oint \frac{\sigma_b}{\mathit{r}}\,d\,a' = P \oint \frac{\sin\theta \sin\theta' \cos\!\!\!\diagup\!\!\!\phi' + \cos\theta \cos\theta'}{\sqrt{r^2 + R^2 - 2\,r\,R\cos\theta'}}\,R^2 \sin\theta'\,d\theta'\,d\phi'$$

$$= 2\pi\,P\,R^2 \cos\theta \int_0^\pi \frac{\cos\theta' \sin\theta'\,d\theta'}{\sqrt{r^2 + R^2 - 2\,r\,R\cos\theta'}}$$

$$= \frac{2\pi\,P}{3\,r^2}\left(R^3 + r^3 - |R^3 - r^3|\right)\cos\theta = \frac{4\pi\,P}{3}\frac{r_<^3}{r^2}\cos\theta \quad \Leftarrow \quad r_< = \min\left(r\,,\,R\right)$$

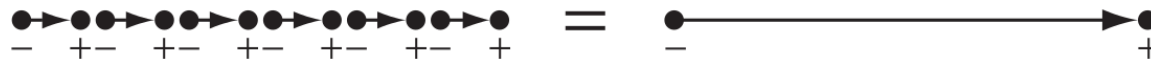$$\Rightarrow \quad \Phi\left(r\,,\,\theta\right) = \frac{P}{3\,\epsilon_0}\frac{r_<^3}{r^2}\cos\theta = \begin{cases} \dfrac{P}{3\,\epsilon_0}\,r\cos\theta = \dfrac{P\,z}{3\,\epsilon_0} & \text{for } r < R \;\; \text{inside} \\[4mm] \dfrac{P}{3\,\epsilon_0}\dfrac{R^3}{r^2}\cos\theta & \text{for } r \geq R \;\; \text{outside} \end{cases}$$

# Physical Interpretation of Bound Charges

- The field of a polarized object is identical to the field that would be produced by a certain distribution of "bound charges," $\sigma_b$ and $\rho_b$.

- So $\rho_b$ & $\sigma_b$ represent *perfectly genuine accumulations of charge*, not fictitious.



- For a long string of dipoles, the head of one effectively cancels the tail of its neighbor, but at the ends there are 2 charges left over: plus at the right end and minus at the left.

- In fact no single electron made the whole trip—a lot of tiny displacements add up to one large one.

- The net charge at the ends is called a *bound* charge to remind that it cannot be removed; but apart from that, bound charge is no different from any other kind.

- For a given polarization $\mathbf{P}$ in a "tube" of dielectric, the dipole moment of chunk parallel to $\mathbf{P}$ the is $P(Ad)$. In terms of the charge ($q$) at the end, this same dipole moment can be written $q\,d$, $\Rightarrow$ $p = P\,A\,d = q\,d$ $\Rightarrow$ $q = P\,A$

- If the ends have been sliced off perpendicularly, $\sigma_b = \dfrac{q}{A} = P$

- For an oblique cut, the *charge* is still the same,

$$A = A_{end} \cos \theta \quad \Rightarrow \quad \sigma_b = \frac{q}{A_{end}} = P \cos \theta = \mathbf{P} \cdot \hat{\mathbf{n}}$$

- The effect of the polarization, then, is to paint a bound charge $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$ over the surface of the material.

- If the polarization is nonuniform, we get accumulations of bound charge within the material, as well as on the surface.

- A diverging **P** for the figure will result in a pileup of negative charge.

- The net bound charge $\int \rho_b \, \mathrm{d}\tau$ in a given volume is equal and opposite to the amount that has been pushed out through the surface.

$$\int_V \rho_b \, \mathrm{d}\tau = -\oint_S \mathbf{P} \cdot \mathrm{d}\mathbf{a} = -\int_V \nabla \cdot \mathbf{P} \, \mathrm{d}\tau \quad \Rightarrow \quad \rho_b = -\nabla \cdot \mathbf{P}$$

$$Q_b = \int_V \rho_b \, d\tau' + \oint_S \sigma_b \, da' = -\int_V \nabla' \cdot \mathbf{P} \, d\tau' + \oint_S \mathbf{P} \cdot d\mathbf{a}'$$

$$= -\oint_S \mathbf{P} \cdot d\mathbf{a}' + \oint_S \mathbf{P} \cdot d\mathbf{a}' = 0 \quad \Rightarrow \quad \text{The total bound charge is 0.}$$

<u>Example 4.3</u>: For 2 spheres of charge: a positive sphere and a negative one. Without polarization the two are superimposed and cancel completely. But when the material is uniformly polarized, all the plus charges move slightly *upward* (the $z$ direction), and all the minus charges move slightly *downwa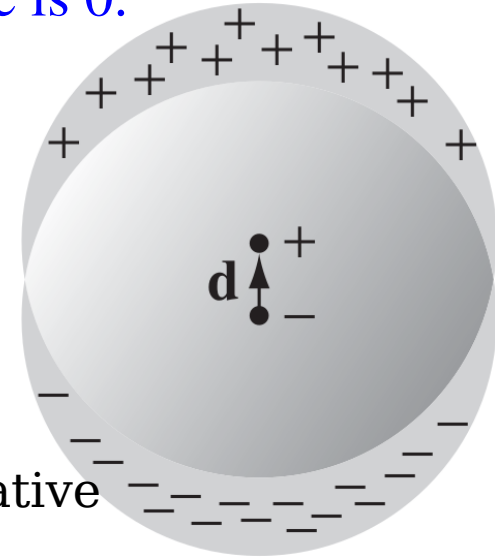rd*. The 2 spheres no longer overlap perfectly: at the top there's a cap of leftover positive charge and at the bottom a cap of negative charge. This "leftover" charge is the bound surface charge $\sigma_b$.



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For the field in the region of overlap between 2 uniformly charged spheres

Prob. 2.18 $\Rightarrow$ $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \dfrac{\rho}{3\,\epsilon_0}(\mathbf{r}_+ - \mathbf{r}_-) \ \Leftarrow\ \mathbf{E}_\pm = \pm\dfrac{\rho\,\mathbf{r}_\pm}{3\,\epsilon_0}$



$$= -\frac{\rho\,\mathbf{d}}{3\,\epsilon_0} = -\frac{1}{4\,\pi\,\epsilon_0}\frac{q\,\mathbf{d}}{R^3} = -\frac{\mathbf{P}}{3\,\epsilon_0} \quad \Leftarrow \quad \mathbf{p} = q\,\mathbf{d} = \frac{4\,\pi}{3}R^3\mathbf{P}$$

For points outside, we have a dipole with potential $\ \Phi = \dfrac{1}{4\,\pi\,\epsilon_0}\dfrac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$

These answers agree with the results of Ex. 4.2.

# The Field Inside a Dielectric

- In developing the theory of bound charges, we assumed the pure kind—perfect dipole. And yet, an actual polarized dielectric consists of *physical* dipoles, albeit extremely tiny ones.

- What is more, we represent discrete molecular dipoles by a continuous density function $\mathbf{P}$. *Outside* the dielectric there is no real problem about this. *Inside* the dielectric, however, such a procedure is open to serious challenge.

- The electric field inside matter must be complicated, on the microscopic level. Moreover, as the atoms move about, the field alters. This true **microscopic** field would be impossible to calculate, nor would it be of much interest if you could.

- We ignore the microscopic bumps and wrinkles in the electric field inside matter, and concentrate on the **macroscopic** field.

- This is defined as the *average* field over regions large enough to contain many thousands of atoms, and yet small enough to ensure that we do not wash out any significant large-scale variations in the field.

- The macroscopic field is also what people mean when they speak of "the" field inside matter.

- To calculate the macroscopic field at some point $\mathbf{r}$ within a dielectric, we must average the true (microscopic) field over an appropriate volume.

- The macroscopic field at $\mathbf{r}$ consists of 2 parts: the average field over the sphere due to all charges *outside*, plus the average due to all charges *inside*: $\mathbf{E} = \mathbf{E}_{\text{out}} + \mathbf{E}_{\text{in}}$



- The average field (over a sphere), produced by charges *outside*, is equal to the field they produce at the center, so $\mathbf{E}_{\text{out}}$ is the field at

$\mathbf{r}$ due to the dipoles exterior to the sphere $\Rightarrow \Phi_{\text{out}} = \dfrac{1}{4\pi\epsilon_0} \displaystyle\int_{\text{outside}} \dfrac{\mathbf{P}(\mathbf{r}')\cdot\hat{\mathbb{r}}}{\mathbb{r}^2} \, d\tau' \quad (!)$

- The dipoles *inside* the sphere are too close to treat in the same fashion. But all we need is their
  *average* field, Prob. 3.47 $\Rightarrow \mathbf{E}_{\text{in}} = -\dfrac{1}{4\pi\epsilon_0}\dfrac{\mathbf{p}}{R^3} = -\dfrac{\mathbf{P}}{3\epsilon_0} \quad \Leftarrow \quad \mathbf{p} = \dfrac{4\pi}{3}R^3\mathbf{P}$
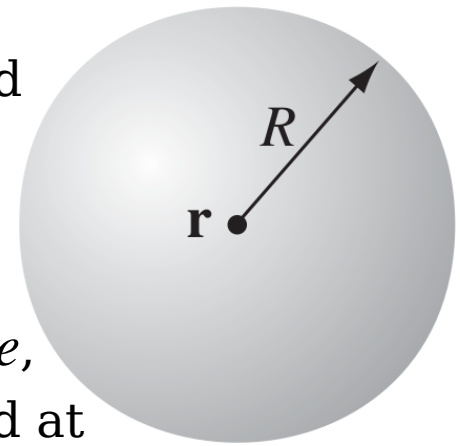
*regardless* of the details of the charge distribution within the sphere.

- With a different thought, the sphere is so small that $\mathbf{P}$ does not vary much over its volume, the term left out of the integral in (!) corresponds to the field at the

center of a uniformly polarized sphere, ie, $-\dfrac{\mathbf{P}}{3\epsilon_0}$, same as $\mathbf{E}_{\text{in}}$ above, and

$\Rightarrow \quad \Phi = \dfrac{1}{4\pi\epsilon_0} \displaystyle\int \dfrac{\mathbf{P}(\mathbf{r}')\cdot\hat{\mathbb{r}}}{\mathbb{r}^2} \, d\tau'$

- Without realizing it, we have been correctly calculating the averaged, macroscopic field, for points inside the dielectric.

<u>Problem 3.47</u>: Show that the average field inside a sphere of radius $R$, due to all

the charge within the sphere, is $\mathbf{E}_{ave} = -\dfrac{1}{4\pi\epsilon_0}\dfrac{\mathbf{p}}{R^3}$, $\mathbf{p}$ is the total dipole moment.

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The average field $\mathbf{E}$ due to a point charge $q$ at $\mathbf{r}$ is, with $\vec{\mathbb{r}} = \mathbf{r} - \mathbf{r}'$



$$\mathbf{E}_{q,ave} = \frac{1}{4\pi R^3/3}\int \mathbf{E}\,d\tau' = \frac{3}{4\pi R^3}\frac{-1}{4\pi\epsilon_0}\int \frac{q}{\mathbb{r}^2}\hat{\mathbb{r}}\,d\tau'$$

The electric field at $\mathbf{r}$ due to the uniform charge density $\rho$ over the sphere

$$\mathbf{E}_\rho = \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{\mathbb{r}^2}\hat{\mathbb{r}}\,d\tau' \quad\Rightarrow\quad \mathbf{E}_{q,ave} = \mathbf{E}_\rho \quad\text{if } \rho \equiv -\frac{q}{4\pi R^3/3}$$

From Gauss's theorem (+ the shell theorem),
(Notice $\mathbf{r}$ is a fixed, not random, direction!)
$$\mathbf{E}_\rho = \frac{\rho\,\mathbf{r}}{3\epsilon_0} = -\frac{q\,\mathbf{r}}{4\pi\epsilon_0 R^3} = -\frac{\mathbf{p}_q}{4\pi\epsilon_0 R^3}$$

If there are many charges *inside* the sphere,

$$\mathbf{E}_{ave} \equiv \sum_i \mathbf{E}_{q_i,ave} = \sum_i \frac{-\mathbf{p}_{q_i}}{4\pi\epsilon_0 R^3} = -\frac{\mathbf{p}}{4\pi\epsilon_0 R^3} \quad\Leftarrow\quad \mathbf{p} \equiv \sum_i \mathbf{p}_{q_i}$$

The same argument, but with $q$ *outside* the sphere, with the outer shell theorem,

$$\mathbf{E}_{ave} = \mathbf{E}_\rho = \frac{1}{4\pi\epsilon_0 r^2}\frac{4\pi}{3}R^3\rho\,\hat{\mathbf{r}} = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

- It all revolves around the curious fact that the **average** field over *any* sphere (due to the charge inside) is the same as the field at the center of a *uniformly polarized* sphere with the same total dipole moment.

- So no matter how crazy the actual microscopic charge configuration, we can replace it by a nice smooth distribution of perfect dipoles, if all we want is the macroscopic (average) field.

- The macroscopic field is certainly independent of the geometry of the averaging region, not limited to spherical shape.

- For a physical electric dipole, one can show $\mathbf{E}_{ave,dip} = \dfrac{1}{\frac{4\pi}{3}R^3} \displaystyle\int_{sphere} \mathbf{E}_{dip} \, d\tau = 0$ contradicting the result of Problem 3.47.

- To correct this, inside an infinitesimal sphere for a pure electric dipole, the average electric field can be written as

$$\mathbf{E}_{ave} = -\frac{\mathbf{p}}{4\pi\epsilon_0 R^3} = \frac{1}{4\pi R^3/3} \int \left(-\frac{\mathbf{p}}{3\epsilon_0}\right) \delta^3(\mathbf{r}) \, d\tau$$

$$\Rightarrow \quad \mathbf{E}_{dip}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{3(\hat{\mathbf{r}}\cdot\mathbf{p})\hat{\mathbf{r}} - \mathbf{p}}{r^3} - \frac{\mathbf{p}}{3\epsilon_0}\delta^3(\mathbf{r}) \quad \Leftarrow \quad \text{Problem 3.48}$$

$$\Rightarrow \quad \mathbf{E}_{ave,dip}(0) \simeq \frac{1}{V} \int_{V} \mathbf{E}_{dip}(0) \, d^3x = \frac{3}{4\pi R^3}\left(-\frac{\mathbf{p}}{3\epsilon_0}\right) = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3} = -\frac{\mathbf{P}}{3\epsilon_0}$$

# The Electric Displacement
## Gauss's Law in the Presence of Dielectrics

● The effect of polarization is to produce accumulations of (bound) charge, $\rho_b = -\nabla \cdot \mathbf{P}$ within the dielectric and $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$ on the surface.

● Within the dielectric, the total charge density $\rho = \rho_b + \rho_f \quad \Leftarrow \quad \rho_f$ : free charge

$$\Rightarrow \quad \epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f \quad \Rightarrow \quad \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

$$\Rightarrow \quad \text{electric displacement: } \boxed{\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}} \quad \Rightarrow \quad \boxed{\nabla \cdot \mathbf{D} = \rho_f} \quad \Rightarrow \quad \oint \mathbf{D} \cdot \mathrm{d}\boldsymbol{a} = Q_{f_{enc}} \quad (\#)$$

● The integral form is a useful way to express Gauss's law, in the context of dielectrics, because *it makes reference only to free charges*, and free charge is the stuff we control.

● When we put the free charge in place, a certain polarization automatically ensues, and this polarization produces the bound charge.

● In a typical problem, therefore, we know $\rho_f$, but we do not (initially) know $\rho_b$; (#) lets us go right to work with the information at hand.

● Whenever the requisite symmetry is present, we can immediately calculate $\mathbf{D}$ by the standard Gauss's law methods.

Example 4.4



Gaussian surface

- It seems that we left out the surface bound charge $\sigma_b$ in the derivation.

- We cannot apply Gauss's law precisely *at* the surface of a dielectric, for here $\rho_b = -\nabla \cdot \mathbf{P}$ blows up, taking the divergence of $\mathbf{E}$ with it.

- In fact we picture the edge of the dielectric as having some finite thickness, within which the polarization tapers off to 0, then there is *no* surface bound charge; $\rho_b$ varies rapidly but smoothly within this "skin," and Gauss's law can be safely applied *everywhere*.

**Physical**

**Idealized**



"Property"

1

2

Transition layer

$h$

$x$

"Property"

1

2

Surface of discontinuity

$x$

## A Deceptive Parallel

● The "similar" behavior of **D** and **E** in the related equations doesn't mean that **D** is "just like" **E**.

● There is no "Coulomb's law" for **D**: $\mathbf{D}(\mathbf{r}) \neq \dfrac{1}{4\pi} \displaystyle\int \dfrac{\hat{\mathbf{r}}}{\mathbf{r}^2} \rho_f(\mathbf{r}') \, d\tau'$, thus the conclusion is false. And the parallel between **E** and **D** is more subtle than that.

● The curl of **E** is always 0, but the curl of **D** is *not* always 0,

$$\nabla \times \mathbf{D} = \epsilon_0 \nabla \times \mathbf{E} + \nabla \times \mathbf{P} = \nabla \times \mathbf{P} \neq 0 \quad \text{in general} \quad (\&)$$

● Because $\nabla \times \mathbf{D} \neq 0$, **D** cannot be expressed as the gradient of a scalar—there is no "potential" for **D**.

● To compute **D**, first look for symmetry. If the problem exhibits spherical, cylindrical, or plane symmetry, you can get **D** directly from (#) by the usual Gauss's law methods.

● This is because in such cases $\nabla \times \mathbf{P}$ is automatically 0.

● If the requisite symmetry is absent, you'll have to think of another approach, and, in particular, you must *not* assume that **D** is determined exclusively by the free charge.

## Boundary Conditions

- (#) tells us the discontinuity in the component perpendicular to an interface:

$$\oint \mathbf{D} \cdot \mathrm{d}\,\boldsymbol{a} = Q_{f_{\text{enc}}} \quad \Rightarrow \quad D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f$$

- (&) gives the discontinuity in parallel components: $\mathbf{D}^{\parallel}_{\text{above}} - \mathbf{D}^{\parallel}_{\text{below}} = \mathbf{P}^{\parallel}_{\text{above}} - \mathbf{P}^{\parallel}_{\text{below}}$

- In the presence of dielectrics, the above 2 equations are sometimes more useful than the corresponding boundary conditions on $\mathbf{E}$:

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0}$$

$$\mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = 0$$

- In general, the following conditions are used

$$\text{Normal}: \quad D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f$$

$$\text{Tangential}: \quad \mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = 0$$

# Linear Dielectrics
## Susceptibility, Permittivity, Dielectric Constant

- For *nonlinear* dielectrics: $P_i = \sum_j \alpha_{ij} E_j + \sum_{jk} \beta_{ijk} E_j E_k + \cdots$

- The polarization of a dielectric ordinarily results from an electric field, which lines up the atomic or molecular dipoles. For many substances, the polarization is *proportional* to the field, provided $\mathbf{E}$ is not too strong:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \quad \Leftarrow \quad \chi_e : \text{electric susceptibility}$$

- Materials that obey this relation are called **linear dielectrics**. $\mathbf{E}$ in the eqn is the *total* field; it may be due to free charges and to the polarization itself.

- If we put a piece of dielectric into an *external* field $\mathbf{E}_0$, we cannot compute $\mathbf{P}$ directly from the equation; the external field will polarize the material, and this polarization will produce its own field, which then contributes to the total field, and this in turn modifies the polarization, etc.

- The simplest approach is to begin with the *displacement*, at least in those cases where $\mathbf{D}$ can be deduced directly from the free charge distribution.

- In linear media $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (1 + \chi_e) \mathbf{E}$

$$\Rightarrow \quad \mathbf{D} = \epsilon \mathbf{E} \quad \Leftarrow \quad \epsilon \equiv \epsilon_0 (1 + \chi_e) \quad \text{permittivity} \quad \Rightarrow \quad \epsilon_0 : \text{permittivity of free space}$$

$$\Rightarrow \quad \text{relative permittivity} \quad \epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0} \geq 1 \quad \text{dielectric constant}$$

- $\epsilon_r$ can be a function of space coordinates. If $\epsilon_r$ is independent of position, the medium is said to be *homogenous*.

- A linear, homogeneous, and isotropic medium is called a *simple medium.* The relative permittivity of a simple medium is a constant.

- A *lossy* medium can be represented by a complex dielectric constant, whose imaginary part provides a measure of power loss in the medium and, in general, is frequency-dependent.

- For *anisotropic* materials the dielectric constant is different for different directions of the electric field, and **D** and **E** vectors generally have different directions; permittivity is a tensor.

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \Rightarrow \begin{bmatrix} D_{x'} \\ D_{y'} \\ D_{z'} \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \begin{bmatrix} E_{x'} \\ E_{y'} \\ E_{z'} \end{bmatrix} \Leftarrow \begin{array}{c} \textit{biaxial} \\ \text{choosing the principal} \\ \text{axes to get rid of the} \\ \text{off-diagonal terms} \end{array}$$

$$\Rightarrow \begin{array}{l} D_{x'} = \epsilon_1 E_{x'} \\ D_{y'} = \epsilon_2 E_{y'} \\ D_{z'} = \epsilon_3 E_{z'} \end{array} \Rightarrow \begin{array}{c} \epsilon_1 = \epsilon_2 \\ \textit{uniaxial} \end{array} \Rightarrow \begin{array}{c} \epsilon_1 = \epsilon_2 = \epsilon_3 \\ \text{isotropic} \end{array}$$

| Material | Dielectric Constant | Material | Dielectric Constant |
|---|---|---|---|
| Vacuum | 1 | Benzene | 2.28 |
| Helium | 1.000065 | Diamond | 5.7-5.9 |
| Neon | 1.00013 | Salt | 5.9 |
| Hydrogen ($H_2$) | 1.000254 | Silicon | 11.7 |
| Argon | 1.000517 | Methanol | 33.0 |
| Air (dry) | 1.000536 | Water | 80.1 |
| Nitrogen ($N_2$) | 1.000548 | Ice (-30° C) | 104 |
| Water vapor (100° C) | 1.00589 | $KTaNbO_3$ (0° C) | 34,000 |



**Example 4.5**: A metal sphere of radius $a$ carries a charge $Q$. It is surrounded, out to radius $b$, by linear dielectric material of permittivity . Find the potential at the center (relative to infinity).

$$\text{spherically symmetric} + \oint \mathbf{D} \cdot d\mathbf{a} = Q \;\Rightarrow\; \mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \quad r > a; \quad \mathbf{E} = \mathbf{P} = \mathbf{D} = 0 \text{ for } r < a$$

$$\Rightarrow \mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & a < r < b \\[2mm] \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & r \geq b \end{cases}$$

$$\Phi = -\int_{\infty}^{0} \mathbf{E}\cdot d\boldsymbol{\ell} = \int_{0}^{\infty}\mathbf{E}\cdot d\mathbf{r} \;\Leftarrow\; d\boldsymbol{\ell} = -d\mathbf{r}$$
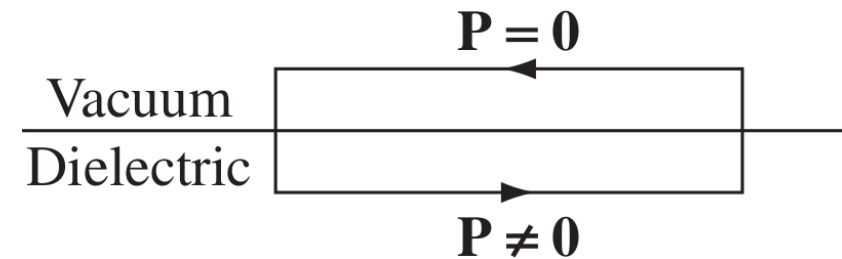
$$\Rightarrow \; = -\int_{\infty}^{b}\frac{Q\,dr}{4\pi\epsilon_0 r^2} - \int_{b}^{a}\frac{Q\,dr}{4\pi\epsilon r^2} - \int_{a}^{0} 0\cdot dr$$

$$= \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right)$$

$$\Rightarrow \quad \mathbf{P} = \epsilon_0 \, \chi_e \, \mathbf{E} = \frac{\epsilon_0 \, \chi_e \, Q}{4 \, \pi \, \epsilon \, r^2} \, \hat{\mathbf{r}} \quad \Rightarrow \quad \rho_b = - \nabla \cdot \mathbf{P} = 0 \, , \quad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \begin{vmatrix} \dfrac{\epsilon_0 \, \chi_e \, Q}{4 \, \pi \, \epsilon \, b^2} \, , & \text{outer surface} \\[2ex] -\dfrac{\epsilon_0 \, \chi_e \, Q}{4 \, \pi \, \epsilon \, a^2} \, , & \text{inner surface} \end{vmatrix}$$

- Although $\mathbf{P}$ & $\mathbf{D}$ are now proportional to $\mathbf{E}$, it doesn't mean that their curls, like $\mathbf{E}$'s, vanish.



$\mathbf{P} = 0$

Vacuum

Dielectric

$\mathbf{P} \neq 0$

- The line integral of $\mathbf{P}$ around a closed path that *straddles the boundary between one type of material and another* need not be 0, although the integral of $\mathbf{E}$ around the same loop must be. It is because the different proportionality factor $\epsilon_0 \chi_e$ on the 2 sides.

- At the interface between a polarized dielectric and the vacuum, $\mathbf{P}$ is 0 on one side but not on the other. Around this loop $\oint \mathbf{P} \cdot d\boldsymbol{\ell} \neq 0$, and, by Stokes' theorem, $\nabla \times \mathbf{P}$ cannot vanish everywhere within the loop.

- If the space is *entirely* filled with a homogeneous linear dielectric, this objection is void; in this rather special circumstance $\nabla \cdot \mathbf{D} = \rho_f$ & $\nabla \times \mathbf{D} = 0$

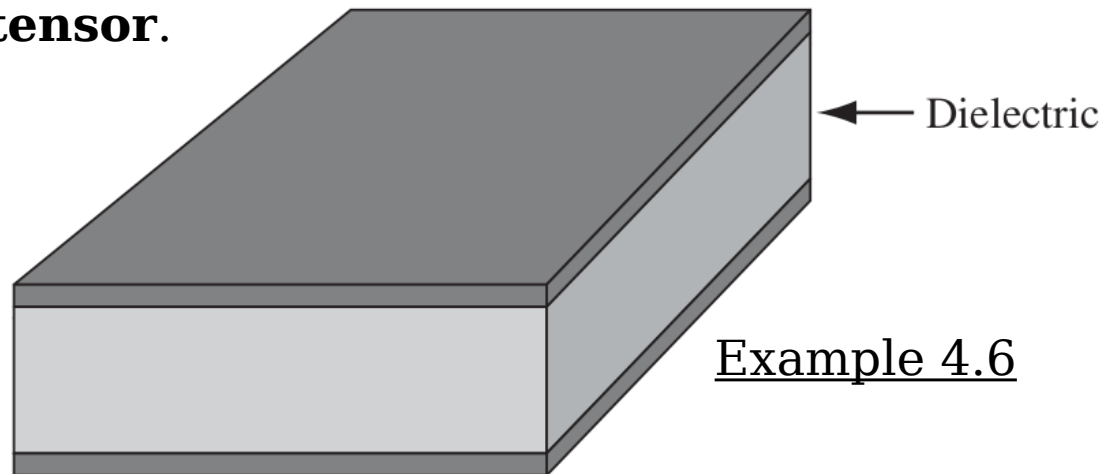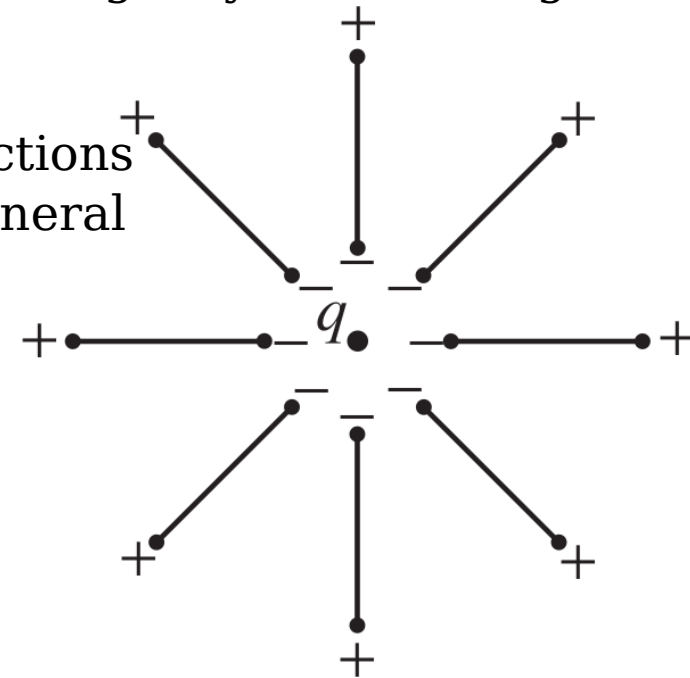as if $\mathbf{D} = \epsilon_0 \, \mathbf{E}_{\text{vac}} \Leftarrow \mathbf{E}_{\text{vac}}$ : electric field from the same free charge in vacuum $\quad \Rightarrow \quad \mathbf{E} = \dfrac{\mathbf{D}}{\epsilon} = \dfrac{\mathbf{E}_{\text{vac}}}{\epsilon_r}$

- When all space is filled with a homogeneous linear dielectric, the field everywhere is simply reduced by a factor of one over the dielectric constant.

- If a free charge $q$ is embedded in a large dielectric, the field is $\mathbf{E} = \dfrac{1}{4\pi\epsilon}\dfrac{q}{r^2}\hat{\mathbf{r}}$

- The force exerted on nearby charges is reduced accordingly. This is because the polarization of the medium partially "shields" the charge, by surrounding it with bound charge of the opposite sign.

- A *crystal* is generally easier to polarize in some directions than in others, so the proportion is replaced by the general linear relation
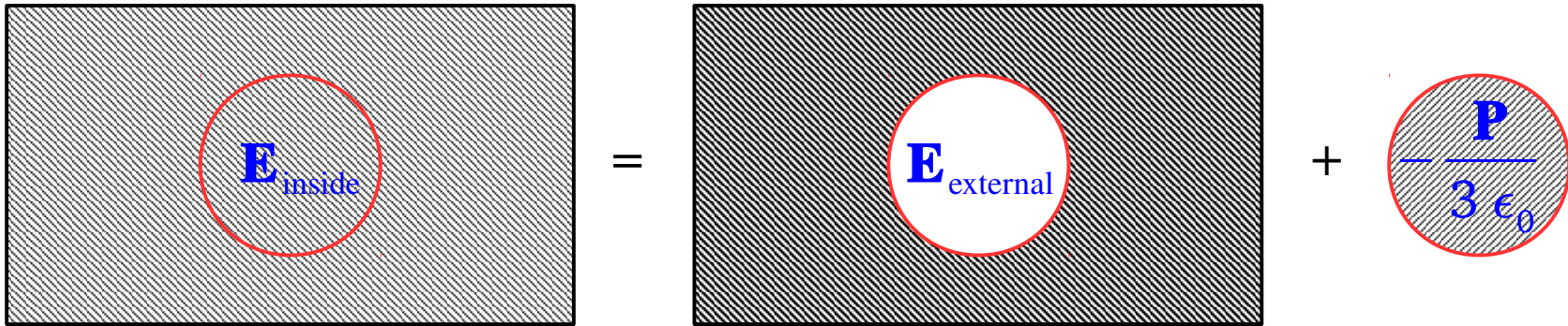
$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \epsilon_0 \begin{bmatrix} \chi_{e,xx} & \chi_{e,xy} & \chi_{e,xz} \\ \chi_{e,yx} & \chi_{e,yy} & \chi_{e,yz} \\ \chi_{e,zx} & \chi_{e,zy} & \chi_{e,zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

- The 9 coefficients, $\chi_{e,xx}$, $\chi_{e,xy}$, ..., constitute the **susceptibility tensor**.

Dielectric

Example 4.6

# Atomic/Molecular Polarizability and Electric Susceptibility

● Consider the relation between mircoscopically molecular properties, ie, the atomic/molecular polarizability $\alpha$, and the macroscopically defined parameter, the electric susceptibility $\chi_e$:

●
$$\mathbf{E}_{inside} \quad = \quad \mathbf{E}_{external} \quad + \quad -\frac{\mathbf{P}}{3\,\epsilon_0}$$

$$\mathbf{P} = \epsilon_0\,\chi_e\,\mathbf{E}_{inside}\,, \quad \frac{\mathbf{P}}{n} \Leftarrow \mathbf{p} = \alpha\,\mathbf{E}_{external} \quad \Rightarrow \quad \mathbf{E}_{inside} = \mathbf{E}_{external} + \left(-\frac{\mathbf{P}}{3\,\epsilon_0}\right),\quad n = \frac{N}{V}$$

$$\Rightarrow \quad \frac{\mathbf{P}}{\epsilon_0\,\chi_e} = \frac{\mathbf{P}}{n\,\alpha} - \frac{\mathbf{P}}{3\,\epsilon_0} \quad \Rightarrow \quad \chi_e = \epsilon_r - 1 = \frac{3\,n\,\alpha}{3\,\epsilon_0 - n\,\alpha}$$

$$\Rightarrow \quad \boxed{\alpha = \frac{3\,\epsilon_0}{n}\,\frac{\epsilon_r - 1}{\epsilon_r + 2}} \quad \text{Clausius-Mossotti equation}$$

● The relation holds best for dilute substances such as gases. For liquids & solids, the equation is only approximately valid, especially if the dielectric constant is large.

# Boundary Value Problems with Linear Dielectrics

● In a (homogeneous isotropic) linear dielectric, the bound charge density $\rho_b$ is

proportional to the free charge density $\rho_f$: $\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot \dfrac{\epsilon_0 \chi_e \mathbf{D}}{\epsilon} = -\dfrac{\chi_e}{1+\chi_e}\rho_f$

● Unless free charge is actually embedded in the material, $\rho=0$, and any net charge must reside at the surface. Within such a dielectric, the potential obeys Laplace's equation.

● It is convenient to rewrite the boundary conditions in a way that makes reference only to the free charge,

$$D_{\text{above}}^{\perp} - D_{\text{below}}^{\perp} = \epsilon_{\text{above}} E_{\text{above}}^{\perp} - \epsilon_{\text{below}} E_{\text{below}}^{\perp} = \sigma_f$$

The potential is continuous cross the interface

$$\Rightarrow \quad \epsilon_{\text{above}} \frac{\partial \Phi_{\text{above}}}{\partial n} - \epsilon_{\text{below}} \frac{\partial \Phi_{\text{below}}}{\partial n} = -\sigma_f$$

$$\Phi_{\text{above}} - \Phi_{\text{below}} = 0$$

**Example 4.7**: A sphere of homogeneous linear dielectric material is placed in an otherwise uniform electric field $\mathbf{E}_0$. Find the electric field inside/outside the sphere.

You can compare the problem with Ex. 3.8 (uncharged conducting sphere).

To solve Laplace's equation,

$$\left[ \begin{array}{l} \Phi_{\text{in}}(r,\theta), \quad r \leq R \\ \Phi_{\text{out}}(r,\theta), \quad r \geq R \end{array} \right., \quad \text{choose } \Phi_{\text{in}}(r=0)=0$$

$$\Rightarrow \left[ \begin{array}{lll} \text{(i)} & \Phi_{\text{in}} = \Phi_{\text{out}}, & \text{at } r=R \\[2mm] \text{(ii)} & \epsilon \dfrac{\partial \Phi_{\text{in}}}{\partial r} = \epsilon_0 \dfrac{\partial \Phi_{\text{out}}}{\partial r}, & \text{at } r=R \\[2mm] \text{(iii)} & \Phi_{\text{out}} \to -E_0 r \cos\theta, & \text{for } r \gg R \end{array} \right.$$



$$\Rightarrow \quad \Phi_{\text{in}}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

$$\Phi_{\text{out}}(r,\theta) = -E_0 r \cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \quad \Leftarrow \text{ (iii)}$$

$$\text{(i)} \Rightarrow \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = -E_0 R \cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

$$\text{(ii)} \Rightarrow \epsilon_r \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos\theta) = -E_0 \cos\theta - \sum_{\ell=0}^{\infty} (\ell+1)\frac{B_\ell}{R^{\ell+2}} P_\ell(\cos\theta)$$

$$\Rightarrow \quad A_1 \, R = - E_0 \, R + \frac{B_1}{R^2} \, , \qquad A_\ell \, R^\ell \quad = \quad \frac{B_\ell}{R^{\ell+1}} \qquad \text{for} \ \ \ell \neq 1$$

$$\epsilon_r \, A_1 = - E_0 - \frac{2 \, B_1}{R^3} \, , \quad \epsilon_r \, \ell \, A_\ell \, R^{\ell-1} = - \frac{\ell+1}{R^{\ell+2}} \, B_\ell \quad \text{for} \ \ \ell \neq 1$$

$$\Rightarrow \quad A_1 = - \frac{3}{\epsilon_r + 2} \, E_0 \, , \quad B_1 = \frac{\epsilon_r - 1}{\epsilon_r + 2} \, R^3 \, E_0 \, , \quad A_\ell = B_\ell = 0 \quad \text{for} \ \ \ell \neq 1$$

$$\Rightarrow \quad \Phi_{\text{in}} = - \frac{3 \, E_0}{\epsilon_r + 2} \, r \cos \theta = - \frac{3 \, E_0}{\epsilon_r + 2} \, z \, , \quad \mathbf{E}_{\text{in}} = \frac{3}{\epsilon_r + 2} \, \mathbf{E}_0 \ \ \Leftarrow \ \ \textit{uniform}$$

$$\Phi_{\text{out}} = - E_0 \, z + \frac{\epsilon_r - 1}{\epsilon_r + 2} \, \frac{R^3}{r^2} \, E_0 \cos \theta \, , \quad \mathbf{E}_{\text{out}} = \mathbf{E}_0 + \frac{\epsilon_r - 1}{\epsilon_r + 2} \, \frac{R^3}{r^3} \left[ 3 \left( \hat{\mathbf{r}} \cdot \mathbf{E}_0 \right) \hat{\mathbf{r}} - \mathbf{E}_0 \right]$$

$$\Rightarrow \quad \mathbf{p}_{\text{induced}} = 4 \, \pi \, \epsilon_0 \, \frac{\epsilon_r - 1}{\epsilon_r + 2} \, R^3 \, \mathbf{E}_0 \ \ \Rightarrow \ \ \mathbf{P} = \frac{\mathbf{p}_{\text{induced}}}{V_{\text{sphere}}} = 3 \, \frac{\epsilon_r - 1}{\epsilon_r + 2} \, \epsilon_0 \, \mathbf{E}_0 = 3 \, \epsilon_0 \left( \mathbf{E}_0 - \mathbf{E}_{\text{in}} \right)$$

$$\Rightarrow \quad \rho_b = - \nabla \cdot \mathbf{P} = 0 \, , \quad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \mathbf{P} \cdot \hat{\mathbf{r}} = 3 \, \frac{\epsilon_r - 1}{\epsilon_r + 2} \, \epsilon_0 \, E_0 \cos \theta$$

Example: **A Long, Dielectric Cylinder in an Electric Field**:
Consider a long, dielectric cylinder of permittivity $\epsilon$ in a uniform electric field that is normal to its axis.



The "general" solution for $\nabla^2 \Phi = 0$ in the cylindrical coordinates

$$\Phi(r, \phi) = A_0 \ln r + B_0 + \sum_{n=1}^{\infty}\left[\left(A_n r^n + \frac{B_n}{r^n}\right)\cos n\phi + \left(C_n r^n + \frac{D_n}{r^n}\right)\sin n\phi\right]$$

$(i)$ Set $\Phi_{in}(r=0) = 0 \implies A_0 = B_0 = 0$, $(ii)$ $\Phi(r,\phi) = \Phi(r,-\phi) \implies C_n = D_n = 0$

$$\Phi_{in} = \sum_{n=1}^{\infty} A_n r^n \cos n\phi$$

$(iii)$ $\Phi_{out}(r \to \infty) \to -E_0 x = -E_0 r \cos\phi \implies$

$$\Phi_{out} = \sum_{n=1}^{\infty} \frac{B_n}{r^n} \cos n\phi - E_0 r \cos\phi$$

Boundary conditions: $\Phi_{in|r=R} = \Phi_{out|r=R}$, $\quad \epsilon \left.\dfrac{\partial \Phi_{in}}{\partial r}\right|_{r=R} = \epsilon_0 \left.\dfrac{\partial \Phi_{in}}{\partial r}\right|_{r=R}$

$$\implies \sum_{n=1}^{\infty} A_n R^n \cos n\phi = -E_0 R \cos\phi + \sum_{n=1}^{\infty} \frac{B_n}{R^n} \cos n\phi$$

$$\epsilon \sum_{n=1}^{\infty} n A_n R^{n-1} \cos n\phi = -\epsilon_0 E_0 \cos\phi - \epsilon_0 \sum_{n=1}^{\infty} n \frac{B_n}{R^{n+1}} \cos n\phi$$

$$\Rightarrow \quad A_1 R = -E_0 R + \frac{B_1}{R}, \qquad A_n R^n = \frac{B_n}{R^n} \quad \text{for } n > 1$$

$$\epsilon A_1 = -\epsilon_0 \left( E_0 + \frac{B_1}{R^2} \right), \quad \epsilon A_n R^{n-1} = -\epsilon_0 \frac{B_n}{R^{n+1}} \quad \text{for } n > 1$$

$$\Rightarrow \quad A_1 = -\frac{2}{\epsilon_r + 1} E_0, \quad B_1 = \frac{\epsilon_r - 1}{\epsilon_r + 1} R^2 E_0 \quad \Leftarrow \quad \epsilon_r = \frac{\epsilon}{\epsilon_0}, \quad A_n = B_n = 0 \quad \text{for } n > 1$$

$$\Phi_{\text{in}} = -\frac{2 E_0}{\epsilon_r + 1} r \cos\theta = -\frac{2 E_0}{\epsilon_r + 1} x, \quad \mathbf{E}_{\text{in}} = \frac{2}{\epsilon_r + 1} \mathbf{E}_0 \quad \Leftarrow \quad uniform$$

$$\Rightarrow$$

$$\Phi_{\text{out}} = -E_0 x + \frac{\epsilon_r - 1}{\epsilon_r + 1} \frac{R^2}{r} E_0 \cos\theta, \quad \mathbf{E}_{\text{out}} = \mathbf{E}_0 + \frac{\epsilon_r - 1}{\epsilon_r + 1} \frac{R^2}{r^2} \left[ 2 \left( \hat{\mathbf{r}} \cdot \mathbf{E}_0 \right) \hat{\mathbf{r}} - \mathbf{E}_0 \right]$$

Example: **Point Charge on a Plane Interface**:
A point charge $q$ is placed on the plane interface of 2 homogeneous infinite dielectrics 1 and 2 with permittivities $\epsilon_1$ and $\epsilon_2$, respectively.



Let $\Phi_1$ and $\Phi_2$ are in region 1 and 2

$$\Rightarrow \quad \Phi_1 = C_1 \frac{q}{r} + B_1 , \quad \Phi_2 = C_2 \frac{q}{r} + B_2$$

$$\Phi_{1,2}(r \to \infty) = 0 \quad \Rightarrow \quad B_1 = B_2 = 0$$

The potential is continuous at the boundary $\Rightarrow C_1 = C_2 = C \Rightarrow \Phi_1 = \Phi_2 = \Phi$

$$\mathbf{D} = -\epsilon \nabla \Phi = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} C \frac{q}{r^2} \hat{\mathbf{r}} \text{ in region } \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \oint_{\mathcal{S}} \mathbf{D} \cdot \mathrm{d}\boldsymbol{a} = q \quad \Rightarrow \quad C = \frac{1}{2\pi(\epsilon_1 + \epsilon_2)}$$

$$\Rightarrow \quad \Phi = \frac{1}{2\pi(\epsilon_1 + \epsilon_2)} \frac{q}{r} \quad \Rightarrow \quad \mathbf{E} = \frac{1}{2\pi(\epsilon_1 + \epsilon_2)} \frac{q}{r^2} \hat{\mathbf{r}} \quad \Rightarrow \quad \mathbf{D}_{1,2} = \frac{\epsilon_{1,2}}{2\pi(\epsilon_1 + \epsilon_2)} \frac{q}{r^2} \hat{\mathbf{r}}$$

$$\Rightarrow \quad \mathbf{P}_i = \mathbf{D}_i - \epsilon_0 \mathbf{E} = \frac{\epsilon_i - \epsilon_0}{2\pi(\epsilon_1 + \epsilon_2)} \frac{q}{r^2} \hat{\mathbf{r}} , \quad i = 1, 2 \quad \Rightarrow \quad \rho_b = -\nabla \cdot \mathbf{P} = 0 \text{ for } r \neq 0$$

$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = 0$ on the interface, but $\sigma_b \neq 0$ close to the charge surface $r = d \to 0$

$$\sigma_{b,i} = \mathbf{P}_i \cdot (-\hat{\mathbf{r}}) = \frac{\epsilon_0 - \epsilon_i}{2\pi(\epsilon_1 + \epsilon_2)} \frac{q}{d^2} \quad \Rightarrow \quad q_b = \int \sigma_b \, \mathrm{d}a = \frac{2\epsilon_0 q}{\epsilon_1 + \epsilon_2} - q \quad \Rightarrow \quad \begin{aligned} Q &= q + q_b \\ &= \frac{2\epsilon_0}{\epsilon_1 + \epsilon_2} q \end{aligned}$$

<u>Example</u>$'$: **A Conducting Charged Sphere Between 2 Dielectrics**:

Same as previous example but with a conducting charged sphere instead of a point charge.



The same argument leads to

$$\Phi = \frac{1}{2\pi(\epsilon_1 + \epsilon_2)}\frac{q}{r} \quad \Rightarrow \quad \mathbf{D}_{1,2} = \frac{\epsilon_{1,2}}{2\pi(\epsilon_1 + \epsilon_2)}\frac{q}{r^2}\,\hat{\mathbf{r}}$$

The free surface charge density

$$\sigma_{f,i} \equiv -\epsilon_i \frac{\partial \Phi}{\partial r}\Big|_R = D_{i,r}(r=R) = \frac{\epsilon_i}{2\pi(\epsilon_1 + \epsilon_2)}\frac{q}{R^2}, \quad i=1,2$$
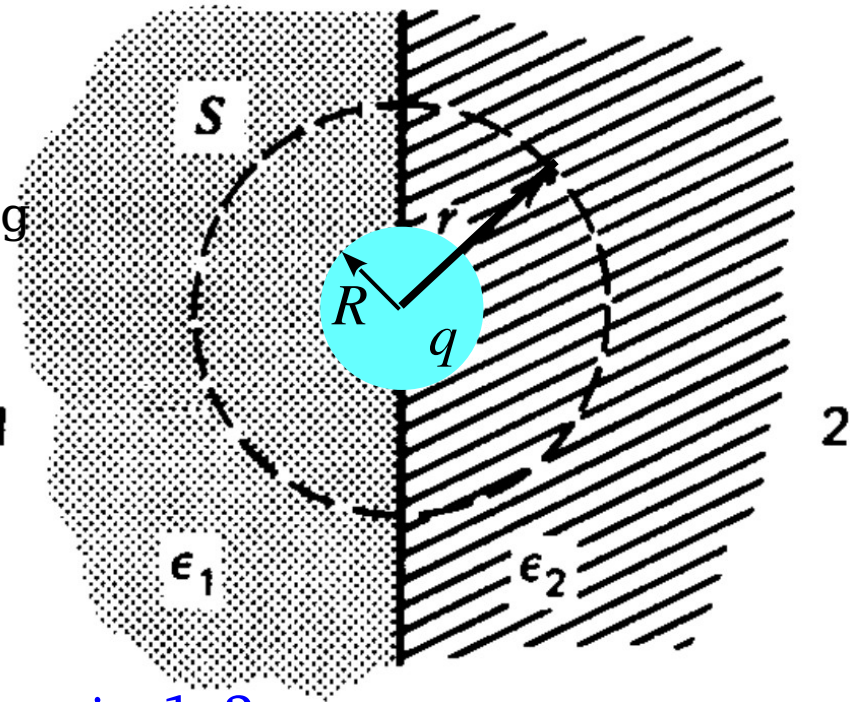
$$\Rightarrow \quad Q_f = \oint \sigma_f \, da = 2\pi R^2 (\sigma_{f,1} + \sigma_{f,2}) = q$$

$$\mathbf{P}_i = \mathbf{D}_i - \epsilon_0 \mathbf{E} = \frac{\epsilon_i - \epsilon_0}{2\pi(\epsilon_1 + \epsilon_2)}\frac{q}{r^2}\,\hat{\mathbf{r}} \quad \Rightarrow \quad \rho_b = -\nabla\cdot\mathbf{P} = 0 \text{ for } r > R$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = 0 \text{ on the interface of the 2 media, but } \sigma_{b,i} = \mathbf{P}_i\cdot(-\hat{\mathbf{r}}) = \frac{\epsilon_0 - \epsilon_i}{2\pi(\epsilon_1 + \epsilon_2)}\frac{q}{R^2}$$

$$\Rightarrow \quad q_b = \int \sigma_b \, da = 2\pi R^2 (\sigma_{b,1} + \sigma_{b,2}) = \left(\frac{2\epsilon_0}{\epsilon_1 + \epsilon_2} - 1\right) q$$

$$\Rightarrow \quad \text{Total charge } Q = q + q_b = \frac{2\epsilon_0}{\epsilon_1 + \epsilon_2}\,q \quad \Leftarrow \quad \text{giving the screening effect where the total charge appears to be less than } q$$

**Problem 4.8**: Suppose the entire region below the plane $z=0$ in the figure is filled with uniform linear dielectric material of susceptibility $\chi_e$. Calculate the force on a point charge $q$ situated a distance $d$ above the origin.

$$\rho_b = -\nabla \cdot \mathbf{P} \;\propto\; \nabla \cdot \mathbf{D} = \rho_f = 0$$

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P_z = \epsilon_0 \chi_e E_z(r, z=0^-)$$

$E_z$ is from (1) $q$ and (2) $\sigma_b$; $\;E_z = E_{z,q} + E_{z,\sigma}$

$$E_{z,q} = \frac{-1}{4\pi\epsilon_0} \frac{q\cos\theta}{r^2+d^2} = \frac{-1}{4\pi\epsilon_0} \frac{q\,d}{(r^2+d^2)^{3/2}}$$

$$E_{z,\sigma} = -\frac{\sigma_b}{2\,\epsilon_0} \;\Rightarrow\; \sigma_b = \epsilon_0 \chi_e E_z = -\epsilon_0 \chi_e \left( \frac{1}{4\pi\epsilon_0} \frac{q\,d}{(r^2+d^2)^{3/2}} + \frac{\sigma_b}{2\,\epsilon_0} \right)$$

$$\Rightarrow \sigma_b = -\frac{1}{2\pi} \frac{\chi_e}{\chi_e+2} \frac{q\,d}{(r^2+d^2)^{3/2}} \;\Rightarrow\; q_b = \int \sigma_b \,\mathrm{d}a = -\frac{\chi_e}{\chi_e+2} q \quad \begin{array}{l}\textit{total}\\ \text{bound charge}\end{array}$$

$$\Rightarrow \mathbf{E}_\sigma = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{\mathbb{r}}}{\mathbb{r}^2} \sigma_b \,\mathrm{d}a \;\Rightarrow\; \mathbf{E}_\sigma(\mathbf{x}=d\,\hat{\mathbf{z}}) = -\frac{1}{4\pi\epsilon_0} \frac{\chi_e}{\chi_e+2} \frac{q}{4\,d^2} \hat{\mathbf{z}}$$
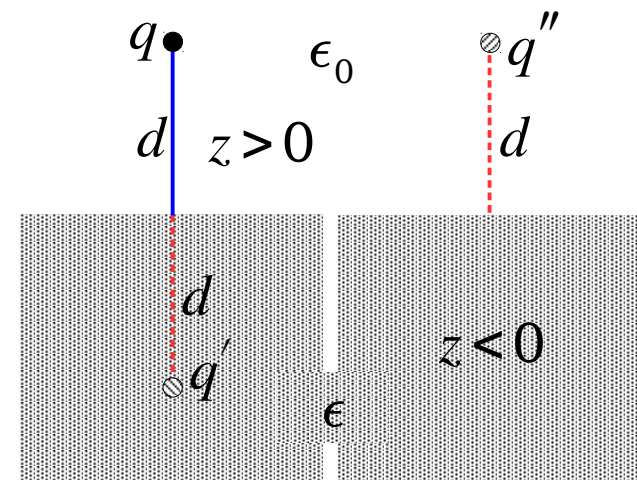
$$\Rightarrow \mathbf{F} = -\frac{1}{4\pi\epsilon_0} \frac{\chi_e}{\chi_e+2} \frac{q^2}{4\,d^2} \hat{\mathbf{z}}$$

$$\mathbf{x} = d\,\hat{\mathbf{z}}\,, \quad \mathbf{x}' = \mathbf{r} = r\left(\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}\right) \quad \Rightarrow \quad \vec{\mathbb{r}} = \mathbf{x} - \mathbf{x}' = d\,\hat{\mathbf{z}} - \mathbf{r}$$

$$\Rightarrow \quad \mathbb{r}^2 = r^2 + d^2\,, \quad \hat{\mathbb{r}} = \frac{d\,\hat{\mathbf{z}} - \mathbf{r}}{\sqrt{r^2 + d^2}}\,, \quad \text{and} \quad \sigma_b = -\frac{1}{2\pi}\frac{\chi_e}{\chi_e + 2}\frac{q\,d}{\left(r^2 + d^2\right)^{3/2}}$$

$$\Rightarrow \quad \int \frac{\hat{\mathbb{r}}}{\mathbb{r}^2}\,\sigma_b\,\mathrm{d}a = -\frac{\chi_e\,q}{2\pi\left(\chi_e + 2\right)}\int_0^\infty\int_0^{2\pi}\frac{d\,\hat{\mathbf{z}} - \mathbf{r}}{\left(r^2 + d^2\right)^{3/2}}\frac{d}{\left(r^2 + d^2\right)^{3/2}}\,r\,\mathrm{d}\theta\,\mathrm{d}r$$

$$= -\frac{\chi_e\,q\,d^2\,\hat{\mathbf{z}}}{\chi_e + 2}\int_0^\infty\frac{\mathrm{d}r^2}{2\left(r^2 + d^2\right)^3} = -\frac{\chi_e\,q\,d^2\,\hat{\mathbf{z}}}{2\left(\chi_e + 2\right)}\int_0^\infty\frac{\mathrm{d}\left(r^2 + d^2\right)}{\left(r^2 + d^2\right)^3}$$

$$= -\frac{\chi_e\,q\,d^2\,\hat{\mathbf{z}}}{2\left(\chi_e + 2\right)}\cdot\left(-\frac{1}{2}\right)\cdot\frac{1}{\left(r^2 + d^2\right)^2}\bigg|_0^\infty$$

$$= \frac{\chi_e\,q\,d^2\,\hat{\mathbf{z}}}{4\left(\chi_e + 2\right)}\left(\frac{1}{\infty^4} - \frac{1}{d^4}\right) = -\frac{\chi_e}{\chi_e + 2}\frac{q}{4\,d^2}\,\hat{\mathbf{z}}$$

As in the conducting plane, it is nicer to use the *method of images*. If we replace the *dielectric* by a single point charge $q'$ at the image position $(r=0,-d)$, from the area $z>0$, and $q''$ at $(r=0,+d)$ from the area $z<0$, then
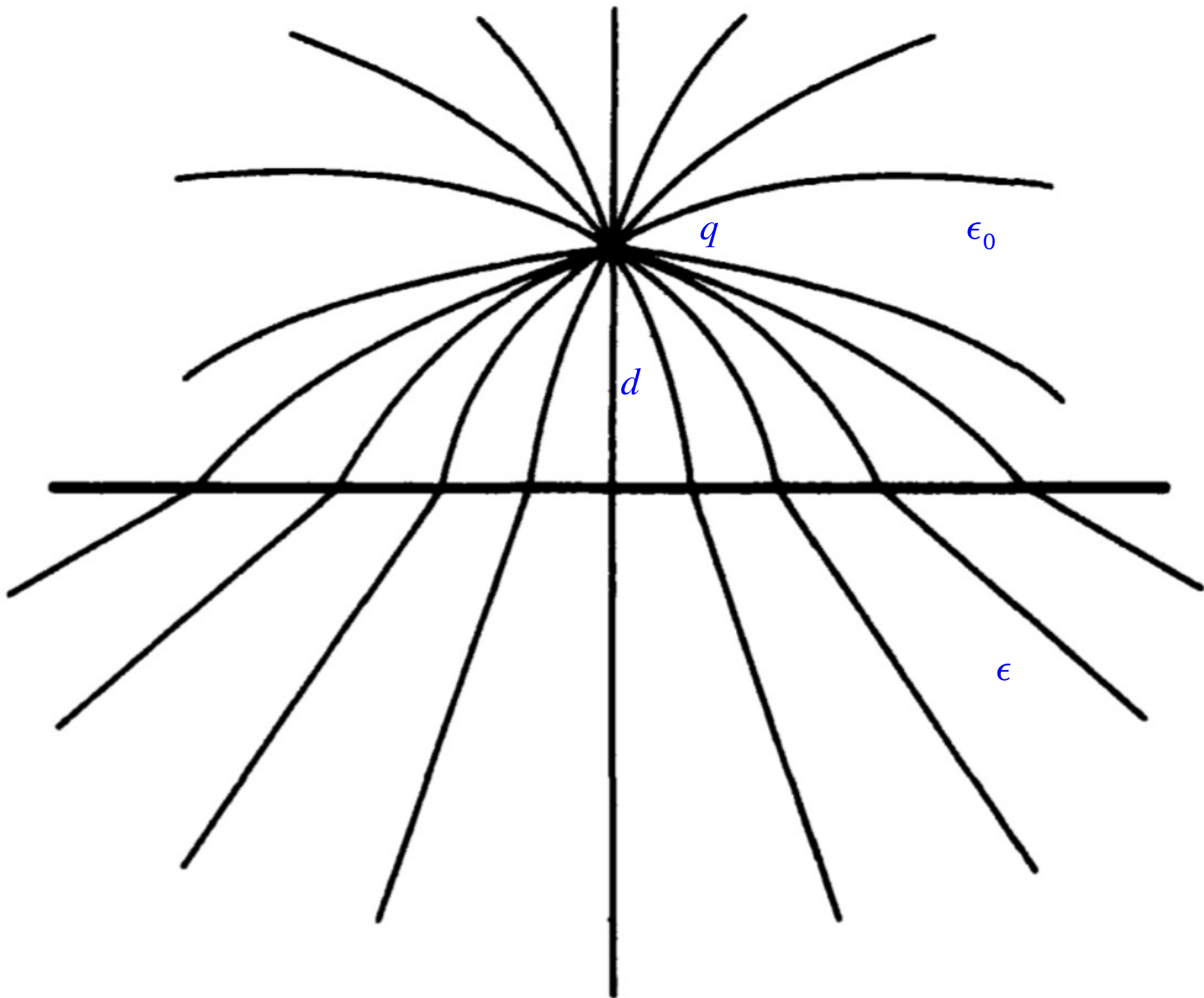


$$\Phi = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\left(\dfrac{q}{\sqrt{r^2+(z-d)^2}}+\dfrac{q'}{\sqrt{r^2+(z+d)^2}}\right) & \text{for } z>0 \\[2em] \dfrac{1}{4\pi\epsilon}\dfrac{q''}{\sqrt{r^2+(z-d)^2}}, \quad \text{where } \epsilon=\epsilon_r\,\epsilon_0 & \text{for } z<0 \end{cases}$$

$$D_z(r,0^+)=D_z(r,0^-)$$
$$\Rightarrow \quad E_r(r,0^+)=E_r(r,0^-)$$

boundary condition

$$\Rightarrow \begin{array}{l} \epsilon_0\,E_z(r,0^+)=\epsilon\,E_z(r,0^-) \\ E_r(r,0^+)=E_r(r,0^-) \end{array} \Rightarrow \begin{array}{l} -\epsilon_0\,\partial_z\,\Phi(r,0^+)=-\epsilon\,\partial_z\,\Phi(r,0^-) \\ -\partial_r\,\Phi(r,0^+)=-\partial_r\,\Phi(r,0^-) \end{array} \Rightarrow \begin{array}{l} q-q'=q'' \\ \epsilon_r(q+q')=q'' \end{array}$$

$$\Rightarrow \quad q'=q_b, \quad q''=2\,\frac{\chi_e+1}{\chi_e+2}\,q \quad \Rightarrow \quad \mathbf{F}=\frac{1}{4\pi\epsilon_0}\frac{q\,q_b}{(2d)^2}\,\hat{\mathbf{z}}=-\frac{1}{4\pi\epsilon_0}\frac{\chi_e}{\chi_e+2}\frac{q^2}{4\,d^2}\,\hat{\mathbf{z}}$$

Selected problems: 9, 10, 13, 15, 21, 24, 38

# Energy in Dielectric Systems

- It takes work to charge up a capacitor: $W = \dfrac{1}{2} C V^2$

- If the capacitor is filled with linear dielectric, its capacitance exceeds the vacuum value by a factor of the dielectric constant, $C = \epsilon_r \, C_{\text{vac}}$

- The work necessary to charge a dielectric-filled capacitor is increased by the same factor. You have to pump on more (free) charge, to achieve a given potential, because part of the field is canceled off by the bound charges.

- A general stored energy formula for electrostatic systems: $W = \dfrac{\epsilon_0}{2} \displaystyle\int E^2 \, d\tau \quad (*)$

- For the $linear$-dielectric-filled capacitor the formula should be changed to

$$W = \frac{1}{2} \int \epsilon \, E^2 \, d\tau = \frac{\epsilon_0}{2} \int \epsilon_r \, E^2 \, d\tau \;\;\Rightarrow\;\; \boxed{W = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, d\tau} \quad (@)$$

<u>Proof:</u> As $\rho_f$ is increased by an amount $\delta\rho_f$, the work done on the incremental $free$ charge is

$$\delta W = \int \Phi \, \delta\rho_f \, d\tau = \int \Phi \, \nabla \cdot \delta\mathbf{D} \, d\tau \;\;\Leftarrow\;\; \nabla \cdot \mathbf{D} = \rho_f, \;\; \delta\rho_f = \nabla \cdot \delta\mathbf{D}, \;\; \mathbf{D} = \epsilon \mathbf{E}$$

$$= \int \nabla \cdot (\Phi \, \delta\mathbf{D}) \, d\tau + \int \mathbf{E} \cdot \delta\mathbf{D} \, d\tau \;\;\Leftarrow\;\; \Phi \nabla \cdot \delta\mathbf{D} = \nabla \cdot (\Phi \, \delta\mathbf{D}) - \nabla\Phi \cdot \delta\mathbf{D}$$

$$= \delta\left( \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, d\tau \right) \;\;\Leftarrow\;\; \delta\mathbf{D} \cdot \mathbf{E} = \epsilon \, \delta\mathbf{E} \cdot \mathbf{E} = \frac{1}{2} \delta(\epsilon \, E^2) = \frac{1}{2} \delta(\mathbf{D} \cdot \mathbf{E})$$

- The energy of a system here is the work required to assemble the system.

- When dielectrics are involved, there are 2 quite different ways one might construe this process:

1. Bring in all the charges (free and bound) and fix each one in its final location. Then Eq. (*) is the formula of energy to assemble the system. But this won't include the work needed in stretching and twisting the dielectric molecules.

2. With the unpolarized dielectric, bring in the free charges & allow the dielectric to respond as it fits. Then Eq. (@) is the formula of energy to assemble the system. In this case the "spring" energy is included because the force applied to the free charge depends on the disposition of the bound charge.

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Example 4.9: A sphere of radius $R$ is filled with material of dielectric constant $\epsilon_r$ and uniform embedded free charge $\rho_f$. What is the energy of this configuration?

---

$$\oint \mathbf{D} \cdot \mathrm{d}\,\boldsymbol{a} = Q_{f_{\text{env}}}$$

$$\rho_f = \frac{Q_f}{4\pi R^3/3}$$

$$\Rightarrow \quad \mathbf{D}(r) = \begin{cases} \dfrac{\rho_f}{3}\, r\,\hat{\mathbf{r}}, & r < R \\[2mm] \dfrac{\rho_f}{3}\dfrac{R^3}{r^2}\,\hat{\mathbf{r}}, & r \geq R \end{cases}$$

$$\Rightarrow \quad \mathbf{E}(r) = \begin{cases} \dfrac{\rho_f}{3\,\epsilon_0\,\epsilon_r}\, r\,\hat{\mathbf{r}}, & r < R \\[2mm] \dfrac{\rho_f}{3\,\epsilon_0}\dfrac{R^3}{r^2}\,\hat{\mathbf{r}}, & r \geq R \end{cases}$$

- The purely *electrostatic* energy (*) is

$$W_1 = \frac{\epsilon_0}{2}\left[\left(\frac{\rho_f}{3\,\epsilon_0\,\epsilon_r}\right)^2 \int_0^R r^2\, 4\pi r^2\, \mathrm{d}\,r + \left(\frac{\rho_f\, R^3}{3\,\epsilon_0}\right)^2 \int_R^\infty \frac{4\pi r^2}{r^4}\, \mathrm{d}\,r\right] = \frac{2\pi}{9\,\epsilon_0}\, \rho_f^2\, R^5\, \frac{5\,\epsilon_r^2 + 1}{5\,\epsilon_r^2}$$

- The *total* energy (@):

$$W_2 = \frac{1}{2} \left( \frac{\rho_f}{3} \frac{\rho_f}{3\,\epsilon_0\,\epsilon_r} \int_0^R r^2\, 4\,\pi\,r^2\,\mathrm{d}r + \frac{\rho_f\,R^3}{3} \frac{\rho_f\,R^3}{3\,\epsilon_0} \int_R^\infty \frac{4\,\pi\,r^2}{r^4}\,\mathrm{d}r \right)$$

$$= \frac{2\,\pi}{9\,\epsilon_0}\,\rho_f^2\,R^5\,\frac{5\,\epsilon_r+1}{5\,\epsilon_r} > W_1 \quad \Leftarrow \quad \epsilon_r > 1$$

Bring in $\mathrm{d}q$, filling out the sphere layer by layer. When we have reached radius $r'$,

work to bring $\mathrm{d}q$ in from $\infty$ to $r'$

$$\mathbf{E}(r) = \begin{cases} \dfrac{\rho_f}{3\,\epsilon_0\,\epsilon_r}\, r\,\hat{\mathbf{r}}, & r < r' \\[2mm] \dfrac{\rho_f}{3\,\epsilon_0\,\epsilon_r}\,\dfrac{r'^3}{r^2}\,\hat{\mathbf{r}}, & r' < r < R \\[2mm] \dfrac{\rho_f}{3\,\epsilon_0}\,\dfrac{r'^3}{r^2}\,\hat{\mathbf{r}}, & r \ge R \end{cases} \Rightarrow$$

$$dW = -dq \left( \int_\infty^R \mathbf{E}\cdot\mathrm{d}\boldsymbol{\ell} + \int_R^{r'} \mathbf{E}\cdot\mathrm{d}\boldsymbol{\ell} \right)$$

$$= -dq \left( \frac{\rho_f\,r'^3}{3\,\epsilon_0} \int_\infty^R \frac{\mathrm{d}r}{r^2} + \frac{\rho_f\,r'^3}{3\,\epsilon_0\,\epsilon_r} \int_R^{r'} \frac{\mathrm{d}r}{r^2} \right)$$

$$= \frac{\rho_f\,r'^3}{3\,\epsilon_0} \left[ \frac{1}{R} + \frac{1}{\epsilon_r}\left( \frac{1}{r'} - \frac{1}{R} \right) \right] dq$$

- This increases the radius ($r'$): $\mathrm{d}q = \rho_f\,4\,\pi\,r'^2\,\mathrm{d}r'$, so the *total* work done, in going from $r'=0$ to $r'=R$, is

$$W = \int \mathrm{d}W = \frac{4\,\pi\,\rho_f^2}{3\,\epsilon_0} \left( \frac{\epsilon_r - 1}{R\,\epsilon_r} \int_0^R r'^5\,\mathrm{d}r' + \frac{1}{\epsilon_r} \int_0^R r'^4\,\mathrm{d}r' \right) = \frac{2\,\pi}{9\,\epsilon_0}\,\rho_f^2\,R^5\,\frac{5\,\epsilon_r+1}{5\,\epsilon_r} = W_2$$

$\Rightarrow$ energy stored in the springs $\quad W_{\text{spring}} = W_2 - W_1 = \dfrac{2\,\pi}{45\,\epsilon\,\epsilon_r}\,\rho_f^2\,R^5\,(\epsilon_r - 1) \quad \Leftarrow \quad \epsilon \equiv \epsilon_r\,\epsilon_0$

To confirm this in an explicit model, imagine the dielectric as a collection of tiny proto-dipoles, each consisting of $+q$ & $-q$ attached to a spring of constant $k$ and equilibrium length 0, so without any field the positive & negative ends coincide. With the field turned on, the electric force is balanced by the spring force

$$q\,E = k\,d \quad \Leftarrow \quad \mathbf{E} = \frac{\rho_f}{3\,\epsilon}\mathbf{r} \quad \Rightarrow \quad k = \frac{\rho_f}{3\,\epsilon\,d^2}P\,r\,\mathrm{d}\tau \quad \Leftarrow \quad p = q\,d = P\,\mathrm{d}\tau$$

$$\Rightarrow \quad \mathrm{d}\,W_{\text{spring}} = \frac{1}{2}\,k\,d^2 = \frac{\rho_f}{6\,\epsilon}P\,r\,\mathrm{d}\tau \quad \Rightarrow \quad W_{\text{spring}} = \frac{\rho_f}{6\,\epsilon}\int P\,r\,\mathrm{d}\tau$$
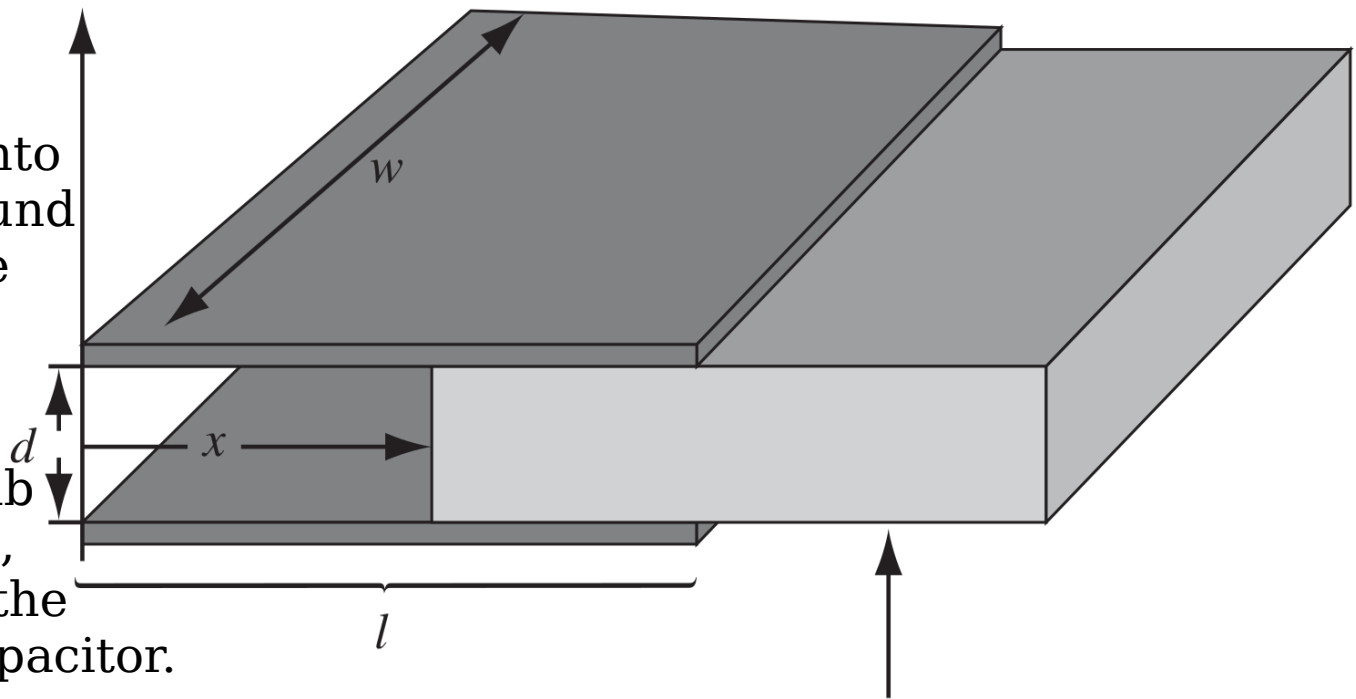
$$\mathbf{P} = \epsilon_0\,\chi_e\,\mathbf{E} = \epsilon_0\,\chi_e\,\frac{\rho_f}{3\,\epsilon}\mathbf{r} = \frac{\epsilon_r - 1}{3\,\epsilon_r}\rho_f\,\mathbf{r} \quad \Leftarrow \quad \epsilon = \epsilon_r\,\epsilon_0$$

$$\Rightarrow \quad W_{\text{spring}} = \frac{\rho_f}{6\,\epsilon}\frac{(\epsilon_r - 1)\,\rho_f}{3\,\epsilon_r}4\,\pi\int_0^R r^4\,\mathrm{d}\,r = \frac{2\,\pi}{45\,\epsilon\,\epsilon_r}\rho_f^2\,R^5\,(\epsilon_r - 1)$$

- To get Eq (@), linearity is necessasry. For nonlinear dielectrics, the eqn is false.

- For *dissipative* systems the whole notion of "stored energy" loses its meaning, because the work done depends not only on the final configuration but on *how it got there*.

- If the molecular "springs" are allowed to have some *friction*, then $W_{\text{spring}}$ can be made as large as you like.

## Forces on Dielectrics

● A dielectric is attracted into an electric field, for the bound charge tends to accumulate near the free charge of the opposite sign.

● Consider the case of a slab of linear dielectric material, partially inserted between the plates of a parallel-plate capacitor.
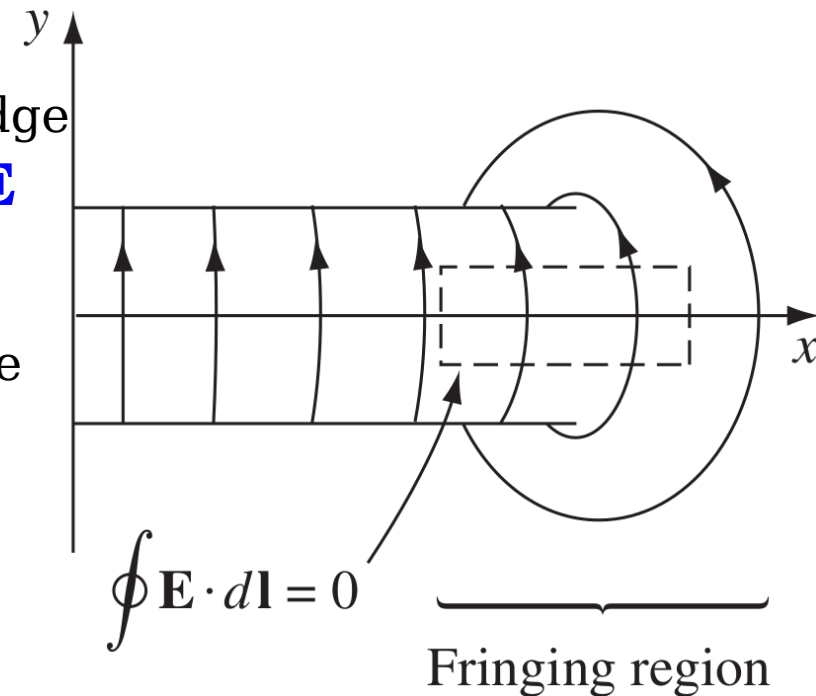
● Ideally the field is uniform inside a parallel-plate capacitor, and 0 outside. If this were true, there would be no net force on the dielectric. A **fringing field** around the edges is responsible for the effect.

● The field could *not* terminate abruptly at the edge of the capacitor, for if it did, the line integral of **E** around the closed loop would not be 0.

● It is this nonuniform fringing field that pulls the dielectric into the capacitor.

● If the dielectric is pulled out an distance $dx$

$$dW_a = F_{app}\, dx = -F\, dx \quad \Rightarrow \quad F = -\frac{dW_a}{dx}$$

$$C = \epsilon_0 \frac{w\,x}{d} + \epsilon\, \frac{w\,(\ell - x)}{d} = \frac{\epsilon_0\, w}{d}\,(\epsilon_r\, \ell - \chi_e\, x), \qquad W_a = \frac{1}{2}\, C\, V^2 = \frac{1}{2}\, \frac{Q^2}{C}$$

$$\Rightarrow \quad F = -\frac{\mathrm{d}\,W_a}{\mathrm{d}\,x} = \frac{Q^2}{2\,C^2}\, \frac{\mathrm{d}\,C}{\mathrm{d}\,x} = \frac{V^2}{2}\, \frac{\mathrm{d}\,C}{\mathrm{d}\,x} = -\frac{\epsilon_0\, \chi_e\, w}{2\,d}\, V^2 \quad \Leftarrow \quad \frac{\mathrm{d}\,C}{\mathrm{d}\,x} = -\frac{\epsilon_0\, \chi_e\, w}{d}$$

● The $-$ sign shows the force to $-x$ way; the dielectric is pulled *into* the capacitor.

● It is a common error to use the eqn with $V$ constant, rather than the one with $Q$ constant, in computing the force, $\quad F = -\dfrac{V^2}{2}\, \dfrac{\mathrm{d}\,C}{\mathrm{d}\,x} \quad \Leftarrow \quad$ wrong sign in computing total force by the system

● It is *possible* to maintain the capacitor at a fixed potential, by connecting it up to a battery. But in that case the *battery also does work* as the dielectric moves;

$$\mathrm{d}\,W_a = F_{\mathrm{app}}\, \mathrm{d}\,x + V\, \mathrm{d}\,Q \quad \Rightarrow \quad F = -\frac{\mathrm{d}\,W_a}{\mathrm{d}\,x} + V\, \frac{\mathrm{d}\,Q}{\mathrm{d}\,x} = -\frac{V^2}{2}\, \frac{\mathrm{d}\,C}{\mathrm{d}\,x} + V^2 \frac{\mathrm{d}\,C}{\mathrm{d}\,x} = \frac{V^2}{2}\, \frac{\mathrm{d}\,C}{\mathrm{d}\,x}$$

● The force on the dielectric cannot depend on whether you hold $Q$ constant or $V$ constant—it is determined entirely by the distribution of charge, free and bound.

● We are able to determine the force *without knowing anything about the fringing fields that are ultimately responsible for it*, by $\nabla \times \mathbf{E} = 0$.

● The energy stored in the fringing fields themselves stays constant, as the slab moves; what does change is the energy well inside the capacitor, where the field is nice and uniform.

<u>Example</u>: **Force on Plates of Parallel Capacitor — Constant Charge**:
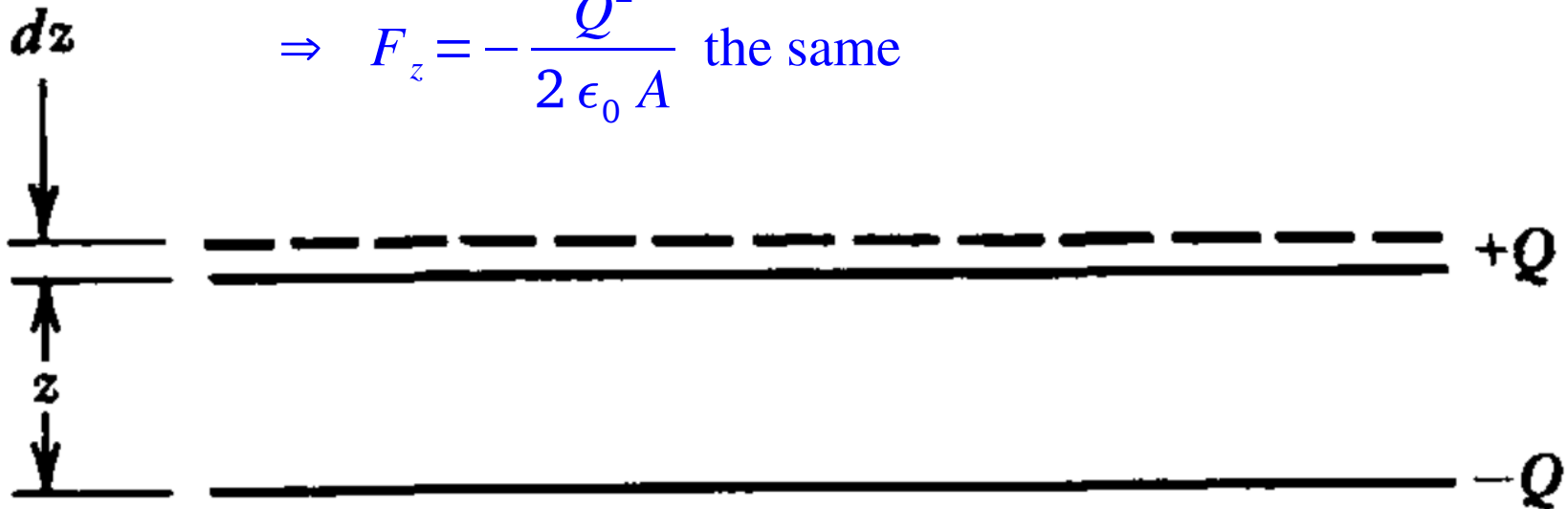
1: $Q = \text{const} \quad \Rightarrow \quad U = W_a = \dfrac{Q^2}{2\,C} \quad \Leftarrow \quad C = \epsilon_0 \dfrac{A}{z}$

$\Rightarrow \quad F_z \equiv -\dfrac{\mathrm{d}\,U}{\mathrm{d}\,z}\Big|_Q = -\dfrac{Q^2}{2}\dfrac{\mathrm{d}}{\mathrm{d}\,z}\dfrac{1}{C} = -\dfrac{Q^2}{2\,\epsilon_0\,A}$

2: Consider a small virtual displacement $\mathrm{d}z$ of the upper plate of the capacitor while keeping $Q$ constant (the system isolated).

$$-F_z\,\mathrm{d}\,z \Leftarrow -\mathrm{d}\,W = +\mathrm{d}\,U = U(z+\mathrm{d}\,z) - U(z) = \dfrac{Q^2}{2}\left(\dfrac{1}{C(z+\mathrm{d}\,z)} - \dfrac{1}{C(z)}\right)$$

$$= \dfrac{Q^2}{2}\left(\dfrac{z+\mathrm{d}\,z}{\epsilon_0\,A} - \dfrac{z}{\epsilon_0\,A}\right) = \dfrac{Q^2}{2\,\epsilon_0\,A}\,\mathrm{d}\,z$$

$$\Rightarrow \quad F_z = -\dfrac{Q^2}{2\,\epsilon_0\,A} \quad \text{the same}$$

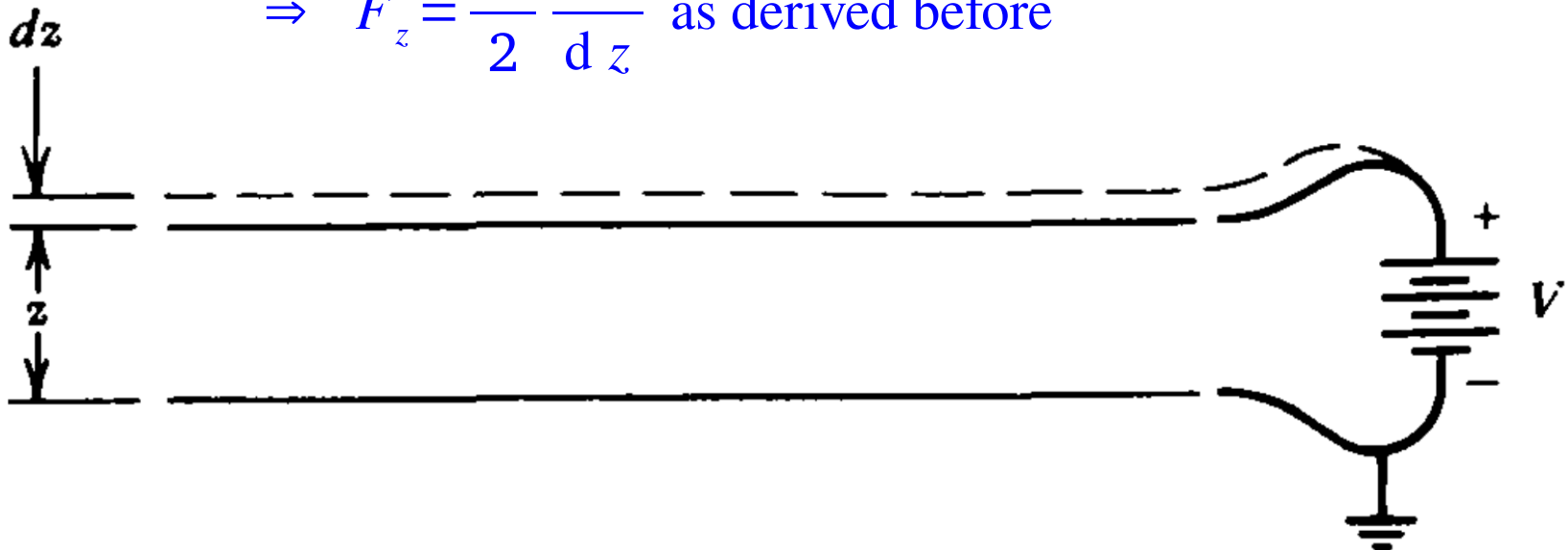Example: **Force on Plates of Parallel Capacitor — Constant Charge**:

1: $V = \text{const} \quad \Rightarrow \quad U = W_a = \dfrac{1}{2} C V^2 \quad \Leftarrow \quad C = \epsilon_0 \dfrac{A}{z}$

$\Rightarrow \quad F_z \equiv \dfrac{d U}{d z}\bigg|_V = \dfrac{V^2}{2} \dfrac{d C}{d z} = \dfrac{V^2}{2} \left( -\dfrac{\epsilon_0 A}{z^2} \right) = -\dfrac{C^2 V^2}{2 \epsilon_0 A} = -\dfrac{Q^2}{2 \epsilon_0 A}$
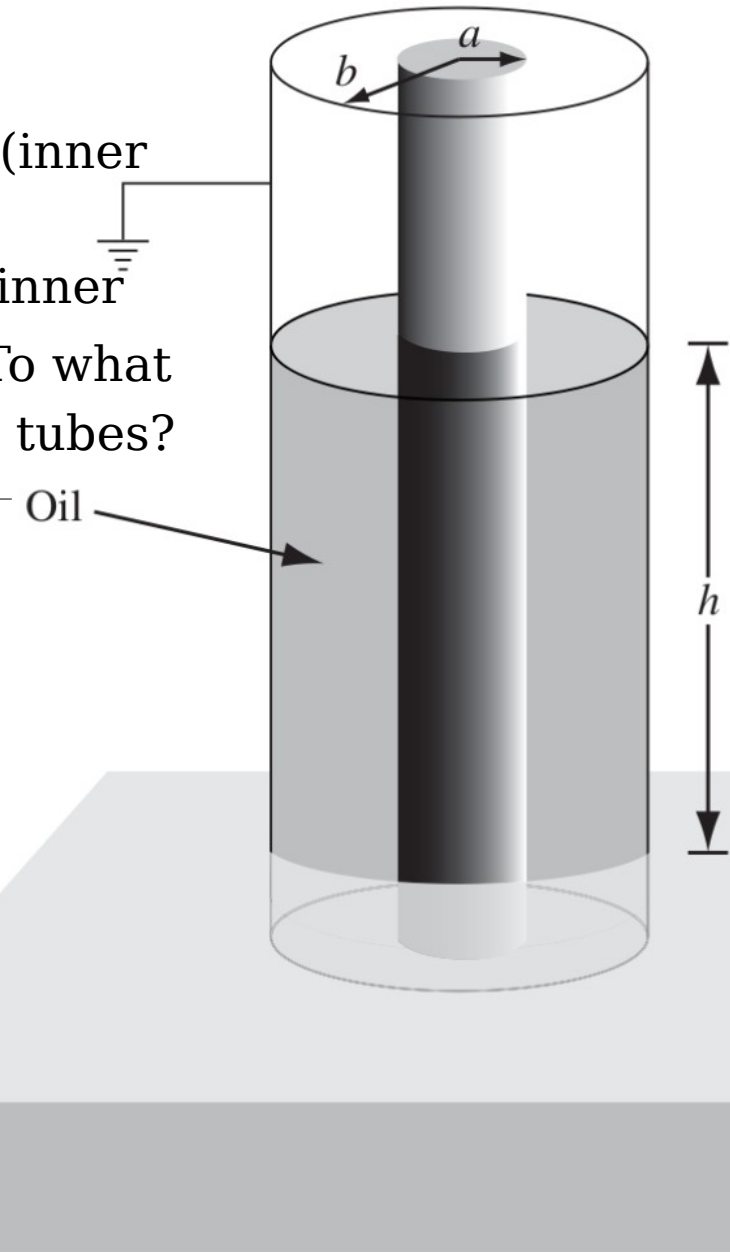
2: Consider a small virtual displacement $dz$ of the upper plate of the capacitor while keeping $Q$ constant (the system isolated).

$$\dfrac{V^2}{2} d C \Leftarrow d U = - F_z d z + V d Q = - F_z d z + V^2 d C$$

$$\Rightarrow \quad F_z = \dfrac{V^2}{2} \dfrac{d C}{d z} \quad \text{as derived before}$$

Problem 4.28: 2 long coaxial cylindrical metal tubes (inner radius $a$, outer radius $b$) stand vertically in a tank of dielectric oil (susceptibility $\chi_e$, mass density $\rho$). The inner tube is maintained at $V$, the outer one is grounded. To what height ($h$) does the oil rise, in the space between the tubes?

Oil

$h$

To find the capacitance $C$ as a function of $h$ out of $L$

$$E_{air} = \frac{\lambda_{air}}{2\pi\epsilon_0 r}, \quad E_{oil} = \frac{\lambda_{oil}}{2\pi\epsilon r}$$

$$\Rightarrow \quad V = \frac{\lambda_{air}}{2\pi\epsilon_0}\ln\frac{b}{a} = \frac{\lambda_{oil}}{2\pi\epsilon}\ln\frac{b}{a}$$

$$\Rightarrow \quad C_{oil} = \frac{2\pi\epsilon h}{\ln b - \ln a}, \quad C_{air} = \frac{2\pi\epsilon_0(L-h)}{\ln b - \ln a}$$

$$\Rightarrow \quad C = C_{air} + C_{oil} = 2\pi\frac{\epsilon_0(\chi_e h + L)}{\ln b - \ln a}$$

$$\Rightarrow \quad F = \frac{V^2}{2}\frac{dC}{dh} = \frac{\pi\epsilon_0\chi_e V^2}{\ln b - \ln a} = -F_{app} = -F_{mg} = \rho\pi(b^2-a^2)h g$$

$$\Rightarrow \quad h = \frac{\epsilon_0\chi_e V^2}{\rho g(b^2-a^2)(\ln b - \ln a)}$$