Chapter 3 Potentials

Laplace's Equation

Introduction

- The primary task of electrostatics is to find the electric field of a given stationary charge distribution: $\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\mathbb{r}}}{\mathbb{r}^2}\rho(\mathbf{r}')\,d\tau'$
- Unfortunately, integrals of this type are difficult to calculate for except the simplest charge configurations. So the best strategy is to calculate the potential,

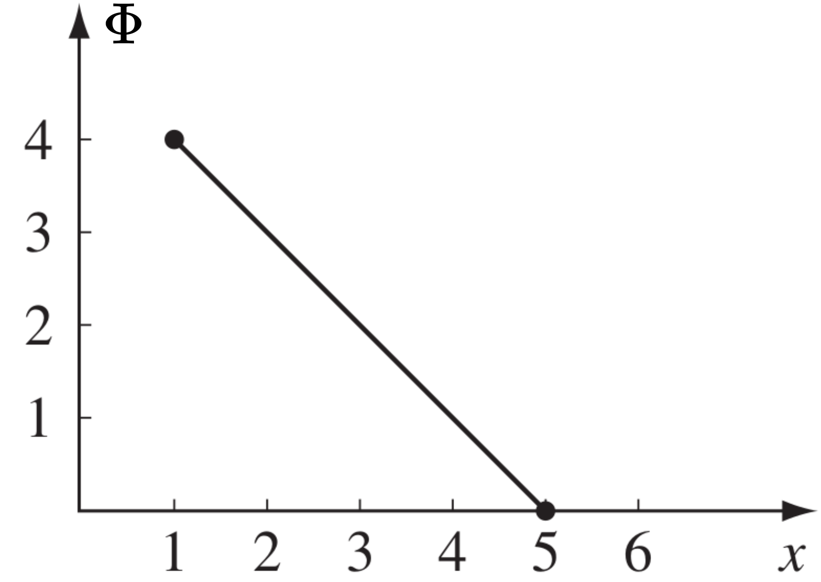
$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathbb{r}}\,d\tau'$$

- Even this integral is not easy to handle analytically. Moreover, in problems involving conductors, charge is free to move around, the only certain thing is the *total* charge of each conductor.
- It is fruitful to recast the problem in differential form, ie, Poisson's equation, $\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$
- We are often interested in finding the potential in a region where $\rho=0$. In this case, Poisson's equation reduces to Laplace's equation:

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

- Its solutions are called **harmonic functions**.

Laplace's Equation in 1d



- $\frac{d^2\Phi}{dx^2}=0 \quad \Rightarrow \quad \Phi(x)=m\,x+b$ straight line
- It contains 2 undetermined constants (m & b), as is appropriate for a 2nd-order (ordinary) differential equation.
- 2 features of this result:

1. $\Phi(x)$ is the *average* of $\Phi(x+a)$ and $\Phi(x-a)$, for any a: $\Phi(x)=\frac{\Phi(x+a)+\Phi(x-a)}{2}$

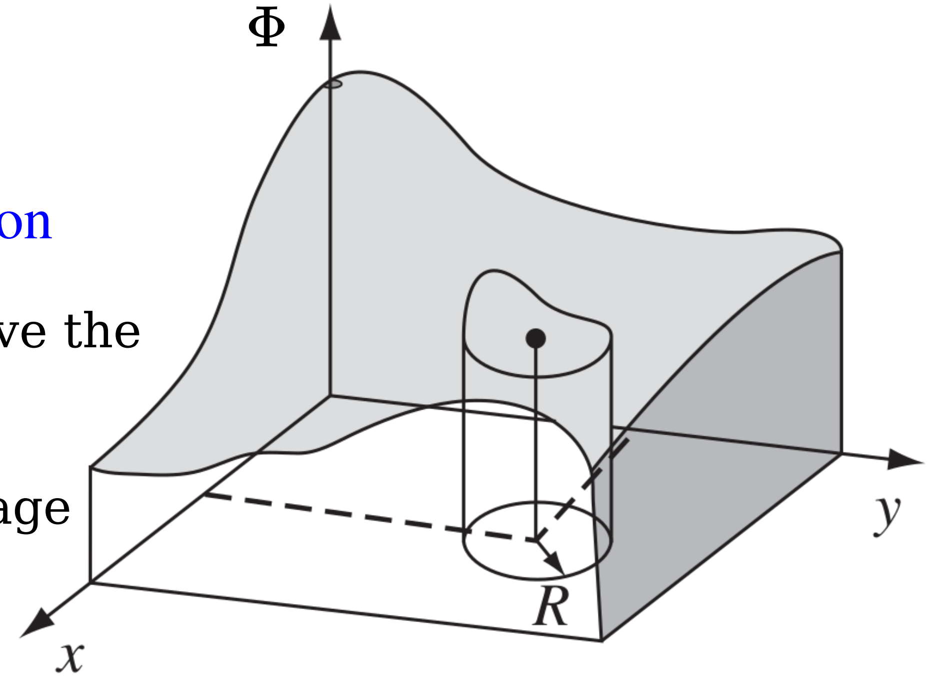
- Laplace's equation is a kind of averaging instruction; it tells you to assign to the point x the average of the values to the left and to the right of x.

2. Laplace's equation tolerates *no local maxima or minima*; extreme values of Φ must occur at the end points.

- If there *were* a local maximum, Φ would be greater at that point than on either side, and therefore could not be the average.
- One expects the 2nd derivative to be negative at a maximum and positive at a minimum. Since Laplace's equation requires, on the contrary, that the 2nd derivative is 0, it seems reasonable that solutions should exhibit no extrema.

Laplace's Equation in 2d

- $\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$ *partial* differential equation
- Harmonic functions (the solution) in 2d have the same properties in 1d:



1. The value of Φ at a point (x, y) is the average of those *around* the point.

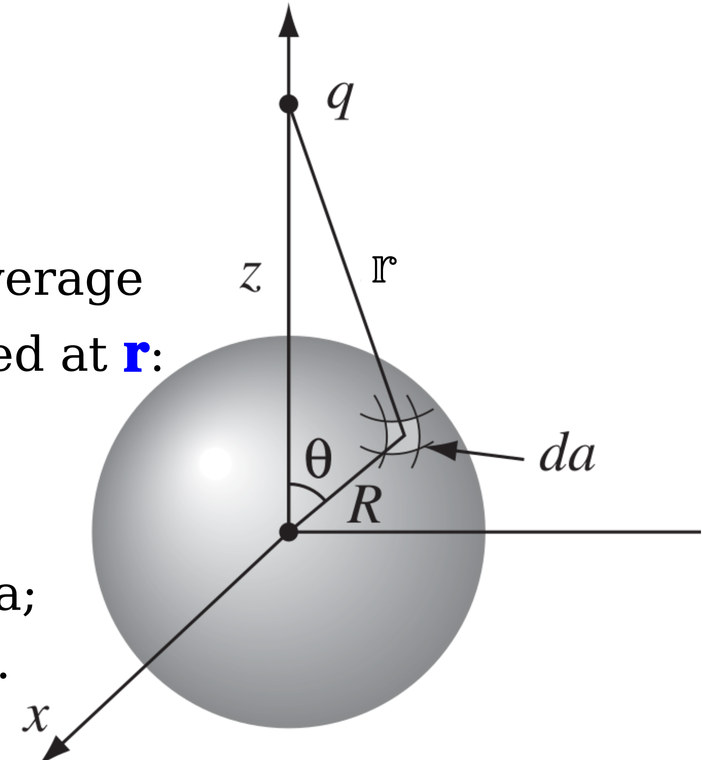
- Draw a circle of any radius R about the point (x, y), the average value of Φ on the circle is equal to the value at the center: $\Phi(x, y) = \frac{1}{2\pi R} \oint_{\text{circle}} \Phi \, d\ell$
- This suggests the **method of relaxation** for computing: Start with specified values for Φ at the boundary, and guesses for Φ on a grid of interior points, then reassign to each point the average of its nearest neighbors iteratively until it forms a numerical solution to Laplace's equation.

2. Φ has no local maxima or minima; all extrema occur at the boundaries.

- From a geometrical point of view, just as a straight line is the shortest distance between 2 points, so a harmonic function in 2d minimizes the surface area spanning the given boundary line.

Laplace's Equation in 3d

- The same 2 properties of the solutions remain true:



1. **Mean value theorem**: The value of Φ at $\mathbf{r}$ is the average value of Φ over a spherical surface of radius R centered at $\mathbf{r}$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} \Phi \, \mathrm{d}a$$

2. As a consequence, Φ has no local maxima or minima; the extreme values of Φ must occur at the boundaries.

<u>Proof</u>: find the average Φ over a spherical surface of R due to a point charge q located outside the sphere: $\Phi(\text{surface}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathbb{r}} \;\Leftarrow\; \mathbb{r}^2 = z^2 + R^2 - 2zR\cos\theta$

$$\Rightarrow \quad \Phi_{\text{ave}} = \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\int \frac{R^2 \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\phi}{\sqrt{z^2+R^2-2zR\cos\theta}} = \frac{q}{4\pi\epsilon_0}\left.\frac{\sqrt{z^2+R^2-2zR\cos\theta}}{2zR}\right|_0^{\pi}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{(z+R)-(z-R)}{2zR} = \frac{q}{4\pi\epsilon_0 z}$$

exact the potential due to q at the *center* of the sphere! By the superposition principle, the same goes for any *collection* of charges outside the sphere: their average potential over the sphere is equal to the net potential at the center.

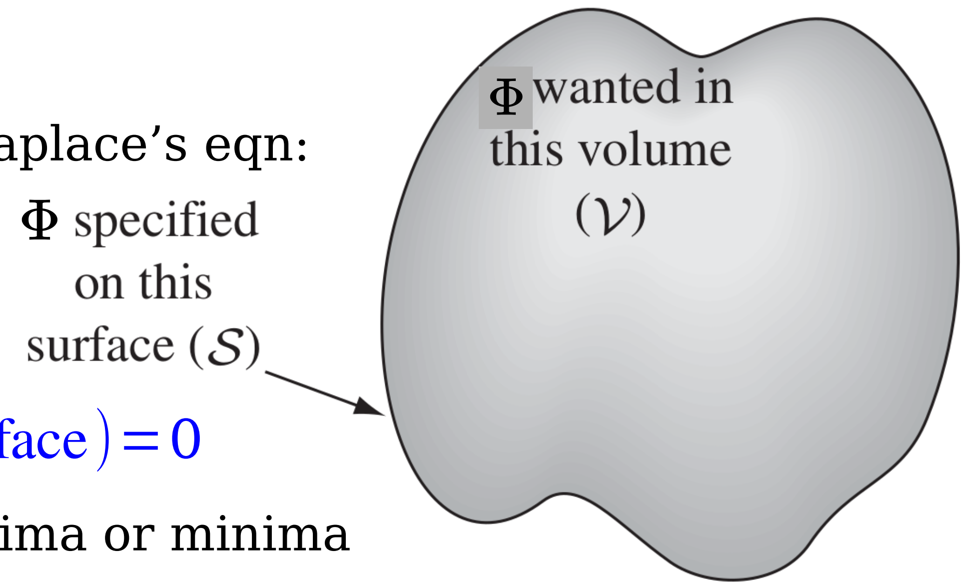
Boundary Conditions and Uniqueness Theorems

- Laplace's equation does not by itself determine Φ; in addition, suitable boundary conditions must be supplied.

- What are appropriate boundary conditions, sufficient to determine the answer and yet not so strong as to generate inconsistencies?

- The *proof* that a proposed set of boundary conditions will suffice is usually presented in the form of a **uniqueness theorem**.

1st uniqueness theorem: (Dirichlet boundary conditions)
The solution to Laplace's equation in some volume $\mathcal{V}$ is uniquely determined if Φ is specified on the boundary surface $\mathcal{S}$.



Proof: Suppose there were 2 solutions to Laplace's eqn:

$$\nabla^2\Phi = 0 \;\text{ and }\; \nabla^2\Phi' = 0$$

the difference $\Phi'' \equiv \Phi' - \Phi$

$$\Rightarrow \quad \nabla^2\Phi'' = \nabla^2\Phi' - \nabla^2\Phi = 0 \;\text{ and }\; \Phi''(\text{surface}) = 0$$

But Laplace's equation allows no local maxima or minima —all extrema occur on the boundaries,

$$\Rightarrow \quad \Phi''_{\max} = \Phi''_{\min} = 0 \quad \Rightarrow \quad \Phi'' = 0 \quad \Rightarrow \quad \Phi' = \Phi$$

Example 3.1

- It doesn't matter *how* you come by your solution; if (a) it satisfies Laplace's equation and (b) it has the correct value on the boundaries, then it's *right*.

- If there was some charge inside the region in question, in which case Φ obeys Poisson's equation, the argument is the same,

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \text{ and } \nabla^2 \Phi' = -\frac{\rho}{\epsilon_0}$$

$$\Rightarrow \quad \nabla^2 \Phi'' = \nabla^2 \Phi' - \nabla^2 \Phi = -\frac{\rho}{\epsilon_0} + \frac{\rho}{\epsilon_0} = 0 \quad \Rightarrow \quad \Phi'' = \Phi' - \Phi = 0 \quad \Rightarrow \quad \Phi' = \Phi$$

Corollary:
The potential in a volume $\mathcal{V}$ is uniquely determined if
(a) the charge density throughout the region, and
(b) the value of Φ on all boundaries,
are specified.

Conductors and the 2^{nd} Uniqueness Theorem

- The *simplest* way to set the boundary conditions for an electrostatic problem is to specify the value of Φ on all surfaces surrounding the region of interest.

- In the laboratory, we have conductors connected to batteries, which maintain a given potential, or to **ground** for $\Phi=0$.

- There are other circumstances in which we do not know the *potential* at the boundary, but rather the *charges* on various conducting surfaces.

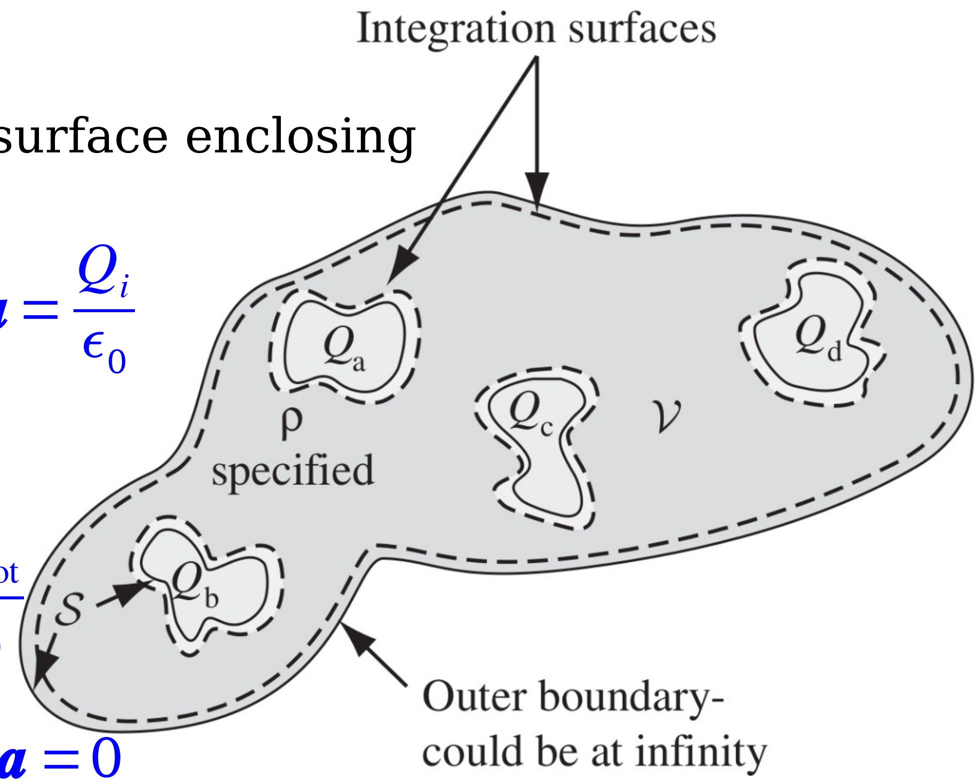
- Assume there is some specified charge density ρ in the region between the conductors. Is the electric field now uniquely determined? Or are there a number of different ways the charges could arrange themselves on their respective conductors, each leading to a different field?

2^{nd} uniqueness theorem: (Neumann boundary condition)
In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is given.

Proof: Suppose there are 2 fields satisfying the conditions of the problem. Both obey Gauss's law in the space between the conductors:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \cdot \mathbf{E}' = \frac{\rho}{\epsilon_0}$$

And both obey Gauss's law for a Gaussian surface enclosing each conductor:



$$\oint_{i^{\text{th}}\ \text{conducting surface}} \mathbf{E}\cdot d\boldsymbol{a} = \frac{Q_i}{\epsilon_0}, \quad \oint_{i^{\text{th}}\ \text{conducting surface}} \mathbf{E}'\cdot d\boldsymbol{a} = \frac{Q_i}{\epsilon_0}$$

For the outer boundary

$$\oint_{\text{outer boundary}} \mathbf{E}\cdot d\boldsymbol{a} = \frac{Q_{\text{tot}}}{\epsilon_0}, \quad \oint_{\text{outer boundary}} \mathbf{E}'\cdot d\boldsymbol{a} = \frac{Q_{\text{tot}}}{\epsilon_0}$$

$$\text{Let } \mathbf{E}'' \equiv \mathbf{E}' - \mathbf{E} \quad \Rightarrow \quad \nabla\cdot\mathbf{E}'' = 0, \quad \oint \mathbf{E}''\cdot d\boldsymbol{a} = 0$$

We don't know how the charge Q_i distributes itself over the i^{th} conductor, but we do know that each conductor is an equipotential, and hence Φ'' is a *constant* over each conducting surface.

$$\nabla\cdot(\Phi''\mathbf{E}'') = \Phi''\,\nabla\cdot\mathbf{E}'' + \mathbf{E}''\cdot\nabla\Phi'' = -E''^2 \quad \Leftarrow \quad \mathbf{E}'' = -\nabla\Phi''$$

$$\Rightarrow \quad -\int_{\mathcal{V}} E''^2\, d\tau \Leftarrow \int_{\mathcal{V}} \nabla\cdot(\Phi''\mathbf{E}'')\, d\tau = \oint_{\mathcal{S}} \Phi''\mathbf{E}''\cdot d\boldsymbol{a}$$

$$\Phi_i'' = \text{const} \quad \Rightarrow \oint_{\mathcal{S}} \Phi''\mathbf{E}''\cdot d\boldsymbol{a} = \sum \Phi_i'' \oint_{\mathcal{S}_i} \mathbf{E}''\cdot d\boldsymbol{a} = 0 \quad \Rightarrow \int_{\mathcal{V}} E''^2\, d\tau = 0$$

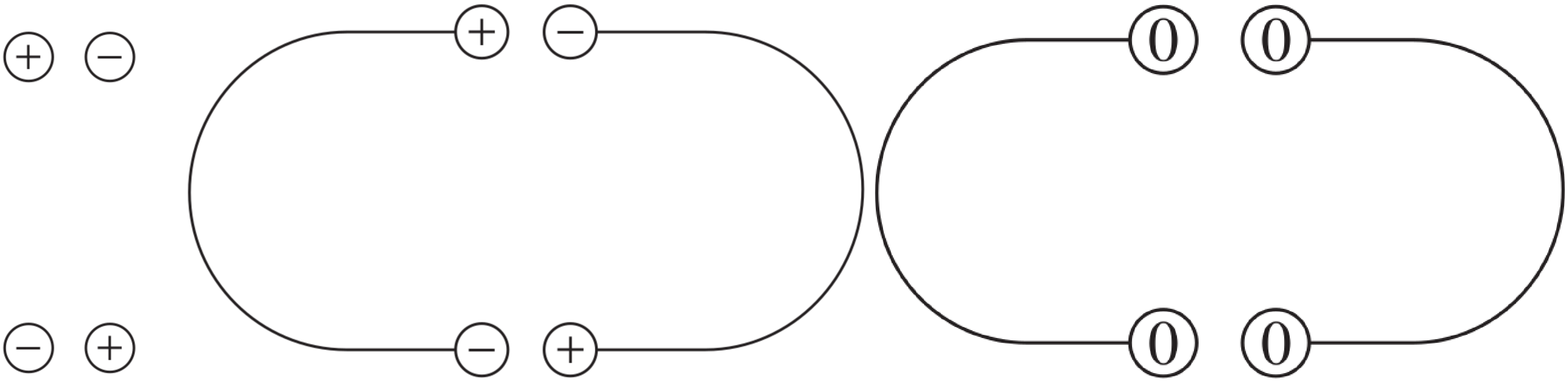
$$\Rightarrow \quad E'' = 0 \ \text{ everywhere } \Rightarrow \quad \mathbf{E} = \mathbf{E}'$$

- Consider Purcell's example: 4 conductors with charges $\pm Q$, situated so that the plusses are near the minuses. what happens if we join them in pairs, by tiny wire?

- One might guess that *nothing* will happen—the configuration looks stable. But it's wrong and *impossible*.

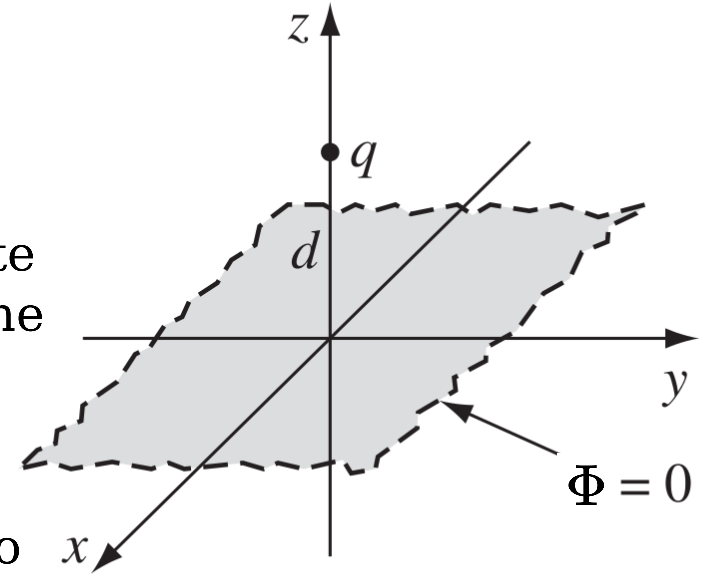
- For there are now 2 conductors, and the total charge on each is 0. One possible way to distribute 0 charge over these conductors is to have no accumulation of charge anywhere, and hence 0 field everywhere.

- By the 2nd uniqueness theorem, this must be the solution: The charge will flow down the tiny wires, canceling itself off.



The Method of Images

The Classic Image Problem



- A point charge q is held a distance d above an infinite grounded conducting plane. What is the potential in the region above the plane?

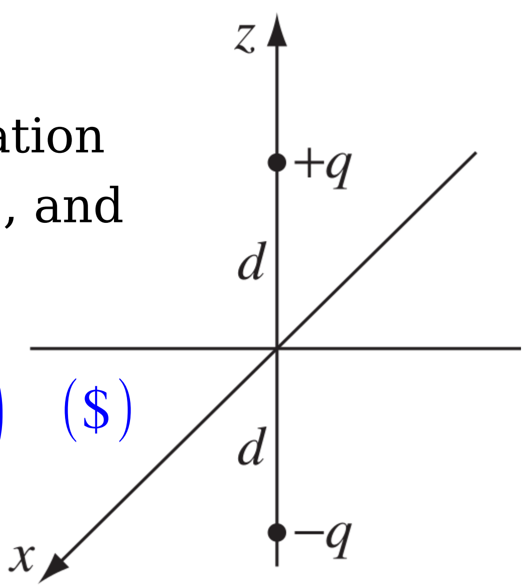
- Our problem is to solve Poisson's eqn in the region $z>0$, with a single point charge q at $(0, 0, d)$, subject to the boundary conditions:

  1. $\Phi=0$ when $z=0$ (the conducting plane is grounded),

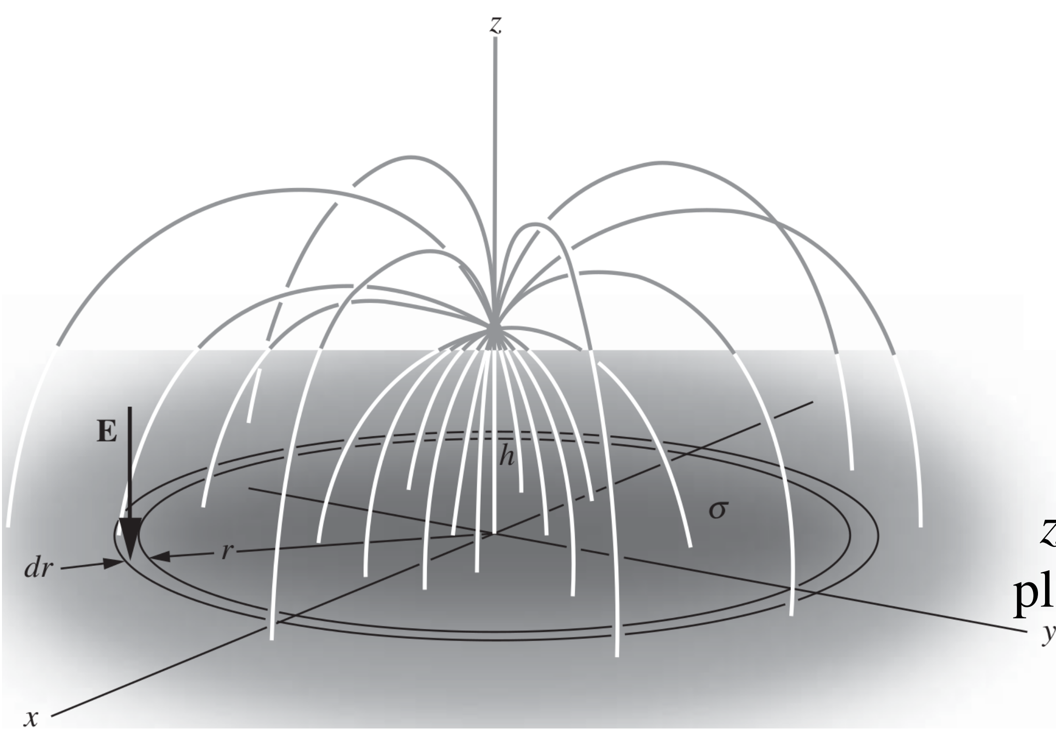
  2. $\Phi\to 0$ far from the charge (for $x^2+y^2+z^2 \gg d^2$.)

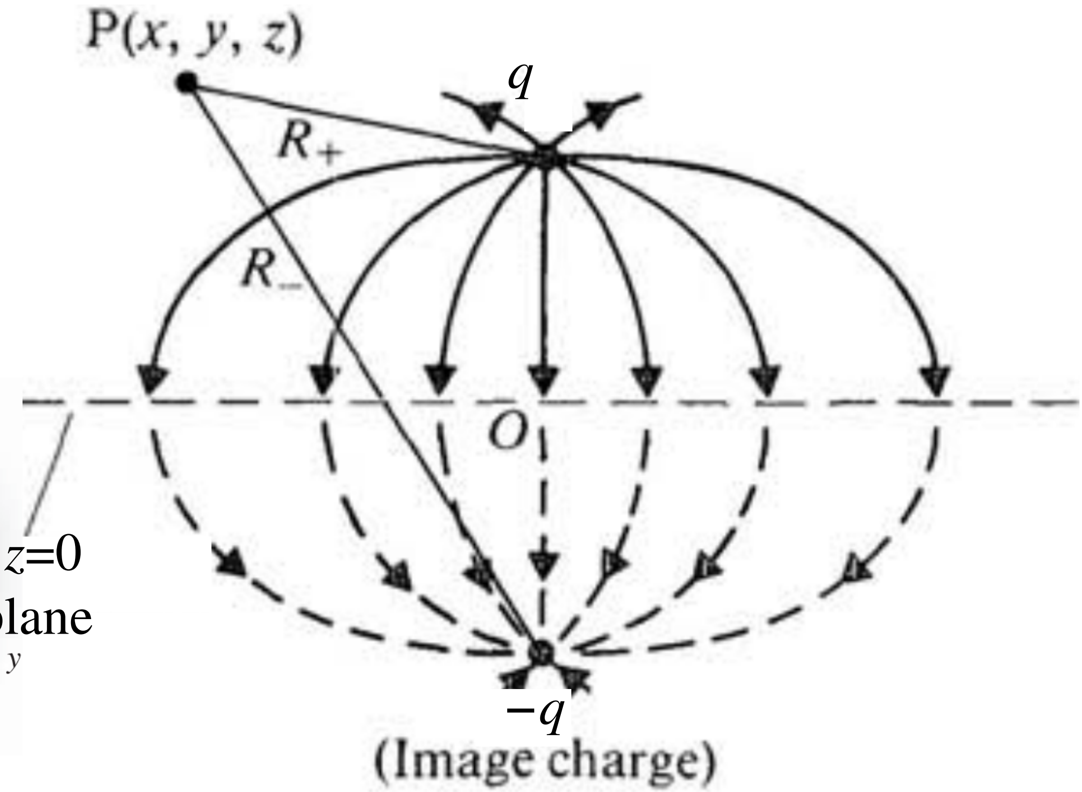
- The 1st uniqueness theorem guarantees that there is only one function that meets these requirements.

- Consider a *completely different* situation. This new configuration consists of 2 point charges, $+q$ at $(0, 0, d)$ and $-q$ at $(0, 0, -d)$, and no conducting plane



$$\Phi(x, y, z) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right) \quad (\$)$$





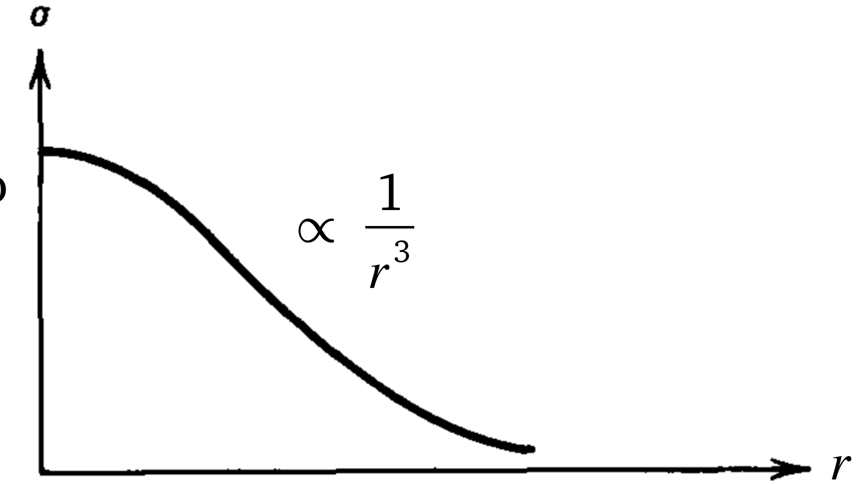
- It follows: 1. $\Phi(z=0)=0$, 2. $\Phi\to 0$ for $x^2+y^2+z^2 \gg d^2$; and the only charge in the region $z>0$ is the point charge q at $(0,0,d)$. But these are precisely the conditions of the original problem!

- Evidently the 2nd configuration happens to produce exactly the same potential as the 1st configuration, in the "upper" region $z \geq 0$. So the potential of a point charge above an infinite grounded conductor is given by ($) for $z \geq 0$.

- The uniqueness theorem plays crucial role here: without it, no one would believe this solution, since it was obtained for a completely different charge distribution. But the theorem certifies it: If it satisfies Poisson's equation in the region of interest, with the correct value at the boundaries, then it must be right.

Induced Surface Charge

- Knowing the potential, it is straightforward to compute the surface charge induced on the conductor:



$$\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial z}\Big|_{z=0} \quad \Leftarrow \quad \sigma = -\epsilon_0 \frac{\partial \Phi}{\partial n}$$

$$= -\frac{\epsilon_0}{4\pi\epsilon_0}\left(\frac{-q(z-d)}{\left(x^2+y^2+(z-d)^2\right)^{3/2}} + \frac{q(z+d)}{\left(x^2+y^2+(z+d)^2\right)^{3/2}}\right)_{z=0} = -\frac{q\,d}{2\pi\left(x^2+y^2+d^2\right)^{3/2}}$$

- As expected, the induced charge is negative (assuming q is positive) and greatest at $x=y=0$.

- The *total* induced charge

$$Q = \int \sigma \, \mathrm{d}a \quad \Rightarrow \quad \mathrm{d}a = r\,\mathrm{d}r\,\mathrm{d}\phi, \quad \sigma = \frac{-q\,d}{2\pi\left(r^2+d^2\right)^{3/2}} \quad \Leftarrow \quad r^2 = x^2+y^2 \quad \text{in the polar coordinates}$$

$$= \int_0^\infty \int_0^{2\pi} \frac{-q\,d}{2\pi\left(r^2+d^2\right)^{3/2}}\, r\,\mathrm{d}\phi\,\mathrm{d}r = \frac{q\,d}{\sqrt{r^2+d^2}}\Bigg|_0^\infty = -q \quad \text{as expected}$$

- q is attracted toward the plane, because of the negative induced charge.

Force and Energy

- Since the potential around q is the same as in the image problem (with q & $-q$ but no conductor), so also is the field, and the force: $\mathbf{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\hat{\mathbf{z}}$
- Energy is *not* the same in the 2 problems. With the 2 point charges and no conductor, $W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d}$. But for a single charge and conducting plane, the energy is *half* of this: $W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}$
- For $W = \frac{\epsilon_0}{2}\int E^2\, \mathrm{d}\tau \Leftarrow \frac{1}{2}\int \rho\,\Phi\,\mathrm{d}\tau$, in the image case, both the upper region ($z>0$) and the lower region ($z<0$) contribute—and by symmetry they contribute equally. But in the original case, only the upper region contains a nonzero field, and hence the energy is half as great.

- One could also determine the energy by calculating the work required to bring q in from ∞

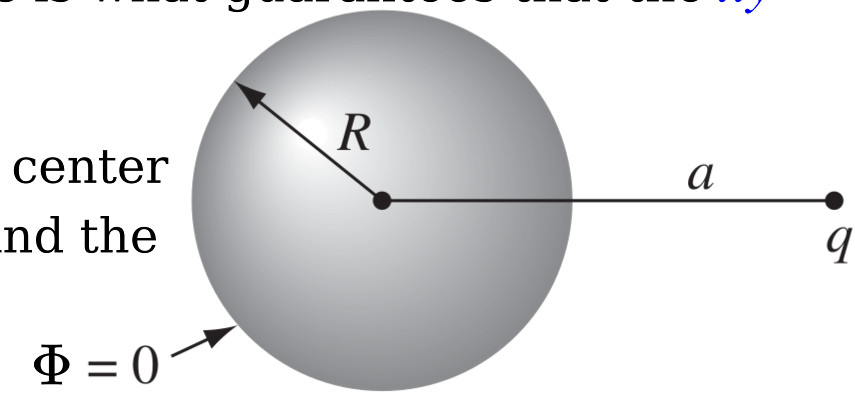
$$W = \int_\infty^d \mathbf{F}\cdot \mathrm{d}\boldsymbol{\ell} = \int_\infty^d \frac{1}{4\pi\epsilon_0}\frac{q^2}{4z^2}\hat{\mathbf{z}}\cdot \mathrm{d}z\,\hat{\mathbf{z}} = \frac{q^2}{4\pi\epsilon_0}\left(-\frac{1}{4z}\right)_\infty^d = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}$$

- As I move q toward the conductor, I do work only on q, not on the conductor. By contrast, if I bring in 2 point charges, I do work on both of them, and the total is twice as great.

Other Image Problems

- *Any* stationary charge distribution near a grounded conducting plane can be treated in the same way, by introducing its mirror image—hence the name **method of images**.

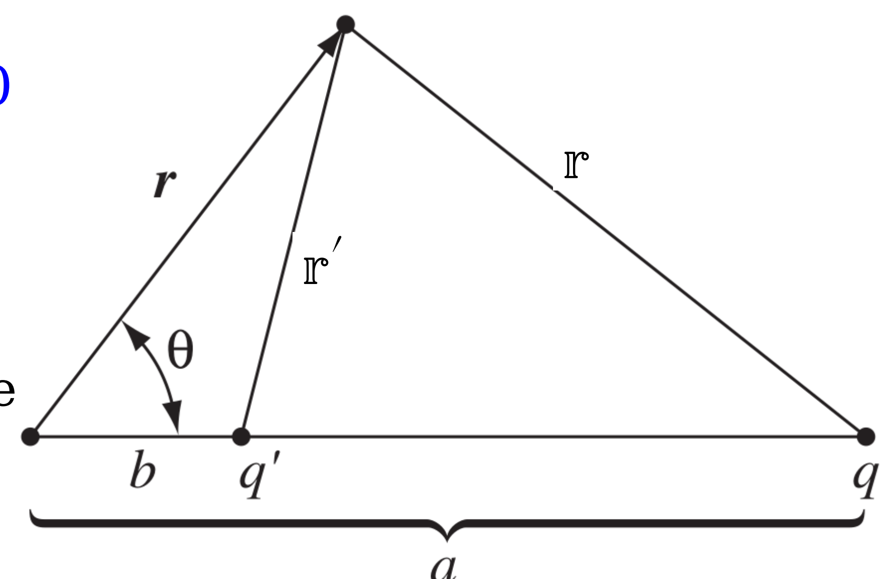
- The image charges have the *opposite* sign; this is what guarantees that the xy plane will be at potential 0.



<u>Example 3.2:</u> q is situated a distance a from the center of a grounded conducting sphere of radius R. Find the potential outside the sphere.

- Consider the *completely different* configuration, consisting of q with another point charge $q' = -\frac{R}{a} q$ placed a distance $b = \frac{R^2}{a}$ from the sphere center.

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right) \quad \Rightarrow \quad \Phi(r=R)=0$$



It fits the boundary conditions for our original problem, in the exterior region.

- $b<R$, so the "image" charge q' is safely inside the sphere—*you cannot put image charges in the region where you are calculating* Φ.

Try to find q' and b

$$\Phi(\mathbf{r}) = \Phi_{\text{real}} + \Phi_{\text{image}} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{(x-a)^2+y^2+z^2}} + \frac{q'}{\sqrt{(x-b)^2+y^2+z^2}}\right) \Rightarrow \quad \Phi_{r=R} = 0$$

$$\mathbf{r}_1 = (+R, 0,0) \quad \Rightarrow \quad 0 = \Phi(\mathbf{r}_1) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{a-R} + \frac{q'}{R-b}\right)$$

$$\mathbf{r}_2 = (-R, 0,0) \quad \Rightarrow \quad 0 = \Phi(\mathbf{r}_2) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{a+R} + \frac{q'}{R+b}\right)$$

$$\Rightarrow \quad \frac{q}{a-R} + \frac{q'}{R-b} = 0, \qquad \frac{q}{a+R} + \frac{q'}{R+b} = 0$$

$$\Rightarrow \quad \frac{R-b}{R-a} = -\frac{R+b}{R+a} \quad \Rightarrow \quad b = \frac{R^2}{a} \quad \Rightarrow \quad q' = -\frac{R}{a} q$$

$$\Rightarrow \quad \Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{(x-a)^2+y^2+z^2}} - \frac{Rq/a}{\sqrt{(x-R^2/a)^2+y^2+z^2}}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{r^2+a^2-2ar\cos\theta}} - \frac{Rq}{\sqrt{a^2r^2+R^4-2R^2ar\cos\theta}}\right)$$

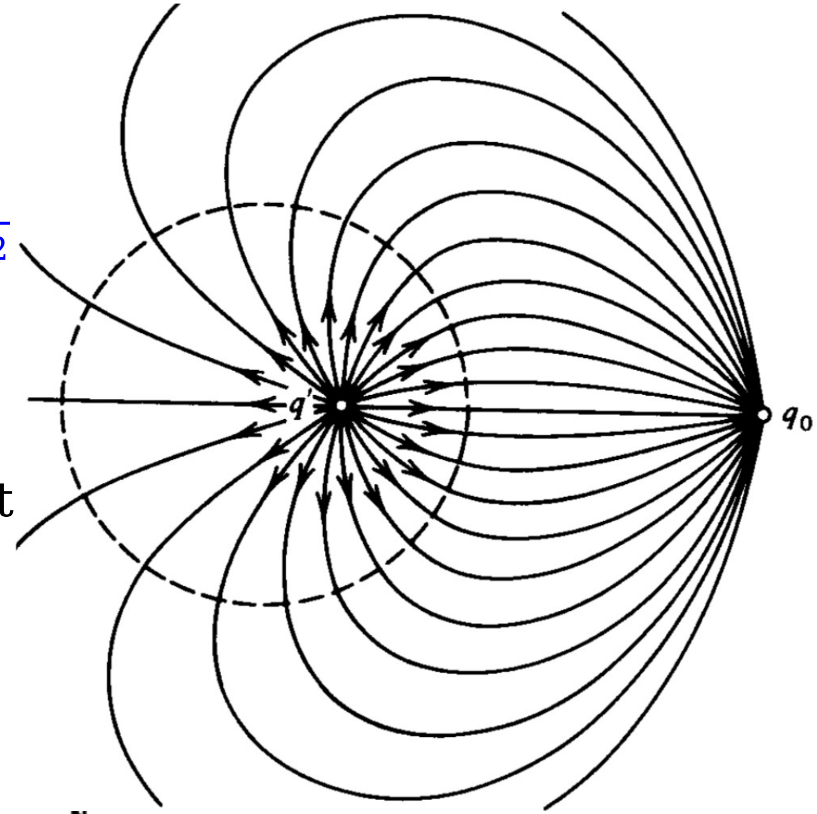
- The force of attraction between the charge and the sphere is

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a-b)^2} = -\frac{1}{4\pi\epsilon_0}\frac{aRq^2}{(a^2-R^2)^2}$$

- The induced surface charge on the sphere

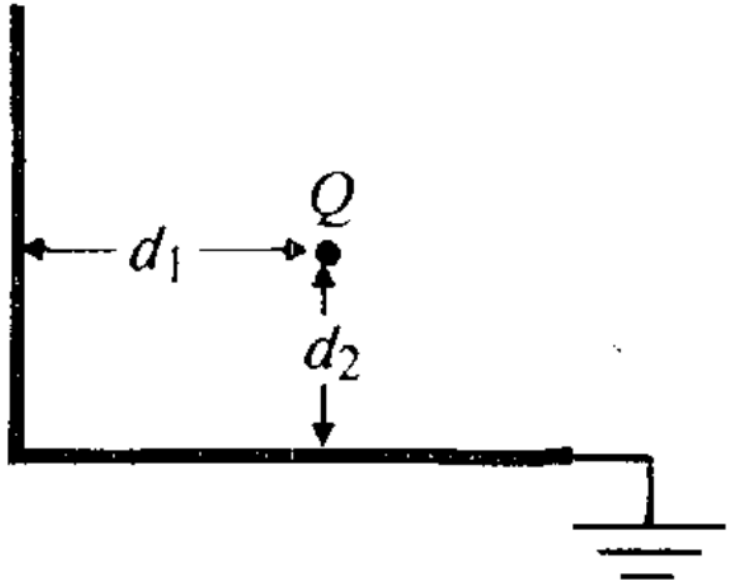
$$\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial r}\Big|_{r=R} = \frac{q}{4\pi R} \frac{R^2 - a^2}{\left(a^2 + R^2 - 2aR\cos\theta\right)^{3/2}}$$

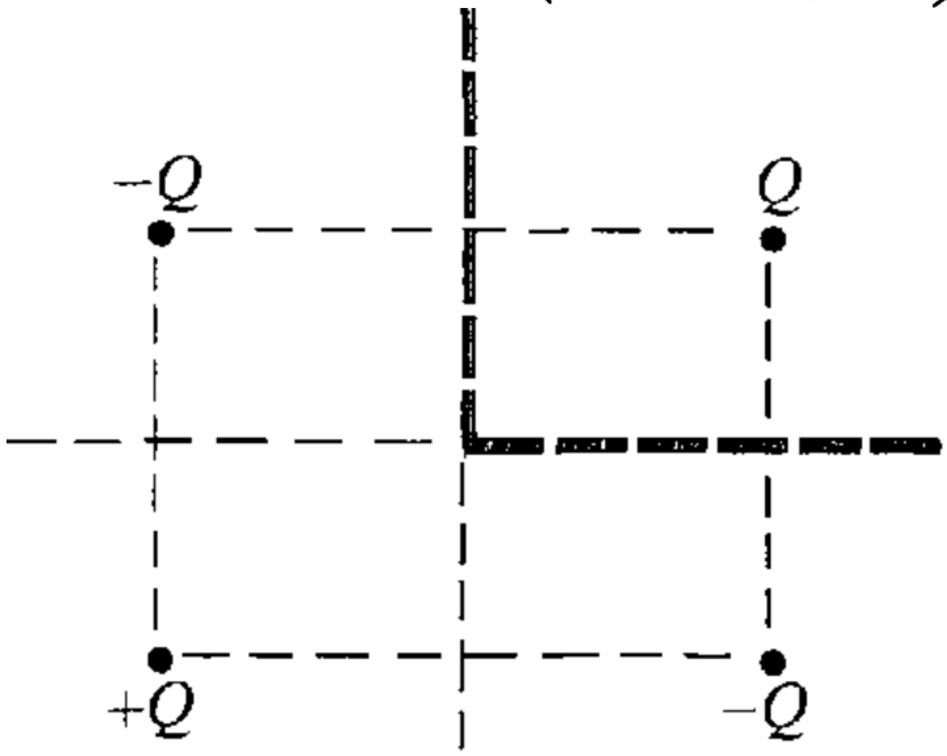
$$\Rightarrow \quad Q = \int \sigma \, \mathrm{d}a = -\frac{R}{a} q = q'$$

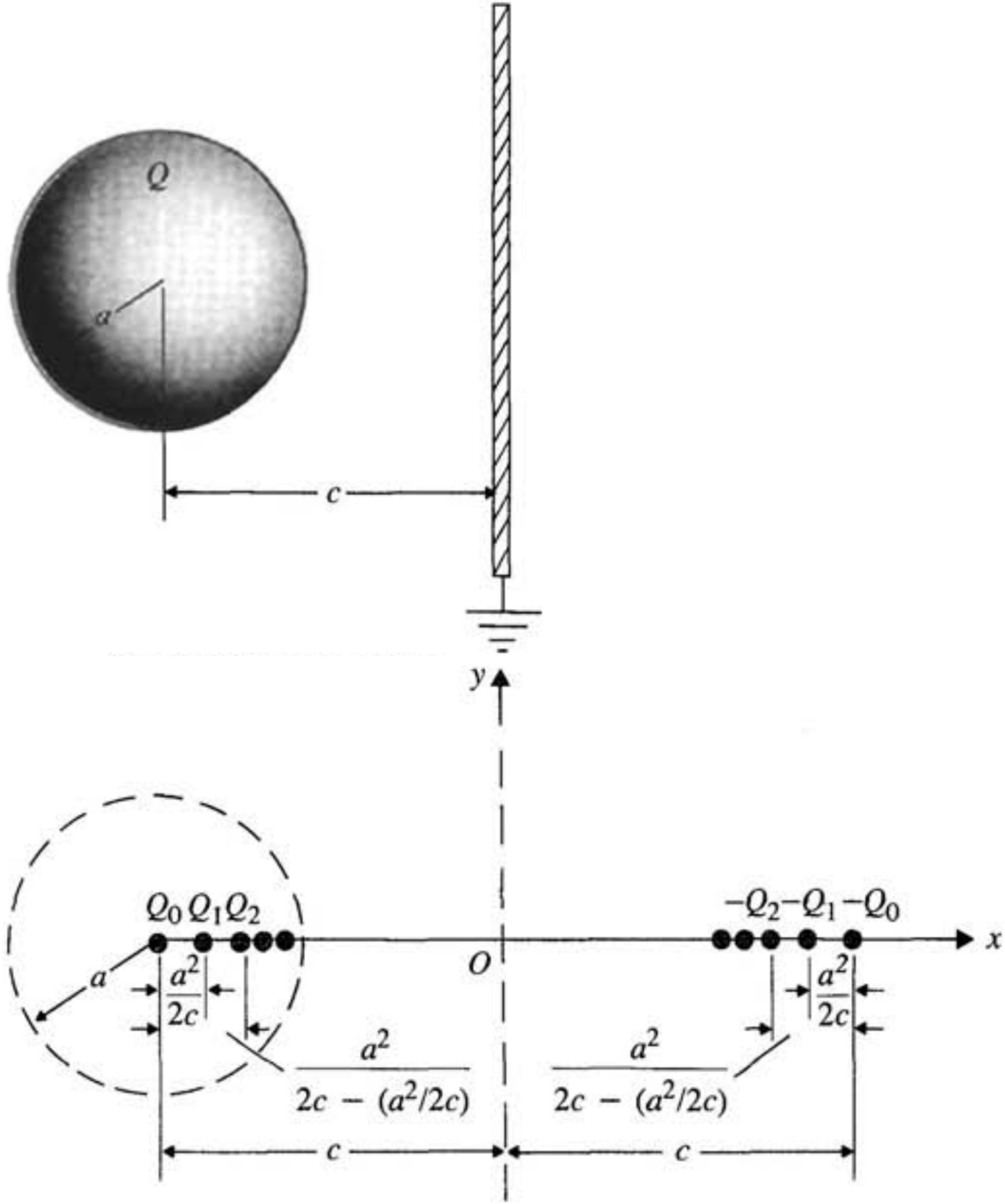


- The key of the method of images is to figure out the right "auxiliary" configuration, and for most shapes this is forbiddingly complicated, if not Impossible.

- Other cases:







Q
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Q_0 Q_1 Q_2
$-Q_2$ $-Q_1$ $-Q_0$
x
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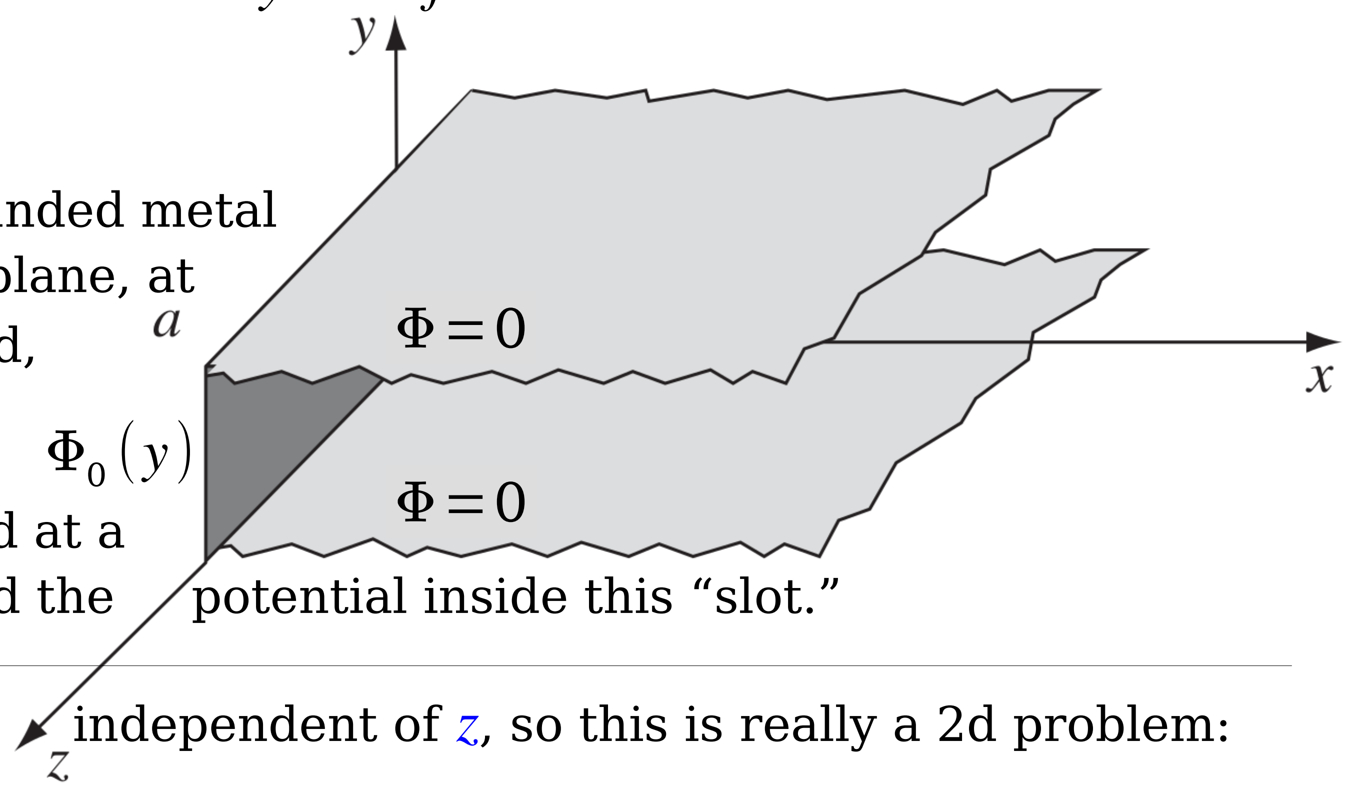
$\frac{a^2}{2c}$
$\frac{a^2}{2c}$
$\frac{a^2}{2c - (a^2/2c)}$
$\frac{a^2}{2c - (a^2/2c)}$
c
c

Separation of Variables

- The **separation of variables** method is applicable in circumstances where the potential (Φ) or the charge density (σ) is specified on the boundaries of some region, and we are asked to find the potential in the interior.

- The basic strategy is very simple: *We look for solutions that are products of functions, each of which depends on only one of the coordinates.*

Cartesian Coordinates



Example 3.3: 2 infinite grounded metal plates lie parallel to the xz-plane, at $y=0$ and at $y=a$. The left end, at $x=0$, is closed off with an infinite strip insulated from the 2 plates, and maintained at a specific potential $\Phi_0(y)$. Find the potential inside this "slot."

The configuration is independent of z, so this is really a 2d problem:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$ subject to the boundary conditions,

(i) $\Phi=0$ when $y=0$, (ii) $\Phi=0$ when $y=a$,

(iii) $\Phi=\Phi_0(y)$ when $x=0$, (iv) $\Phi\to 0$ as $x\to\infty$.

(iv), although not explicitly stated in the problem, is necessary on physical grounds: as it gets farther and farther away from the strip at $x=0$, $\Phi\to 0$.

Since the potential is specified on all boundaries, the answer is uniquely determined.

$$\Phi(x,y)=X(x)Y(y) \;\Rightarrow\; Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}=0 \;\Rightarrow\; \frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=0$$

$f(x)+g(y)=0 \;\Rightarrow\; f$ and g must both be constant.

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}=-\frac{1}{Y}\frac{d^2Y}{dy^2}=k^2 \;\Rightarrow\; \frac{d^2X}{dx^2}=k^2X,\quad \frac{d^2Y}{dy^2}=-k^2Y \;\Leftarrow\; \text{boundary consideration}$$

$$\Rightarrow \begin{cases} X=Ae^{kx}+Be^{-kx} \\ Y=C\sin ky+D\cos ky \end{cases} \Rightarrow\; V(x,y)=\left(Ae^{kx}+Be^{-kx}\right)\left(C\sin ky+D\cos ky\right)$$

$$\text{(iv)} \;\Rightarrow\; V(x,y)=e^{-kx}\left(C\sin ky+D\cos ky\right) \;\Rightarrow\; \Phi(x,y)=Ce^{-kx}\sin ky \;\Leftarrow\; \text{(i)}$$

$$\text{(ii)} \;\Rightarrow\; \sin ka=0 \;\Rightarrow\; k=\frac{n\pi}{a},\; n\in\mathbb{N} \;\Rightarrow\; \text{infinite solutions can be constructed}$$

$$\Phi_1,\Phi_2,\cdots \;\Rightarrow\; \Phi=\alpha_1\Phi_1+\alpha_2\Phi_2+\cdots \;\Rightarrow\; \nabla^2\Phi=\alpha_1\nabla^2\Phi_1+\alpha_2\nabla^2\Phi_2+\cdots=0$$

$$\text{general solution}\;\; \Phi(x,y)=\sum_{n=1}^{\infty}C_n e^{-\frac{n\pi x}{a}}\sin\frac{n\pi y}{a} \;\Rightarrow\; \Phi(0,y)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi y}{a}=\Phi_0(y)$$

Fourier sine series ↑ (iii)

$$\Rightarrow \sum_{n=1}^{\infty} C_n \int_0^a \sin\frac{n\pi y}{a}\sin\frac{n'\pi y}{a}\,dy = \int_0^a \Phi_0(y)\sin\frac{n'\pi y}{a}\,dy$$

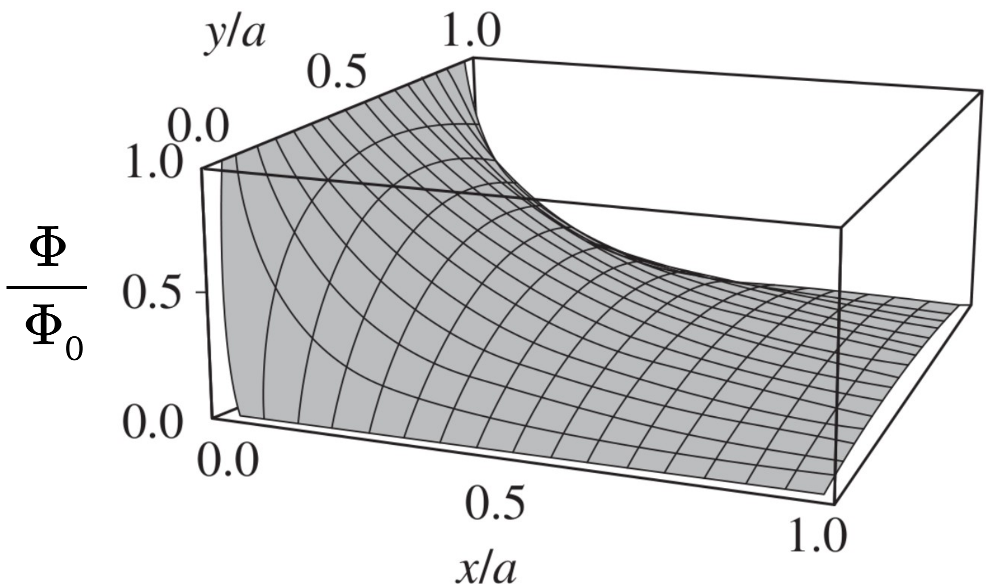
$$\Rightarrow C_n = \frac{2}{a}\int_0^a \Phi_0(y)\sin\frac{n\pi y}{a}\,dy \quad \Leftarrow \quad \int_0^a \sin\frac{n\pi y}{a}\sin\frac{n'\pi y}{a}\,dy = \frac{a}{2}\delta_{nn'}$$

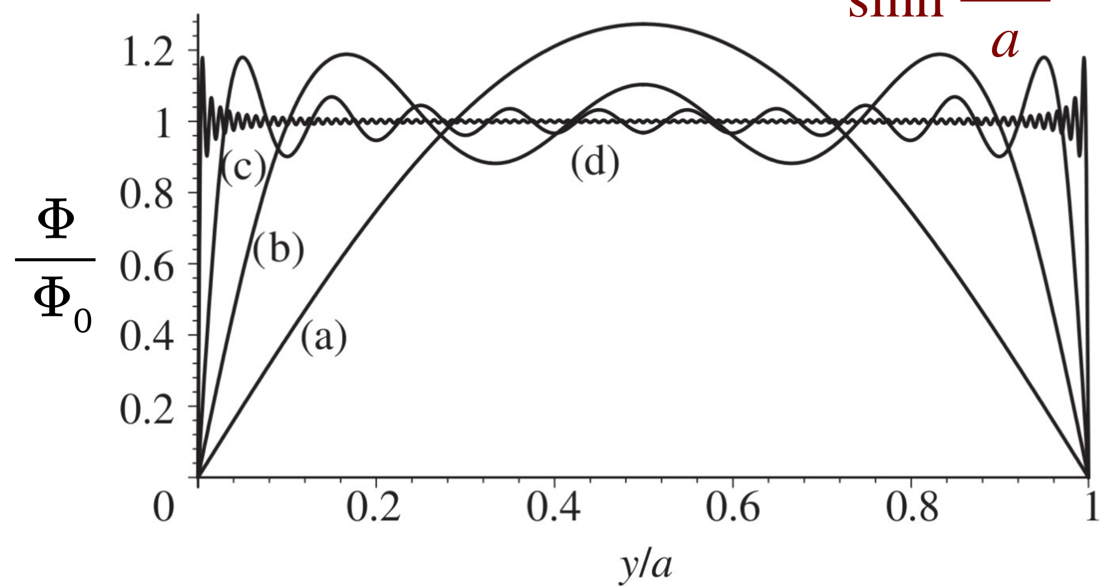
If $\Phi_0(y) = \Phi_0 = \text{const}$

$$\Rightarrow C_n = \frac{2\Phi_0}{a}\int_0^a \sin\frac{n\pi y}{a}\,dy = 2\Phi_0\frac{1-\cos n\pi}{n\pi} = \begin{cases} 0 & \text{, if } n \text{ is even} \\ \dfrac{4\Phi_0}{n\pi} & \text{, if } n \text{ is odd} \end{cases}$$

analytical solution

$$\Rightarrow \Phi(x,y) = \frac{4\Phi_0}{\pi}\sum_{n=1,3,5\cdots}^{\infty}\frac{e^{-\frac{n\pi x}{a}}}{n}\sin\frac{n\pi y}{a} \quad \text{vs} \quad \Phi(x,y) = \frac{2\Phi_0}{\pi}\tan^{-1}\frac{\sin\frac{\pi y}{a}}{\sinh\frac{\pi x}{a}}$$





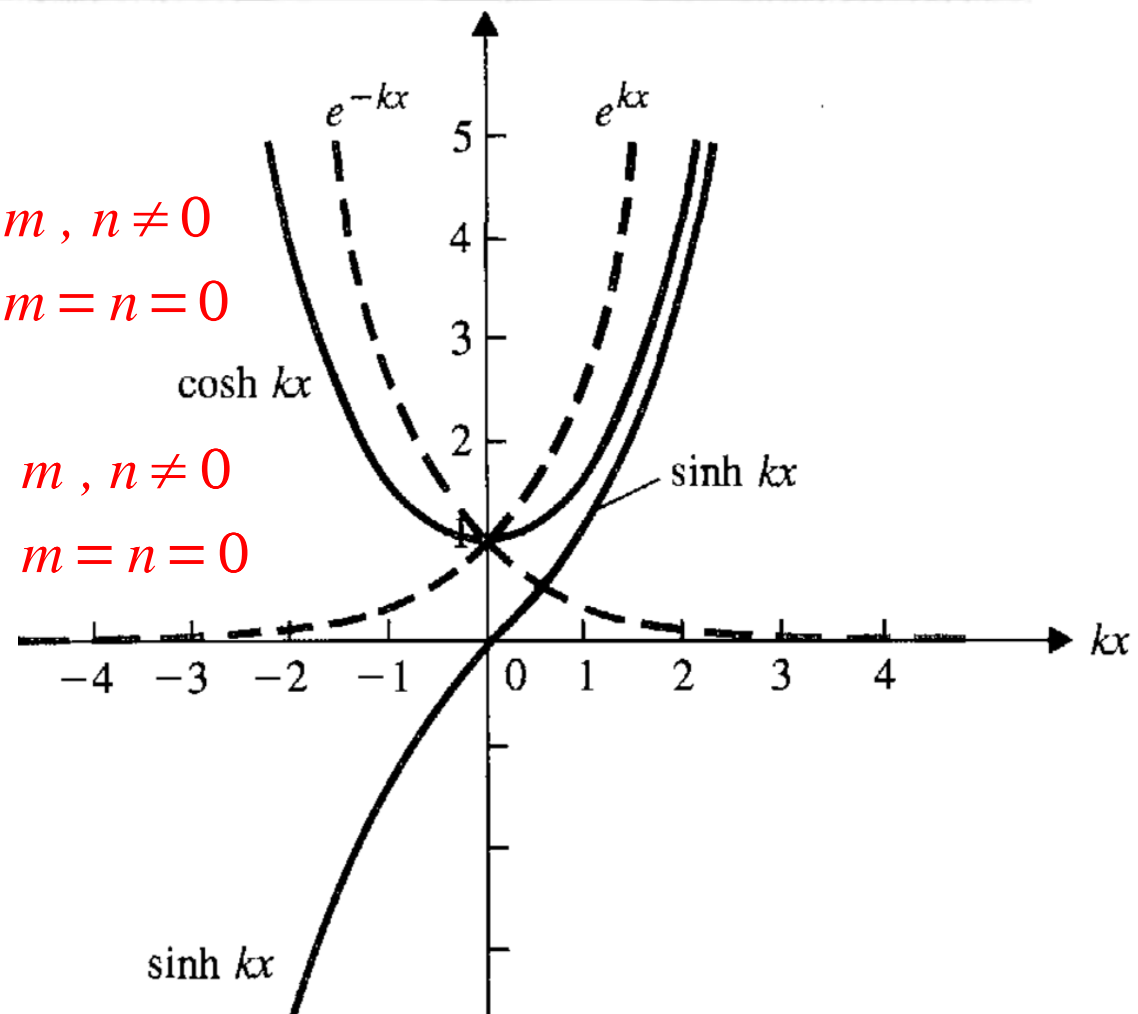
Possible Solutions of $X''(x) + k_x^2 X(x) = 0$

k_x^2	k_x	$X(x)$	Exponential forms† of $X(x)$
0	0	$A_0 x + B_0$	
+	k	$A_1 \sin kx + B_1 \cos kx$	$C_1 e^{jkx} + D_1 e^{-jkx}$
−	jk	$A_2 \sinh kx + B_2 \cosh kx$	$C_2 e^{kx} + D_2 e^{-kx}$

$$\int_0^{2\pi} \sin mx \sin nx \, dx = \begin{cases} \pi \delta_{mn} & \text{for } m, n \neq 0 \\ 0 & \text{for } m = n = 0 \end{cases}$$

$$\int_0^{2\pi} \cos mx \cos nx \, dx = \begin{cases} \pi \delta_{mn} & \text{for } m, n \neq 0 \\ 2\pi & \text{for } m = n = 0 \end{cases}$$

$$\int_0^{2\pi} \sin mx \cos nx \, dx = 0$$



- The success of this method hinged on 2 extraordinary properties of the separable solutions: **completeness** and **orthogonality**.

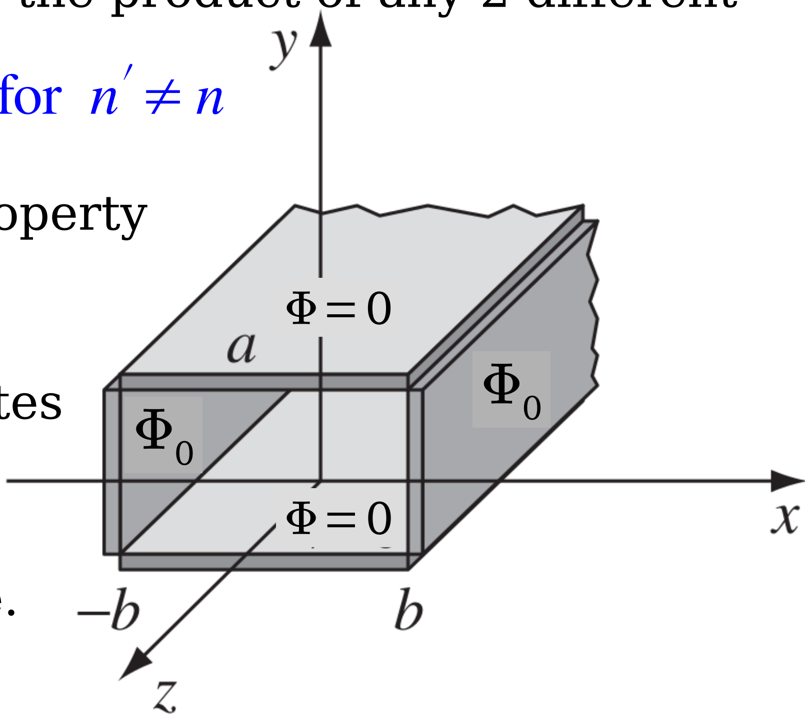
- A set of functions $f_n(y)$ is said to be **complete** if any other function $f(y)$ can be expressed as a linear combination of them: $f(y) = \sum_{n=1}^{\infty} C_n f_n(y)$

- $\sin \frac{n\pi y}{a}$'s are complete on the interval $0 \le y \le a$.

- It is guaranteed by Dirichlet's theorem that the solution can be obtained, given the proper choice of the coefficients C_n.

- A set of functions is orthogonal if the integral of the product of any 2 different members of the set is 0: $\int_0^a f_n(y) f_{n'}(y)\, dy = 0 \quad \text{for } n' \neq n$

- The sine functions are orthogonal. this is the property allowing to solve for the coefficients C_n.



Example 3.4: 2 infinitely-long grounded metal plates at $y=0$ and $y=a$, are connected at $x=\pm b$ by metal strips maintained at a constant potential Φ_0. Find the potential inside the resulting rectangular pipe.

Independent of z, $\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}=0$ subject to the boundary conditions,

(i) $\Phi=0$ when $y=0$, (ii) $\Phi=0$ when $y=a$, (iii) $\Phi=\Phi_0$ when $x=\pm b$,

$$\Phi(x,y)=\left(A e^{kx}+B e^{-kx}\right)\left(C \sin ky+D \cos ky\right)$$

$$=\cosh kx\left(C \sin ky+D \cos ky\right) \Leftarrow \left[\begin{array}{ll}\Phi(x,y)=\Phi(-x,y) & \text{symmetric}\\ e^{kx}+e^{-kx}=2\cosh kx & \end{array}\right.$$

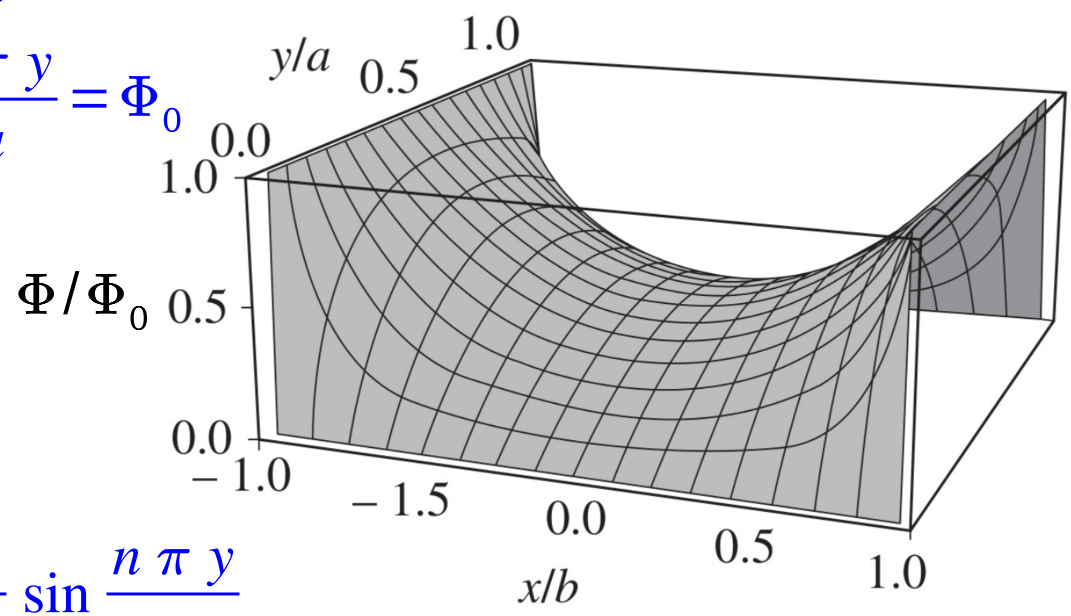
$$=C \cosh \frac{n\pi x}{a} \sin \frac{n\pi y}{a} \Leftarrow D=0, \quad k=\frac{n\pi}{a}$$

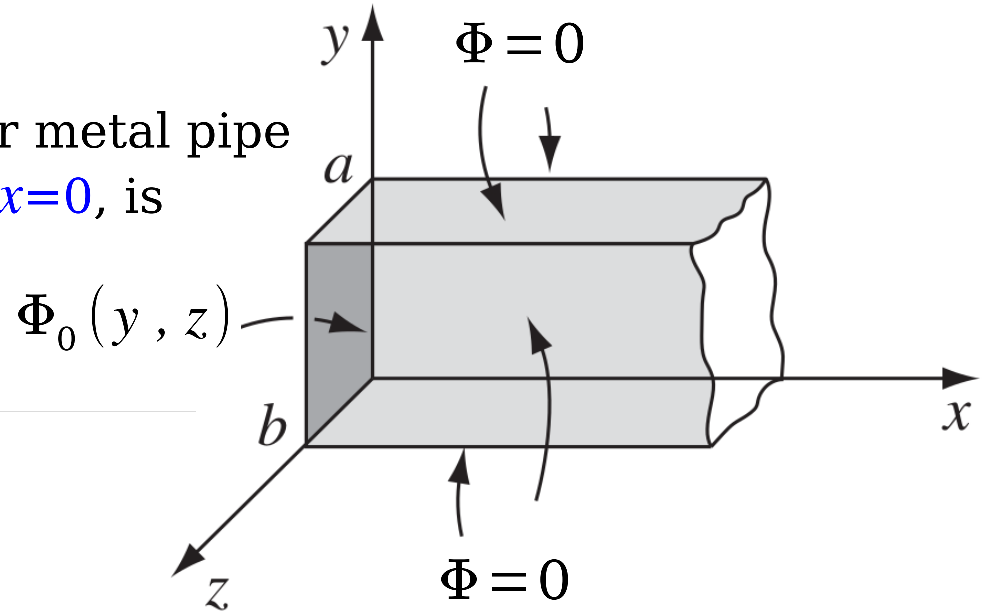
$$\Rightarrow \Phi(x,y)=\sum_{n=1}^{\infty} C_n \cosh \frac{n\pi x}{a} \sin \frac{n\pi y}{a} \quad \text{general solution}$$

$$\Rightarrow \Phi(b,y)=\sum_{n=1}^{\infty} C_n \cosh \frac{n\pi b}{a} \sin \frac{n\pi y}{a}=\Phi_0$$

$$\Rightarrow C_n \cosh \frac{n\pi b}{a}=\left[\begin{array}{ll} 0 & ,n \ \text{even}\\ \frac{4\Phi_0}{n\pi} & ,n \ \text{odd}\end{array}\right.$$

$$\Rightarrow \Phi(x,y)=\frac{4\Phi_0}{\pi}\sum_{n=1,3\cdots}^{\infty}\frac{\cosh \frac{n\pi x}{a}}{n \cosh \frac{n\pi b}{a}} \sin \frac{n\pi y}{a}$$





Example 3.5: An infinitely long rectangular metal pipe (sides a & b) is grounded, but one end, at $x=0$, is maintained at a specified potential $\Phi_0(y,z)$. Find the potential inside the pipe.

A 3d problem, $\dfrac{\partial^2\Phi}{\partial x^2}+\dfrac{\partial^2\Phi}{\partial y^2}+\dfrac{\partial^2\Phi}{\partial z^2}=0$

subject to the boundary conditions

(i) $\Phi=0$ when $y=0$, (ii) $\Phi=0$ when $y=a$,

(iii) $\Phi=0$ when $z=0$, (iv) $\Phi=0$ when $z=b$,

(v) $\Phi\to 0$ as $x\to\infty$, (vi) $\Phi=\Phi_0(y,z)$ when $x=0$

$$\Phi(x,y,z)=X(x)Y(y)Z(z) \Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0$$

$$\Rightarrow \frac{d^2X}{dx^2}=(k^2+\ell^2)X, \quad \frac{d^2Y}{dy^2}=-k^2Y, \quad \frac{d^2Z}{dz^2}=-\ell^2 Z \Leftarrow$$ boundary consideration

$$\Rightarrow X=Ae^{\sqrt{k^2+\ell^2}\,x}+Be^{-\sqrt{k^2+\ell^2}\,x}, \quad Y=C\sin ky+D\cos ky, \quad Z=E\sin\ell z+F\cos\ell z$$

$$(v)\Rightarrow A=0, \quad (i)\Rightarrow D=0, \quad (iii)\Rightarrow F=0, \quad (ii)\Rightarrow k=\frac{n\pi}{a}, \quad (iv)\Rightarrow \ell=\frac{m\pi}{b}$$

$$\Rightarrow \Phi_{nm}(x,y,z)=Ce^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x}\sin\frac{n\pi y}{a}\sin\frac{m\pi z}{b}$$

$$\Rightarrow \quad \Phi(x,y,z)=\sum_{n,m}\Phi_{nm}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}\,e^{-\pi x\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}}\sin\frac{n\pi y}{a}\sin\frac{m\pi z}{b} \quad \text{general solution}$$

$$\Rightarrow \quad \Phi(0,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}\sin\frac{n\pi y}{a}\sin\frac{m\pi z}{b}=\Phi_0(y,z)$$

$$\Rightarrow \quad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}\int_0^a\sin\frac{n\pi y}{a}\sin\frac{n'\pi y}{a}\,dy\int_0^b\sin\frac{m\pi z}{b}\sin\frac{m'\pi z}{b}\,dz$$

$$=\int_0^a\int_0^b\Phi_0(y,z)\sin\frac{n'\pi y}{a}\sin\frac{m'\pi z}{b}\,dy\,dz$$

$$\Rightarrow \quad C_{nm}=\frac{4}{ab}\int_0^a\int_0^b\Phi_0(y,z)\sin\frac{n\pi y}{a}\sin\frac{m\pi z}{b}\,dy\,dz$$

If $\Phi_0(y,z)=\Phi_0=\text{const}$

$$\Rightarrow \quad C_{nm}=\frac{4\Phi_0}{ab}\int_0^a\sin\frac{n\pi y}{a}\,dy\int_0^b\sin\frac{m\pi z}{b}\,dz=\begin{cases}0, & \text{if } n \text{ or } m \text{ is even}\\ \dfrac{16\Phi_0}{mn\pi^2}, & \text{if } n \text{ and } m \text{ are odd}\end{cases}$$

$$\Rightarrow \quad \Phi(x,y,z)=\frac{16\Phi_0}{\pi^2}\sum_{n,m=1,3,5\cdots}\frac{e^{-\pi x\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}}}{nm}\sin\frac{n\pi y}{a}\sin\frac{m\pi z}{b}$$

Spherical Coordinates

- For round objects, spherical coordinates are more natural.
- $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Phi}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 \Phi}{\partial^2 \phi}=0$$
- Assume the problem has **azimuthal symmetry**, so that Φ *is independent of* ϕ

$$\Rightarrow \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Phi}{\partial \theta}\right)=0$$

$$\Phi(r,\theta)=R(r)\,\Theta(\theta) \quad \Rightarrow \quad \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)=0$$

$$\Rightarrow \quad \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)=\ell(\ell+1), \quad \frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)=-\ell(\ell+1)$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)=\ell(\ell+1)R$$

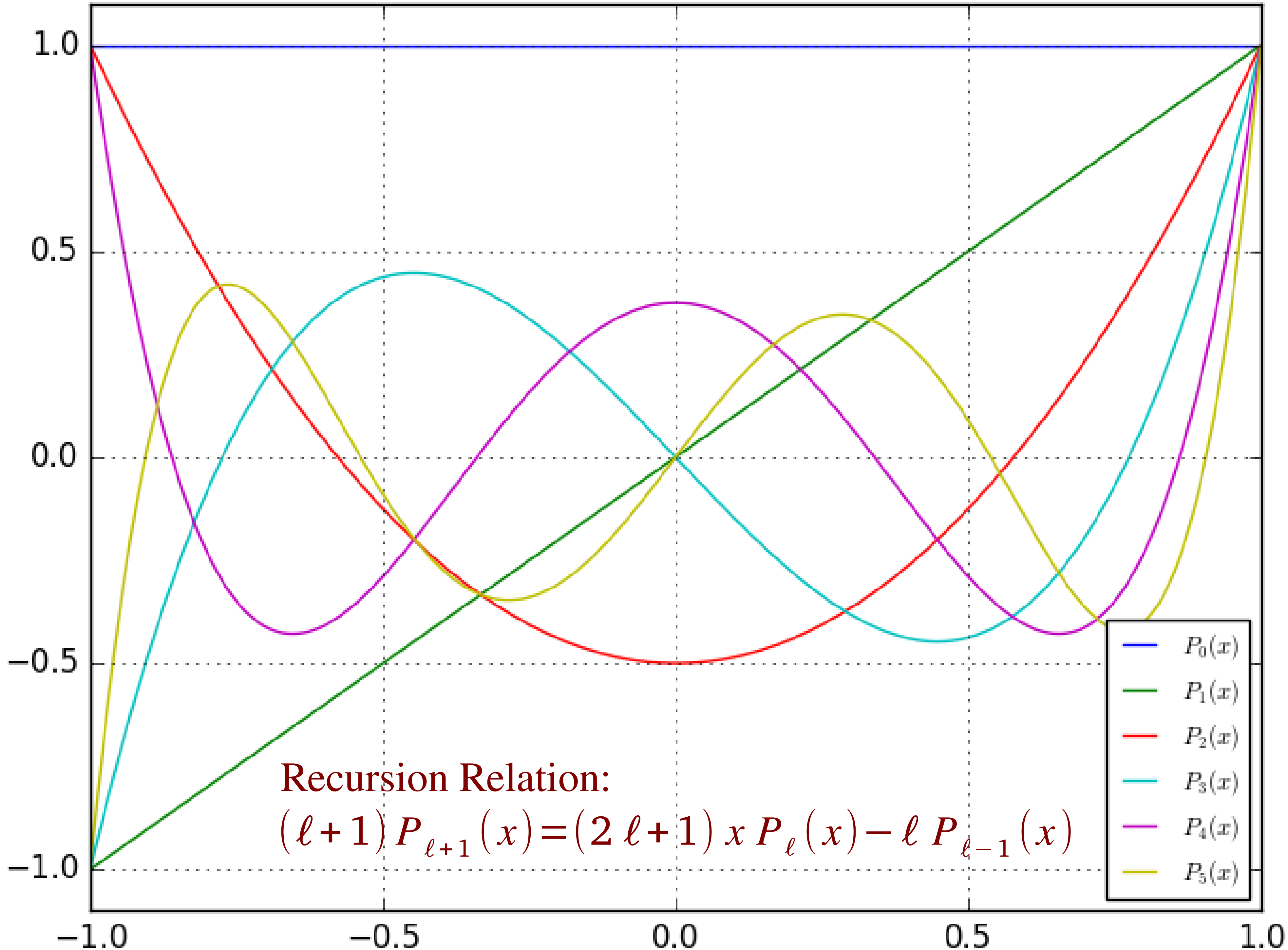
$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)=-\ell(\ell+1)\,\Theta\sin\theta$$

$$\Rightarrow \quad \left[\begin{array}{l} R(r)=A\,r^{\ell}+\dfrac{B}{r^{\ell+1}} \\ \Theta(\theta)=P_\ell(\cos\theta) \end{array}\right.$$

- The solutions of Θ are **Legendre polynomials** in the variable $\cos\theta$.

- $P_\ell(x)$ is most conveniently defined by the **Rodrigues formula**:

$$P_\ell(x)\equiv\frac{1}{2^\ell\,\ell!}\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}(x^2-1)^\ell$$


1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0
$P_0(x)$
$P_1(x)$
$P_2(x)$
$P_3(x)$
$P_4(x)$
$P_5(x)$
Recursion Relation:
$(\ell+1)\,P_{\ell+1}(x)=(2\,\ell+1)\,x\,P_\ell(x)-\ell\,P_{\ell-1}(x)$

$$P_0(x)=1\,,\qquad P_1(x)=x\,,\qquad P_2(x)=\frac{3x^2-1}{2}$$

$$P_3(x)=\frac{5x^3-3x}{2}\,,\quad P_4(x)=\frac{35x^4-30x^2+3}{8}\,,\quad P_5(x)=\frac{63x^5-70x^3+15x}{8}$$

- $P_\ell(x)$ is (as the name suggests) an ℓ^{th}-order *polynomial* in x; it contains only *even* powers, if ℓ is even, and *odd* powers, if ℓ is odd. And $P_\ell(1)=1$ for all ℓ.

- The Rodrigues formula works only for nonnegative integer values of ℓ. And it provides us with only 1 solution. A 2nd-order differential equation should possess 2 independent solutions, for every value of ℓ.

- It turns out that the other solutions blow up at $\theta=0$ and/or $\theta=\pi$, and are unacceptable on physical grounds.

- For instance, the 2nd solution for $\ell=0$ is $\Theta(\theta)=\ln\tan\dfrac{\theta}{2}$

- $\Phi_\ell(r,\theta)=\left(A\,r^\ell+\dfrac{B}{r^{\ell+1}}\right)P_\ell(\cos\theta)$ for each ℓ

$\Rightarrow$ general solution $$\Phi(r,\theta)=\sum\Phi_\ell(r,\theta)=\sum_{\ell=0}^{\infty}\left(A_\ell\,r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

Proof for Orthogonality

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\ell(\ell+1)P=0 \Rightarrow \frac{d}{dx}\left[(1-x^2)\frac{dP(x)}{dx}\right]+\ell(\ell+1)P(x)=0 \Leftarrow x=\cos\theta$$

$$\int_{-1}^{1} P_{\ell'}\left[\frac{d}{dx}\left((1-x^2)\frac{dP_\ell}{dx}\right)+\ell(\ell+1)P_\ell\right]dx=0$$

$$\Rightarrow \int_{-1}^{1}\left((x^2-1)\frac{dP_\ell}{dx}\frac{dP_{\ell'}}{dx}+\ell(\ell+1)P_\ell P_{\ell'}\right)dx=0 \Leftarrow \text{ integration by parts}$$

$$\ell\leftrightarrow\ell' \Rightarrow [\ell(\ell+1)-\ell'(\ell'+1)]\int_{-1}^{1}P_{\ell'}P_\ell\, dx=0 \Rightarrow \int_{-1}^{1}P_{\ell'}(x)P_\ell(x)\,dx=0 \text{ for } \ell\neq\ell'$$

Use Rodrigues' formula to determine the value for $\ell=\ell'$

$$N_\ell\equiv\int_{-1}^{1}P_\ell^2(x)\,dx=\frac{1}{4^\ell(\ell!)^2}\int_{-1}^{1}\left(\frac{d^\ell}{dx^\ell}(x^2-1)^\ell\right)\left(\frac{d^\ell}{dx^\ell}(x^2-1)^\ell\right)dx$$

$$=\frac{(-1)^\ell}{4^\ell(\ell!)^2}\int_{-1}^{1}(x^2-1)^\ell\frac{d^{2\ell}}{dx^{2\ell}}(x^2-1)^\ell\,dx=\frac{(2\ell)!}{4^\ell(\ell!)^2}\int_{-1}^{1}(1-x^2)^\ell\,dx \Leftarrow \begin{array}{c}\text{integration by parts}\\ +\text{direct differentiation}\end{array}$$

$$(1-x^2)^\ell=(1-x^2)(1-x^2)^{\ell-1}=(1-x^2)^{\ell-1}+\frac{x}{2\ell}\frac{d}{dx}(1-x^2)^\ell$$

$$\Rightarrow N_\ell=\frac{2\ell-1}{2\ell}N_{\ell-1}+\frac{(2\ell-1)!}{4^\ell(\ell!)^2}\int_{-1}^{1}x\,d(1-x^2)^\ell=\frac{2\ell-1}{2\ell}N_{\ell-1}-\frac{1}{2\ell}N_\ell \Leftarrow \begin{array}{l}\text{integration}\\ \text{by parts}\end{array}$$

$$\Rightarrow (2\ell+1)N_\ell=(2\ell-1)N_{\ell-1}=\cdots=(2\cdot 0+1)N_0 \Rightarrow N_\ell=\frac{2}{2\ell+1} \Leftarrow N_0=2 \Leftarrow P_0=1$$

$$\Rightarrow \int_{-1}^{1}P_\ell(x)P_{\ell'}(x)\,dx=\frac{2}{2\ell+1}\delta_{\ell\ell'} \quad \text{orthogonality condition}$$

Example 3.6 & 3.7: The potential $\Phi_0(\theta)$ is specified on the surface of a hollow sphere, of radius R. Find the potential *inside* and *outside* the sphere.

Inside the sphere, all $B_\ell = 0$ —otherwise the potential would blow up at the origin; Outside the sphere $A_\ell = 0$ for all ℓ, or else Φ would not go to 0 at ∞,

$$\Rightarrow \begin{array}{l} \Phi_{\text{in}}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \\ \Phi_{\text{out}}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \end{array} \Rightarrow \left. \begin{array}{l} \Phi_{\text{in}}(R,\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) \\ \Phi_{\text{out}}(R,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta) \end{array} \right] = \Phi_0(\theta)$$

The Legendre polynomials (like the sines) constitute a complete set of *orthogonal* functions, on the interval $-1 \le x \le 1$ $(0 \le \theta \le \pi)$,

$$\Rightarrow \int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, \mathrm{d}x = \int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta) \sin\theta\, \mathrm{d}\theta = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

$$\Rightarrow \begin{bmatrix} A_{\ell'} R^{\ell'} \\ \dfrac{B_{\ell'}}{R^{\ell'+1}} \end{bmatrix} \frac{2}{2\ell'+1} = \int_0^\pi \Phi_0(\theta) P_{\ell'}(\cos\theta) \sin\theta\, \mathrm{d}\theta$$

$$\Rightarrow \begin{bmatrix} A_\ell \\ B_\ell \end{bmatrix} = \frac{2\ell+1}{2} \begin{bmatrix} R^{-\ell} \\ R^{\ell+1} \end{bmatrix} \int_0^\pi \Phi_0(\theta) P_\ell(\cos\theta) \sin\theta\, \mathrm{d}\theta$$

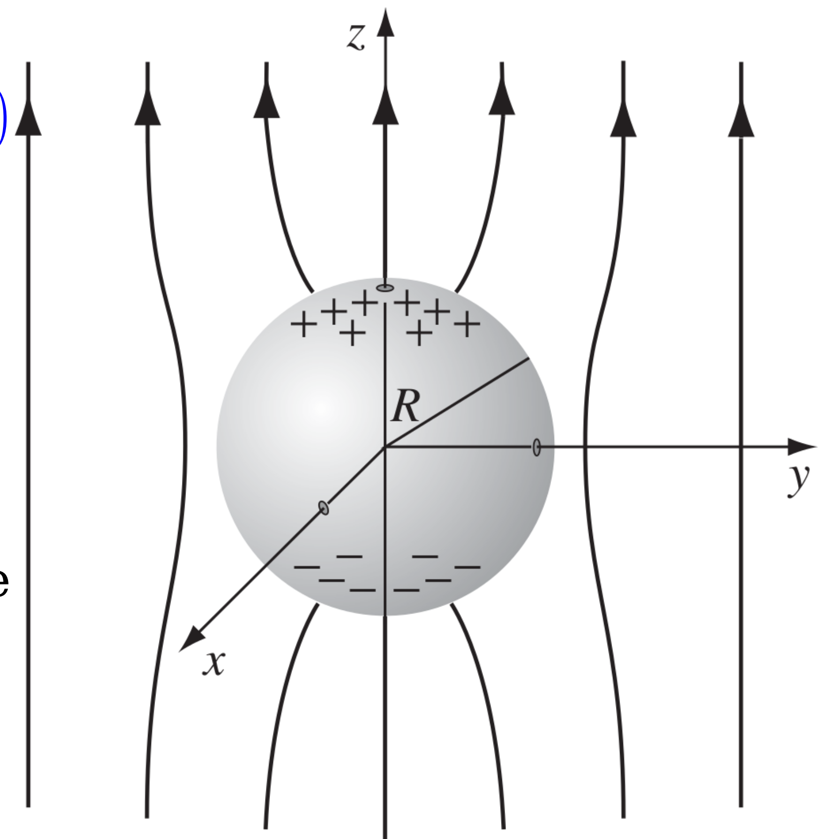
If $\Phi_0(\theta) = k\sin^2\frac{\theta}{2} \Rightarrow \Phi_0(\theta) = k\frac{1-\cos\theta}{2} = k\frac{P_0(\cos\theta) - P_1(\cos\theta)}{2}$

$$\Rightarrow A_0 = \frac{k}{2}, \quad B_0 = \frac{k}{2}R, \quad A_1 = -\frac{k}{2R}, \quad B_1 = -\frac{k}{2}R^2, \quad A_\ell = B_\ell = 0 \text{ for } \ell > 1$$

$$\Rightarrow \Phi(r,\theta) = \begin{cases} \frac{k}{2}\left(r^0 P_0(\cos\theta) - \frac{r^1}{R}P_1(\cos\theta)\right) = \frac{k}{2}\left(1 - \frac{r}{R}\cos\theta\right), & r < R \\ \frac{k}{2}\left(\frac{R}{r^1}P_0(\cos\theta) - \frac{R^2}{r^2}P_1(\cos\theta)\right) = \frac{k}{2}\frac{R}{r}\left(1 - \frac{R}{r}\cos\theta\right), & r \geq R \end{cases}$$

$$= \frac{k}{2}\frac{R}{r_>}\left(1 - \frac{r_<}{r_>}\cos\theta\right) \Leftarrow \begin{matrix} r_< = \min(r, R) \\ r_> = \max(r, R) \end{matrix}$$

<u>Example 3.8</u>: An uncharged metal sphere of radius R is placed in a uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$. The field will push positive charge to the northern surface of the sphere, and, symmetrically, negative charge to the southern surface. This induced charge, in turn, distorts the field in the neighborhood of the sphere. Find the potential in the region outside the sphere.



The sphere is an equipotential $\Rightarrow$ Let $\Phi\left(r=R\right)=0$

By symmetry the entire xy-plane is at potential 0 $\Rightarrow$ $\Phi\left(z=0\right)=0$

Far from the sphere the field is $E_0\hat{\mathbf{z}}$ $\Rightarrow$ $\Phi \rightarrow -E_0 z + C = -E_0 z \Leftarrow \Phi\left(z=0\right)=0$

The boundary conditions for this problem (i) $\Phi(r{=}R)=0$, (ii) $\Phi(r \gg R) \rightarrow -E_0 r\cos\theta$

$$(\text{i}) \Rightarrow A_\ell R^\ell + \frac{B_\ell}{R^{\ell+1}} = 0 \Rightarrow B_\ell = -A_\ell R^{2\ell+1} \Rightarrow \Phi = \sum_{\ell=0}^{\infty} A_\ell \left(r^\ell - \frac{R^{2\ell+1}}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

$$(\text{ii}) \Rightarrow \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) = -E_0 r\cos\theta \Rightarrow A_1 = -E_0, \text{ all other } A_\ell = 0$$

$$\Rightarrow \Phi(r,\theta) = -E_0\left(r - \frac{R^3}{r^2} \right)\cos\theta$$

The 1st term $-E_0 r\cos\theta$ is due to the external field; the contribution attributable to the induced charge is $E_0\dfrac{R^3}{r^2}\cos\theta$

The induced charge density $\sigma(\theta) = -\epsilon_0 \dfrac{\partial\Phi}{\partial r}\Big|_{r=R} = \epsilon_0 E_0\left(1+2\dfrac{R^3}{r^3}\right)\cos\theta\Big|_{r=R} = 3\epsilon_0 E_0\cos\theta$

$$\Rightarrow \begin{cases} \sigma > 0 \text{ for } 0 \le \theta \le \pi/2 & \text{the northern hemisphere} \\ \sigma < 0 \text{ for } \pi/2 \le \theta \le \pi & \text{the southern hemisphere} \end{cases}$$

Example 3.9: A specified charge density $\sigma_0(\theta)$ is glued over the surface of a spherical shell of radius R. Find the potential inside and outside the sphere.

By direct integration, $\Phi = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{\sigma_0}{\mathbb{r}}\, \mathrm{d}a$, but separation of variables is easier:

$$\Rightarrow \begin{cases} \Phi_{\text{in}}(r,\theta) = \displaystyle\sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta), \quad r \le R \quad \text{inside} \\ \Phi_{\text{out}}(r,\theta) = \displaystyle\sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta), \quad r \ge R \quad \text{outside} \end{cases}$$

Condition: $\Phi_{\text{in}}(r=R) = \Phi_{\text{out}}(r=R) \Leftarrow$ potential being *continuous*

$$\Rightarrow \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta) \Rightarrow B_\ell = A_\ell R^{2\ell+1}$$

The radial derivative of Φ has a discontinuity at the surface: $\left(\dfrac{\partial \Phi_{\text{out}}}{\partial r} - \dfrac{\partial \Phi_{\text{in}}}{\partial r}\right)_{r=R} = -\dfrac{\sigma_0(\theta)}{\epsilon_0}$

$$\Rightarrow -\sum_{\ell=0}^{\infty} (\ell+1)\frac{B_\ell}{R^{\ell+2}} P_\ell(\cos\theta) - \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos\theta) = -\frac{\sigma_0(\theta)}{\epsilon_0}$$

$$\Rightarrow \sum_{\ell=0}^{\infty} (2\ell+1) A_\ell R^{\ell-1} P_\ell(\cos\theta) = \frac{\sigma_0(\theta)}{\epsilon_0}$$

$$\Rightarrow \quad A_\ell = \frac{1}{2\epsilon_0 R^{\ell-1}} \int_0^\pi \sigma_0(\theta)\, P_\ell(\cos\theta) \sin\theta \, d\theta$$

If $\sigma_0(\theta) = k\cos\theta = k\, P_1(\cos\theta) \quad \Rightarrow$ all the A_ℓ's are 0 except

$$A_1 = \frac{k}{2\epsilon_0} \int_0^\pi P_1^2(\cos\theta) \sin\theta \, d\theta = \frac{k}{3\epsilon_0} \quad \Rightarrow \quad \begin{cases} \Phi_{\text{in}}(r,\theta) = \dfrac{k}{3\epsilon_0}\, r \cos\theta, & r \le R \\ \Phi_{\text{out}}(r,\theta) = \dfrac{k}{3\epsilon_0} \dfrac{R^3}{r^2} \cos\theta, & r \ge R \end{cases}$$

$$\Rightarrow \quad \Phi(r,\theta) = \frac{k}{3\epsilon_0} \frac{r_<^3}{r^2} \cos\theta \quad \text{where} \quad r_< = \min(r, R)$$

If $\sigma_0(\theta)$ is the induced charge on a metal sphere in an external $E_0 \hat{\mathbf{z}} \;\Rightarrow\; k = 3\epsilon_0 E_0$ then the potential inside is $E_0 r \cos\theta = E_0 z$, and the field is $-E_0 \hat{\mathbf{z}}$ —exactly right to cancel off the external field.

Outside the sphere the potential due to this surface charge is $E_0 \dfrac{R^3}{r^2} \cos\theta$

Cylindrical Coordinates

- For axis-symmetric objects, cylindrical coordinates are more natural.
- $$\nabla^2 \Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$ A general solution requires the knowlege of **Bessel functions**.

- When the length of the cylindrical geometry is large to its radius, the potential may be considered to be independent of z

$$\frac{\partial^2 \Phi}{\partial z^2} = 0 \quad \Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0 \quad \Leftarrow \quad \text{2D problem}$$

$$\Rightarrow \quad \Phi(r, \phi) = R(r)\,\Psi(\phi) \quad \Rightarrow \quad \frac{r}{R}\frac{d}{d r}\left(r \frac{d R}{d r} \right) + \frac{1}{\Psi}\frac{d^2 \Psi}{d \phi^2} = 0$$

$$\Rightarrow \quad \frac{r}{R}\frac{d}{d r}\left(r \frac{d R}{d r} \right) = k^2, \quad \frac{1}{\Psi}\frac{d^2 \Psi}{d \phi^2} = -k^2 \quad \Leftarrow \quad k \text{ is a constant}$$

If the range of ϕ is unrestricted $\Rightarrow$ $k = n \in \mathbb{Z}$

$$\Rightarrow \quad \begin{array}{l} r^2 \dfrac{d^2 R}{d r^2} + r \dfrac{d R}{d r} - n^2 R = 0 \\ \dfrac{d^2 \Psi}{d \phi^2} + n^2 \Psi = 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} R_n(r) = A\, r^n + \dfrac{B}{r^n} \\ \Psi_n(\phi) = C \sin n\phi + D \cos n\phi \end{array}$$

$$\Rightarrow \quad \Phi_n = \left(A_n r^n + \frac{B_n}{r^n} \right) \sin n\phi + \left(C_n r^n + \frac{D_n}{r^n} \right) \cos n\phi, \quad n \neq 0$$

- If the region of interest includes the cylindrical axis $r=0$, the terms with the $\frac{1}{r^n}$ factor cannot exist. If the region of interest includes the point at ∞, the terms with the r^n factor cannot exist, since the potential must be 0 as $r\to\infty$.

- $k=0 \Rightarrow$

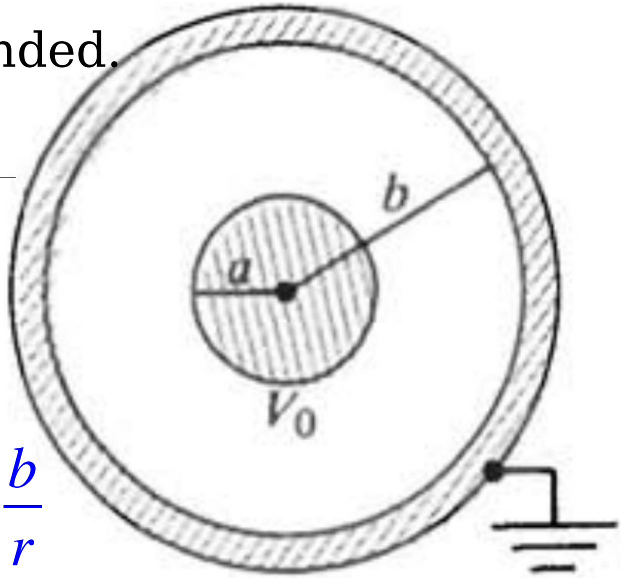
$$\frac{d}{dr}\left(r\frac{dR}{dr}\right)=0 \qquad R(r)=A_0\ln r+B_0$$

$$\frac{d^2\Psi}{d\phi^2}=0 \Rightarrow \Psi(\phi)=C_0\phi+D_0 = D_0 \Leftarrow C_0=0 \qquad \Rightarrow \Phi_0=A_0\ln r+B_0$$

$$\Rightarrow \Phi(r,\phi)=\sum_{n=0}^{\infty}\Phi_n=\Phi_0+\sum_{n=1}^{\infty}\Phi_n$$

$$=A_0\ln r+B_0+\sum_{n=1}^{\infty}\left[\left(A_n r^n+\frac{B_n}{r^n}\right)\sin n\phi+\left(C_n r^n+\frac{D_n}{r^n}\right)\cos n\phi\right]$$

Example: For a very long coaxial cable, the inner conductor is of radius a and of potential Φ_0. The outer conductor is of radius b and grounded. Find the potential in the space between the conductors.

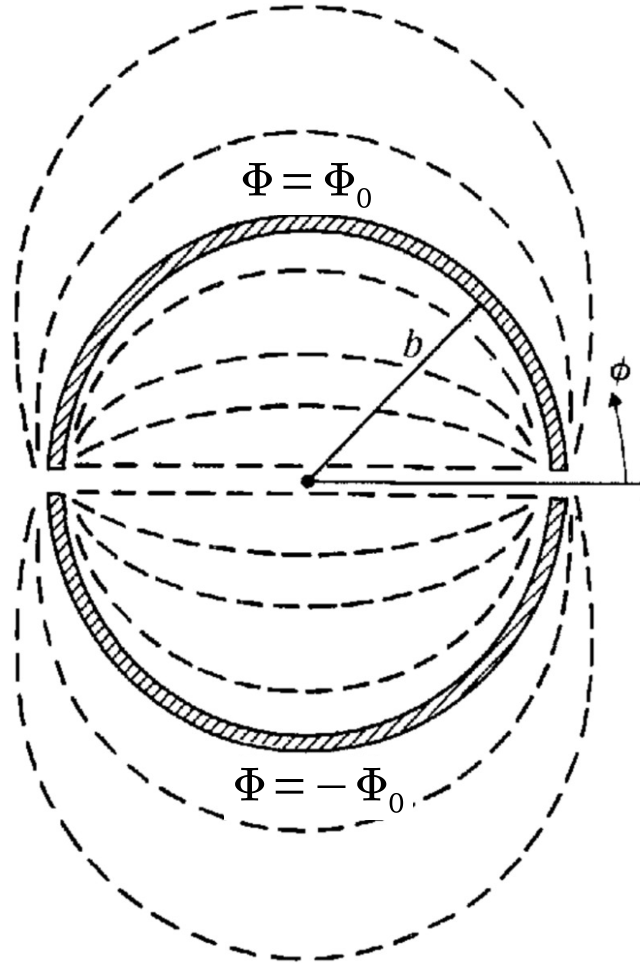


No z-dependence, and no ϕ-dependence by symmetry

$$\Rightarrow k=0 \Rightarrow \Phi=A_0\ln r+B_0 \Rightarrow \begin{array}{l}\Phi(b)=0 \;=A_0\ln b+B_0\\ \Phi(a)=\Phi_0=A_0\ln a+B_0\end{array}$$

$$\Rightarrow A_0=-\frac{\Phi_0}{\ln(b/a)},\; B_0=\frac{\Phi_0\ln b}{\ln(b/a)} \Rightarrow \Phi(r)=\frac{\Phi_0}{\ln(b/a)}\ln\frac{b}{r}$$

<u>Example</u>: A long conducting circular tube of radius b is split in 2 halves. The upper half is kept at $\Phi=\Phi_0$ and the lower half at $\Phi=-\Phi_0$. Find the potential both inside and outside the tube.



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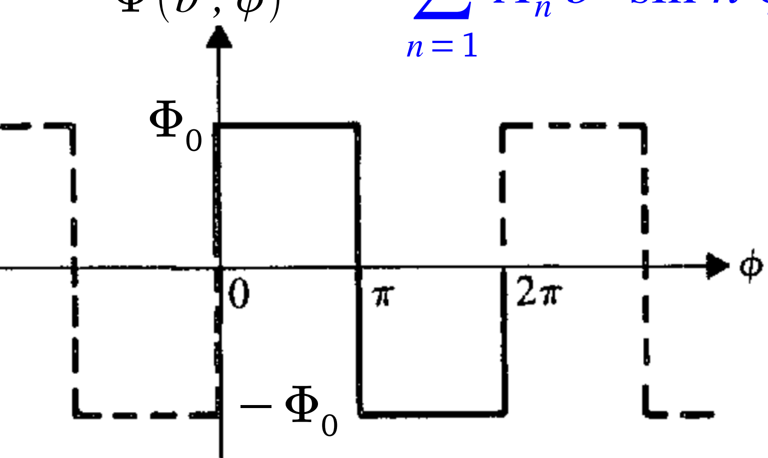
z-independence $\Rightarrow$ $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial \phi^2}=0$$

$$\Phi(b,\phi)=\begin{cases}\Phi_0 & \text{for } 0<\phi<\pi \\ -\Phi_0 & \text{for } \pi<\phi<2\pi\end{cases}$$ $\Rightarrow$ $\Phi(r,\phi)$ is an odd function of ϕ

$\Rightarrow$ Only the terms with $\sin n\phi$ in $\Phi(r,\phi)$ survive.

Inside the tube, $r<b$, the r^{-n} factor terms cannot exist,

$$\Rightarrow \quad \Phi(r,\phi)=\sum_{n=1}^{\infty} A_n r^n \sin n\phi$$



$$\Rightarrow \quad \sum_{n=1}^{\infty} A_n b^n \sin n\phi=\begin{cases}\Phi_0 & \text{for } 0<\phi<\pi \\ -\Phi_0 & \text{for } \pi<\phi<2\pi\end{cases} \quad \Rightarrow \quad A_n=\begin{cases}\dfrac{4\Phi_0}{n\pi b^n} & \text{for odd } n \\ 0 & \text{for even } n\end{cases}$$

$$\Rightarrow \quad \Phi(r,\phi)=\frac{4\Phi_0}{\pi}\sum_{\text{odd } n}^{\infty}\left(\frac{r}{b}\right)^n\frac{\sin n\phi}{n}, \qquad r<b$$

$$=\frac{4\Phi_0}{\pi}\sum_{m=0}^{\infty}\left(\frac{r}{b}\right)^{2m+1}\frac{\sin(2m+1)\phi}{2m+1}, \qquad r<b$$

Outside the tube, $r>b$, the r^n factor terms cannot exist,

$$\Rightarrow \quad \Phi(r,\phi)=\sum_{n=1}^{\infty}\frac{B_n}{r^n}\sin n\phi$$

$$\Rightarrow \quad \Phi(b,\phi)=\sum_{n=1}^{\infty}\frac{B_n}{b^n}\sin n\phi=\begin{cases}\Phi_0 & \text{for } 0<\phi<\pi\\ -\Phi_0 & \text{for } \pi<\phi<2\pi\end{cases} \quad \Rightarrow \quad B_n=\begin{cases}\dfrac{4\Phi_0 b^n}{n\pi} & \text{for odd } n\\ 0 & \text{for even } n\end{cases}$$

$$\Rightarrow \quad \Phi(r,\phi)=\frac{4\Phi_0}{\pi}\sum_{\text{odd } n}^{\infty}\left(\frac{b}{r}\right)^n\frac{\sin n\phi}{n}, \qquad r>b$$

$$=\frac{4\Phi_0}{\pi}\sum_{m=0}^{\infty}\left(\frac{b}{r}\right)^{2m+1}\frac{\sin(2m+1)\phi}{2m+1}, \quad r>b$$

General expression:

$$\Phi(r,\phi)=\frac{4\Phi_0}{\pi}\sum_{\text{odd } n}^{\infty}\left(\frac{r_<}{r_>}\right)^n\frac{\sin n\phi}{n}=\frac{4\Phi_0}{\pi}\sum_{m=0}^{\infty}\left(\frac{r_<}{r_>}\right)^{2m+1}\frac{\sin(2m+1)\phi}{2m+1}$$

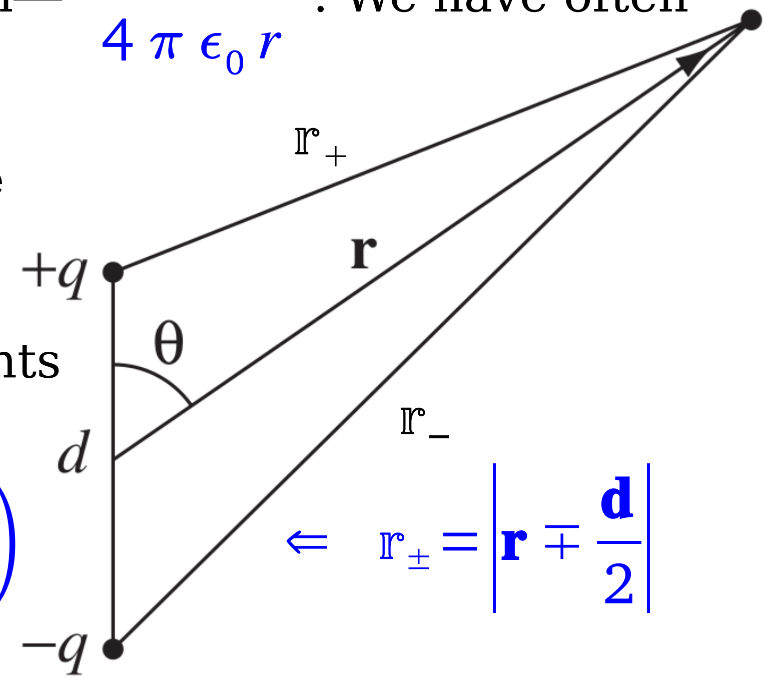
$$\text{where } r_{\lessgtr}=\begin{matrix}\min\\ \max\end{matrix}(r,b)$$

Multipole Expansion

Approximate Potentials at Large Distances

- If you are far away from a localized charge distribution, it "looks" like a point charge, and the potential is—to good approximation— $\dfrac{Q}{4\pi\epsilon_0 r}$. We have often used this as a check on formulas for Φ.

- An **electric dipole** consists of 2 equal & opposite charges ($\pm q$) separated by a distance d.



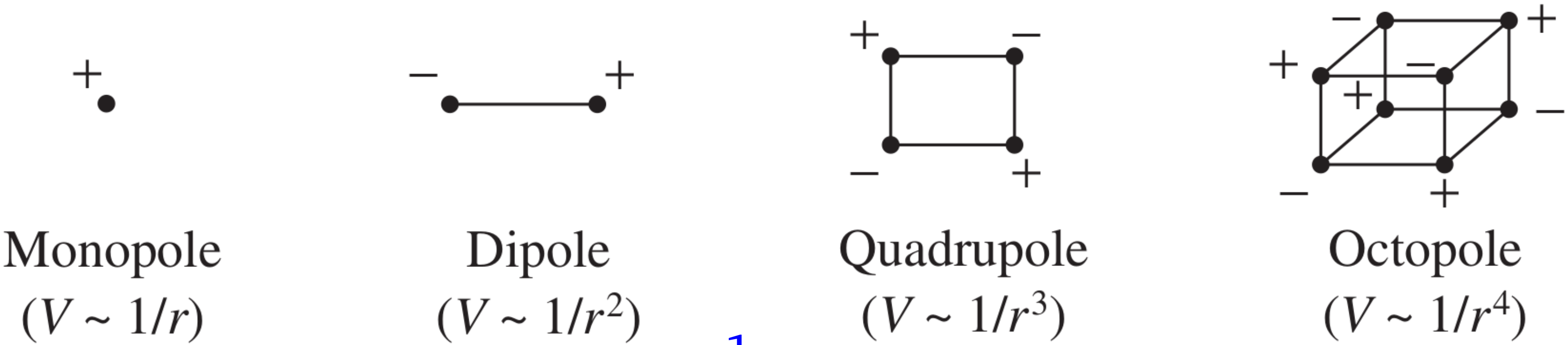
Example 3.10: ind the approximate potential at points far from the dipole.

- $$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} + \frac{-q}{\mathscr{r}_-}\right) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-}\right) \qquad \Leftarrow \quad \mathscr{r}_\pm = \left|\mathbf{r} \mp \frac{\mathbf{d}}{2}\right|$$

$$\mathscr{r}_\pm^2 = r^2 \mp r\,d\cos\theta + \frac{d^2}{4} = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

$$r \gg d \quad \Rightarrow \quad \frac{1}{\mathscr{r}_\pm} \simeq \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

$$\Rightarrow \quad \frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} \simeq \frac{d}{r^2}\cos\theta \quad \Rightarrow \quad \Phi(\mathbf{r}) \simeq \frac{1}{4\pi\epsilon_0}\,\frac{q\,d\cos\theta}{r^2}$$

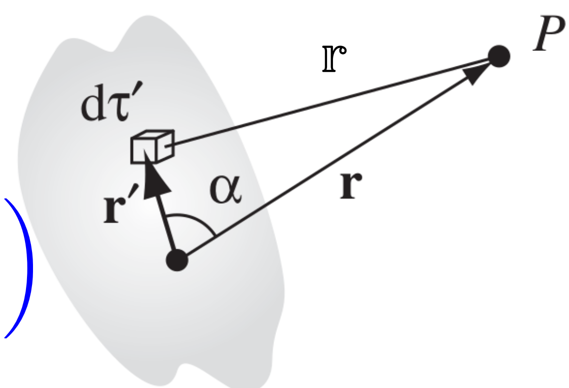


- The potential of a dipole goes like $\dfrac{1}{r^2}$ at large r; it falls off more rapidly than that of a point charge.

- If we put together a pair of equal & opposite dipoles to make a **quadrupole**, the potential goes like $\dfrac{1}{r^3}$; for back-to-back *quadrupoles* (an **octopole**), it goes like $\dfrac{1}{r^4}$; and so on.

- For an electric **monopole** (point charge), whose potential goes like $\dfrac{1}{r}$.

- The potential at $\mathbf{r}$ for any localized charge distribution



$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\, d\tau' \quad \Leftarrow \quad \begin{aligned} \mathscr{r}^2 &= r^2 + r'^2 - 2\, r\, r' \cos\alpha \\ &= r^2\left(1 + \frac{r'^2}{r^2} - 2\,\frac{r'}{r}\cos\alpha\right) \end{aligned}$$

$$\Rightarrow \quad \mathscr{r} = r\sqrt{1+\epsilon} \quad \Leftarrow \quad \epsilon \equiv \frac{r'}{r}\left(\frac{r'}{r} - 2\cos\alpha\right)$$

$$\Rightarrow \quad \frac{1}{\mathbb{r}} = \frac{1}{r}\frac{1}{\sqrt{1+\epsilon}} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots\right)$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\frac{r'}{r}\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\frac{r'^2}{r^2}\left(\frac{r'}{r} - 2\cos\alpha\right)^2 - \frac{5}{16}\frac{r'^3}{r^3}\left(\frac{r'}{r} - 2\cos\alpha\right)^3 + \cdots\right]$$

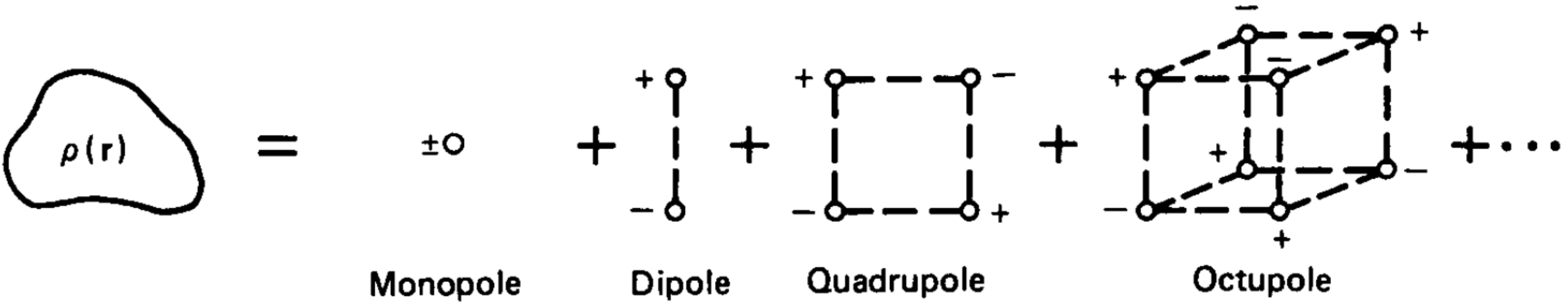
$$= \frac{1}{r}\left[1 + \frac{r'}{r}\cos\alpha + \left(\frac{r'}{r}\right)^2 \frac{3\cos^2\alpha - 1}{2} + \left(\frac{r'}{r}\right)^3 \frac{5\cos^3\alpha - 3\cos\alpha}{2} + \cdots\right]$$

$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha) \quad \Rightarrow \frac{1}{\mathbb{r}}$$

: the *generating function* for Legendre polynomials

$$\Rightarrow \quad \Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int r'^n P_n(\cos\alpha)\,\rho(\mathbf{r}')\,d\tau' \qquad (@)$$

$$\Rightarrow \quad \Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\alpha\,\rho(\mathbf{r}')\,d\tau' + \frac{1}{r^3}\int r'^2\,\frac{3\cos^2\alpha - 1}{2}\,\rho(\mathbf{r}')\,d\tau' + \cdots\right)$$



- This is the desired result—the **multipole expansion** of Φ in powers of $\frac{1}{r}$.
- The 1st term ($n=0$) is the monopole contribution ($\frac{1}{r}$); the 2nd ($n=1$) is the dipole ($\frac{1}{r^2}$); the 3rd is quadrupole; the 4th is octopole; and so on.
- α is the angle of $\mathbf{r}$ & $\mathbf{r}'$, the integrals depend on the direction to the field point.
- For the potential along the z'-axis, then α is the usual polar angle θ'.
- (@) is *exact*, but it is *useful* primarily as an approximation scheme: the lowest nonzero term in the expansion provides the approximate potential at large r, and the successive terms tell us how to improve the approximation if greater precision is required.

The Monopole and Dipole Terms

- The multipole expansion is dominated (at large r) by the monopole term:

$$\Phi_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \quad \Leftarrow \quad Q = \int \rho \, \mathrm{d}\tau \quad \text{total charge}$$

- For a *point* charge *at the origin*, Φ_{mon} is the *exact* potential, not merely a 1st approximation at large r; in this case, all the higher multipoles vanish.

- If the total charge is 0, the dominant term in the potential will be the dipole:

$$\Phi_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r' \rho(\mathbf{r}') \cos\alpha \, \mathrm{d}\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}\cdot\int \mathbf{r}' \rho(\mathbf{r}') \, \mathrm{d}\tau' \quad \Leftarrow \quad \hat{\mathbf{r}}\cdot\mathbf{r}' = r'\cos\alpha$$

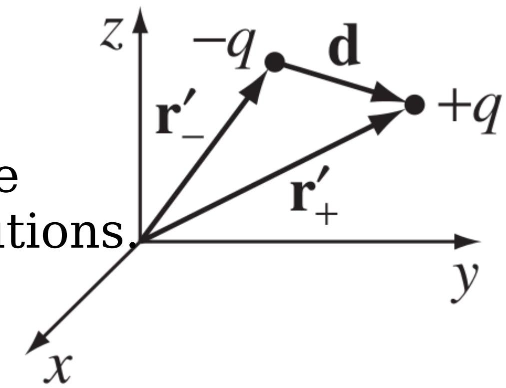
$$\Rightarrow \quad \Phi_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \quad \Leftarrow \quad \mathbf{p} \equiv \int \mathbf{r}' \rho(\mathbf{r}') \, \mathrm{d}\tau' \quad \text{dipole moment}$$

- The dipole moment is determined by the geometry (size, shape, and density) of the charge distribution.

- The dipole moment of a collection of *point* charges is $\mathbf{p} = \sum_{i=1}^{n} q_i \mathbf{r}_i'$

- For a physical dipole (equal & opposite charges, $\pm q$), $\mathbf{p} = q\,\mathbf{r}_+' - q\,\mathbf{r}_-'$

$$= q(\mathbf{r}_+' - \mathbf{r}_-') = q\,\mathbf{d}$$



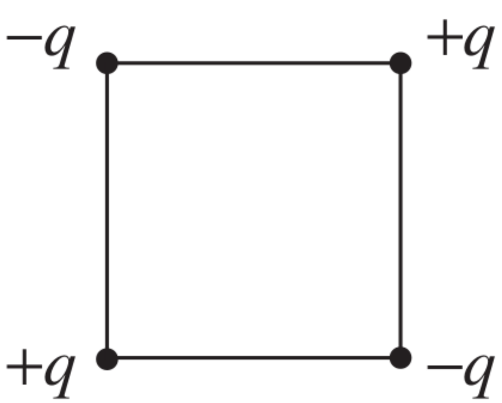
- The result in Ex 3.10 is only the approximate potential of the physical dipole—evidently there are higher multipole contributions.

- As you go farther and farther away, Φ_{dip} becomes a better and better approximation, since the higher terms die off more rapidly with increasing r.

- Dipole moments are vectors, and they add accordingly: if you have 2 dipoles, $\mathbf{p}_1$ and $\mathbf{p}_2$, the total dipole moment is $\mathbf{p}_1+\mathbf{p}_2$.

- With 4 charges at the corners of a square, the net dipole moment is 0.

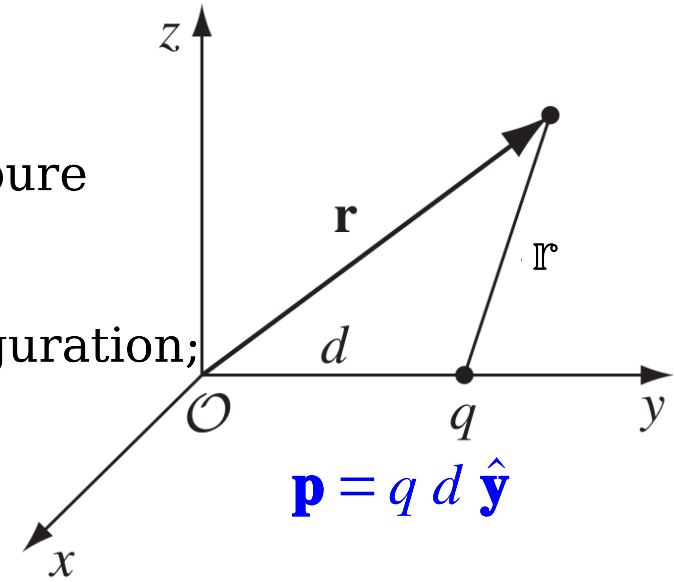
- This is a *quadrupole*, and its potential is dominated by the quadrupole term in the multipole expansion.



- The total potential can be expressed as

$$\Phi(\mathbf{r})=\frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r}+\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}+\frac{1}{2r^3}\sum_{i,j}Q_{ij}\frac{x_i x_j}{r^2}+\cdots\right)$$

Origin of Coordinates in Multipole Expansions



- If a point charge is *not* at the origin, it's no longer a pure monopole.

- The monopole potential is not $\dfrac{q}{4\pi\epsilon_0 r}$ for this configuration; rather, the exact potential is $\dfrac{q}{4\pi\epsilon_0 \mathfrak{r}}$.

$$\mathbf{p} = q\,d\,\hat{\mathbf{y}}$$

- The multipole expansion is a series in inverse powers of r (the distance to the *origin*), and when we expand $\dfrac{1}{r}$, we get *all* powers, not just the 1st. So moving the origin can radically alter a multipole expansion.

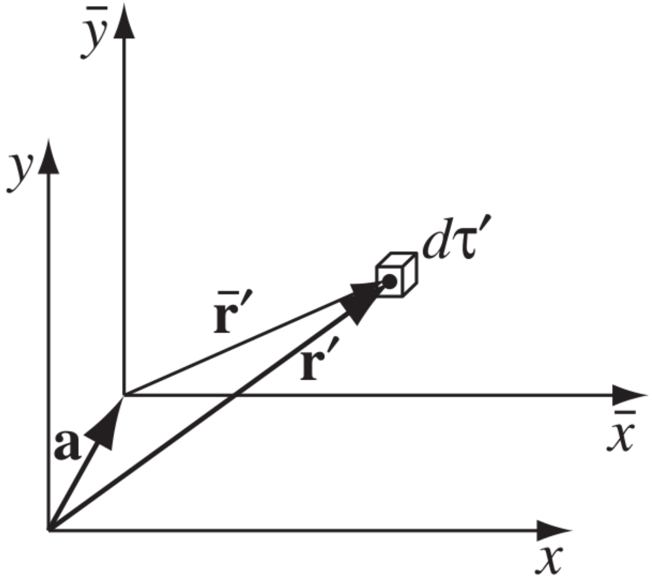
- The **monopole moment** Q does not change, since the total charge is obviously independent of the coordinate system. But the other multipoles are not.

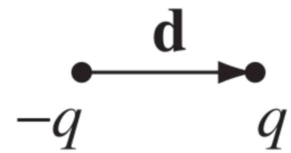
- *If the total charge is 0, the dipole moment is independent of the choice of origin.*

- If we displace the origin by an amount $\mathbf{a}$, the new dipole moment is then

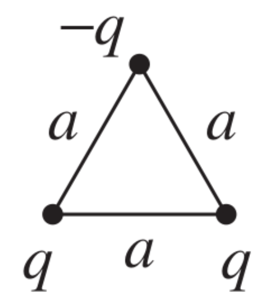
$$\bar{\mathbf{p}} = \int \bar{\mathbf{r}}'\,\rho(\mathbf{r}')\,\mathrm{d}\tau' = \int (\mathbf{r}' - \mathbf{a})\,\rho(\mathbf{r}')\,\mathrm{d}\tau' = \int \mathbf{r}'\,\rho(\mathbf{r}')\,\mathrm{d}\tau' - \mathbf{a}\int \rho(\mathbf{r}')\,\mathrm{d}\tau'$$

$$= \mathbf{p} - Q\,\mathbf{a} \quad\Rightarrow\quad \text{when } Q = 0 \quad\Rightarrow\quad \bar{\mathbf{p}} = \mathbf{p}$$





(a)



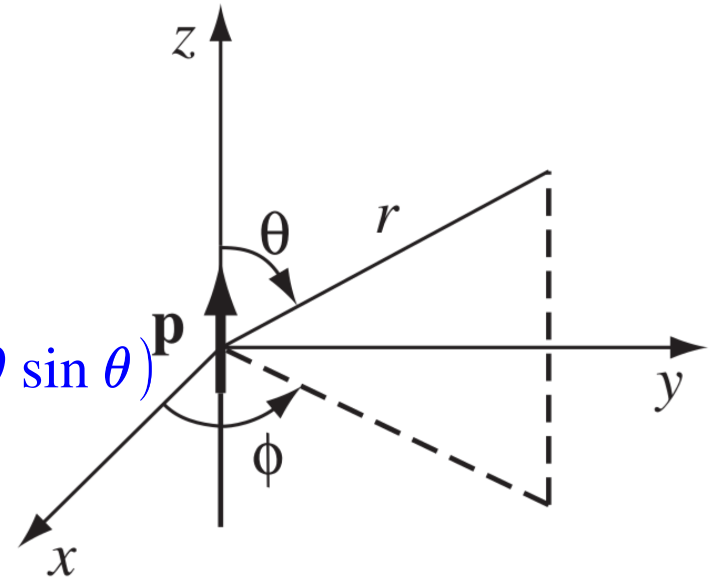
(b)

- If someone asks for the dipole moment in Fig. a, you can answer with confidence "$q\,d$" (because $Q=0$), but if you're asked for the dipole moment in Fig. (b), the appropriate response would be "With respect to *what origin*" (because $Q\neq 0$)?

- **Theorem**: For an arbitrary charge distribution $\rho(\mathbf{r})$ the components of the first nonvanishing multipole are independent of the origin of the coordinate axes, but the values of all higher multipole moments do in general depend on the choice of origin.

The Electric Field of a Dipole

- Let $\mathbf{p}$ is at the origin and points in the z direction,



$$\Phi_{\text{dip}}(r,\theta)=\frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{4\pi\epsilon_0 r^2}=\frac{p\cos\theta}{4\pi\epsilon_0 r^2}\quad\Leftarrow\quad \mathbf{p}=p\,\hat{\mathbf{z}}=p\left(\hat{\mathbf{r}}\cos\theta-\hat{\boldsymbol{\theta}}\sin\theta\right)$$

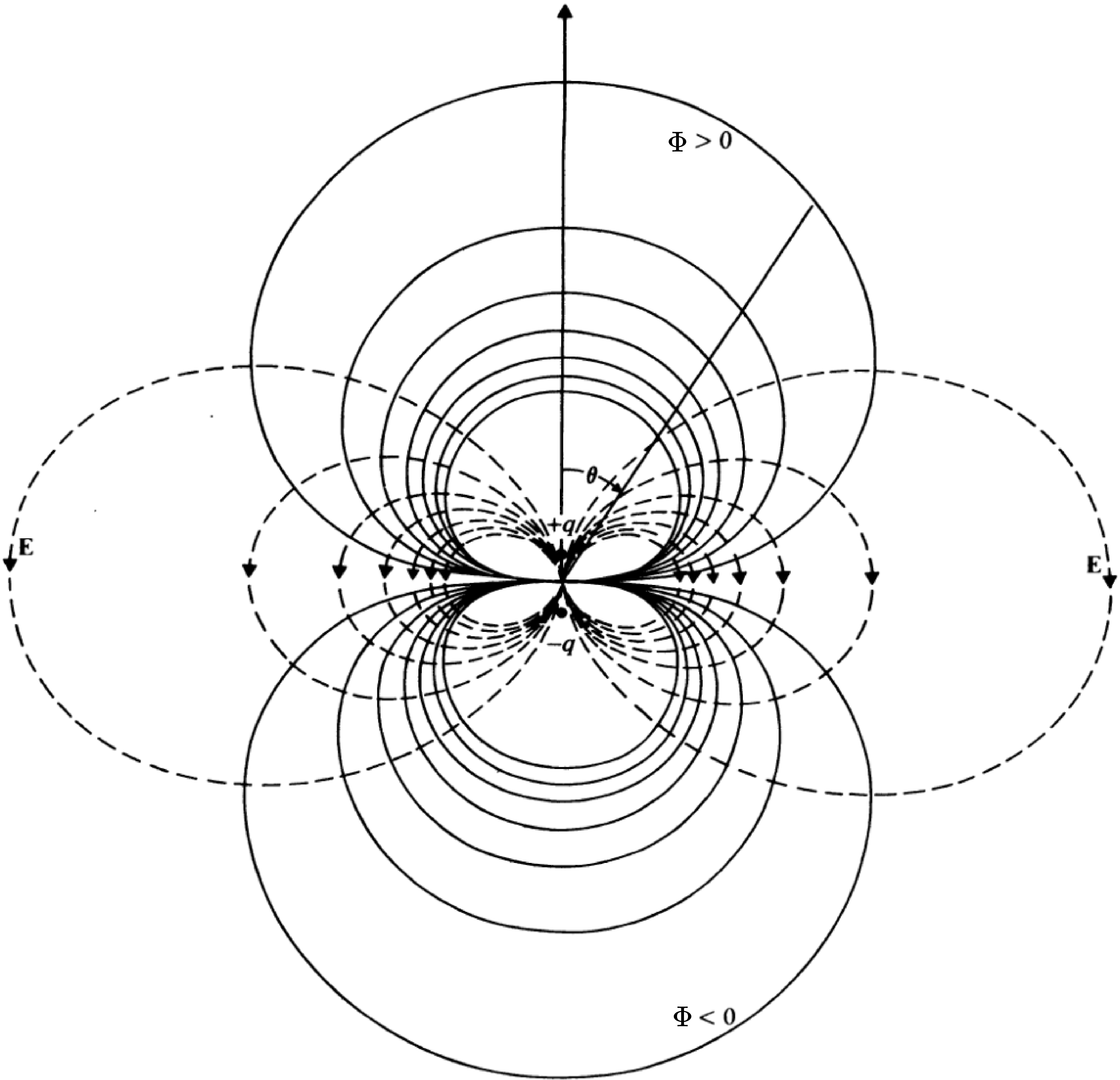
$$\Rightarrow\begin{cases}E_r=-\dfrac{\partial\Phi_{\text{dip}}}{\partial r}=\dfrac{p\cos\theta}{2\pi\epsilon_0 r^3}\\[2ex] E_\theta=-\dfrac{1}{r}\dfrac{\partial\Phi_{\text{dip}}}{\partial\theta}=\dfrac{p\sin\theta}{4\pi\epsilon_0 r^3}\\[2ex] E_\phi=-\dfrac{1}{r\sin\theta}\dfrac{\partial\Phi_{\text{dip}}}{\partial\phi}=0\end{cases}\Rightarrow$$

$$\mathbf{E}_{\text{dip}}(r,\theta)=\frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}}\right)=\frac{1}{4\pi\epsilon_0}\frac{3(\hat{\mathbf{r}}\cdot\mathbf{p})\hat{\mathbf{r}}-\mathbf{p}}{r^3}$$

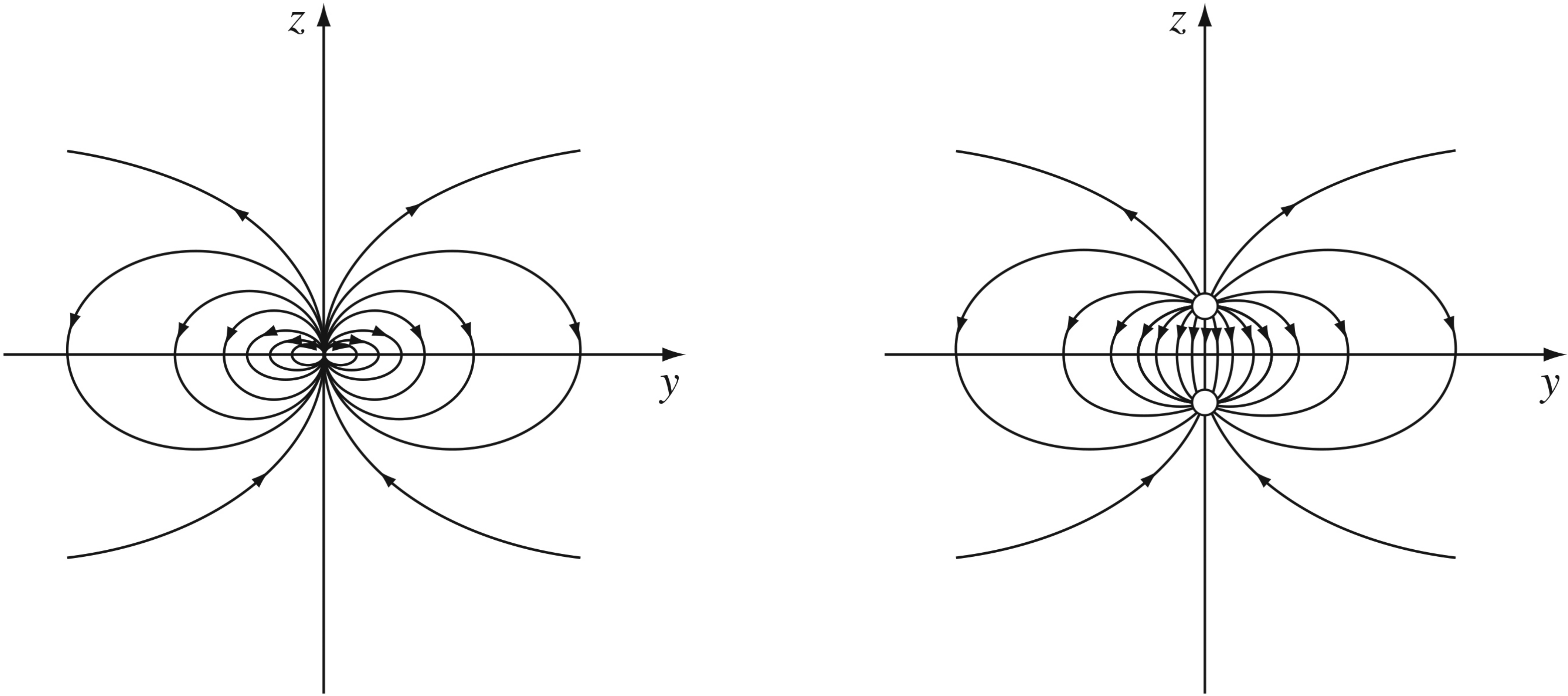
- This formula makes explicit reference to a particular coordinate system (spherical) and assumes a particular orientation for $\mathbf{p}$ (along z). It can be recast in a coordinate-free form (see Prob. 3.36).

- The dipole field falls off as the inverse *cube* of r; the *monopole* field goes as the inverse *square*. Quadrupole fields go like $\frac{1}{r^4}$, octopole like $\frac{1}{r^5}$, and so on.

Selected problems: 5, 11, 16, 19, 32, 43



Φ > 0
θ
+q
E
E
−q
Φ < 0



(a) Field of a "pure" dipole

(b) Field of a "physical" dipole

Quadrupole E & Φ