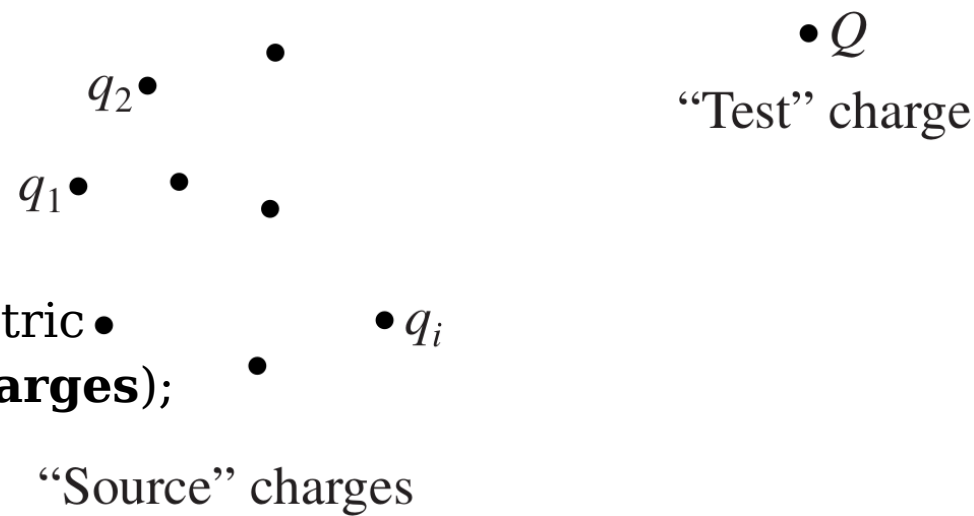


Chapter 2 Electrostatics

The Electric Field

Introduction

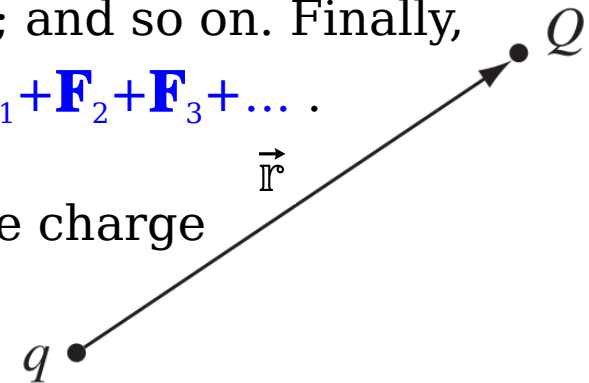
● In electrodynamics, if we have some electric charges, q_1, q_2, q_3, \dots (call them **source charges**); we want to know what force they exert on another charge, Q (call it the **test charge**).



● The solution is facilitated by the **principle of superposition**, the interaction between any 2 charges is completely unaffected by the presence of others.

● To determine the force on Q , we can compute the force \mathbf{F}_1 , due to q_1 alone (ignoring all the others); then compute \mathbf{F}_2 , due to q_2 alone; and so on. Finally, we take the vector sum of all these individual forces: $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \dots$.

● Thus, if we can find the force on Q due to a single source charge q , we are, in principle, done.



● The force on Q depends on the separation distance between the charges, and also depends on their *velocities* and on the *acceleration* of q at some earlier time.

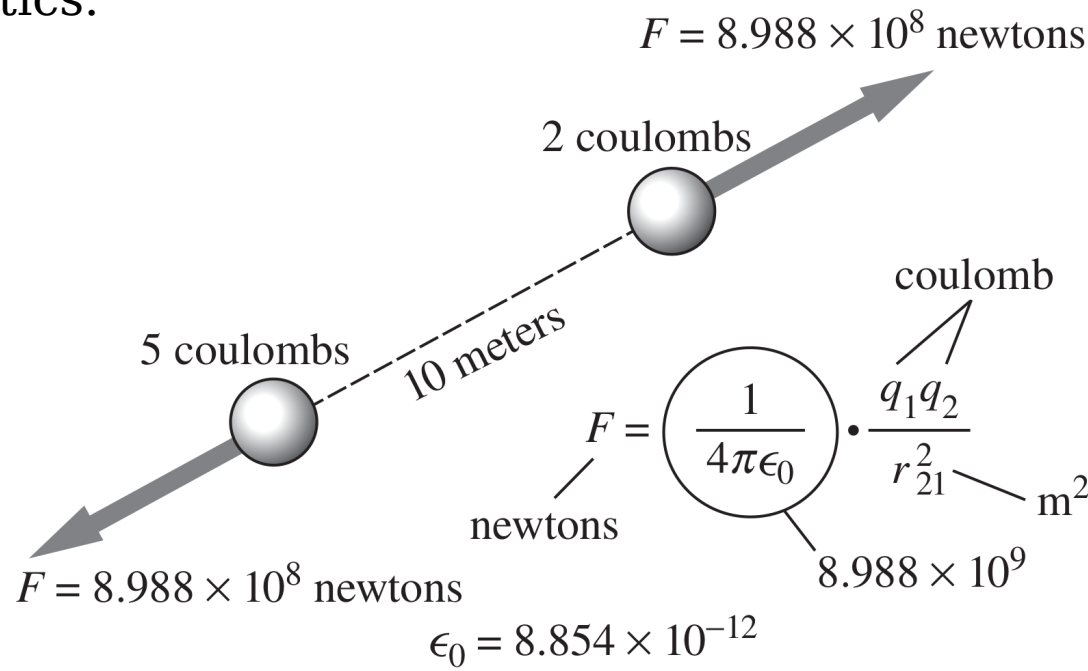
● To begin with, we consider the special case of **electrostatics** in which all the *source charges are stationary* (though the test charge may be moving).

Coulomb's Law

- The force on a test charge Q due to a single point charge q , that is at *rest* a distance away is given by **Coulomb's law**:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \hat{\mathbf{r}} \quad \leftarrow \quad \epsilon_0 = 8.85 \times 10^{-12} \frac{\text{C}^2}{\text{N}\cdot\text{m}^2} \text{ permittivity of free space, } \vec{\mathbf{r}} = \mathbf{r} - \mathbf{r}'$$

- In words, the force is proportional to the product of the charges and inversely proportional to the square of the separation distance.
- The force points along the line from q to Q ; it is repulsive if q and Q have the same sign, and attractive if their signs are opposite.
- Coulomb's law and the principle of superposition constitute the physical input for electrostatics.



The Electric Field

● If we have several point charges q_1, q_2, \dots, q_n , at distances r_1, r_2, \dots, r_n from Q , the total force on Q is evidently

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \dots = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1 Q}{r_1^2} \hat{\mathbf{r}}_1 + \frac{q_2 Q}{r_2^2} \hat{\mathbf{r}}_2 + \dots \right) = \frac{Q}{4\pi\epsilon_0} \left(\frac{q_1}{r_1^2} \hat{\mathbf{r}}_1 + \frac{q_2}{r_2^2} \hat{\mathbf{r}}_2 + \dots \right)$$

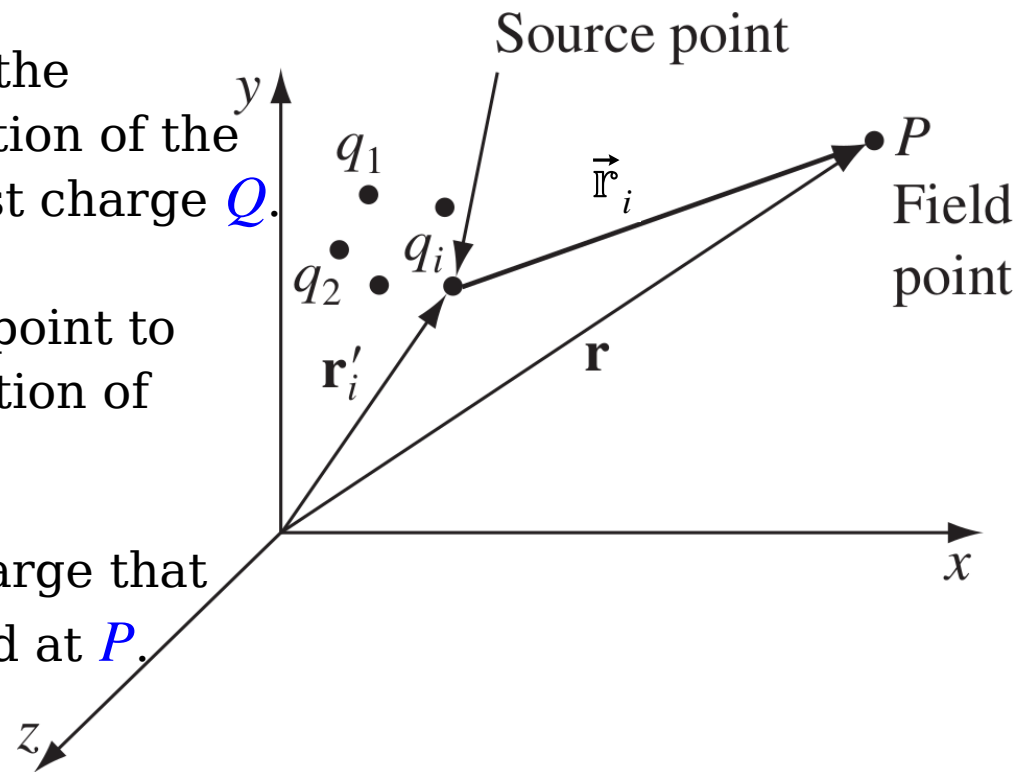
$$\Rightarrow \mathbf{F} = Q\mathbf{E} \quad \leftarrow \quad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \frac{q_i}{r_i^2} \hat{\mathbf{r}}_i \quad \text{the electric field of the source charges}$$

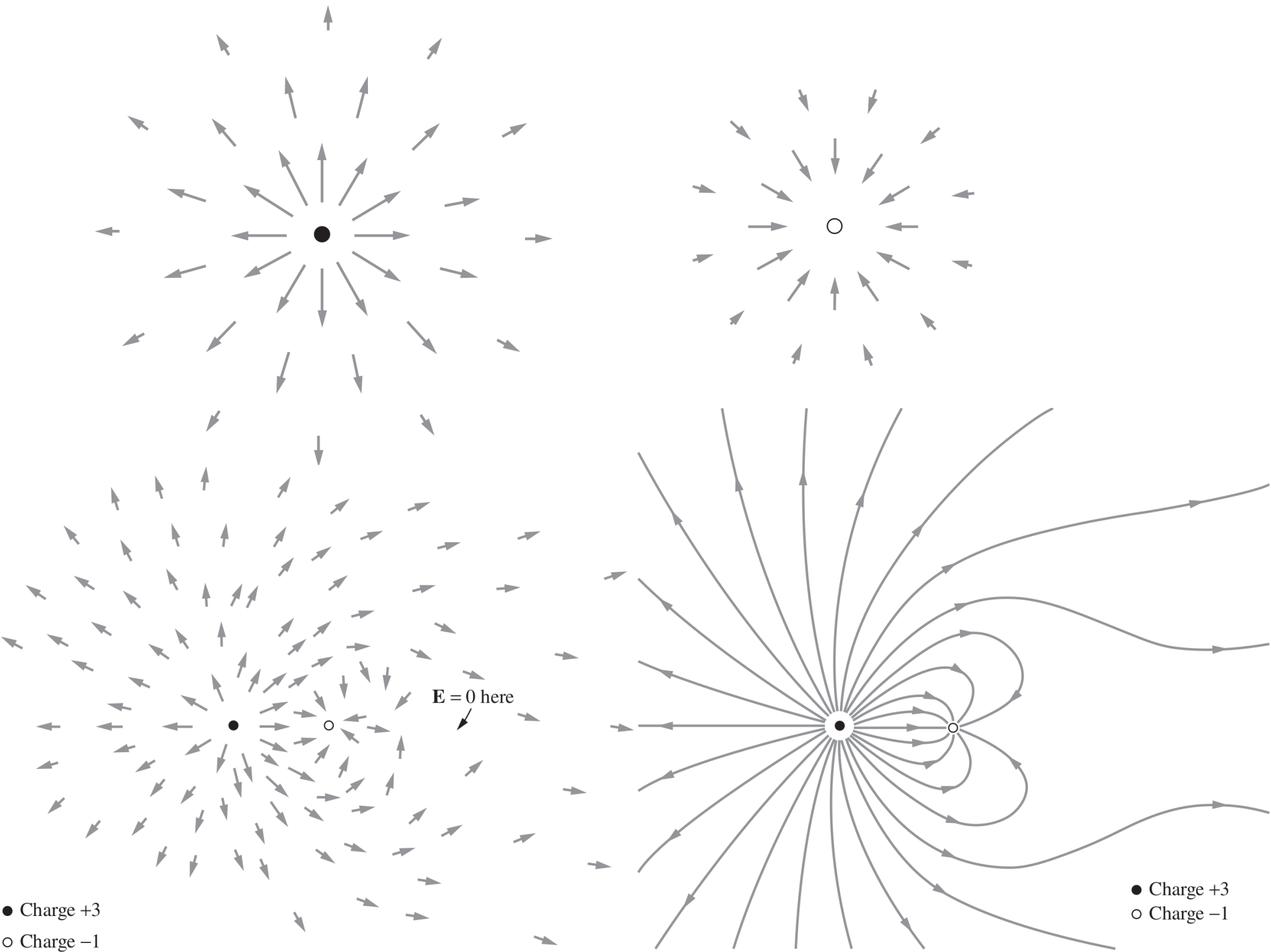
● \mathbf{E} is a function of position (\mathbf{r}), because the separation vectors \vec{r}_i depend on the location of the **field point** P , but no reference to the test charge Q .

● \mathbf{E} is a vector quantity that varies from point to point and is determined by the configuration of source charges.

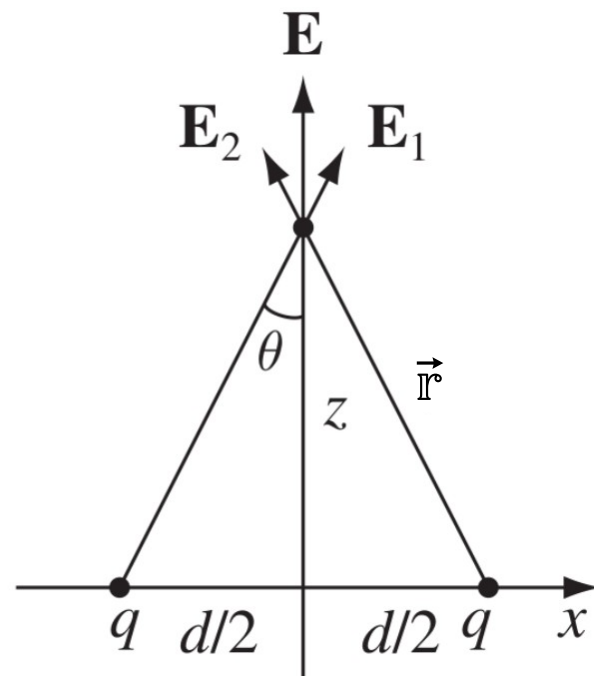
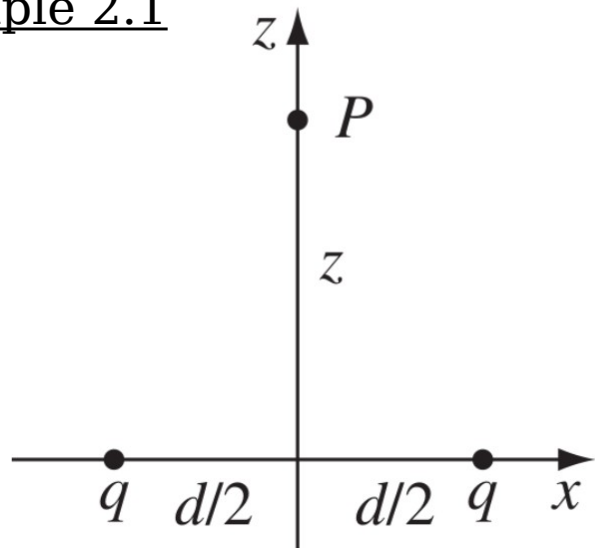
● physically, $\mathbf{E}(\mathbf{r})$ is the force per unit charge that would be exerted on a test charge, placed at P .

● One is encouraged to think of \mathbf{E} as a “real” physical entity, filling the space around electric charges.





Example 2.1



Continuous Charge Distributions

● Our definition of \mathbf{E} assumes that the source of the field is a collection of discrete point charges q_i . If the charge is distributed continuously over some region, then

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{\mathbf{r}}}{r^2} dq$$

● If the charge is spread out along a *line*, with charge-per-unit-length λ , then $dq = \lambda dl'$

● If the charge is smeared out over a *surface*, with charge-per-unit-area σ , then $dq = \sigma da'$

● If the charge fills a *volume*, with charge-per-unit-volume ρ , then $dq = \rho d\tau' \Rightarrow dq \rightarrow \lambda dl' \sim \sigma da' \sim \rho d\tau'$

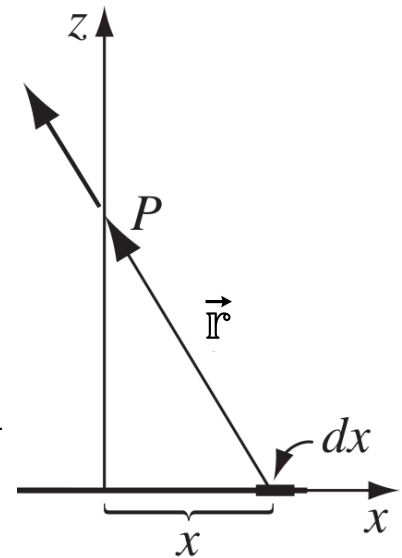
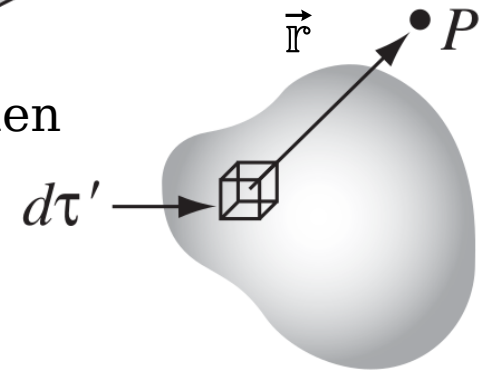
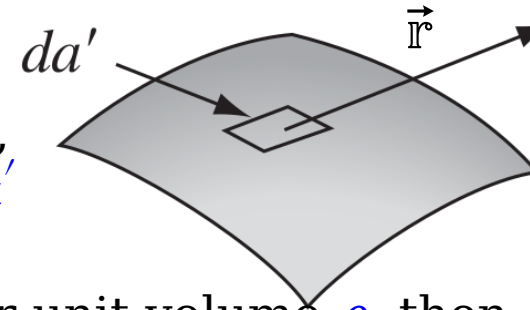
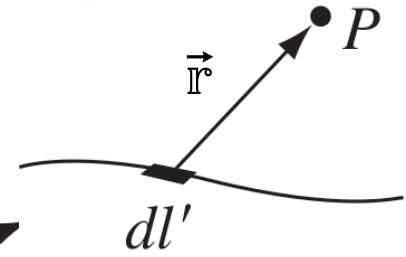
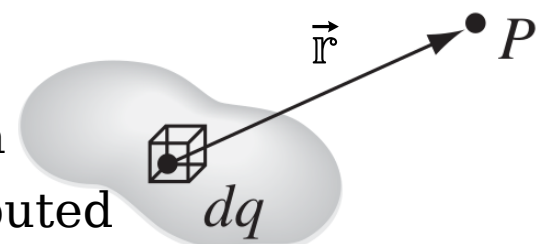
● \mathbf{E} of a line charge:
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\mathbf{r}')}{r^2} \hat{\mathbf{r}} dl'$$

● \mathbf{E} for a surface charge:
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\mathbf{r}')}{r^2} \hat{\mathbf{r}} da'$$

● \mathbf{E} for a volume charge:
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{r^2} \hat{\mathbf{r}} d\tau'$$

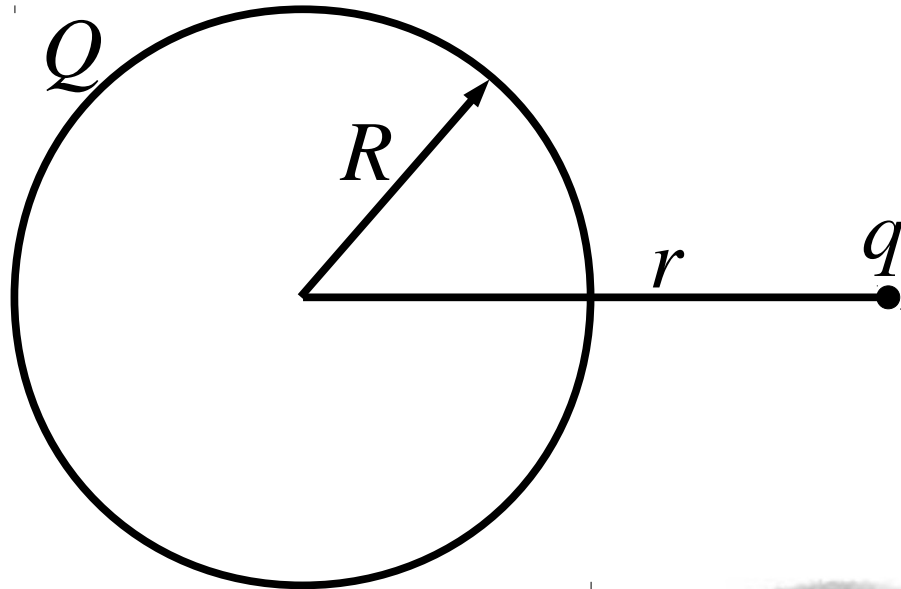
● This eqn itself is often referred to as “Coulomb’s law,” because a volume charge is in a sense the most general and realistic case.

Example 2.2



Shell Theorem

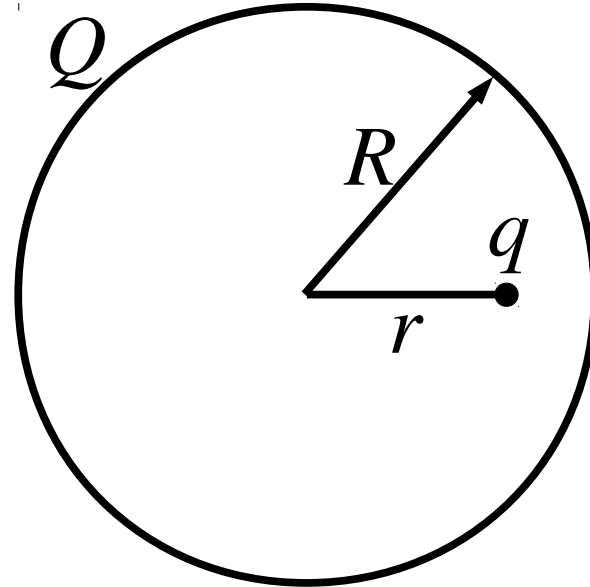
Outside



$$F = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2}$$

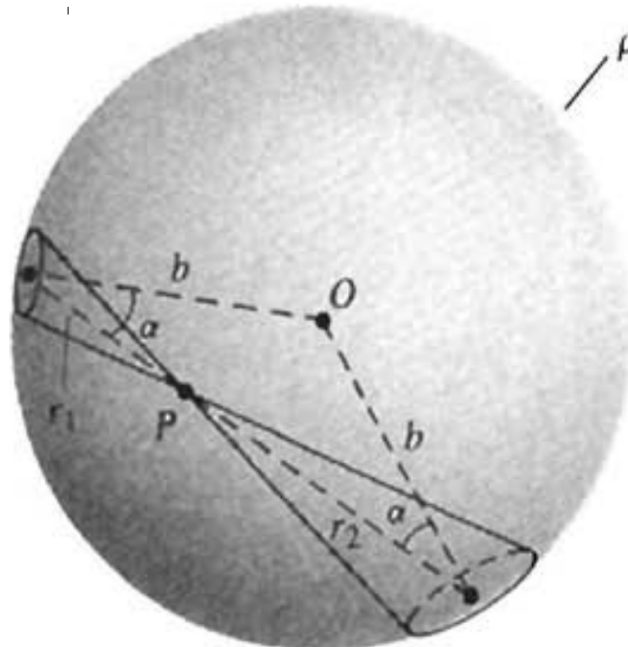
$$d\Omega \equiv \sin\theta \, d\theta \, d\phi$$

Inside



$$F = 0$$

$$\begin{aligned} dF &\propto \frac{q \, dQ}{r^2} = \frac{q\sigma}{r^2} \, da \\ &= \frac{q\sigma r^2 \, d\Omega}{r^2 \cos\alpha} \\ &= \frac{q\sigma \, d\Omega}{\cos\alpha} \end{aligned}$$



Proof of the shell theorem

shell's charge: $Q = 4 \pi R^2 \sigma \Rightarrow \sigma = \frac{Q}{4 \pi R^2}$

ring's charge $dQ = \sigma \times \text{area of ring}$
 $= \sigma \times 2 \pi R \sin \theta \times R d\theta = \frac{Q}{2} \sin \theta d\theta$

law of cosine: $r^2 + R^2 - 2 R r \cos \theta = s^2$
 $\Rightarrow R r \sin \theta d\theta = s ds$

$s^2 + r^2 - 2 s r \cos \phi = R^2 \Rightarrow \cos \phi = \frac{s^2 + r^2 - R^2}{2 r s}$

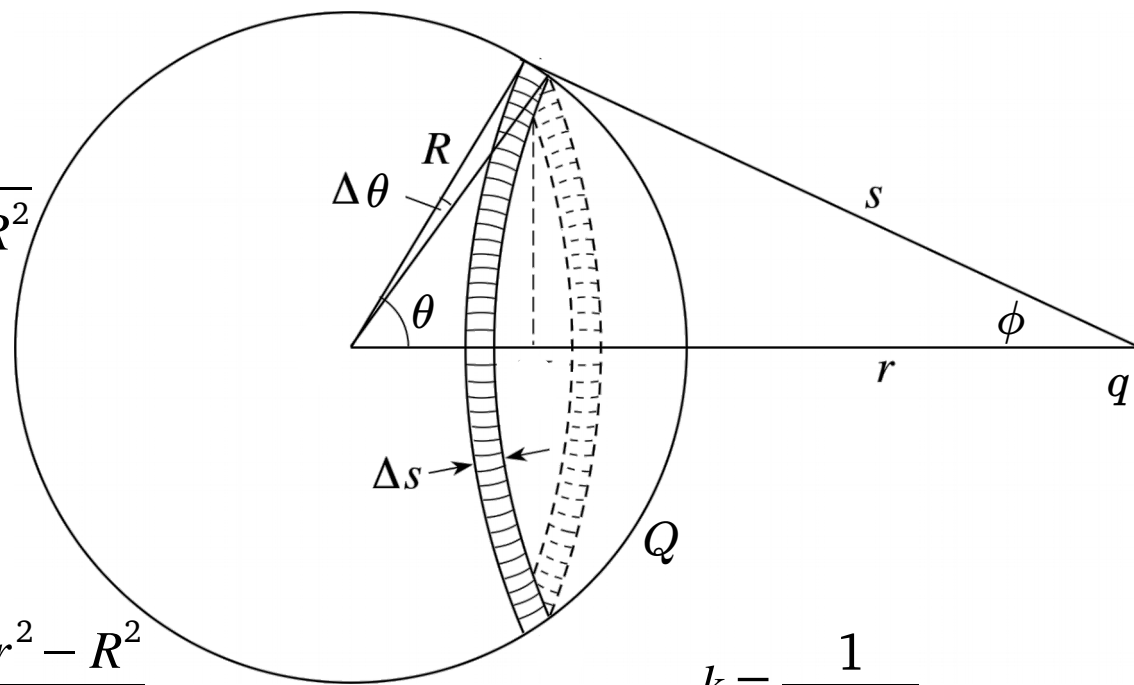
$$k = \frac{1}{4 \pi \epsilon_0}$$

$dF_r = dF \cos \phi = \frac{k q dQ}{s^2} \cos \phi \Leftrightarrow dF = \text{force from the ring} = \frac{k q dQ}{s^2} + \text{symmetry}$

$$\Rightarrow F_r = \int dF_r = k q \int \frac{\cos \phi dQ}{s^2}$$

$$= \frac{k q Q}{2} \int \frac{\sin \theta d\theta}{s^2} \frac{s^2 + r^2 - R^2}{2 r s} = \frac{k q Q}{4 r^2 R} \int \frac{s^2 + r^2 - R^2}{s^2} ds$$

$$= \begin{cases} \frac{k q Q}{4 r^2 R} \int_{r-R}^{r+R} \left(1 + \frac{r^2 - R^2}{s^2} \right) ds = \frac{1}{4 \pi \epsilon_0} \frac{q Q}{r^2} & \text{for } r > R \text{ outside} \\ \frac{k q Q}{4 r^2 R} \int_{R-r}^{r+R} \left(1 + \frac{r^2 - R^2}{s^2} \right) ds = 0 & \text{for } r < R \text{ inside} \end{cases}$$



Divergence and Curl of Electrostatic Fields

Field Lines, Flux, and Gauss's Law

- For a single point charge q , situated at the origin:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

- Because the field falls off like $\frac{1}{r^2}$, the vectors get shorter

as you go farther away from the origin; they always point radially outward.

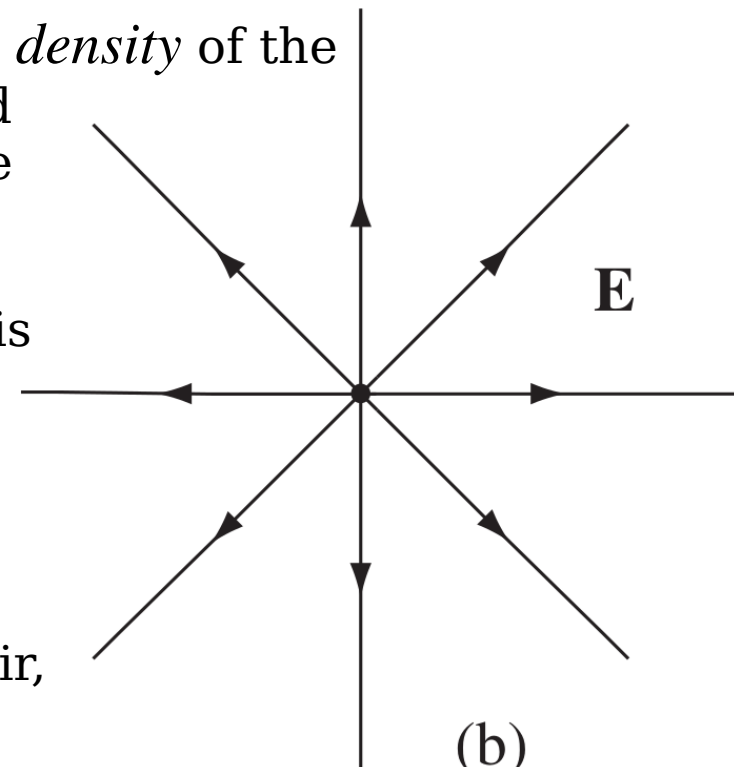
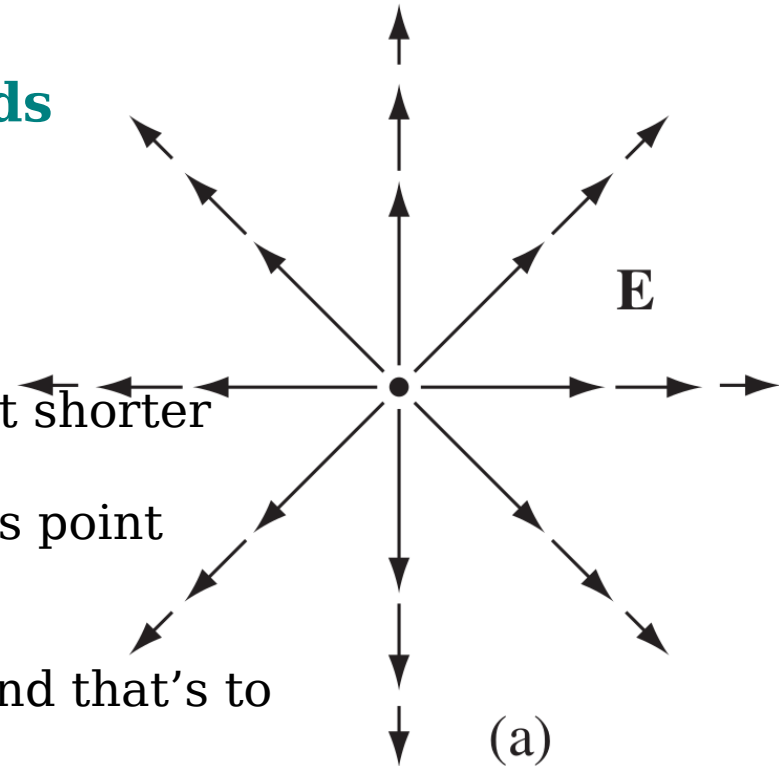
- But there is a nicer way to represent this field, and that's to connect up the arrows, to form **field lines**.

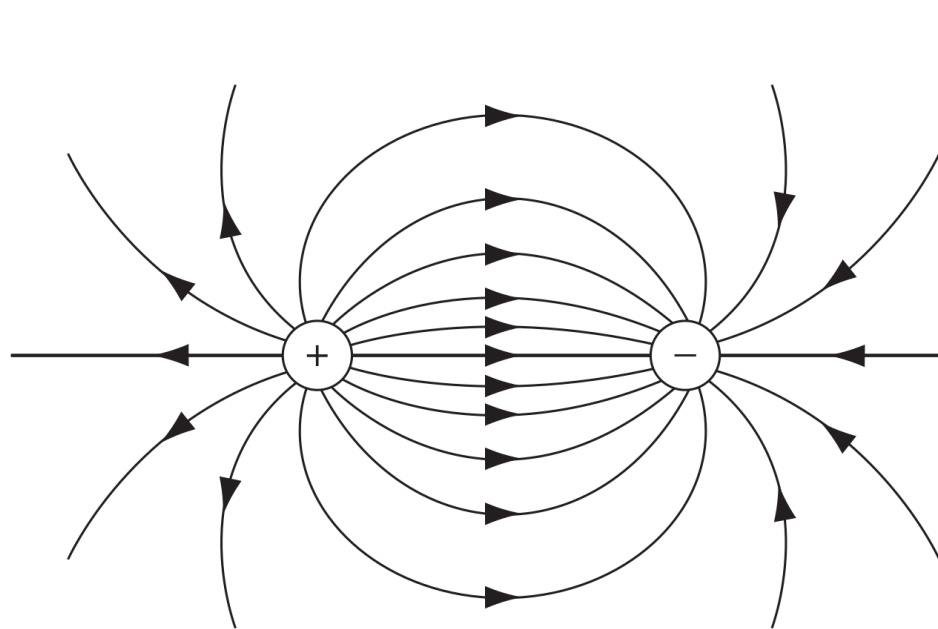
- Then the magnitude of the field is indicated by the *density* of the field lines: it's strong near the center where the field lines are close together, and weak farther out, where they are relatively far apart.

- Imagine the model in 3d, then the density of lines is the total number divided by the area of the sphere

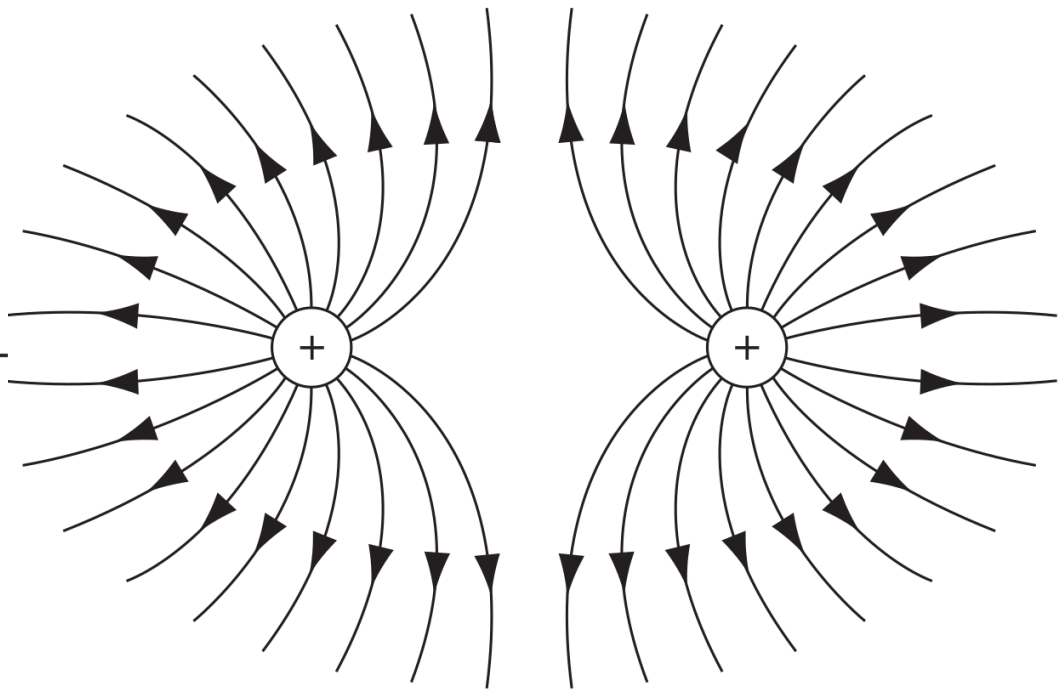
($\frac{n}{4\pi r^2}$), which does go like ($\frac{1}{r^2}$).

- Field lines begin on positive charges and end on negative ones; they cannot simply terminate in midair, though they may extend out to infinity.





Opposite charges

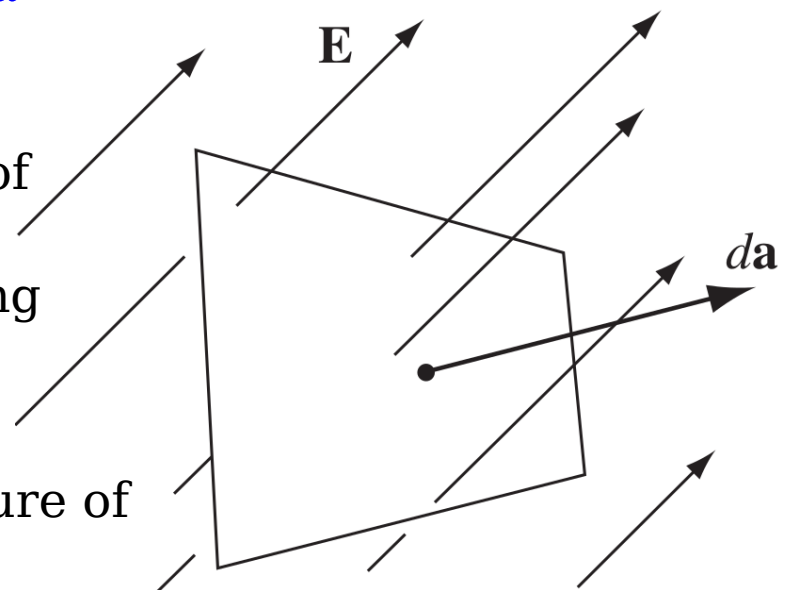


Equal charges

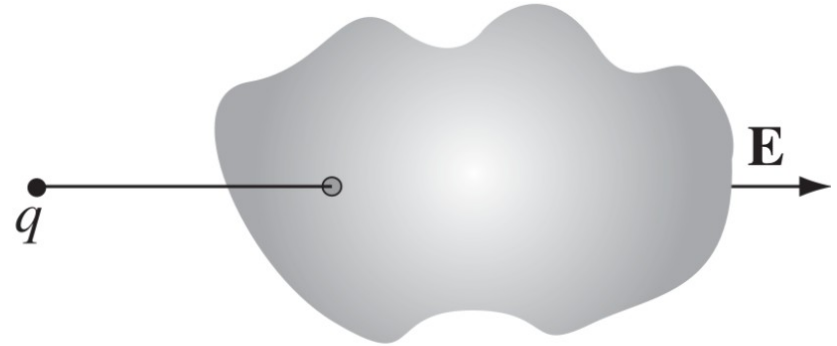
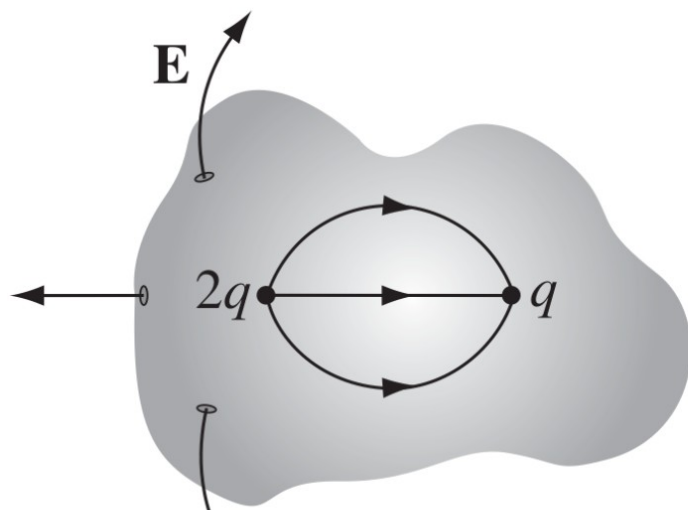
- Field lines can never cross—at the intersection, the field would have 2 different directions at once!

- The *flux* of \mathbf{E} through a surface S : $\Phi_E \equiv \int_S \mathbf{E} \cdot d\mathbf{a}$ is a measure of the “number of field lines” passing through S .

- The field strength is proportional to the density of field lines (the number per unit area), and hence $\mathbf{E} \cdot d\mathbf{a}$ is proportional to the number of lines passing through the infinitesimal area $d\mathbf{a}$.



- So the flux through any *closed* surface is a measure of the total charge inside.



- For the field lines that originate on a positive charge must either pass out through the surface or else terminate on a negative charge inside. But a charge *outside* the surface will contribute nothing to the total flux, since its field lines pass in on one side and out the other. This is the *essence* of **Gauss's law**.

- For a point charge q at the origin, the flux of \mathbf{E} through a spherical surface of

radius r is
$$\oint \mathbf{E} \cdot d\mathbf{a} = \int \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} \cdot r^2 \sin\theta \, d\theta \, d\phi \hat{\mathbf{r}} = \frac{q}{\epsilon_0}$$

- The radius of the sphere cancels out, for while the surface area goes up as r^2 ,

the field goes *down* as $\frac{1}{r^2}$, so the product is constant. In terms of the field-line

picture, it means that the same number of field lines pass through any sphere centered at the origin, regardless of its size.

- In fact, it didn't have to be a sphere—*any* closed surface, whatever its shape, would be pierced by the same number of field lines. Evidently the flux through

any surface enclosing the charge is $\frac{q}{\epsilon_0}$.

- If we have a bunch of charges scattered about, according to the principle of superposition, the total field is the (vector) sum of all the individual fields:

$$\mathbf{E} = \sum_{i=1}^n \mathbf{E}_i \Rightarrow \oint \mathbf{E} \cdot d\mathbf{a} = \sum_{i=1}^n \oint \mathbf{E}_i \cdot d\mathbf{a} = \sum_{i=1}^n \frac{q_i}{\epsilon_0} \Rightarrow \oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

- Gauss's Law all hinges on the $\frac{1}{r^2}$ character of Coulomb's law; without that the crucial cancellation of the r 's in the equation would not take place.

- Gauss's law is an *integral* equation, but we can easily turn it into a *differential* one, by applying the divergence theorem:

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{E} \, d\tau \quad \Rightarrow \quad \int_V \nabla \cdot \mathbf{E} \, d\tau = \int_V \frac{\rho}{\epsilon_0} \, d\tau \quad \Rightarrow \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$Q_{\text{enc}} = \int_V \rho \, d\tau$$

Gauss's law in differential form

- The differential version is tidier, but the integral form has the advantage in that it accommodates point, line, and surface charges more naturally.

The Divergence of \mathbf{E}

$$\bullet \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\hat{\mathbf{r}}}{r^2} \rho(\mathbf{r}') d\tau'$$

$$\Rightarrow \nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) \rho(\mathbf{r}') d\tau' \leftarrow \vec{r} \equiv \mathbf{r} - \mathbf{r}', \nabla \text{ only works on } \mathbf{r}$$

$$= \frac{1}{4\pi\epsilon_0} \int 4\pi \delta^3(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}') d\tau' = \frac{\rho(\mathbf{r})}{\epsilon_0} \leftarrow \nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} = 4\pi \delta^3(\vec{r})$$

$$\bullet \text{To recover the integral form } \int_V \nabla \cdot \mathbf{E} d\tau = \oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \int_V \rho d\tau = \frac{Q_{\text{enc}}}{\epsilon_0}$$

Applications of Gauss's Law

- When symmetry permits, Gauss's law affords by far the quickest and easiest way of computing electric fields.

- Gauss's law is always *true*, but it is not always *useful* if ρ is neither uniform or not symmetrical.

- *Symmetry is crucial* to this application of Gauss's law.

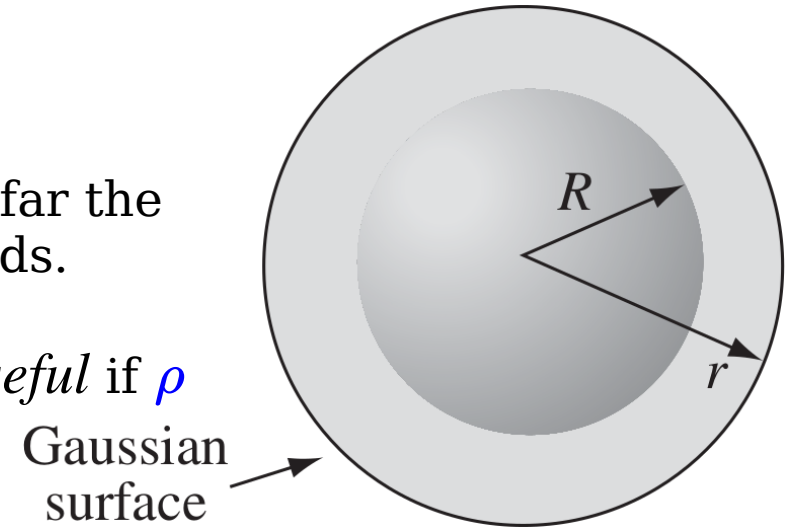
- 3 kinds of symmetry that work:

1. *Spherical symmetry*: Make your Gaussian surface a concentric sphere.

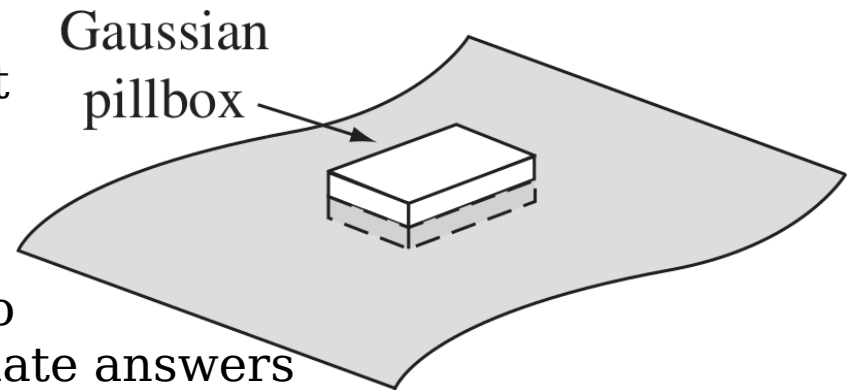
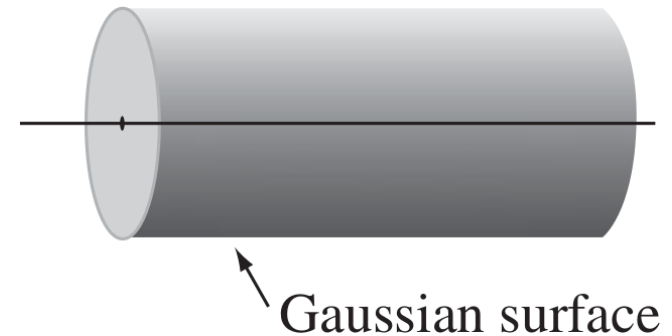
2. *Cylindrical symmetry*: Make your Gaussian surface a coaxial cylinder.

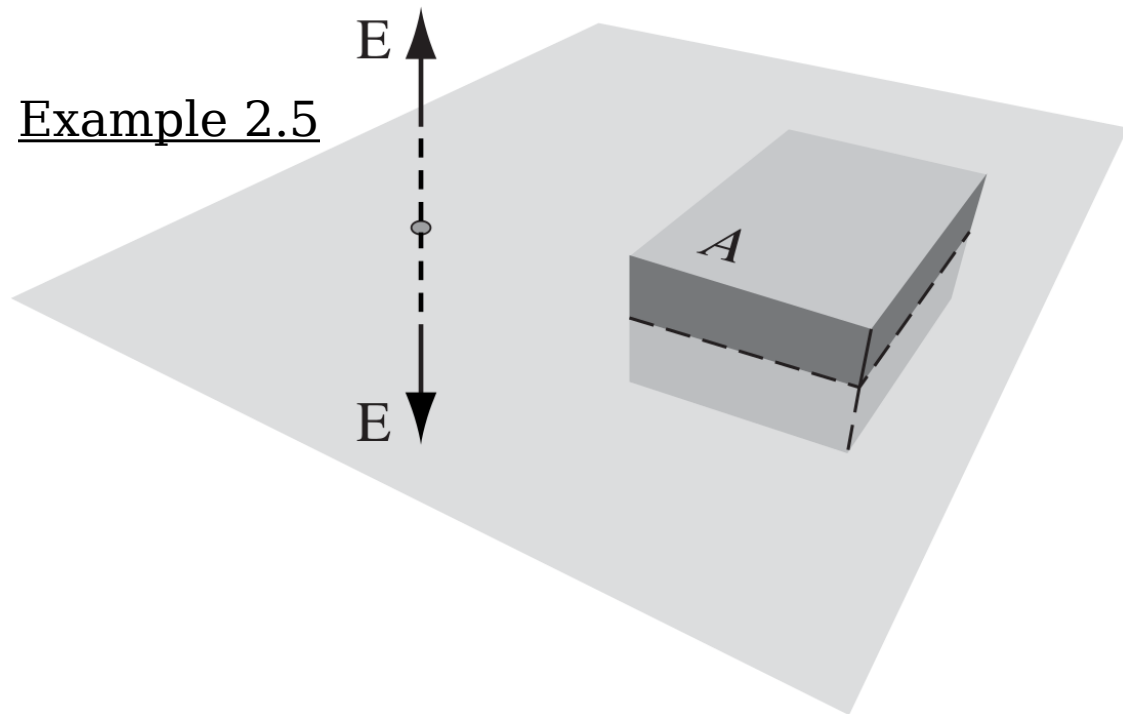
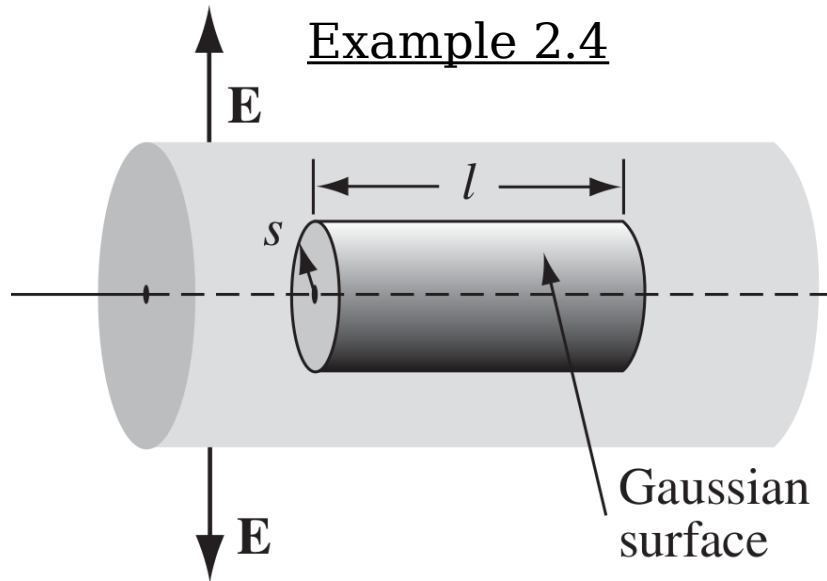
3. *Plane symmetry*: Use a Gaussian "pillbox" that straddles the surface.

- Although (2) and (3) technically require infinitely long cylinders, and planes extending to infinity, we shall often use them to get approximate answers for "long" cylinders or "large" planes, at points far from the edges.

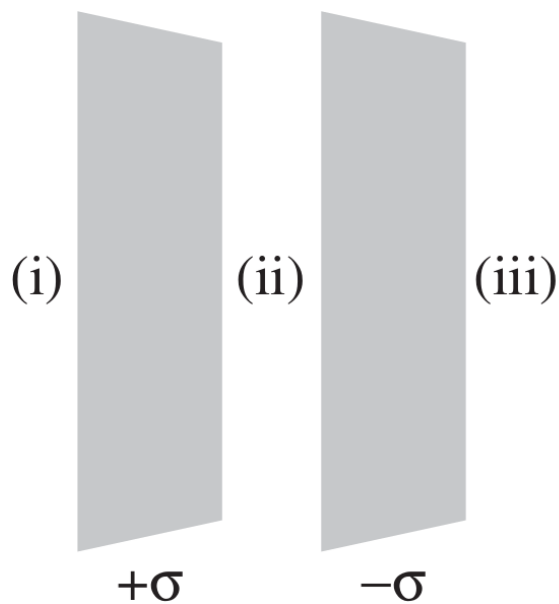


Example 2.3

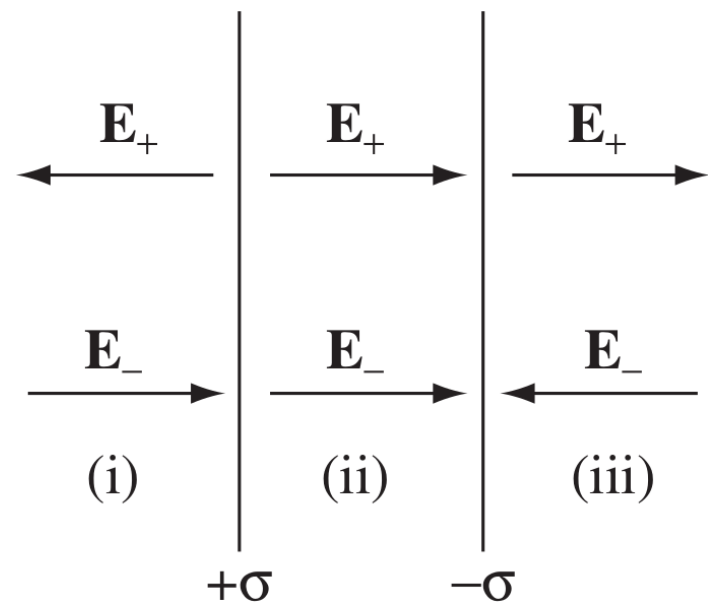




- Although the direct use of Gauss's law to compute \mathbf{E} is limited to cases of symmetry, we can put together *combinations* of objects possessing such symmetry, even though the arrangement as a whole is not symmetrical.



Example 2.6



The Curl of \mathbf{E}

● For a point charge at the origin: $\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$

$$\Rightarrow \mathbf{E} \cdot d\boldsymbol{\ell} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} dr \Leftrightarrow d\boldsymbol{\ell} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin\theta d\phi \hat{\boldsymbol{\phi}}$$

$$\Rightarrow \int_a^b \mathbf{E} \cdot d\boldsymbol{\ell} = \frac{1}{4\pi\epsilon_0} \int_a^b \frac{q}{r^2} dr = -\frac{1}{4\pi\epsilon_0} \frac{q}{r} \Big|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r_a} - \frac{q}{r_b} \right)$$

$$\Rightarrow \oint \mathbf{E} \cdot d\boldsymbol{\ell} = 0 \Leftrightarrow (*) \text{ The integral around a closed path is 0, that is, } r_a = r_b$$

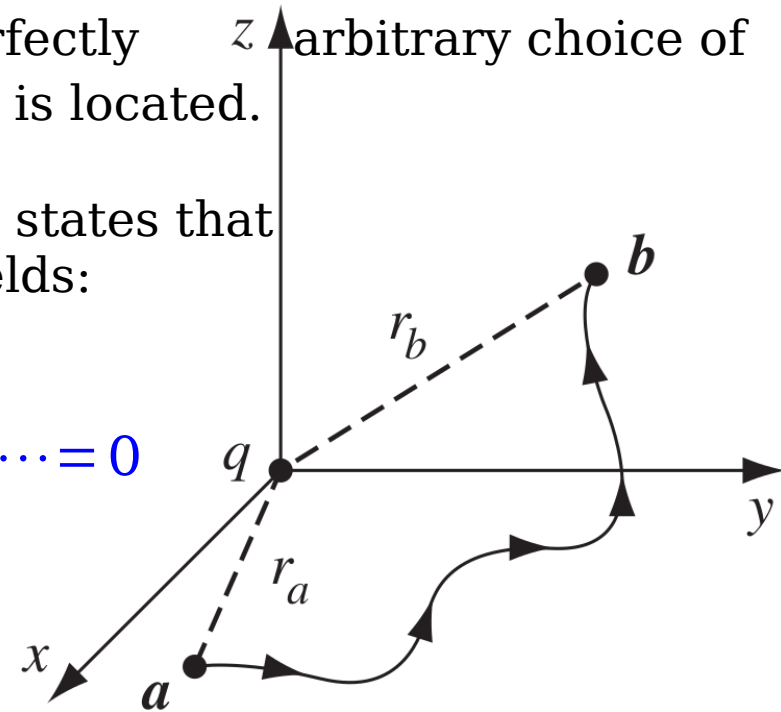
$$\Rightarrow \nabla \times \mathbf{E} = 0 \Leftrightarrow \text{applying Stokes' theorem}$$

● These results make no reference to what is a perfectly arbitrary choice of coordinates; they hold no matter *where* the charge is located.

● For many charges, the principle of superposition states that the total field is a vector sum of their individual fields:

$$\begin{aligned} \mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \dots &\Rightarrow \nabla \times \mathbf{E} = \nabla \times (\mathbf{E}_1 + \mathbf{E}_2 + \dots) \\ &= \nabla \times \mathbf{E}_1 + \nabla \times \mathbf{E}_2 + \dots = 0 \end{aligned}$$

● Thus, these results hold for *any static charge distribution whatever*.

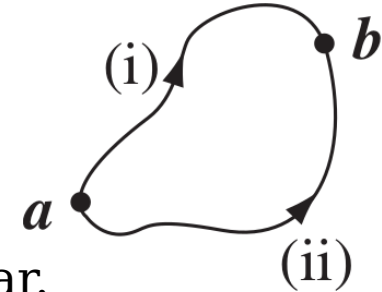


Electric Potential

Introduction to Potential

- The electric field \mathbf{E} is a very special *kind* of vector function, ie, its curl is 0.

- Exploit this special property of electric fields to reduce a *vector* problem (finding \mathbf{E}) to a much simpler *scalar* problem.



- Any vector whose curl is 0 is equal to the gradient of some scalar.

- $\nabla \times \mathbf{E} = 0 \Rightarrow \oint \mathbf{E} \cdot d\ell = 0 \Rightarrow \Phi(\mathbf{r}) \equiv - \int_0^{\mathbf{r}} \mathbf{E} \cdot d\ell$ electric potential $\Leftarrow O$: standard reference

$$\Rightarrow \Phi(\mathbf{b}) - \Phi(\mathbf{a}) = - \int_0^{\mathbf{b}} \mathbf{E} \cdot d\ell + \int_0^{\mathbf{a}} \mathbf{E} \cdot d\ell$$

$$= - \int_0^{\mathbf{b}} \mathbf{E} \cdot d\ell - \int_{\mathbf{a}}^0 \mathbf{E} \cdot d\ell = - \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\ell$$

$$\Phi(\mathbf{b}) - \Phi(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} \nabla \Phi \cdot d\ell \Rightarrow \int_{\mathbf{a}}^{\mathbf{b}} \nabla \Phi \cdot d\ell = - \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\ell \Rightarrow \mathbf{E} = - \nabla \Phi$$

- The electric field is the gradient of a scalar potential.

- Here path independence (or, equivalently, the fact that $\nabla \times \mathbf{E} = 0$) plays a crucial role in this argument. If the line integral of \mathbf{E} depended on the path taken, then the potential definition would be nonsense.

Comments on Potential

The name

- There is a connection between “potential” and “potential energy,” but “potential” and “potential energy” are different terms.
- A surface over which the potential is constant is called an **equipotential**.

Advantage of the potential formulation

- If you know Φ , you can get \mathbf{E} —just take the gradient: $\mathbf{E} = -\nabla\Phi$.
- This is extraordinary for \mathbf{E} is a vector quantity (3 components), but Φ is a scalar (1 component).
- This is because the 3 components of \mathbf{E} are not really independent. They are explicitly interrelated by the very condition we started with, $\nabla \times \mathbf{E} = 0$
$$\Rightarrow \frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}, \quad \frac{\partial E_y}{\partial z} = \frac{\partial E_z}{\partial y}, \quad \frac{\partial E_z}{\partial x} = \frac{\partial E_x}{\partial z} \Rightarrow \mathbf{E} \text{ is a very special kind of vector}$$
- What the potential formulation does is to exploit this feature to maximum advantage, reducing a vector problem to a scalar one, in which there is no need to fuss with components.

The reference point O :

- There is an essential ambiguity in the definition of potential, since the choice of reference point O was arbitrary.

- Changing reference points amounts to adding a constant K to the potential:

$$\Phi'(\mathbf{r}) = - \int_{O'}^{\mathbf{r}} \mathbf{E} \cdot d\boldsymbol{\ell} = - \int_{O'}^O \mathbf{E} \cdot d\boldsymbol{\ell} - \int_O^{\mathbf{r}} \mathbf{E} \cdot d\boldsymbol{\ell} = K + \Phi(\mathbf{r})$$

where K is the line integral of \mathbf{E} from the reference point O to the new one O' .

- Adding a constant to Φ will not affect the potential difference between 2 points:

$$\Phi'(\mathbf{b}) - \Phi'(\mathbf{a}) = \Phi(\mathbf{b}) - \Phi(\mathbf{a}) \quad \text{and} \quad \nabla \Phi' = \nabla \Phi$$

- That's why all such Φ 's, differing only in their choice of reference point, correspond to the same field \mathbf{E} .

- The only quantity of intrinsic interest is the *difference* in altitude between 2 points, and *that* is the same *whatever* your reference level.

- There is a “natural” spot to use for O in electrostatics and that is a point infinitely far from the charge. Ordinarily, we “set the 0 of potential at infinity.”

- But there is one special circumstance in which this convention fails: when the charge distribution itself extends to infinity. The symptom of trouble, in such cases, is that the potential blows up. For a uniformly charged plane

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0} \hat{\mathbf{n}} \quad \Rightarrow \quad \Phi(z) = - \int_{O=\infty}^z \frac{\sigma}{2\epsilon_0} dz = - \frac{\sigma}{2\epsilon_0} (z - \infty) \rightarrow \infty$$

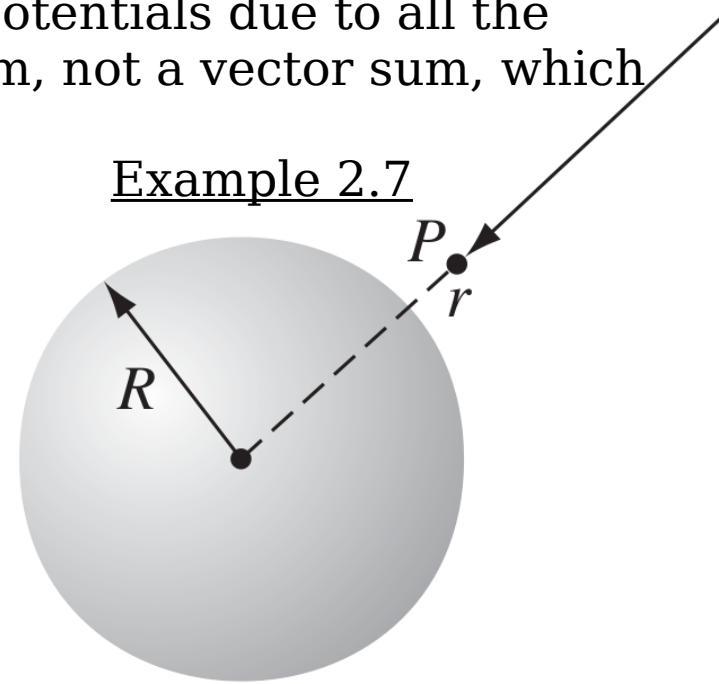
- In “real life” there is no such thing as a charge distribution that goes on forever, and we can *always* use infinity as our reference point.

Potential obeys the superposition principle:

- The original superposition says that the total force on Q is the vector sum of the forces attributable to the source charges individually: $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \dots$
- Dividing through by Q , we see that the electric field obeys the superposition principle: $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \dots$
- Integrating from the common reference point to \mathbf{r} , it follows that the potential also satisfies such a principle: $\Phi = \Phi_1 + \Phi_2 + \dots$
- The potential at any given point is the sum of the potentials due to all the source charges separately. Since it is an ordinary sum, not a vector sum, which makes it a lot easier to work with.

Units of Potential:

- Potential is newton-meters per coulomb, or joules per coulomb. A joule per coulomb is a **volt**.



Poisson's Equation and Laplace's Equation

- $\mathbf{E} = -\nabla \Phi$ + $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ and $\nabla \times \mathbf{E} = 0$

$\Rightarrow \nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$ Poisson's equation $\Rightarrow \nabla^2 \Phi = 0$ Laplace's equation $\Leftarrow \rho = 0$

- $\nabla \times (-\nabla \Phi) = 0 \Rightarrow \nabla \times \mathbf{E} = 0 \Leftrightarrow \mathbf{E} = -\nabla \Phi$

- It takes only *one* differential equation (Poisson's) to determine Φ , because Φ is a scalar; for \mathbf{E} we needed *two*, the divergence and the curl.

The Potential of a Localized Charge Distribution

● Ordinarily it's \mathbf{E} that we're looking for. The idea is that it might be easier to get Φ first, and then calculate \mathbf{E} by taking the gradient.

● Typically we know where the charge ρ is, and we want to find Φ . What we must do, then, is "invert" Poisson's equation.

● With a point charge at the origin: $\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{\hat{\mathbf{r}}}{r^2}$

$$\Rightarrow \mathbf{E} \cdot d\boldsymbol{\ell} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} dr \quad \Leftarrow \quad d\boldsymbol{\ell} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin\theta d\phi \hat{\boldsymbol{\phi}}$$

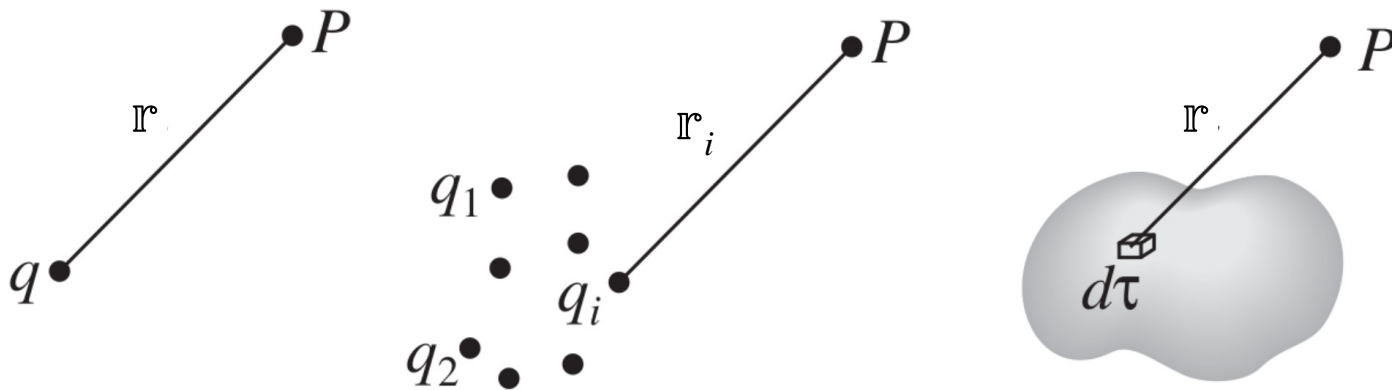
$$\Rightarrow \Phi(r) = - \int_0^r \mathbf{E} \cdot d\boldsymbol{\ell} = - \frac{1}{4\pi\epsilon_0} \int_{\infty}^r \frac{q}{r'^2} dr' = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \Big|_{\infty}^r = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

using ∞ for the reference point kills the lower limit on the integral.

● The sign of Φ (minus sign) in the definition was chosen in order to make the potential of a positive charge come out positive.

● In general, the potential of a point charge q is $\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \quad \Leftarrow \quad r = |\mathbf{r} - \mathbf{r}'|$

● Then the potential of a collection of charges is $\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \frac{q_i}{r_i}$



- For a continuous distribution, $\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{r}$

- For a volume charge,

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{r} d\tau' \Rightarrow \mathbf{E}(\mathbf{r}) = -\nabla\Phi = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{r^2} \hat{\mathbf{r}} d\tau'$$

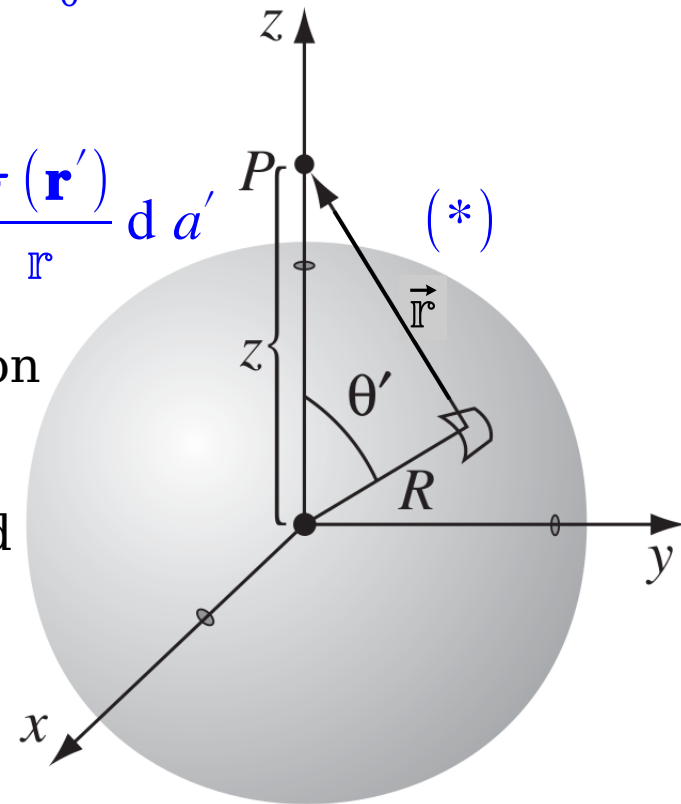
- The potentials of line and surface charges are

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\mathbf{r}')}{r} d\ell' \quad \text{and} \quad \Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\mathbf{r}')}{r} da' \quad (*)$$

- Every equation above is predicated on the assumption that the reference point is at ∞ .

Example 2.8: Find the potential of a uniformly charged spherical shell of radius R .

- Same as in Ex 2.7, but using (*).



$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\mathbf{r}')}{r} da' = \frac{\sigma}{4\pi\epsilon_0} \int \frac{R^2 \sin\theta' d\theta' d\phi'}{|\mathbf{r} - \mathbf{R}|}$$

Choose $\hat{\mathbf{r}} = \hat{\mathbf{z}} \Rightarrow |\mathbf{r} - \mathbf{R}| = \sqrt{r^2 + R^2 - 2rR\cos\theta'}$

$$\begin{aligned} \Rightarrow \Phi(\mathbf{r}) &= \frac{R^2\sigma}{4\pi\epsilon_0} \int_0^\pi \frac{-d\cos\theta'}{\sqrt{r^2 + R^2 - 2rR\cos\theta'}} \int_0^{2\pi} d\phi' \\ &= \frac{2\pi R^2\sigma}{4\pi\epsilon_0} \int_{-1}^1 \frac{dw}{\sqrt{r^2 + R^2 + 2rRw}} \quad \leftarrow w \equiv -\cos\theta' \\ &= \frac{R^2\sigma}{4\epsilon_0 r R} \cdot 2 \sqrt{r^2 + R^2 + 2rRw} \Big|_{-1}^1 = \frac{R\sigma}{2\epsilon_0 r} (r + R - |r - R|) \\ &= \begin{cases} \frac{R\sigma}{2\epsilon_0 r} \cdot 2R = \frac{R^2\sigma}{\epsilon_0 r} = \frac{Q}{4\pi\epsilon_0 r} & \text{for } r \geq R \\ \frac{R\sigma}{2\epsilon_0 r} \cdot 2r = \frac{R\sigma}{\epsilon_0} = \frac{Q}{4\pi\epsilon_0 R} & \text{for } r \leq R \end{cases} \quad \leftarrow Q = 4\pi R^2\sigma \end{aligned}$$

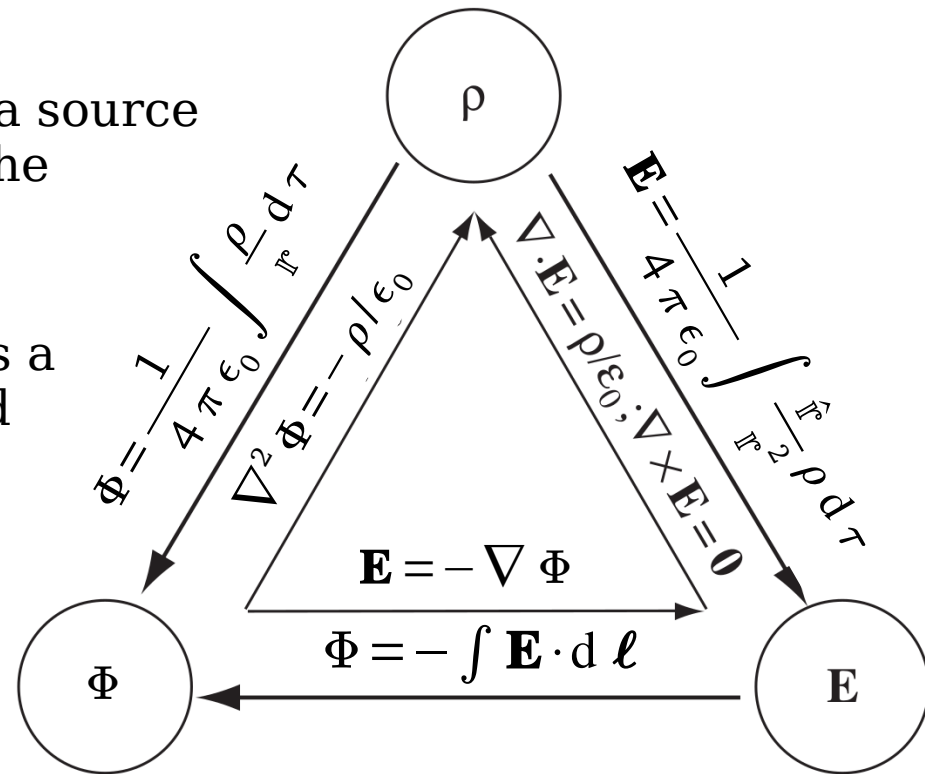
- Although it is easier to obtain \mathbf{E} with Gauss's theorem as in Ex. 2.3, using the potential Φ to obtain \mathbf{E} is much easier in general than deriving \mathbf{E} directly (compared with the proof of the shell theorem).

Boundary Conditions

- In the typical electrostatic problem, given a source charge distribution ρ , and you want to find the electric field \mathbf{E} it produces.

- Unless the symmetry of the problem allows a solution by Gauss's law, it is generally to find the potential first, as an intermediate step.

- Thus the 3 fundamental quantities of electrostatics are: ρ , \mathbf{E} , and Φ . We have derived all 6 formulas interrelating them.

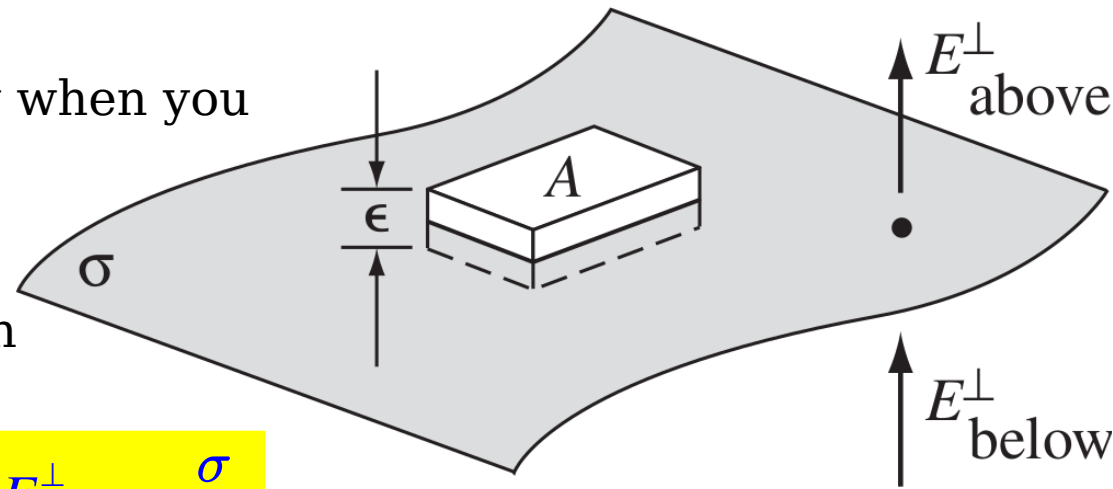


- We began with just 2 experimental observations: (1) **the principle of superposition**, and (2) **Coulomb's law**. From these, all else followed.

- \mathbf{E} always undergoes a discontinuity when you cross a surface charge σ .

- For a wafer-thin Gaussian pillbox, extending just barely over the edge in each direction. Gauss's law says

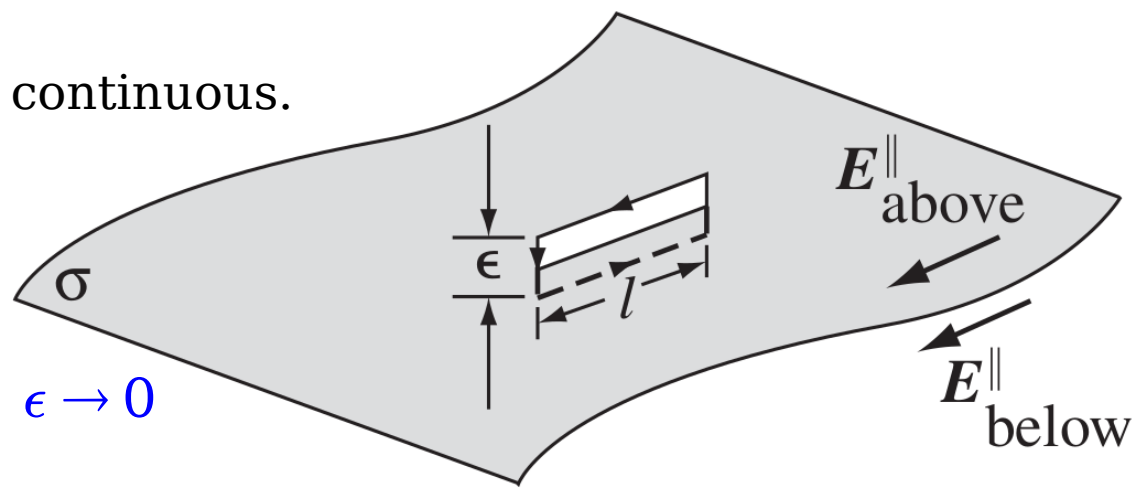
$$\int_s \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0} \Rightarrow E_{\text{above}}^\perp - E_{\text{below}}^\perp = \frac{\sigma}{\epsilon_0}$$



- The normal component of \mathbf{E} is discontinuous by an amount $\frac{\sigma}{\epsilon_0}$ at any boundary.

- If there is *no* surface charge, E^\perp is continuous.

- The *tangential* component of \mathbf{E} , by contrast, is *always* continuous.



$$0 = \oint \mathbf{E} \cdot d\boldsymbol{\ell} = E_{\text{above}}^\parallel \ell - E_{\text{below}}^\parallel \ell \quad \Leftarrow \quad \epsilon \rightarrow 0$$

$$\Rightarrow E_{\text{above}}^\parallel = E_{\text{below}}^\parallel$$

- The boundary conditions on \mathbf{E} can be combined into $\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}}$

- The potential, meanwhile, is continuous (but not smooth) across *any* boundary,

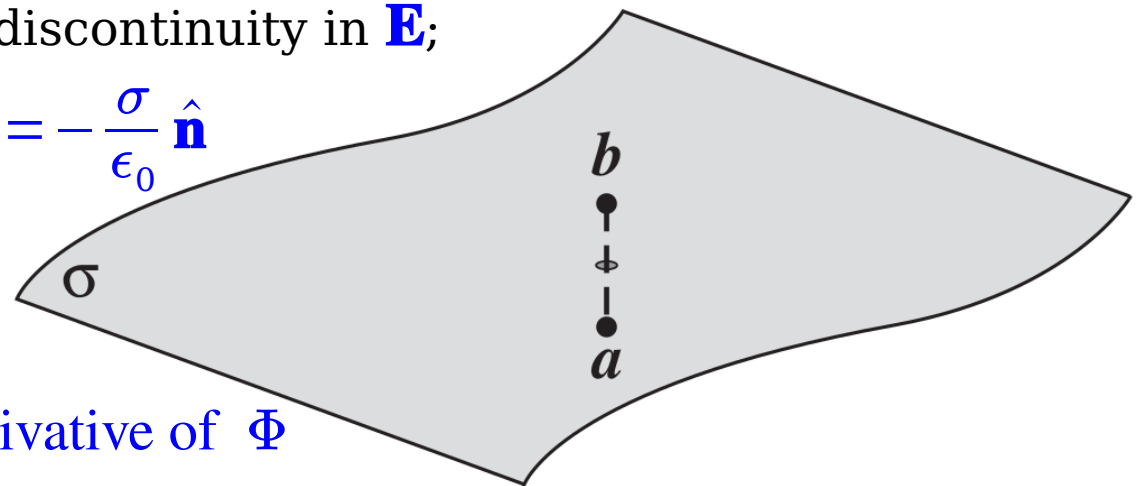
$$\Phi_{\text{above}} - \Phi_{\text{below}} = - \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\boldsymbol{\ell} = 0 \quad \Leftarrow \quad \mathbf{b} - \mathbf{a} \rightarrow 0 \quad \Rightarrow \quad \Phi_{\text{above}} = \Phi_{\text{below}}$$

- But the gradient of Φ inherits the discontinuity in \mathbf{E} ;

$$\mathbf{E} = -\nabla \Phi \Rightarrow \nabla \Phi_{\text{above}} - \nabla \Phi_{\text{below}} = -\frac{\sigma}{\epsilon_0} \hat{\mathbf{n}}$$

$$\Rightarrow \frac{\partial \Phi_{\text{above}}}{\partial n} - \frac{\partial \Phi_{\text{below}}}{\partial n} = -\frac{\sigma}{\epsilon_0}$$

where $\frac{\partial \Phi}{\partial n} \equiv \hat{\mathbf{n}} \cdot \nabla \Phi$ normal derivative of Φ

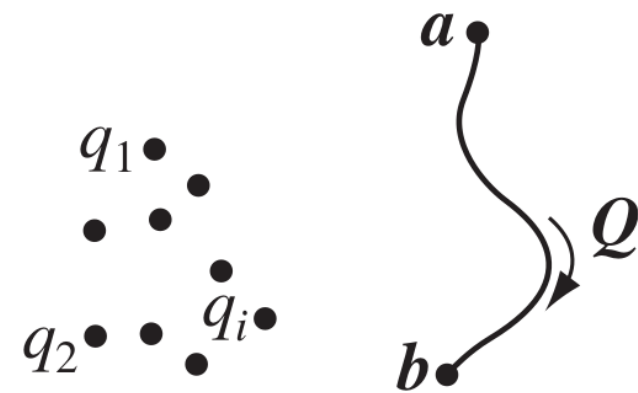


- These boundary conditions relate the fields and potentials *just* above and *just* below the surface.

Work and Energy in Electrostatics

The Work It Takes to Move a Charge

- How much work will you have to do to move a test charge Q from point \mathbf{a} to point \mathbf{b} ?



- What we're interested in here is the *minimum* force you must exert to do the job. The work you do is

$$W_{\text{app}} = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F}_{\text{app}} \cdot d\boldsymbol{\ell} = -Q \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\boldsymbol{\ell} = Q [\Phi(\mathbf{b}) - \Phi(\mathbf{a})]$$

- The answer is independent of the path you take from \mathbf{a} to \mathbf{b} ; in mechanics we would call the electrostatic force “conservative.”

- $\Phi(\mathbf{b}) - \Phi(\mathbf{a}) = \frac{W_{\text{app}}}{Q}$, ie, the potential difference between points \mathbf{a} and \mathbf{b} is

equal to the work per unit charge required to carry a particle from \mathbf{a} to \mathbf{b} .

- If you want to bring Q in from far away and stick it at point \mathbf{r} , the work needed

$$W = Q [\Phi(\mathbf{r}) - \Phi(\infty)] = Q \Phi(\mathbf{r})$$

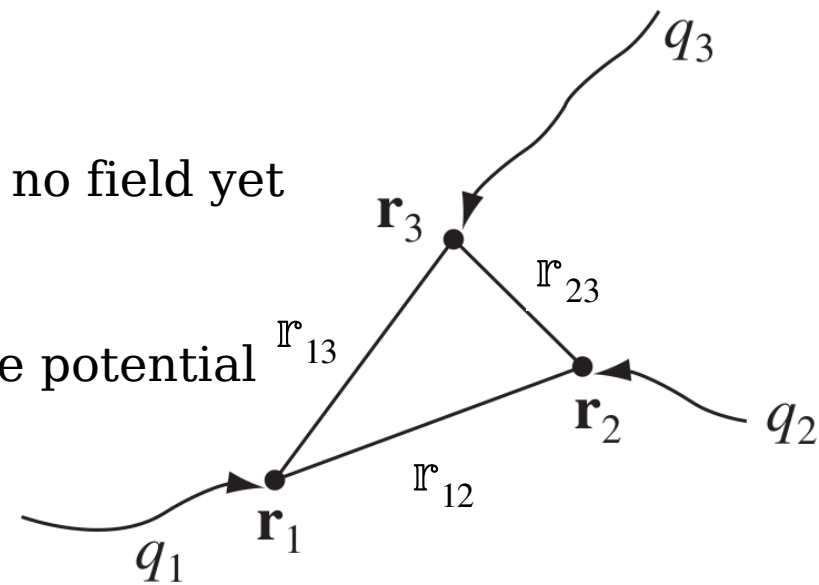
- In this sense, *potential* is potential *energy* (the work it takes to create the system) *per unit charge* (as the *field* is the *force* per unit charge).

The Energy of a Point Charge Distribution

- The 1st charge, q_1 , takes no work, since there is no field yet to fight against.

- Bringing in q_2 will need $q_2 \Phi_1(\mathbf{r}_2)$, where Φ_1 is the potential due to q_1 , and \mathbf{r}_2 is the place we're putting q_2 :

$$W_2 = \frac{1}{4 \pi \epsilon_0} q_2 \frac{q_1}{r_{12}} \quad \leftarrow \quad r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$$



- Now bringing in q_3 requires work $q_3 [\Phi_1(\mathbf{r}_3) + \Phi_2(\mathbf{r}_3)]$: $W_3 = \frac{1}{4 \pi \epsilon_0} q_3 \left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}} \right)$

- The extra work to bring in q_4 will be $W_4 = \frac{1}{4 \pi \epsilon_0} q_4 \left(\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}} \right)$

- The *total* work necessary to assemble the first 4 charges, then, is

$$W = W_2 + W_3 + W_4 = \frac{1}{4 \pi \epsilon_0} \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_1 q_4}{r_{14}} + \frac{q_2 q_3}{r_{23}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}} \right)$$

- The general rule: $W = \frac{1}{4 \pi \epsilon_0} \sum_{i=1}^n \sum_{j>i}^n \frac{q_i q_j}{r_{ij}} = \frac{1}{8 \pi \epsilon_0} \sum_{i=1}^n \sum_{j \neq i}^n \frac{q_i q_j}{r_{ij}}$

- In the form of the most right hand side, the answer plainly does not depend on the *order* in which you assemble the charges, since every pair occurs in the sum.

- Rewrite the equation:
$$W = \frac{1}{2} \sum_{i=1}^n q_i \left(\sum_{j \neq i}^n \frac{1}{4 \pi \epsilon_0} \frac{q_j}{r_{ij}} \right) = \frac{1}{2} \sum_{i=1}^n q_i \Phi(\mathbf{r}_i) \quad (\#)$$

the potential $\Phi(\mathbf{r}_i)$ at point \mathbf{r}_i (the position of q_i) is due to *all* the other charges.

- That's how much work it takes to assemble a configuration of point charges; it's also the amount of work you'd get back if you dismantled the system.
- It also represents energy stored in the configuration (ie, potential energy).

The Energy of a Continuous Charge Distribution

- For a volume charge density ρ , the total work becomes: $W = \frac{1}{2} \int \rho \Phi \, d\tau$ (%)
- The corresponding integrals for line and surface charges would be

$$W = \frac{1}{2} \int \lambda \Phi \, d\ell \quad \text{and} \quad W = \frac{1}{2} \int \sigma \Phi \, da$$

- $W = \frac{1}{2} \int \rho \Phi \, d\tau = \frac{\epsilon_0}{2} \int (\nabla \cdot \mathbf{E}) \Phi \, d\tau \leftarrow \rho = \epsilon_0 \nabla \cdot \mathbf{E}$
 $= \frac{\epsilon_0}{2} \left(- \int \mathbf{E} \cdot \nabla \Phi \, d\tau + \oint \Phi \mathbf{E} \cdot d\mathbf{a} \right) \leftarrow \text{integration by parts}$
 $= \frac{\epsilon_0}{2} \left(\int_{\mathcal{V}} E^2 \, d\tau + \oint_S \Phi \mathbf{E} \cdot d\mathbf{a} \right) \leftarrow \mathbf{E} = -\nabla \Phi$

- The integral of E^2 can only increase as the volume increases (the integrand being positive); evidently the surface integral must decrease correspondingly to

leave the sum intact: $E \propto \frac{1}{r^2}$, $\Phi \propto \frac{1}{r}$, $da \propto r^2 \Rightarrow \oint \Phi \mathbf{E} \cdot d\mathbf{a} \propto \frac{1}{r}$

- Integrating over all space: $W = \frac{\epsilon_0}{2} \int E^2 \, d\tau$ all space (!) $\leftarrow \oint \Phi \mathbf{E} \cdot d\mathbf{a} \rightarrow 0$

Example 2.9: Find the energy of a uniformly charged spherical shell of total charge q and radius R .

$$\begin{aligned} \text{Ans: } W_1 &= \frac{1}{2} \int \Phi \sigma \, d a = \frac{q^2}{32 \pi^2 \epsilon_0 R^3} \int d a \quad \Leftarrow \quad \sigma = \frac{q}{4 \pi R^2}, \quad \Phi(R) = \frac{q}{4 \pi \epsilon_0 R} \\ &= \frac{4 \pi R^2 q^2}{32 \pi^2 \epsilon_0 R^3} = \frac{q^2}{8 \pi \epsilon_0 R} \end{aligned}$$

$$E(r < R) = 0, \quad E(r \geq R) = \frac{q}{4 \pi \epsilon_0 r^2} \Rightarrow E(r) = \frac{q}{4 \pi \epsilon_0 r^2} \Theta(r - R)$$

$$\begin{aligned} W_2 &= \frac{\epsilon_0}{2} \int E^2 \, d \tau = \frac{\epsilon_0}{2} \int_R^\infty \int_0^\pi \int_0^{2\pi} \left(\frac{q}{4 \pi \epsilon_0 r^2} \right)^2 r^2 \sin \theta \, d \phi \, d \theta \, d r \\ &= \frac{\epsilon_0 q^2}{32 \pi^2 \epsilon_0^2} \cdot 4 \pi \cdot \int_R^\infty \frac{d r}{r^2} = \frac{q^2}{8 \pi \epsilon_0} \left(-\frac{1}{r} \right)_R^\infty = \frac{q^2}{8 \pi \epsilon_0 R} \end{aligned}$$

$$\Rightarrow W_1 = W_2$$

- Usually it is easier to calculate W_1 than W_2 , if possible.

Comments on Electrostatic Energy

A perplexing “inconsistency.”

● Eq. (!) clearly implies that the energy of a stationary charge distribution is always *positive*. But Eq. (#) can either be positive or negative. Any conflict?

● The *both* answers are correct, but they speak to slightly different questions.

● Eq. (#) does not take into account the work necessary to *make* the point charges, but only find the work required to bring them together.

● Eq. (!) is more *complete* since it tells you the total energy stored in a charge configuration; however, indicates that the energy of a point charge is in fact ∞ :

$$W = \frac{\epsilon_0}{2(4\pi\epsilon_0)^2} \int \frac{q^2}{r^4} r^2 \sin\theta \, dr \, d\theta \, d\phi = \frac{q^2}{8\pi\epsilon_0} \int_0^\infty \frac{dr}{r^2} \rightarrow \infty \quad \Leftarrow \text{self-energy}$$

● Eq. (#) is more appropriate when you're dealing with point charges, because we prefer to leave out that portion of the total energy that is attributable to the fabrication of the point charges themselves.

● The inconsistency lies between the discrete expression (#) and its continuous one (%). In the discrete equation, $\Phi(\mathbf{r}_i)$ represents the potential due to all the *other* charges *but not* q_i , whereas in the latter, $\Phi(\mathbf{r})$ is the full potential.

● This is because, for a *continuous* distribution, it is difficult to differ the self-energy from the other part. It is better to use Eq. (#) for point charges.

Where is the energy stored?

- Eq. (%) and (!) offer 2 different ways of calculating the same thing. The 1st is an integral over the charge distribution; the 2nd is an integral over the field. These can involve completely different regions.
- For instance, in the case of the spherical shell, the charge is confined to the surface, whereas the electric field is everywhere *outside* this surface.
- Is the energy stored in the field, as in Eq. (!), or in the charge, as in Eq. (%)?
- In the context of radiation theory (and field theory) it is useful (and sometimes essential) to regard the energy as stored in the field, with a density $\frac{\epsilon_0}{2} E^2 =$ energy per unit volume
- In electrostatics it is ok to say it is stored in the charge, with a density $\frac{1}{2} \rho \Phi$

The superposition principle

- Because electrostatic energy is *quadratic* in the fields, it does *not* obey a superposition principle.
- The energy of a compound system is not the sum of the energies of its parts considered separately—there are also “cross terms”:

$$\begin{aligned} W_{\text{tot}} &= \frac{\epsilon_0}{2} \int E^2 d\tau = \frac{\epsilon_0}{2} \int (\mathbf{E}_1 + \mathbf{E}_2)^2 d\tau \\ &= \frac{\epsilon_0}{2} \int (E_1^2 + E_2^2 + 2\mathbf{E}_1 \cdot \mathbf{E}_2) d\tau = W_1 + W_2 + \epsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2 d\tau \end{aligned}$$

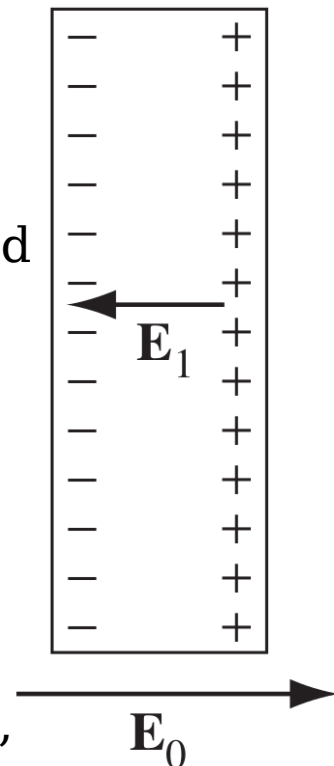
Conductors

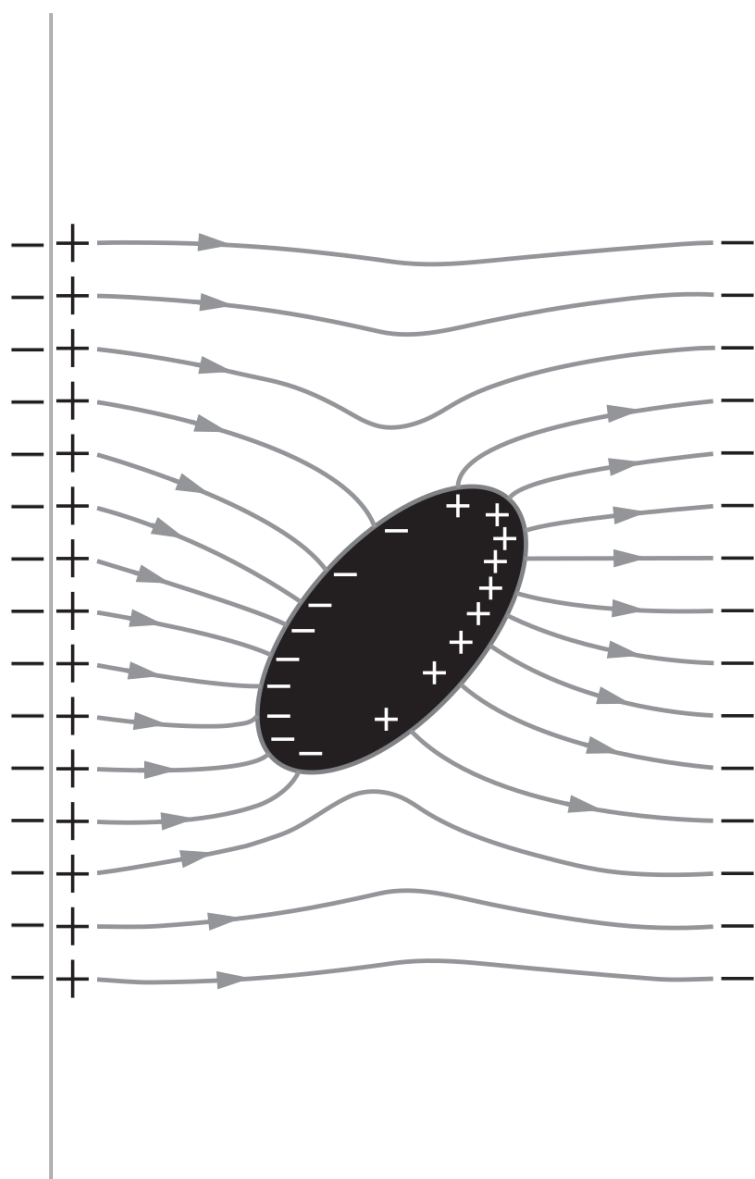
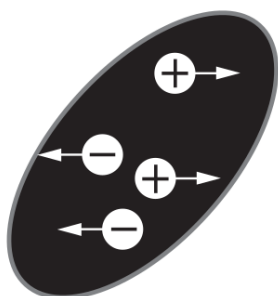
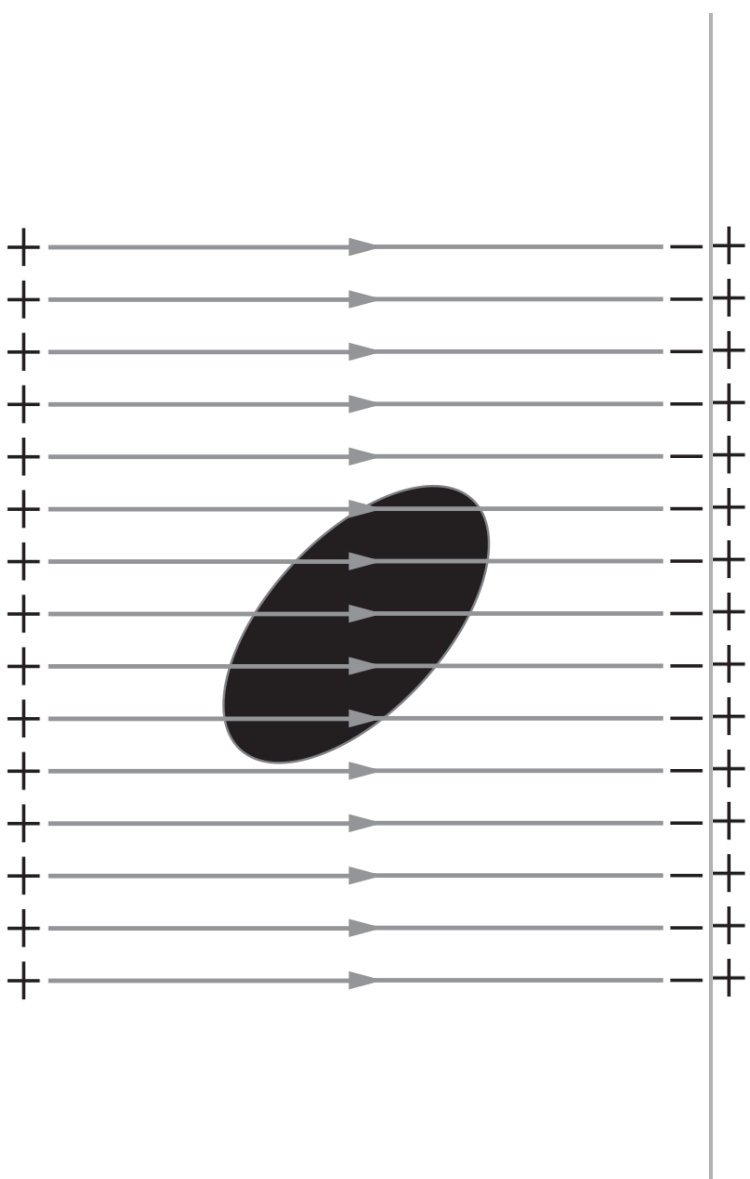
Basic Properties

- In an **insulator**, each electron is on a short leash, attached to a certain atom.
- In a metallic **conductor**, one or more electrons per atom are free to roam.
- In liquid conductors such as salt water, it is ions that do the moving.
- A *perfect* conductor would contain an *unlimited* supply of free charges. In real life there are no perfect conductors, but metals come pretty close, in most cases.
- The basic electrostatic properties of ideal conductors follow:

$E=0$ inside a conductor:

- Because if there were any field, those free charges would move, and it wouldn't be electrostatics any more.
- A better explanation: In an external electric field \mathbf{E}_0 , the field will drive any free positive charges to the right, and negative ones to the left.
- When they come to the edge of the material, the charges pile up. Now, these **induced charges** produce a field of their own, \mathbf{E}_1 , which, is in the *opposite direction* to \mathbf{E}_0 .





- It means that the field of the induced charges *tends to cancel the original field*.
- Charge will continue to flow until this cancellation is complete, and the resultant field inside the conductor is precisely 0.

$\rho=0$ inside a conductor:

- $\mathbf{E} = 0 \Rightarrow \rho = 0 \Leftarrow \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$

- There is still charge around, but exactly as much plus as minus, so the net charge density in the interior is 0.

Any net charge resides on the surface.

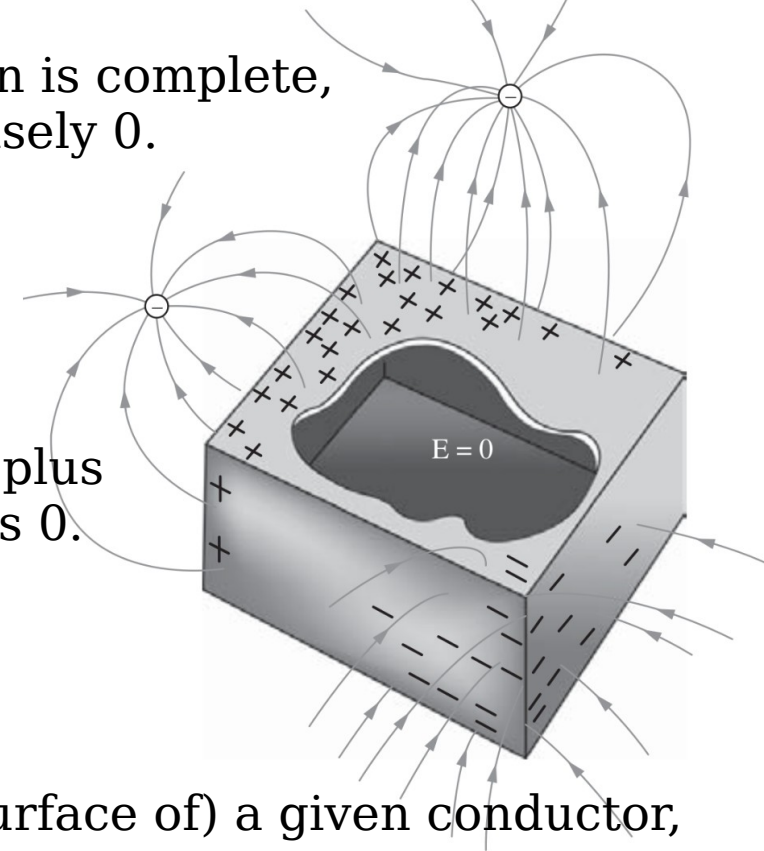
A conductor is an equipotential.

- For if **a** and **b** are any 2 points within (or at the surface of) a given conductor,

$$\Phi(\mathbf{b}) - \Phi(\mathbf{a}) = - \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\boldsymbol{\ell} = 0 \Rightarrow \Phi(\mathbf{b}) = \Phi(\mathbf{a})$$

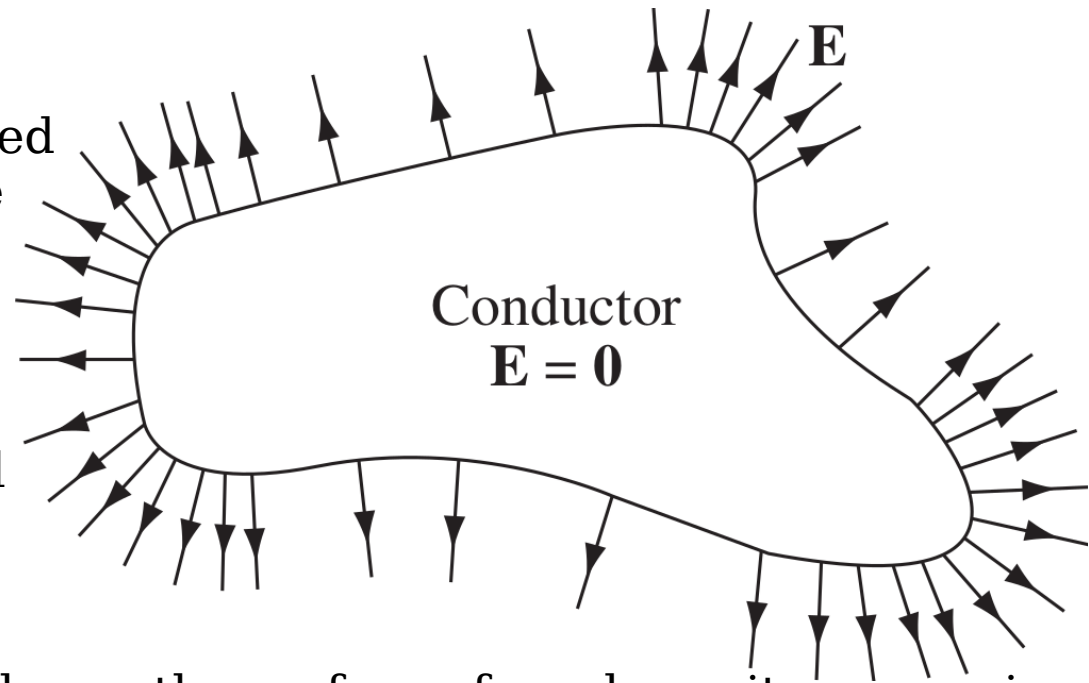
\mathbf{E} is perpendicular to the surface, just outside a conductor:

- In a conductor, charge will immediately flow around the surface until it kills off the tangential component.
- For being perpendicular to the surface, charge cannot flow since it is confined to the conducting object.



- These phenomena can also be phrased in terms of energy. Like any other free dynamical system, the charge on a conductor will seek the configuration that minimizes its potential energy.

- So the electrostatic energy of a solid object is a minimum when that charge is spread over the surface.



- If the charge is uniformly distributed over the surface of a sphere, its energy is

$$\sigma = \frac{q}{4 \pi R^2}, \quad \Phi(R) = \frac{q}{4 \pi \epsilon_0 R}$$

$$\Rightarrow W = \frac{1}{2} \int \sigma \Phi \, d a = \frac{1}{2} \frac{q}{4 \pi R^2} \frac{q}{4 \pi \epsilon_0 R} \times 4 \pi R^2 = \frac{q^2}{8 \pi \epsilon_0 R}$$

- If the charge is uniformly distributed throughout the volume, its energy is greater

$$\rho = \frac{q}{4 \pi R^3 / 3}, \quad \Phi(r) = \frac{q}{8 \pi \epsilon_0 R} \left(3 - \frac{r^2}{R^2} \right), \quad r \in [0, R]$$

$$\Rightarrow W = \frac{1}{2} \int \rho \Phi \, d \tau = \frac{1}{2} \frac{3 q}{4 \pi R^3} \frac{q}{8 \pi \epsilon_0 R} \times 4 \pi \left(r^3 - \frac{r^5}{5 R^2} \right)_0^R = \frac{3 q^2}{20 \pi \epsilon_0 R}$$

Induced Charges

- If you hold a charge $+q$ near an uncharged conductor, the two will attract one another.

- Because q will pull $-$ charges over to the near side and repel $+$ charges to the far side.

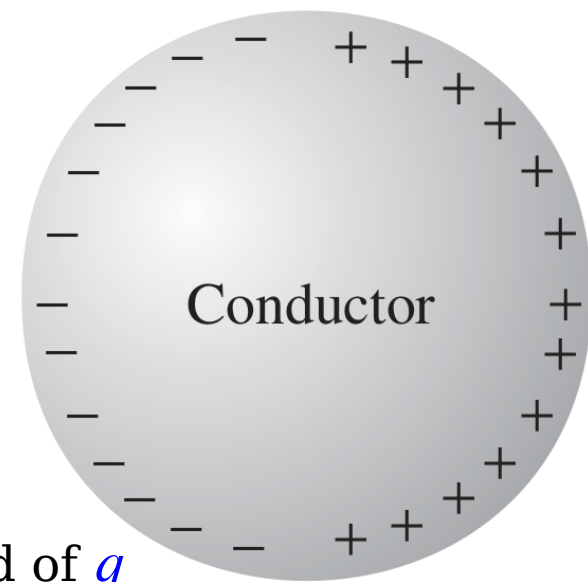
- Imagine that the charge moves around to kill off the field of q for points inside the conductor, where the total field must be 0.

- The negative induced charge is closer to q , so there is a net force of attraction.

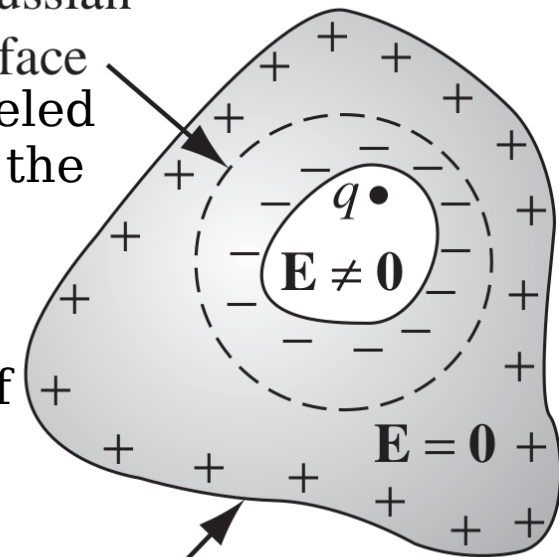
- If you put some charge in a cavity of the conductor, the field in the cavity will not be 0. But the cavity and its contents are electrically isolated from the outside world by the surrounding conductor.

- No external fields penetrate the conductor; they are canceled at the outer surface by the induced charge there. Similarly, the field due to charges within the cavity is canceled by the induced charge on the inner surface.

- The compensating charge left over on the outer surface of the conductor effectively “communicates” the presence of q to the outside world.



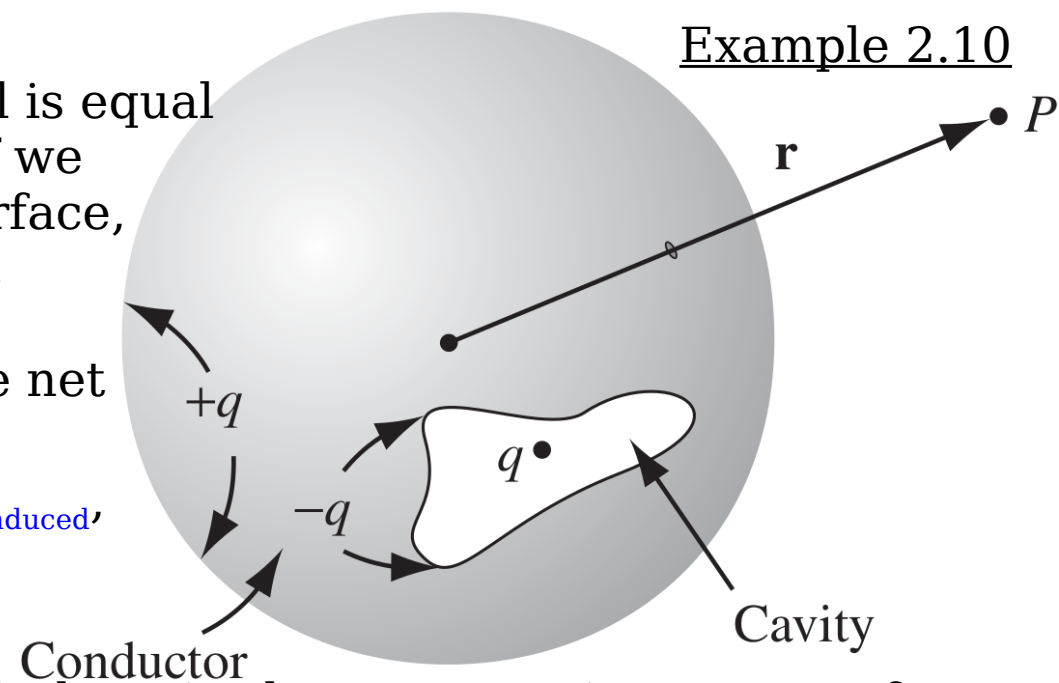
Gaussian surface



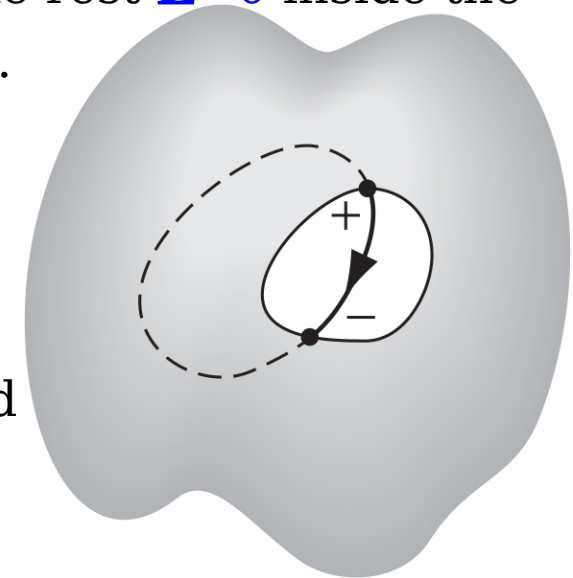
Example 2.10

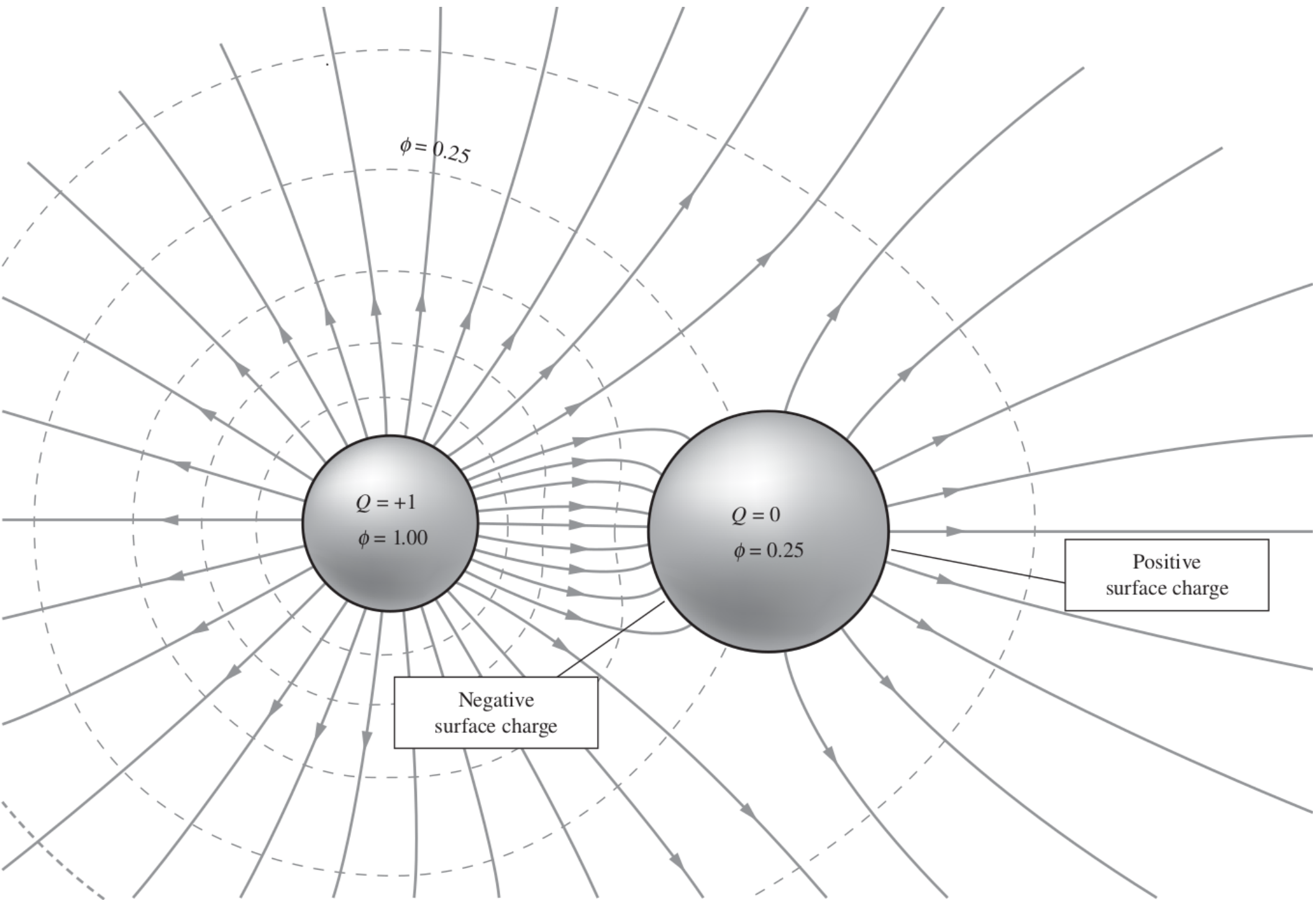
- The charge induced on the cavity wall is equal and opposite to the charge inside, for if we surround the cavity with a Gaussian surface, all points of which are in the conductor,

$\oint \mathbf{E} \cdot d\mathbf{a} = 0$ hence (by Gauss's law) the net enclosed charge must be $0 = Q_{\text{enc}} = q + q_{\text{induced}}$, so $q_{\text{induced}} = -q$.



- If the conductor is electrically neutral, there is charge $+q$ on its outer surface.
- If a cavity has no charge, then the field within the cavity is 0.
- For any field line would have to begin and end on the cavity wall, from $+charge$ to $-charge$. For the field line being part of a closed loop, the rest $\mathbf{E} = 0$ inside the conductor, so $\oint \mathbf{E} \cdot d\boldsymbol{\ell}$ is distinctly *positive*, violating Eq. (*).
- This is why you are relatively safe inside a metal car during a thunderstorm.
- The same principle applies to the placement of sensitive apparatus inside a grounded **Faraday cage** or net, to shield out stray electric fields—**Electrostatic Shielding**.





Surface Charge and the Force on a Conductor

● The field inside a conductor is 0, so the field immediately *outside* is:

$$\mathbf{E} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}} \Rightarrow \sigma = -\epsilon_0 \frac{\partial \Phi}{\partial n}$$

● These equations enable you to calculate the *surface charge* on a conductor, if you can determine \mathbf{E} or Φ .

● In the presence of an electric field, a surface charge will experience a force; the force per unit area, $\mathbf{f} = \sigma \mathbf{E}$.

● But the electric field is *discontinuous* at a surface charge, so we should use the *average* of the two:

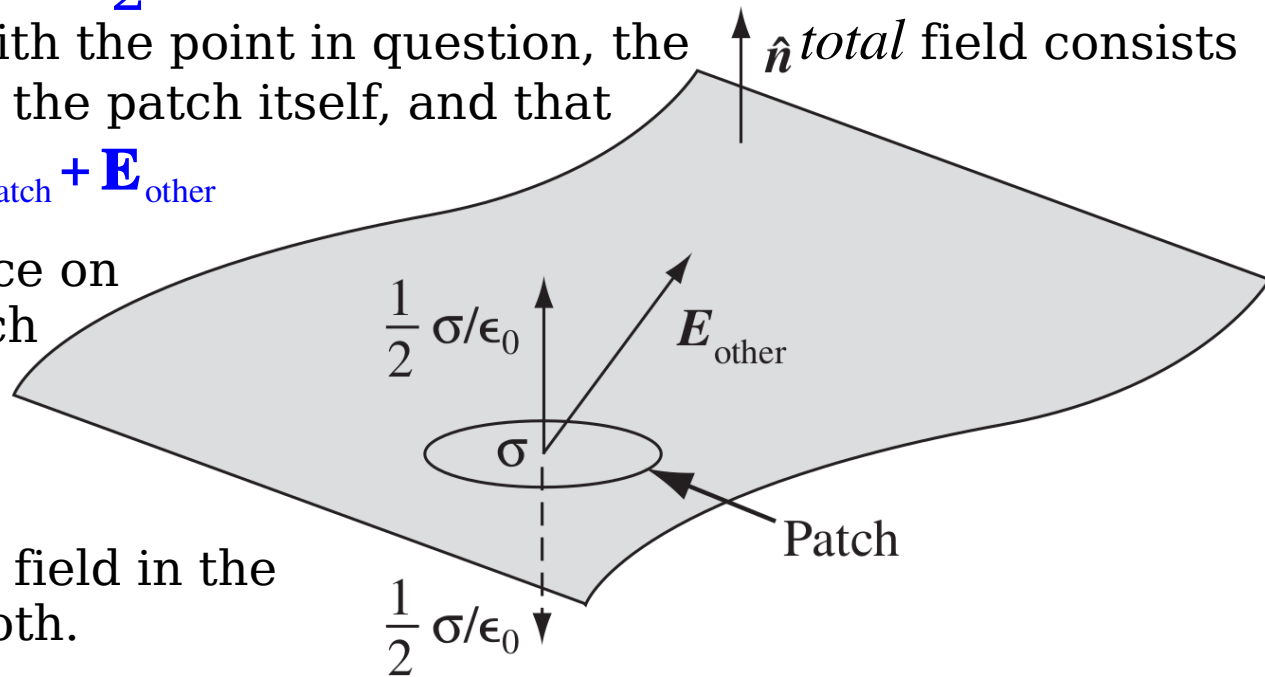
$$\mathbf{f} = \sigma \mathbf{E}_{\text{average}} = \frac{1}{2} \sigma (\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}})$$

● For a tiny patch of surface with the point in question, the $\hat{\mathbf{n}}$ *total* field consists of 2 parts—that attributable to the patch itself, and that

due to everything else: $\mathbf{E} = \mathbf{E}_{\text{patch}} + \mathbf{E}_{\text{other}}$

● The patch cannot exert a force on itself, so the force on the patch is due exclusively to $\mathbf{E}_{\text{other}}$, and this suffers no discontinuity.

● If we removed the patch, the field in the “hole” would be perfectly smooth.



- The discontinuity is due entirely to the charge on the patch, which puts out a Field ($\frac{\sigma}{2\epsilon_0}$) on either side, pointing away from the surface

$$\begin{aligned} \mathbf{E}_{\text{above}} &= \mathbf{E}_{\text{other}} + \frac{\sigma}{2\epsilon_0} \hat{\mathbf{n}} \\ \mathbf{E}_{\text{below}} &= \mathbf{E}_{\text{other}} - \frac{\sigma}{2\epsilon_0} \hat{\mathbf{n}} \end{aligned} \Rightarrow \mathbf{E}_{\text{other}} = \frac{1}{2} (\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}}) = \mathbf{E}_{\text{average}}$$

- Averaging is a device for removing the contribution of the patch itself.
- The argument applies to *any* surface charge, especially, to a conductor,

$$\begin{aligned} \mathbf{E}_{\text{outside}} &= \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}} \\ \mathbf{E}_{\text{inside}} &= 0 \end{aligned} \Rightarrow \mathbf{f} = \sigma \mathbf{E}_{\text{average}} = \frac{\sigma}{2} (\mathbf{E}_{\text{outside}} + \mathbf{E}_{\text{inside}}) = \frac{\sigma^2}{2\epsilon_0} \hat{\mathbf{n}}$$

- This amounts to an outward **electrostatic pressure** on the surface, tending to draw the conductor into the field, regardless of the *sign* of σ

$$P = |\mathbf{f}| = \frac{\sigma^2}{2\epsilon_0} = \frac{\epsilon_0}{2} \left(\frac{\sigma}{\epsilon_0} \right)^2 = \frac{\epsilon_0}{2} E^2$$

Example: Force Between 2 Halves of a Charged Sphere:

A conducting spherical shell of radius R is charged to a potential V . The shell is sliced into 2 separate hemispheres and kept in place. Find the force exerted by the hemispheres on each other.
[Problem 2.42]

- Consider the northern hemisphere:

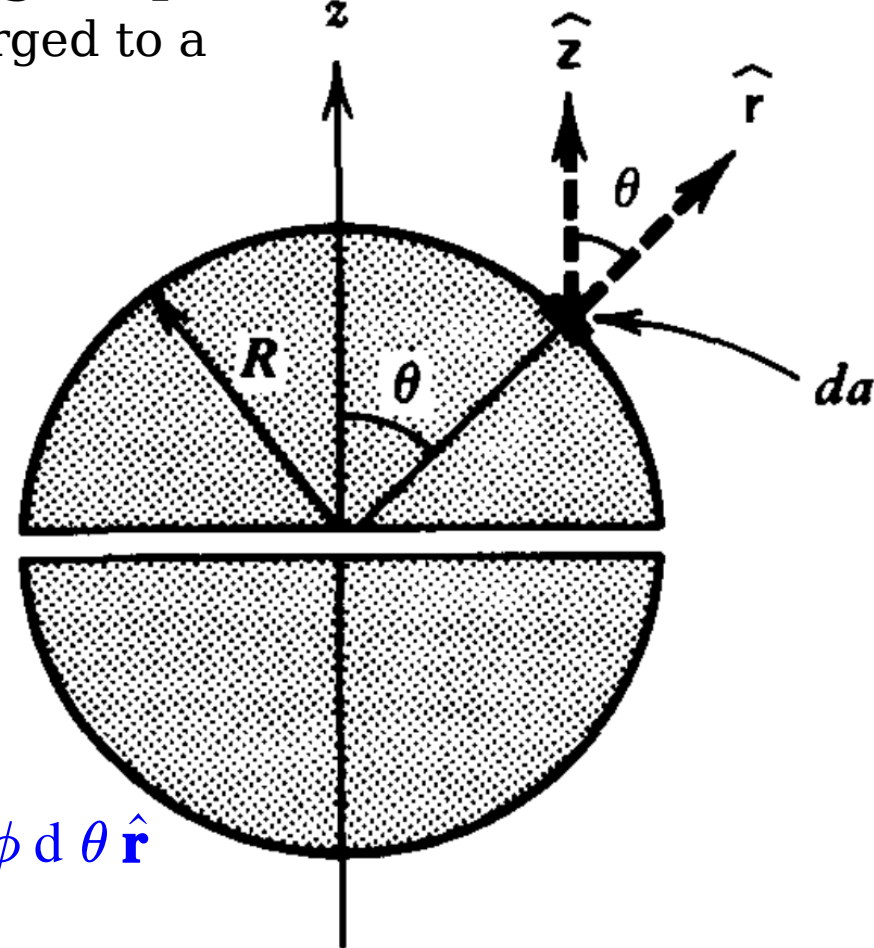
$$Q = C V = 4 \pi \epsilon_0 R V \Rightarrow \sigma = \frac{Q}{4 \pi R^2} = \epsilon_0 \frac{V}{R}$$

$$\mathbf{f} \equiv \frac{\text{force}}{\text{area}} = \sigma \mathbf{E}_{\text{average}} = \frac{\sigma^2}{2 \epsilon_0} \hat{\mathbf{n}}$$

$$\Rightarrow d\mathbf{F} = \mathbf{f} da = \frac{\sigma^2}{2 \epsilon_0} da \hat{\mathbf{r}} = \frac{\epsilon_0 V^2}{2 R^2} R^2 \sin \theta d\phi d\theta \hat{\mathbf{r}}$$

$$\Rightarrow \mathbf{F} = \int d\mathbf{F} = \frac{\epsilon_0}{2} V^2 \int_0^{\pi/2} \int_0^{2\pi} (\sin \theta \cancel{\cos \phi} \hat{\mathbf{x}} + \sin \theta \cancel{\sin \phi} \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}}) \sin \theta d\phi d\theta$$

$$= \pi \epsilon_0 V^2 \hat{\mathbf{z}} \int_0^{\pi/2} \cos \theta \sin \theta d\theta = \frac{\pi}{2} \epsilon_0 V^2 \hat{\mathbf{z}} \leftarrow \text{repulsive}$$



Capacitors

- For 2 conductors, we put charge $+Q$ on one and $-Q$ on the other. Since V is constant over a conductor, the potential difference between them:

$$V = V_+ - V_- = - \int^+ \mathbf{E} \cdot d\ell$$

- \mathbf{E} is *proportional* to Q : $\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{r^2} \hat{r} d\tau$, so if you double ρ , you double \mathbf{E} .

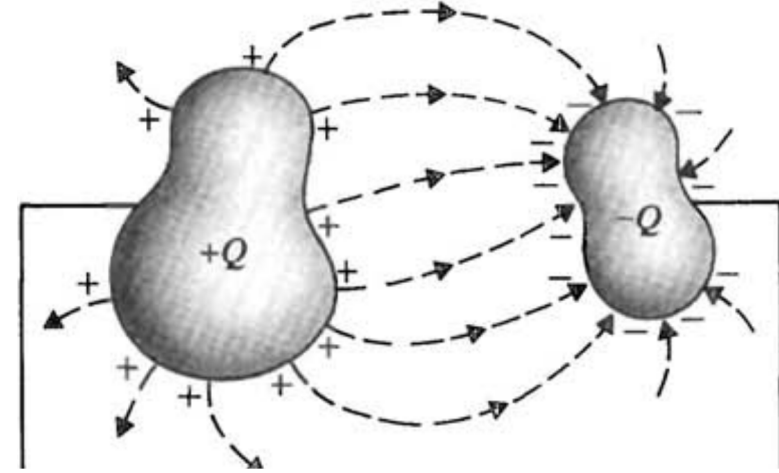
- Since \mathbf{E} is proportional to Q , so also is V . The constant of proportionality is called the **capacitance** of the arrangement: $C \equiv \frac{Q}{V}$

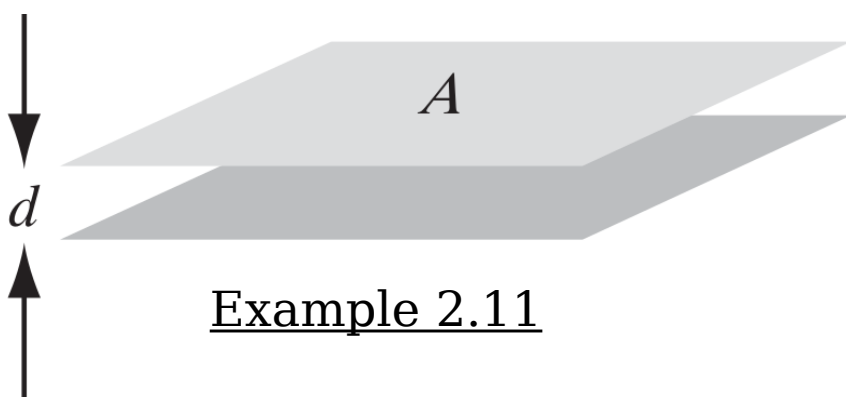
- Capacitance is a purely geometrical quantity, determined by the sizes, shapes, and separation of the 2 conductors.

- In SI units, C is measured in **farads** (F); a farad is a coulomb-per-volt. Actually, this turns out to be inconveniently large; more practical units are the microfarad (10^{-6} F) and the picofarad (10^{-12} F).

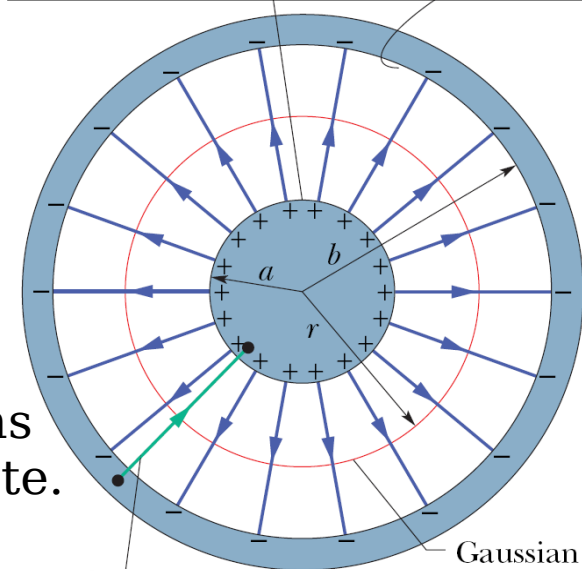
- V is, by definition, the potential of the *positive* conductor; and Q is the charge of the *positive* conductor. So capacitance is an intrinsically positive quantity.

- For the capacitance of a single conductor, the “2nd conductor,” with the negative charge, is an imaginary spherical shell of $r \rightarrow \infty$ surrounding the one conductor. So the capacitance is given by taking V with ∞ as the reference point.





Example 2.11



Example 2.12

● To “charge up” a capacitor, you have to remove electrons from the positive plate and carry them to the negative plate.

● In doing so, you fight against the electric field, which is pulling them back toward the positive conductor and pushing them away from the negative one.

● Let the charge on the positive plate is q , so that the potential difference is $\frac{q}{C}$. The work needed to

transport the next piece of charge, dq , is

$$dW = \frac{q}{C} dq \Rightarrow W = \int_0^Q \frac{q}{C} dq \quad \Leftarrow \text{total work}$$

$$= \frac{Q^2}{2C} = \frac{1}{2} CV^2 \quad \Leftarrow Q = CV$$

Selected problems: 8, 16, 25, 36, 39, 50

