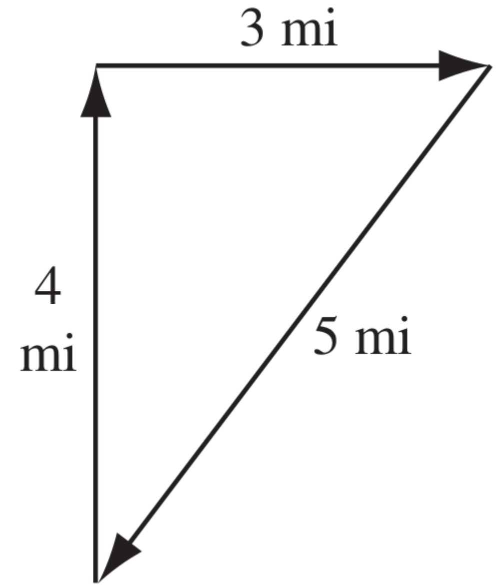
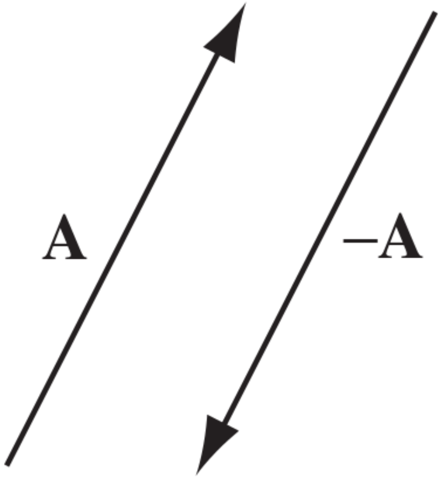
Chapter 1 Vector Analysis



Vector Algebra

Vector Operations



- **Displacements,** straight line segments going from one point to another, have *direction* as well as *magnitude* (length),

- Such objects are called **vectors**: velocity, acceleration, force and momentum are other examples.

- Quantities that have magnitude but no direction are called **scalars**: examples include mass, charge, density, and temperature.

- The magnitude of a vector $\mathbf{A}$ is written $|\mathbf{A}|$ or, more simply, A.

- $-\mathbf{A}$ is a vector with the same magnitude as $\mathbf{A}$ but of opposite direction.

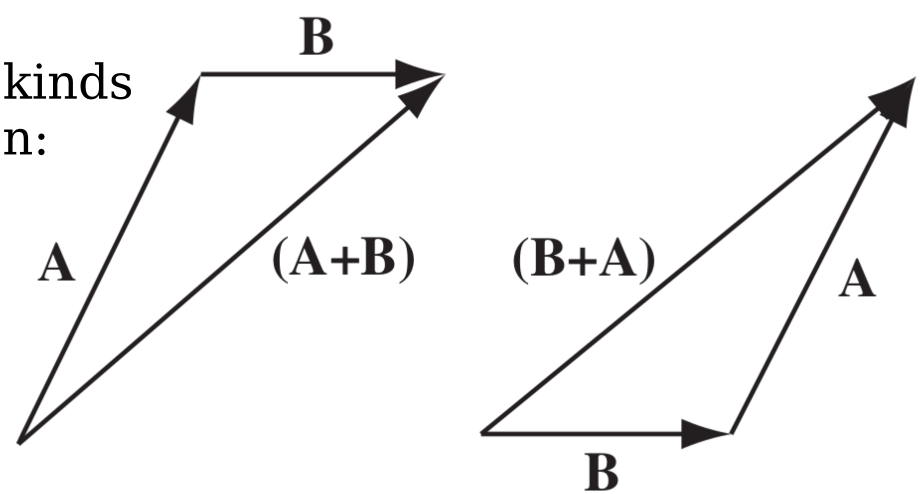
- Define 4 vector operations: addition and 3 kinds of multiplication:

(i) Addition of 2 vectors

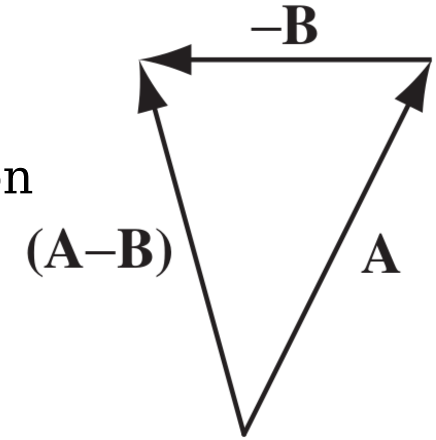
$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A} \qquad \textit{commutative}$$

$$(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C}) \qquad \textit{associative}$$

$$\mathbf{A}-\mathbf{B}=\mathbf{A}+(-\mathbf{B}) \qquad \textit{subtraction}$$



(ii) Multiplication by a scalar: Multiplication of a vector by a positive scalar a multiplies the magnitude but leaves the direction unchanged.



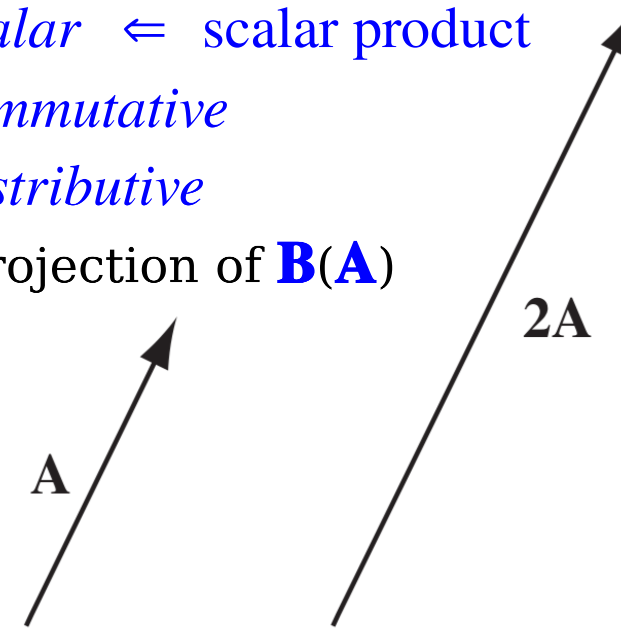
$$a\left(\mathbf{A}+\mathbf{B}\right)=a\,\mathbf{A}+a\,\mathbf{B} \quad \textit{distributive}$$

- If a is negative, the direction is reversed.

(iii) Dot product of 2 vectors: $\mathbf{A}\cdot\mathbf{B}\equiv A\,B\cos\theta$ *a scalar* ⇐ scalar product

$$\mathbf{A}\cdot\mathbf{B}=\mathbf{B}\cdot\mathbf{A} \qquad \textit{commutative}$$

$$\mathbf{A}\cdot\left(\mathbf{B}+\mathbf{C}\right)=\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C} \quad \textit{distributive}$$

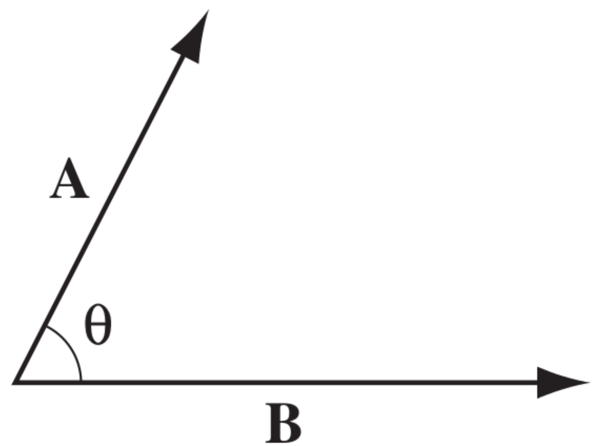


- Geometrically, $\mathbf{A}\cdot\mathbf{B}$ is the product of $\mathbf{A}$($\mathbf{B}$) times the projection of $\mathbf{B}$($\mathbf{A}$) along $\mathbf{A}$($\mathbf{B}$).

- If $\mathbf{A}\parallel\mathbf{B}$, then $\mathbf{A}\cdot\mathbf{B}=A\,B$.

- For any vector $\mathbf{A}$, $\mathbf{A}\cdot\mathbf{A}=A^2$.

- If $\mathbf{A}\perp\mathbf{B}$, then $\mathbf{A}\cdot\mathbf{B}=0$.

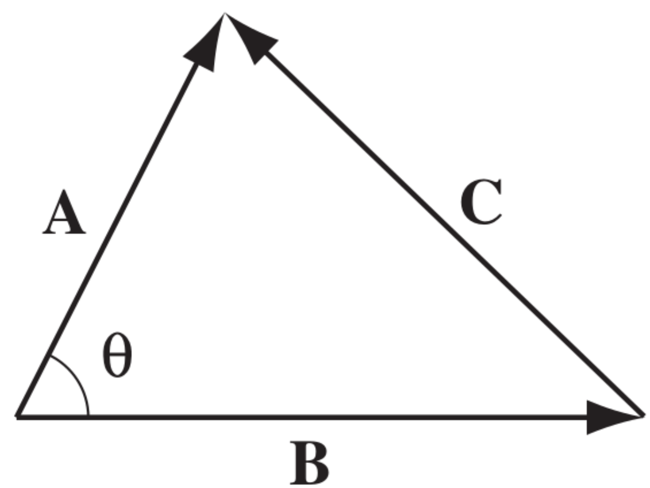


<u>Example 1.1</u>

(iv) Cross product of 2 vectors:

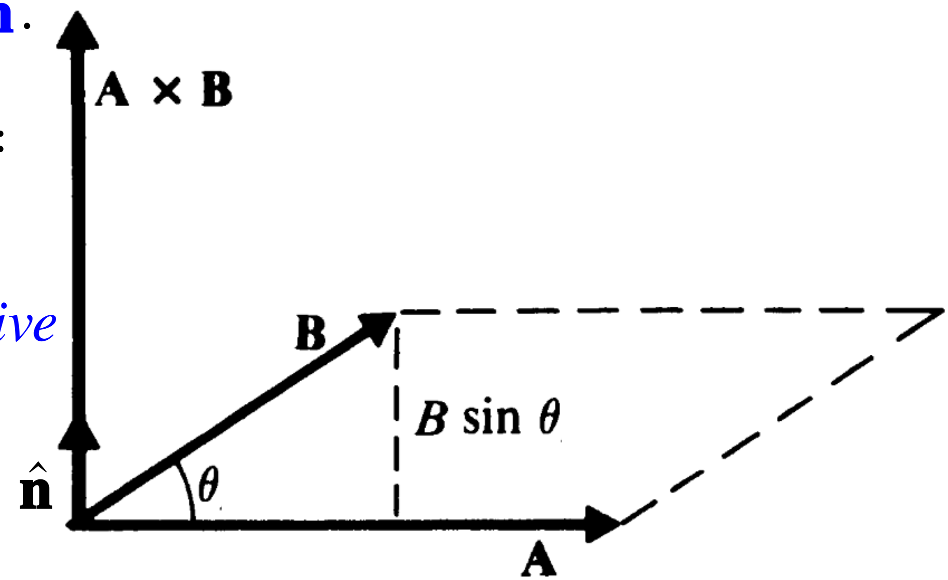
$$\mathbf{A}\times\mathbf{B}\equiv A\,B\sin\theta\,\hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a unit vector ⊥ the plane of $\mathbf{A}$ and $\mathbf{B}$



- There are 2 directions ⊥ any plane: "in" and "out."

- The ambiguity is resolved by the **right-hand rule**: let your fingers point in the direction of the 1st vector and curl around (via the smaller angle) toward the 2nd; then your thumb indicates the direction of $\hat{\mathbf{n}}$.



- $\mathbf{A}\times\mathbf{B}$ is itself a *vector*, ie, **vector product**:

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C})=\mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C} \quad \textit{distributive}$$

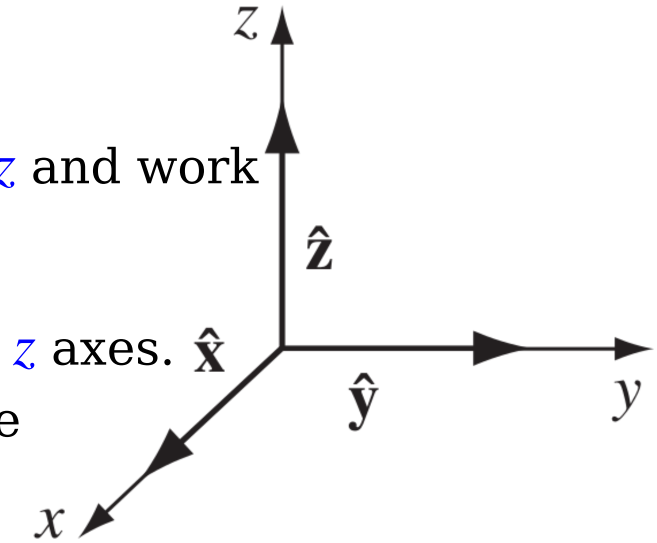
$$\mathbf{A}\times\mathbf{B}=-\mathbf{B}\times\mathbf{A} \quad \textit{anti - commutative}$$

- Geometrically, $|\mathbf{A}\times\mathbf{B}|$ is the area of the parallelogram generated by $\mathbf{A}$ and $\mathbf{B}$.

- If 2 vectors are parallel, their cross product is 0.

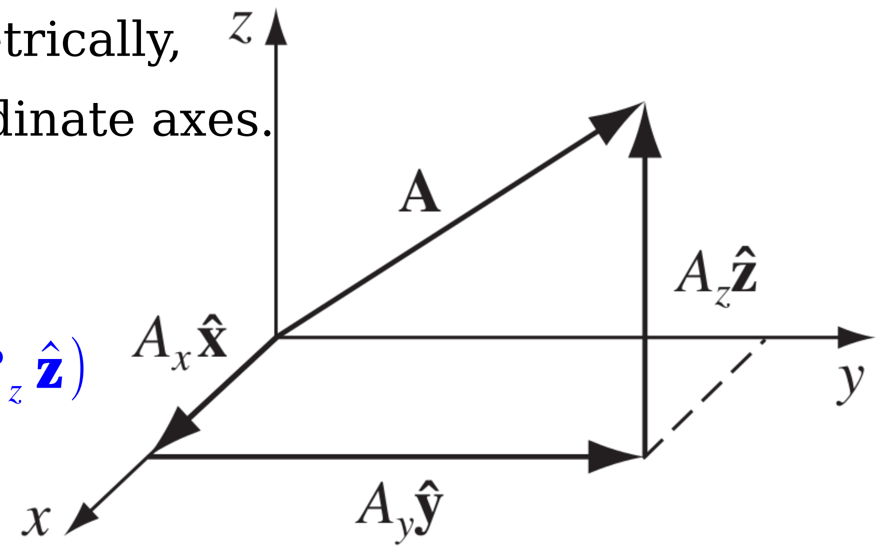
- $\mathbf{A}\times\mathbf{A}=0$ for any vector $\mathbf{A}$.

Vector Algebra: Component Form



- It is often easier to set up Cartesian coordinates x, y, z and work with vector **components**.

- Let $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ be unit vectors parallel to the x, y, and z axes. An arbitrary vector $\mathbf{A}$ can be expanded in terms of these **basis vectors**: $\mathbf{A} = A_x\,\hat{\mathbf{x}} + A_y\,\hat{\mathbf{y}} + A_z\,\hat{\mathbf{z}}$



- A_x, A_y, A_z, are the “components” of $\mathbf{A}$; geometrically, they are the projections of $\mathbf{A}$ along the 3 coordinate axes.

$$A_x = \mathbf{A}\cdot\hat{\mathbf{x}}\,, \quad A_y = \mathbf{A}\cdot\hat{\mathbf{y}}\,, \quad A_z = \mathbf{A}\cdot\hat{\mathbf{z}}$$

- Reformulate the vector operations:

$$\mathbf{A}+\mathbf{B} = \left(A_x\,\hat{\mathbf{x}} + A_y\,\hat{\mathbf{y}} + A_z\,\hat{\mathbf{z}}\right) + \left(B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\,\hat{\mathbf{z}}\right)$$
$$= \left(A_x + B_x\right)\hat{\mathbf{x}} + \left(A_y + B_y\right)\hat{\mathbf{y}} + \left(A_z + B_z\right)\hat{\mathbf{z}}$$

Rule (i): *To add vectors, add like components.*

$$a\,\mathbf{A} = \left(a\,A_x\right)\hat{\mathbf{x}} + \left(a\,A_y\right)\hat{\mathbf{y}} + \left(a\,A_z\right)\hat{\mathbf{z}}$$

Rule (ii): *To multiply by a scalar, multiply each component.*

$$\hat{\mathbf{x}}\perp\hat{\mathbf{y}}\perp\hat{\mathbf{z}} \quad\Rightarrow\quad \hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1\,, \quad \hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{x}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = 0$$
$$\Rightarrow\quad \mathbf{A}\cdot\mathbf{B} = \left(A_x\,\hat{\mathbf{x}} + A_y\,\hat{\mathbf{y}} + A_z\,\hat{\mathbf{z}}\right)\cdot\left(B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\,\hat{\mathbf{z}}\right) = A_x\,B_x + A_y\,B_y + A_z\,B_z$$

Rule (iii): *To calculate the dot product, multiply like components, and add.*

Let $\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}$, $\hat{\mathbf{x}}_2 = \hat{\mathbf{y}}$, $\hat{\mathbf{x}}_3 = \hat{\mathbf{z}}$, $A_1 = A_x$, $A_2 = A_y$, $A_3 = A_z$

$$\Rightarrow \quad \mathbf{A} = A_x\,\hat{\mathbf{x}} + A_y\,\hat{\mathbf{y}} + A_z\,\hat{\mathbf{z}} = A_1\,\hat{\mathbf{x}}_1 + A_2\,\hat{\mathbf{x}}_2 + A_3\,\hat{\mathbf{x}}_3 = \sum_{k=1}^{3} A_k\,\hat{\mathbf{x}}_k$$

$$\hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j = \delta_{ij} \quad \Leftarrow \quad \delta_{ij} = \begin{cases} 1\,, & \text{for } i = j \\ 0\,, & \text{for } i \neq j \end{cases}$$ Kronecker delta

$$\Rightarrow \quad \mathbf{A} \cdot \mathbf{B} = \sum_i A_i\,\hat{\mathbf{x}}_i \cdot \sum_j B_j\,\hat{\mathbf{x}}_j = \sum_i \sum_j A_i\,B_j\,\hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j$$

$$= \sum_i \sum_j A_i\,B_j\,\delta_{ij} = \sum_i^{3} A_i\,B_i = A_1\,B_1 + A_2\,B_2 + A_3\,B_3$$

One can get rid of the annoying summation symbol $\sum$ by allowing one index up and the same index down to represent the summation.

For example, $A^i\,B_i = \sum_i^{3} A_i\,B_i = A_1\,B_1 + A_2\,B_2 + A_3\,B_3$

This is called the Einstein notation. We will only use it in Chapter 12.

$$\mathbf{A}\cdot\mathbf{A} = A_x^2 + A_y^2 + A_z^2 \quad \Rightarrow \quad A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$\hat{\mathbf{x}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}\times\hat{\mathbf{z}} = 0\,,$$

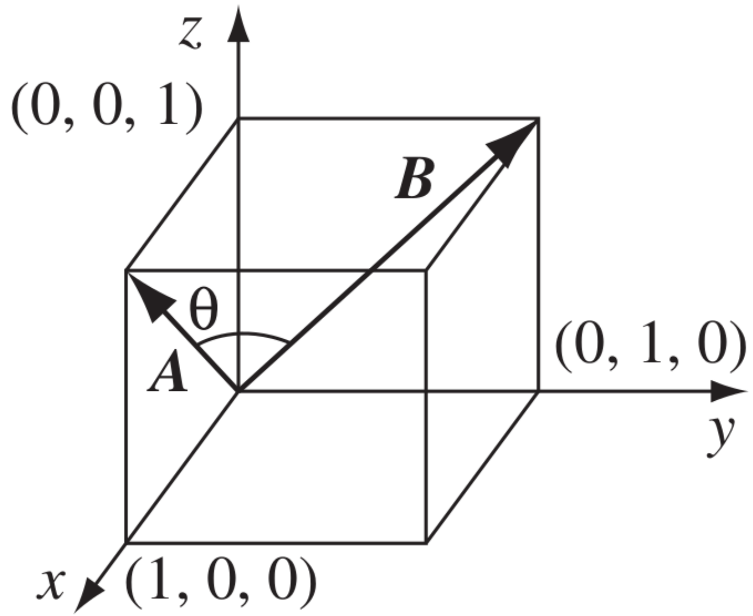
$$\hat{\mathbf{x}}\times\hat{\mathbf{y}} = -\hat{\mathbf{y}}\times\hat{\mathbf{x}} = \hat{\mathbf{z}}\,,\quad \hat{\mathbf{y}}\times\hat{\mathbf{z}} = -\hat{\mathbf{z}}\times\hat{\mathbf{y}} = \hat{\mathbf{x}}\,,\quad \hat{\mathbf{z}}\times\hat{\mathbf{x}} = -\hat{\mathbf{x}}\times\hat{\mathbf{z}} = \hat{\mathbf{y}}$$

$$\Rightarrow \quad \mathbf{A}\times\mathbf{B} = \left(A_x\,\hat{\mathbf{x}} + A_y\,\hat{\mathbf{y}} + A_z\,\hat{\mathbf{z}}\right)\times\left(B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\,\hat{\mathbf{z}}\right)$$

$$= \left(A_y B_z - A_z B_y\right)\hat{\mathbf{x}} + \left(A_z B_x - A_x B_z\right)\hat{\mathbf{y}} + \left(A_x B_y - A_y B_x\right)\hat{\mathbf{z}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Rule (iv): *To calculate the cross product, form the determinant whose 1st row is* $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$, *whose 2nd row is* **A** *(in component form), and whose 3rd row is* **B**.

Example 1.2



Levi-Civita symbol: $\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is } (1,2,3), (2,3,1), \text{ or } (3,1,2) \\ -1 & \text{if } (i, j, k) \text{ is } (1,3,2), (3,2,1), \text{ or } (2,1,3) \\ 0 & \text{if } i = j, \text{ or } j = k, \text{ or } k = i \end{cases}$

$$\Rightarrow \quad \hat{\mathbf{x}}_i \times \hat{\mathbf{x}}_j = \sum_{k=1}^{3} \epsilon_{ijk} \hat{\mathbf{x}}_k$$

$$\Rightarrow \quad \mathbf{A} \times \mathbf{B} = \sum_i A_i \hat{\mathbf{x}}_i \times \sum_j B_j \hat{\mathbf{x}}_j = \sum_{i,j} A_i B_j \hat{\mathbf{x}}_i \times \hat{\mathbf{x}}_j$$

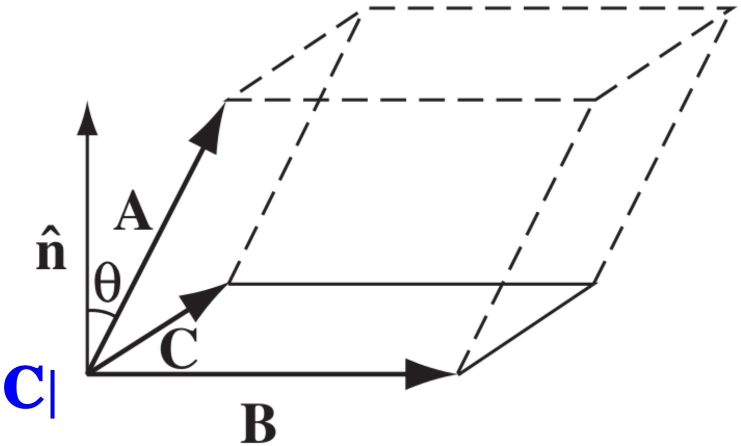
$$= \sum_{i,j} A_i B_j \sum_k \epsilon_{ijk} \hat{\mathbf{x}}_k = \sum_{i,j,k} \epsilon_{ijk} A_i B_j \hat{\mathbf{x}}_k = \begin{vmatrix} \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 & \hat{\mathbf{x}}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$\Rightarrow \quad (\mathbf{A} \times \mathbf{B})_i = \sum_{j,k} \epsilon_{ijk} A_j B_k$$

By the way, if $\mathbb{M}$ is a 3×3 matrix, det $\mathbb{M} = |\mathbb{M}| = \sum_{i,j,k} \epsilon_{ijk} \mathbb{M}_{1i} \mathbb{M}_{2j} \mathbb{M}_{3k}$

Triple Products

(i) Scalar triple product: $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$



- Geometrically, $|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|$ is the volume of the parallelepiped generated by $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, since $|\mathbf{B}\times\mathbf{C}|$ is the area of the base, and $|A\cos\theta|$ is the altitude.

- $$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \quad \Leftarrow \quad \text{cyclic}$$
  $$\mathbf{A}\cdot(\mathbf{C}\times\mathbf{B})=\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C})=\mathbf{C}\cdot(\mathbf{B}\times\mathbf{A})=-\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$$
  $$\Rightarrow \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

(ii) Vector triple product:

- The vector triple product can be simplified by the so-called **BAC-CAB** rule:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \qquad (*)$$

- $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}=-\mathbf{C}\times(\mathbf{A}\times\mathbf{B})=-\mathbf{A}(\mathbf{B}\cdot\mathbf{C})+\mathbf{B}(\mathbf{A}\cdot\mathbf{C})$ entirely different vector

- All *higher* vector products can be similarly reduced, usually by repeated (*)

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$$
$$\mathbf{A}\times[\mathbf{B}\times(\mathbf{C}\times\mathbf{D})]=\mathbf{B}[\mathbf{A}\cdot(\mathbf{C}\times\mathbf{D})]-(\mathbf{A}\cdot\mathbf{B})(\mathbf{C}\times\mathbf{D})$$

$$\sum_{k} \epsilon_{ijk}\,\epsilon_{mnk} = \delta_{im}\,\delta_{jn} - \delta_{in}\,\delta_{jm}\,, \quad \sum_{j,k} \epsilon_{ijk}\,\epsilon_{\ell jk} = 2\,\delta_{i\ell}\,, \quad \sum_{i,j,k} \epsilon_{ijk}\,\epsilon_{ijk} = 6$$

$$\Uparrow \quad \sum_{k} \delta_{ik}\,\delta_{jk} = \delta_{ij}\,, \quad \sum_{k} \delta_{kk} = 3$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \sum_{i} A_i\,(\mathbf{B}\times\mathbf{C})_i = \sum_{i} A_i \sum_{j,k} \epsilon_{ijk}\,B_j\,C_k$$

$$= \sum_{i,j,k} \epsilon_{ijk}\,A_i\,B_j\,C_k = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

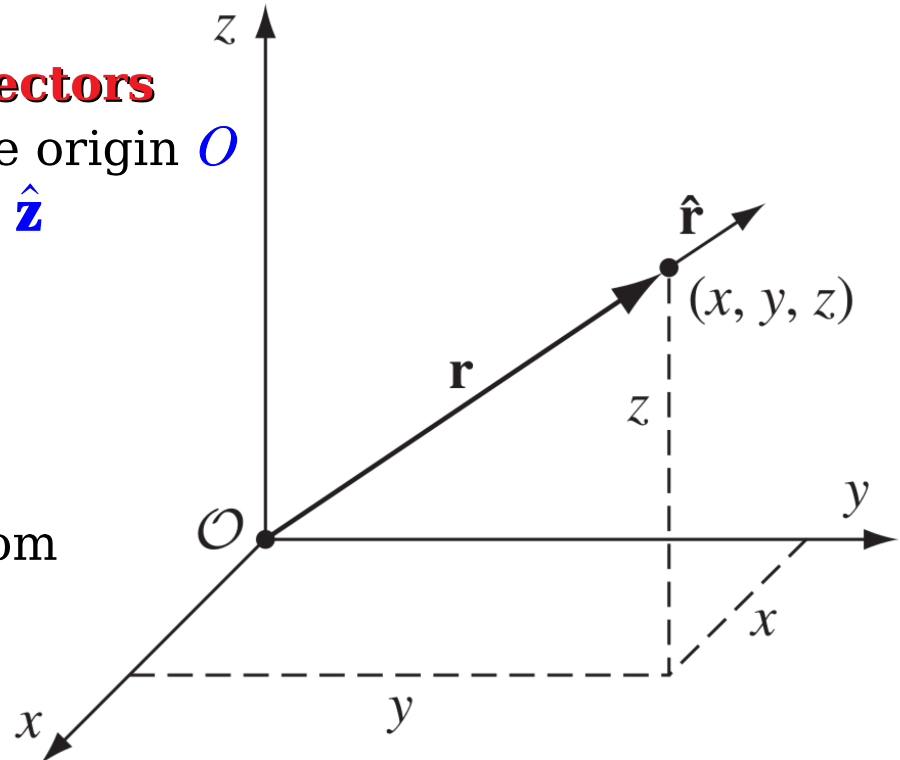
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \sum_{i,j,k} \epsilon_{ijk}\,A_i\,(\mathbf{B}\times\mathbf{C})_j\,\hat{\mathbf{x}}_k = \sum_{i,j,k} \epsilon_{ijk}\,A_i\,\hat{\mathbf{x}}_k \sum_{m,n} \epsilon_{jmn}\,B_m\,C_n$$

$$= \sum_{i,j,k,m,n} \epsilon_{ijk}\,\epsilon_{jmn}\,A_i\,B_m\,C_n\,\hat{\mathbf{x}}_k = -\sum_{i,j,k,m,n} \epsilon_{ikj}\,\epsilon_{mnj}\,A_i\,B_m\,C_n\,\hat{\mathbf{x}}_k$$

$$= -\sum_{i,k,m,n} (\delta_{im}\,\delta_{kn} - \delta_{in}\,\delta_{km})\,A_i\,B_m\,C_n\,\hat{\mathbf{x}}_k = \sum_{i,k} (A_i\,C_i\,B_k\,\hat{\mathbf{x}}_k - A_i\,B_i\,C_k\,\hat{\mathbf{x}}_k)$$

$$= \Big(\sum_{i} A_i\,C_i\Big) \sum_{k} B_k\,\hat{\mathbf{x}}_k - \Big(\sum_{i} A_i\,B_i\Big) \sum_{k} C_k\,\hat{\mathbf{x}}_k = (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\,\mathbf{C}$$

Position, Displacement, and Separation Vectors



- The vector to the location of a point from the origin O is called the **position vector**: $\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$

- Its magnitude, $r = \sqrt{x^2 + y^2 + z^2}$

and the unit vector is $\hat{\mathbf{r}} = \dfrac{\mathbf{r}}{r} = \dfrac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$

- The **infinitesimal displacement** vector, from (x, y, z) to $(x+\mathrm{d}x, y+\mathrm{d}y, z+\mathrm{d}z)$, is

$$\mathrm{d}\boldsymbol{\ell} = \mathrm{d}\mathbf{r} = \mathrm{d}x\,\hat{\mathbf{x}} + \mathrm{d}y\,\hat{\mathbf{y}} + \mathrm{d}z\,\hat{\mathbf{z}}$$

- In electrodynamics, one frequently encounters problems involving 2 points—typically, a **source point** $\mathbf{r}'$, where an electric charge is located, and a **field point** $\mathbf{r}$, at which you are calculating the electric or magnetic field.

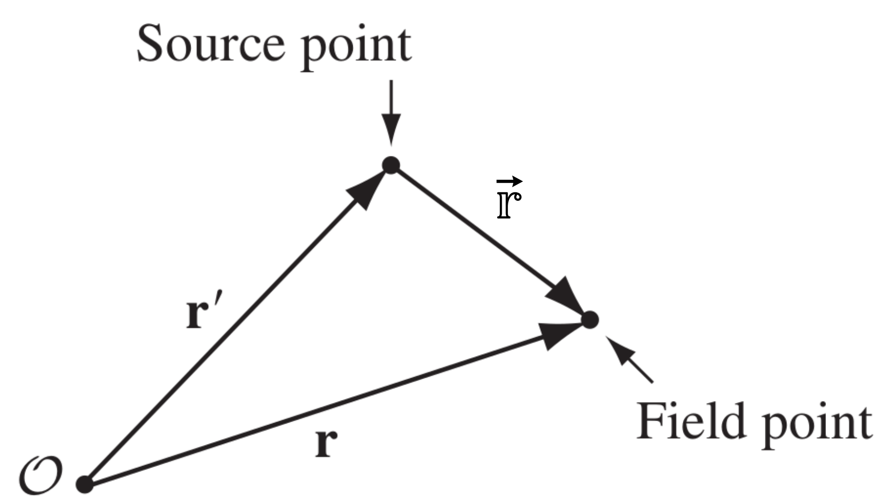
- A short-hand for the **separation vector** from the source point to the field point

$$\vec{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}' \quad \Rightarrow \quad \mathscr{r} = |\mathbf{r} - \mathbf{r}'|, \quad \hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

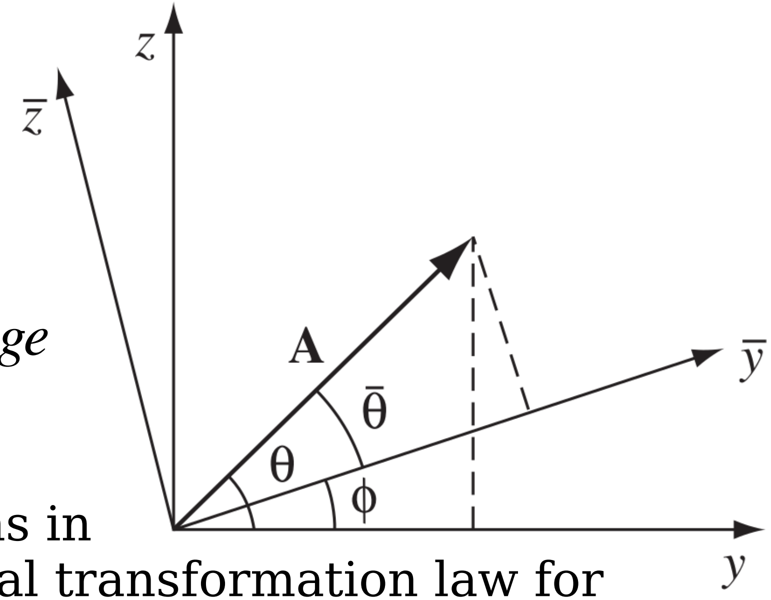
$$\vec{\mathscr{r}} = (x - x')\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + (z - z')\,\hat{\mathbf{z}}$$

$$\Rightarrow \quad \mathscr{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

$$\hat{\mathscr{r}} = \frac{(x - x')\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + (z - z')\,\hat{\mathbf{z}}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}$$



How Vectors Transform



- The definition of a vector as "a quantity with a magnitude and direction" is not satisfactory.

- A vector *should transform properly when you change coordinates*.

- The coordinate frame we use to describe positions in space is arbitrary, but there is a specific geometrical transformation law for converting vector components from one frame to another.

- Let the $\bar{x}$, $\bar{y}$, $\bar{z}$ system is rotated by angle ϕ, relative to x, y, z, about the common $x = \bar{x}$ axes:

$$A_y = A\cos\theta\,,\quad A_z = A\sin\theta$$

$$\Rightarrow \begin{aligned} \bar{A}_y &= A\cos\bar{\theta} = A\cos(\theta-\phi) = A(\cos\theta\cos\phi + \sin\theta\sin\phi) = A_y\cos\phi + A_z\sin\phi \\ \bar{A}_z &= A\sin\bar{\theta} = A\sin(\theta-\phi) = A(\sin\theta\cos\phi - \cos\theta\sin\phi) = -A_y\sin\phi + A_z\cos\phi \end{aligned}$$

$$\Rightarrow \begin{bmatrix} \bar{A}_y \\ \bar{A}_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} A_y \\ A_z \end{bmatrix}$$

- For rotation about an *arbitrary* axis in 3D:

$$\begin{bmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{bmatrix} = \begin{bmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\Rightarrow \quad \bar{A}_i = \sum_{j=1}^{3} R_{ij} A_j$$

- Formally, *a vector is any set of 3 components that transforms in the same manner as a displacement when you change coordinates*. As always, displacement is the *model* for the behavior of all vectors.

- A (2nd-rank) **tensor** is a quantity with 9 components, T_{xx}, T_{xy}, T_{xz}, T_{yx}, ..., T_{zz}, which transform with 2 factors of R:

$$\begin{aligned}\overline{T}_{xx} = R_{xx}\left(R_{xx}T_{xx} + R_{xy}T_{xy} + R_{xz}T_{xz}\right) \\ + R_{xy}\left(R_{xx}T_{yx} + R_{xy}T_{yy} + R_{xz}T_{yz}\right) \\ + R_{xz}\left(R_{xx}T_{zx} + R_{xy}T_{zy} + R_{xz}T_{zz}\right), \cdots\end{aligned} \quad \Rightarrow \quad \overline{T}_{ij} = \sum_{k=1}^{3}\sum_{\ell=1}^{3} R_{ik}R_{j\ell}T_{k\ell}$$

- In general, an n^{th}-rank tensor has n indices and 3^n components, and transforms with n factors of R.

- A vector is a tensor of rank 1, and a scalar is a tensor of rank 0.

Differential Calculus

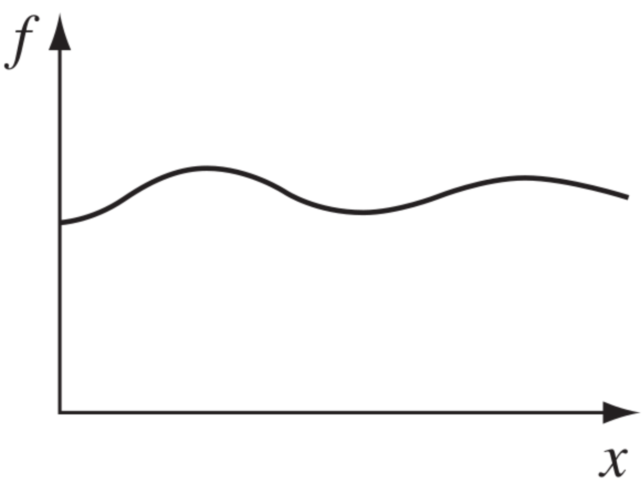
"Ordinary" Derivatives

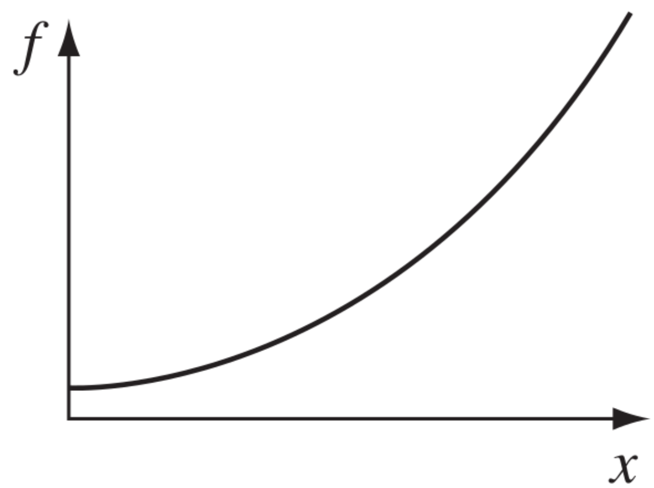
- If we have $f(x)$, what does $\frac{df}{dx}$ do for us? It tells us how rapidly $f(x)$ varies when we change x by a tiny amount, dx $\Rightarrow$ $df = \frac{df}{dx}\, dx$

- If we increment x by dx, then f changes by df; the derivative is the proportionality factor.

- If f varies slowly with x, and the derivative is correspondingly small. If f increases rapidly with x, and the derivative is large.

- *Geometrical Interpretation*: The derivative $\frac{df}{dx}$ is the *slope* of the graph of f vs x.





Gradient

- If we have a function of 3 variables, $T(x,y,z)$, we want to generalize the notion of "derivative" to functions like T, which depend not on *one* but on *3* variables.

- A derivative is supposed to tell us how fast the function varies for a little distance, and on what *direction* we move.

- $$\mathrm{d}T = \frac{\partial T}{\partial x}\mathrm{d}x + \frac{\partial T}{\partial y}\mathrm{d}y + \frac{\partial T}{\partial z}\mathrm{d}z$$

This tells us how T changes when we alter all 3 variables by $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$.

- Rewrite $\mathrm{d}T = \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right)\cdot\left(\mathrm{d}x\,\hat{\mathbf{x}} + \mathrm{d}y\,\hat{\mathbf{y}} + \mathrm{d}z\,\hat{\mathbf{z}}\right) = \nabla T\cdot\mathrm{d}\mathbf{r}$

$$\text{where } \nabla T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \quad \Leftarrow \quad \text{gradient of } T$$

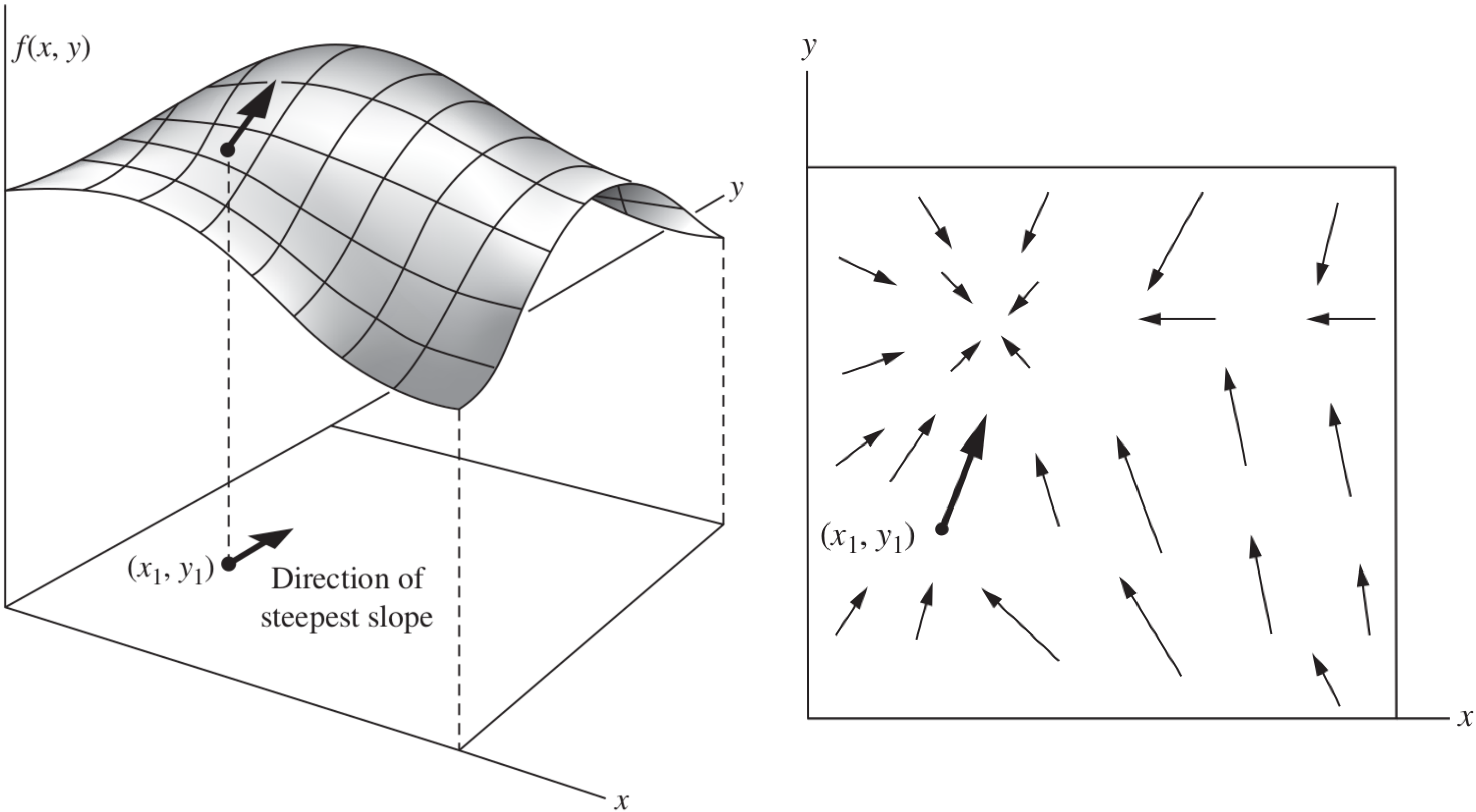
- ∇T is a *vector* quantity, a generalized derivative, with 3 components.

Geometrical Interpretation of the Gradient

- Like any vector, the gradient has *magnitude* and *direction*:

$$\mathrm{d}T = \nabla T\cdot\mathrm{d}\mathbf{r} = |\nabla T|\,|\mathrm{d}\mathbf{r}|\cos\theta$$

- If we fix the *magnitude* $|\mathrm{d}\mathbf{r}|$ and search around in various *directions* (vary θ), the *maximum* change in T evidently occurs when $\theta=0$ (for then $\cos\theta=1$).



f(x, y)
y
x
(x1, y1)
Direction of steepest slope
y
(x1, y1)
x

- For a fixed $|\mathrm{d}\mathbf{r}|$, $\mathrm{d}T$ is greatest when moving in the *same direction* as ∇T.

- *The gradient* ∇T *points in the direction of maximum increase of the function* T.

- *The magnitude* $|\nabla T|$ *gives the slope (increase rate) along this maximal direction*.

- The direction of steepest ascent is the direction of the gradient.

- The direction of max *descent* is opposite to the direction of max *ascent*, while at right angles ($\theta = 90^\circ$) the slope is 0 (the gradient $\perp$ the contour lines).

- If $\nabla T = 0$ at (x, y, z), then $\mathrm{d}T = 0$ for small displacements about the point (x, y, z). This is, then, a **stationary point** of $T(x, y, z)$.

- It could be an extremum, ie, maximum (a summit), a minimum (a valley), a saddle point (a pass), or a “shoulder.”

- If you want to locate the extrema of a function of 3 variables, set its gradient equal to 0.

<u>Example 1.3</u>

The Del Operator

- $\nabla T = \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) T$, the term in parentheses is called **del**:

$$\nabla = \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \quad = \hat{\mathbf{x}}\, \partial_x + \hat{\mathbf{y}}\, \partial_y + \hat{\mathbf{z}}\, \partial_z = \sum_{k=1}^{3} \hat{\mathbf{x}}_k\, \partial_k$$

- ∇ is not a vector, but a **vector operator** that *acts upon* T (a function).
- There are 3 ways the operator ∇ can act:

1. On a scalar function T: ∇T (the gradient);
2. On a vector function $\mathbf{v}$, via the dot product: $\nabla \cdot \mathbf{v}$ (the **divergence**);
3. On a vector function $\mathbf{v}$, via the cross product: $\nabla \times \mathbf{v}$ (the **curl**).

The Divergence

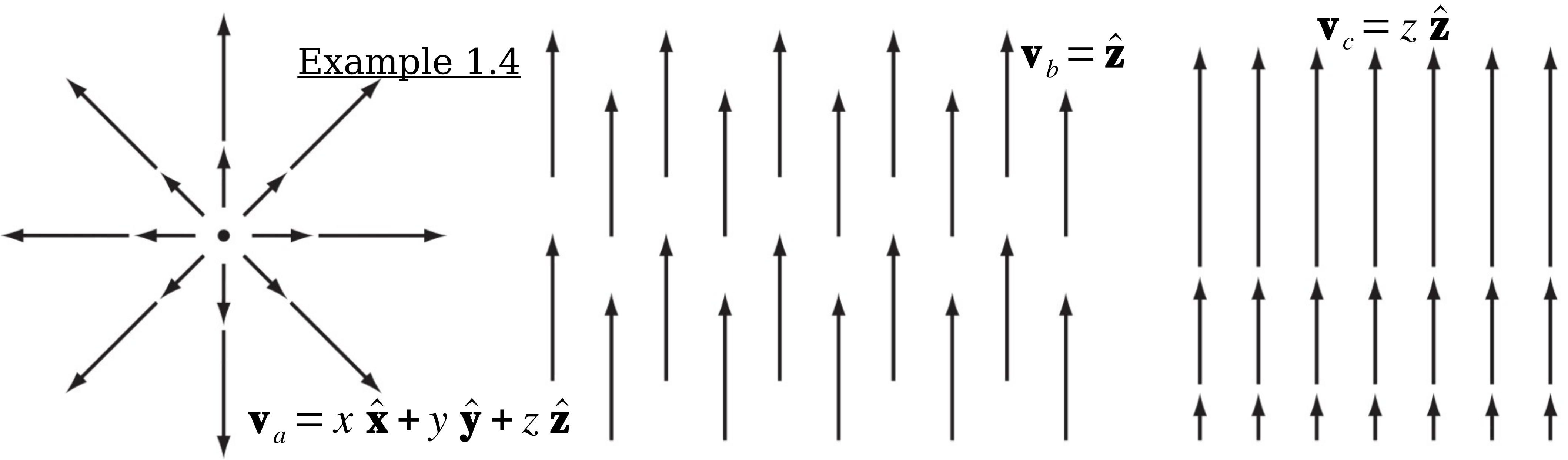
- $\nabla \cdot \mathbf{v} = \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) \cdot \left(v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}} \right) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \sum_{k=1}^{3} \frac{\partial v_k}{\partial x_k}$
- The divergence of a vector function $\mathbf{v}$ is itself a *scalar* $\nabla \cdot \mathbf{v}$.

- *Geometrical Interpretation*: $\nabla \cdot \mathbf{v}$ is a measure of how much the vector $\mathbf{v}$ spreads out (diverges) from the point in question.

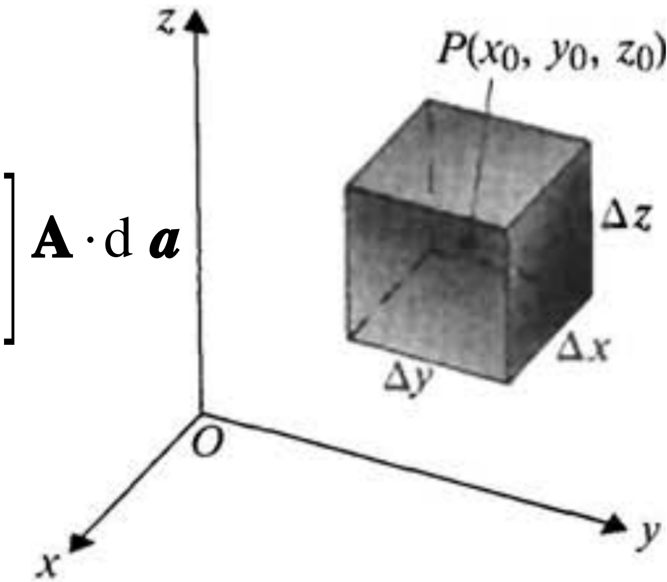
- The vector function in 1st figure has a large (positive) divergence (if the arrows pointed in, it would be a *negative* divergence), the 2nd has 0 divergence, and the 3rd again has a positive divergence.

- A point of positive divergence is a source, or “faucet”; a point of negative divergence is a sink, or “drain.”

Example 1.4



$$\nabla\cdot\mathbf{A}\equiv\lim_{\Delta\tau\to 0}\frac{1}{\Delta\tau}\oint_S \mathbf{A}\cdot\mathrm{d}\boldsymbol{a}$$



$$\oint_S \mathbf{A}\cdot\mathrm{d}\boldsymbol{a}=\left[\int_{\substack{\text{front}\\\text{face}}}+\int_{\substack{\text{back}\\\text{face}}}+\int_{\substack{\text{right}\\\text{face}}}+\int_{\substack{\text{left}\\\text{face}}}+\int_{\substack{\text{top}\\\text{face}}}+\int_{\substack{\text{bottom}\\\text{face}}}\right]\mathbf{A}\cdot\mathrm{d}\boldsymbol{a}$$

$$\int_{\substack{\text{front}\\\text{face}}}\mathbf{A}\cdot\mathrm{d}\boldsymbol{a}=\mathbf{A}_{\substack{\text{front}\\\text{face}}}\cdot\Delta\boldsymbol{a}_{\substack{\text{front}\\\text{face}}}=\mathbf{A}_{\substack{\text{front}\\\text{face}}}\cdot\Delta y\,\Delta z\,\hat{\mathbf{x}}$$

$$=A_x\left(x_0+\Delta x/2\,,\,y_0\,,\,z_0\right)\Delta y\,\Delta z$$

$$A_x\left(x_0\pm\frac{\Delta x}{2}\,,\,y_0\,,\,z_0\right)=A_x\left(x_0\,,\,y_0\,,\,z_0\right)\pm\frac{\Delta x}{2}\left.\frac{\partial A_x}{\partial x}\right|_{(x_0\,,\,y_0\,,\,z_0)}+O\left((\Delta x)^2\right)$$

$$\int_{\substack{\text{back}\\\text{face}}}\mathbf{A}\cdot\mathrm{d}\boldsymbol{a}=\mathbf{A}_{\substack{\text{back}\\\text{face}}}\cdot\Delta\boldsymbol{a}_{\substack{\text{back}\\\text{face}}}=\mathbf{A}_{\substack{\text{back}\\\text{face}}}\cdot\Delta y\,\Delta z\left(-\hat{\mathbf{x}}\right)=-A_x\left(x_0-\frac{\Delta x}{2}\,,\,y_0\,,\,z_0\right)\Delta y\,\Delta z$$

$$\left[\int_{\substack{\text{front}\\\text{face}}}+\int_{\substack{\text{back}\\\text{face}}}\right]\mathbf{A}\cdot\mathrm{d}\boldsymbol{a}=\left[\frac{\partial A_x}{\partial x}+O\left((\Delta x)^2\right)\right]_{(x_0\,,\,y_0\,,\,z_0)}\Delta x\,\Delta y\,\Delta z$$

$$\Rightarrow\quad\left[\int_{\substack{\text{right}\\\text{face}}}+\int_{\substack{\text{left}\\\text{face}}}\right]\mathbf{A}\cdot\mathrm{d}\boldsymbol{a}=\left[\frac{\partial A_y}{\partial y}+O\left((\Delta y)^2\right)\right]_{(x_0\,,\,y_0\,,\,z_0)}\Delta x\,\Delta y\,\Delta z$$

$$\left[\int_{\substack{\text{top}\\\text{face}}}+\int_{\substack{\text{bottom}\\\text{face}}}\right]\mathbf{A}\cdot\mathrm{d}\boldsymbol{a}=\left[\frac{\partial A_z}{\partial z}+O\left((\Delta z)^2\right)\right]_{(x_0\,,\,y_0\,,\,z_0)}\Delta x\,\Delta y\,\Delta z$$

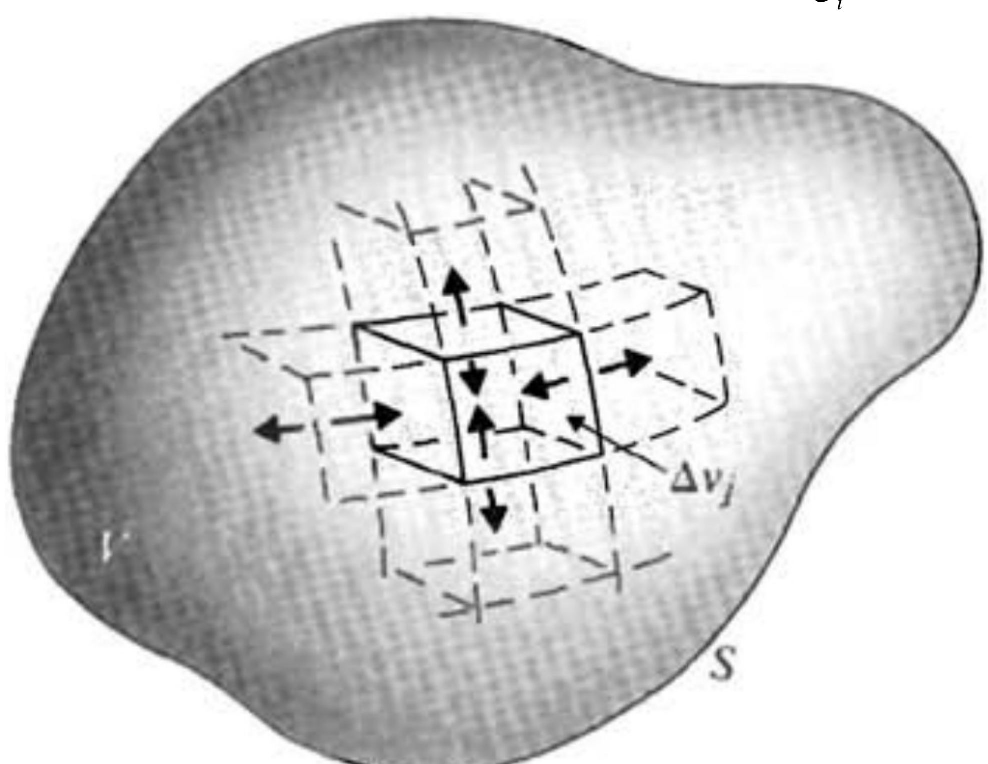
$$\Delta \tau = \Delta x \Delta y \Delta z$$

$$\Rightarrow \oint_S \mathbf{A} \cdot \mathrm{d}\boldsymbol{a} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)_{(x_0, y_0, z_0)} \Delta \tau + \sum_{i=1}^{3} O\left((\Delta x_i)^2 \right) \Delta \tau$$

$$\Rightarrow \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{as} \quad \Delta \tau \to 0 \quad \Leftarrow \quad \Delta x_i \to 0$$

$$\nabla \cdot \mathbf{A}(x_i, y_i, z_i) = \frac{1}{\mathrm{d}\tau_i} \oint_{\mathcal{S}_i} \mathbf{A} \cdot \mathrm{d}\boldsymbol{a}_i \quad \Rightarrow \quad \nabla \cdot \mathbf{A}(x_i, y_i, z_i)\, \mathrm{d}\tau_i = \oint_{\mathcal{S}_i} \mathbf{A} \cdot \mathrm{d}\boldsymbol{a}_i$$

$$\Rightarrow \int_{\mathcal{V}} \nabla \cdot \mathbf{A}\, \mathrm{d}\tau = \oint_{\mathcal{S}} \mathbf{A} \cdot \mathrm{d}\boldsymbol{a}$$



The Curl

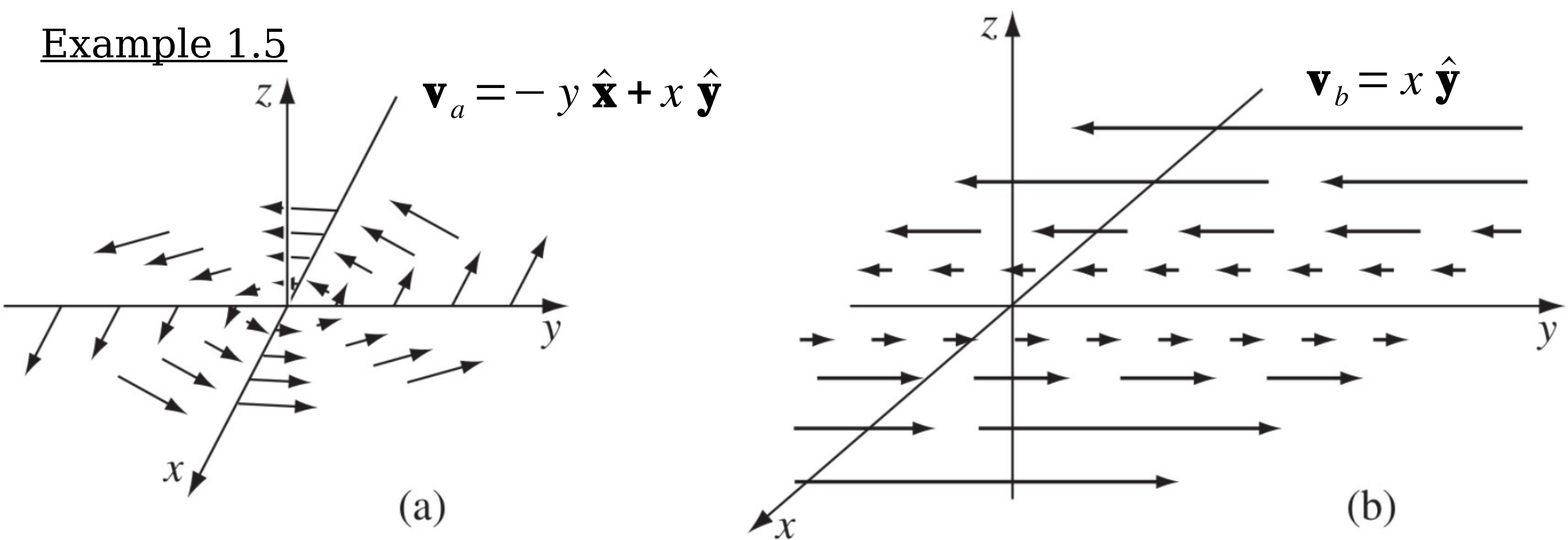
- $\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}}$

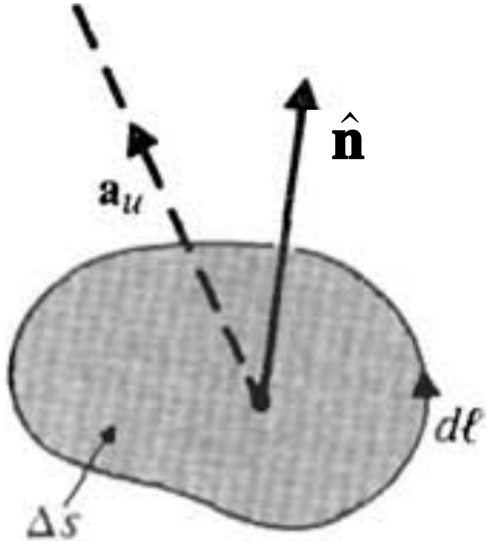
- The curl of a vector function $\mathbf{v}$ is a vector.

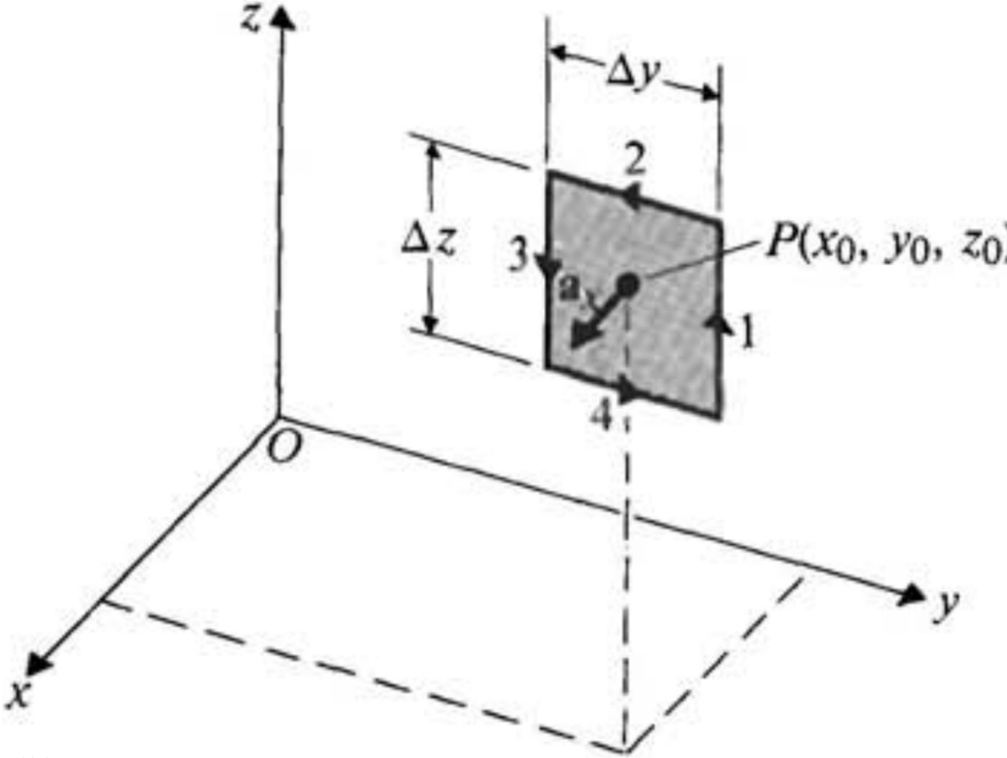
- *Geometrical Interpretation*: $\nabla \times \mathbf{v}$ is a measure of how much the vector $\mathbf{v}$ swirls around the point in question.

- The 3 functions in the above have 0 curl, whereas the functions shown have a substantial curl, pointing in the z direction (with the right-hand rule).

Example 1.5







$$\nabla\times\mathbf{A}\equiv\lim_{\Delta a\to 0}\frac{1}{\Delta a}\left(\hat{\mathbf{n}}\oint_C \mathbf{A}\cdot d\boldsymbol{\ell}\right)_{\max}$$

$$\Rightarrow\quad(\nabla\times\mathbf{A})_i=\lim_{\Delta a_i\to 0}\frac{1}{\Delta a_i}\oint_{C_i}\mathbf{A}\cdot d\boldsymbol{\ell}\quad\Rightarrow\quad(\nabla\times\mathbf{A})_x=\lim_{\Delta y\,\Delta z\to 0}\frac{1}{\Delta y\,\Delta z}\oint_{\substack{\text{sides}\\1,2,3,4}}\mathbf{A}\cdot d\boldsymbol{\ell}$$

$$\text{side 1/3: }\ d\boldsymbol{\ell}=\pm\Delta z\,\hat{\mathbf{z}}\quad\Rightarrow\quad\mathbf{A}\cdot d\boldsymbol{\ell}=\pm A_z\left(x_0,\,y_0\pm\frac{\Delta y}{2},\,z_0\right)\Delta z$$

$$A_z\left(x_0,\,y_0\pm\frac{\Delta y}{2},\,z_0\right)=A_z(x_0,y_0,z_0)\pm\frac{\Delta y}{2}\left.\frac{\partial A_z}{\partial y}\right|_{(x_0,y_0,z_0)}+O\left((\Delta y)^2\right)$$

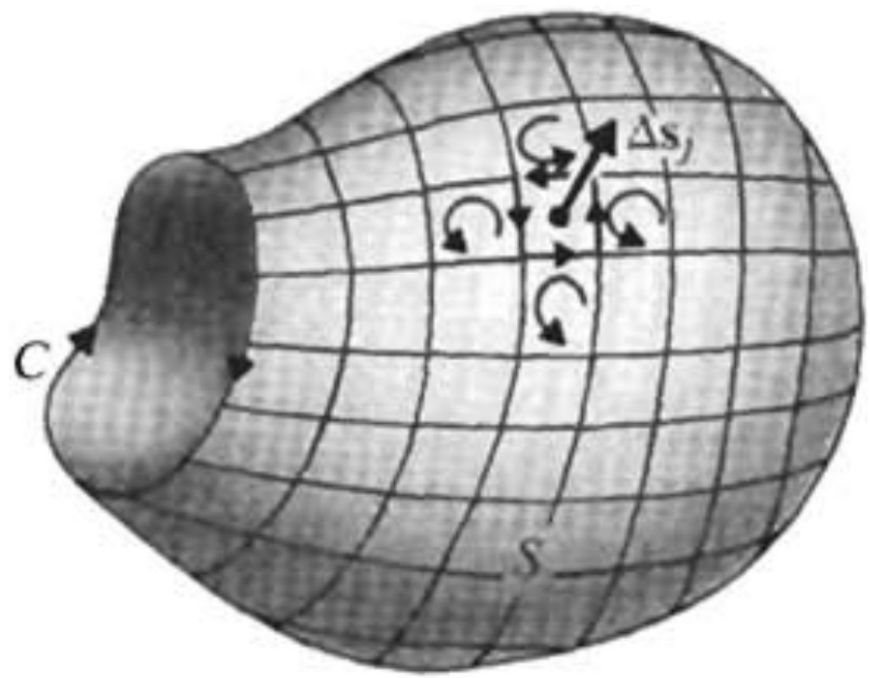
$$\Rightarrow\quad\int_{\text{Sides 1/3}}\mathbf{A}\cdot d\boldsymbol{\ell}=\left[A_z(x_0,y_0,z_0)\pm\frac{\Delta y}{2}\left.\frac{\partial A_z}{\partial y}\right|_{(x_0,y_0,z_0)}+O\left((\Delta y)^2\right)\right](\pm\Delta z)$$

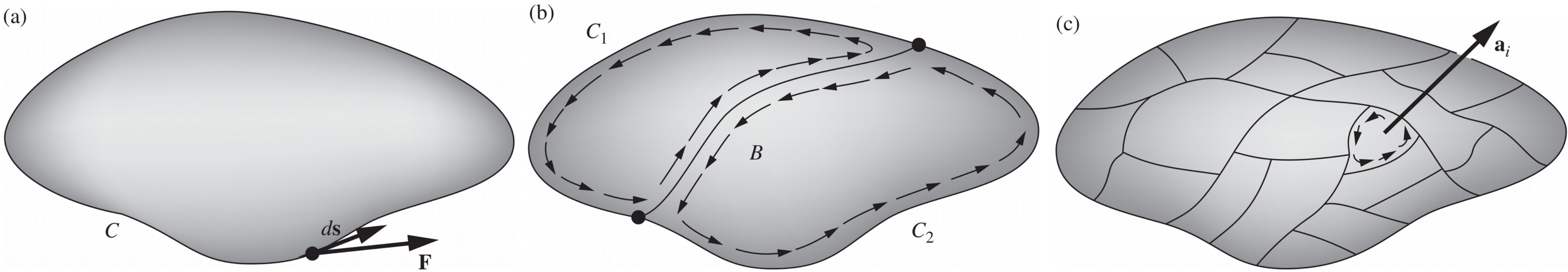
$$\Rightarrow \begin{array}{l} \displaystyle\int_{\substack{\text{Sides}\\ 1\ \&\ 3}} \mathbf{A}\cdot \mathrm{d}\boldsymbol{\ell} = \left[+ \left.\frac{\partial A_z}{\partial y}\right|_{(x_0,\,y_0,\,z_0)} + O\left((\Delta y)^2\right) \right] \Delta y \,\Delta z \\ \displaystyle\int_{\substack{\text{Sides}\\ 2\ \&\ 4}} \mathbf{A}\cdot \mathrm{d}\boldsymbol{\ell} = \left[- \left.\frac{\partial A_y}{\partial z}\right|_{(x_0,\,y_0,\,z_0)} + O\left((\Delta z)^2\right) \right] \Delta y \,\Delta z \end{array} \Rightarrow \left(\nabla\times\mathbf{A}\right)_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

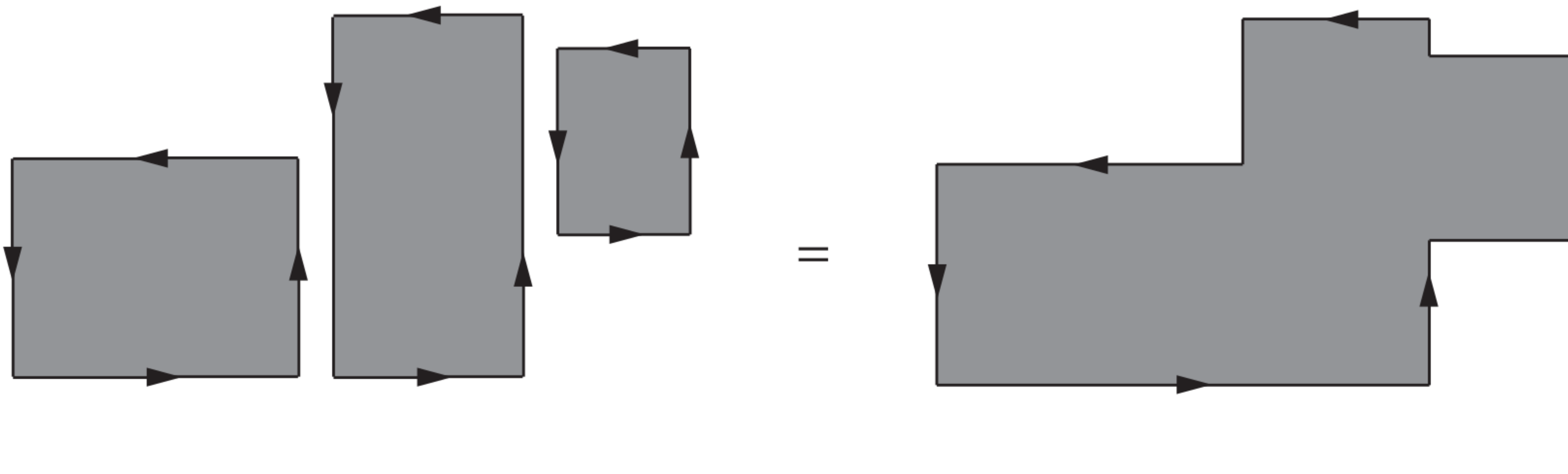
$$\Rightarrow \nabla\times\mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\Rightarrow \left(\nabla\times\mathbf{A}\right)_j \cdot \mathrm{d}\boldsymbol{a}_j = \oint_{C_j} \mathbf{A}\cdot\mathrm{d}\boldsymbol{\ell}$$

$$\Rightarrow \int_S \nabla\times\mathbf{A}\cdot\mathrm{d}\boldsymbol{a} = \oint_C \mathbf{A}\cdot\mathrm{d}\boldsymbol{\ell}$$




(a)
C
ds
F
(b)
C_1
B
C_2
(c)
$\mathbf{a}_i$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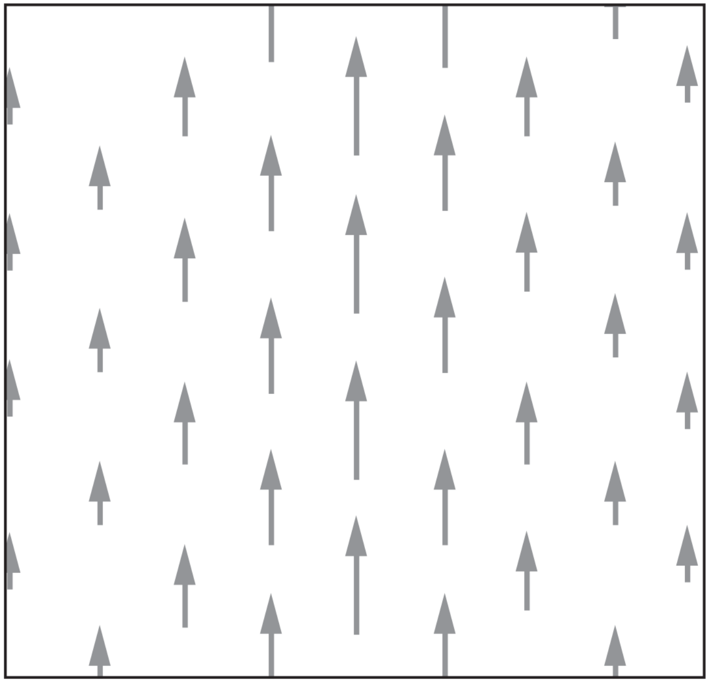
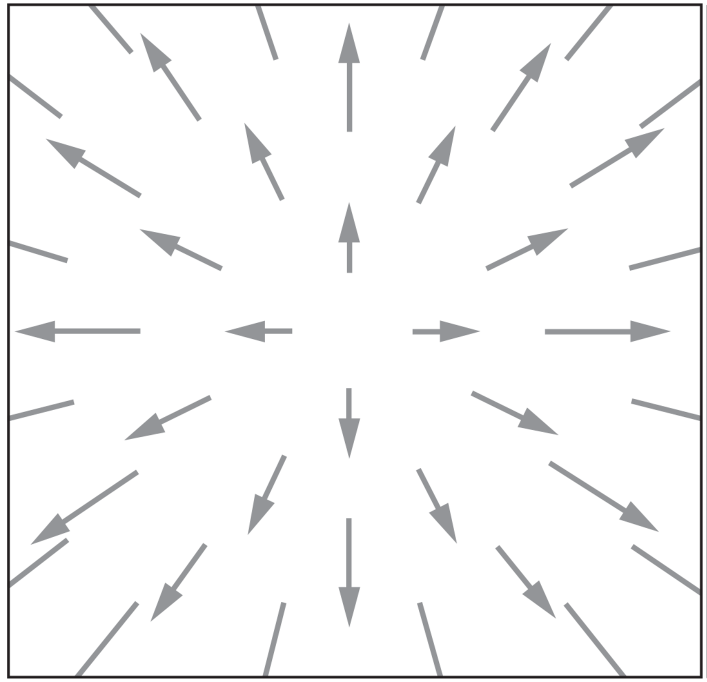
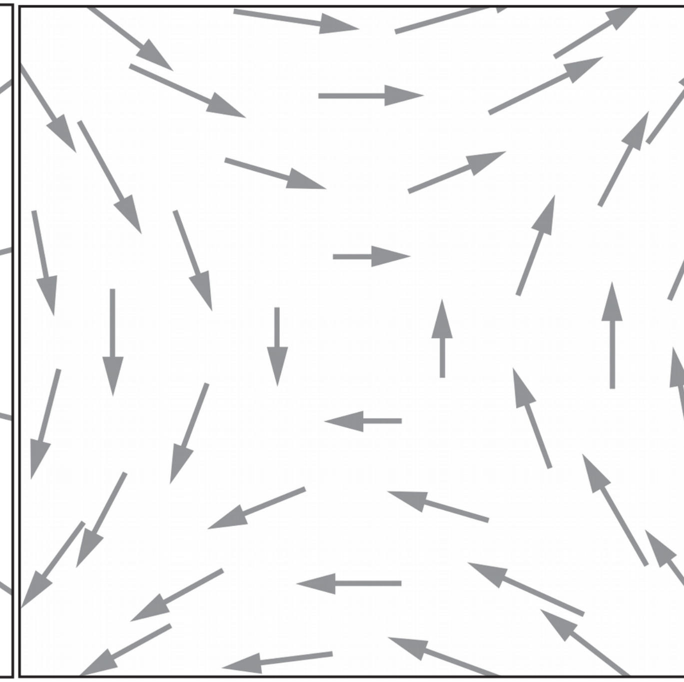


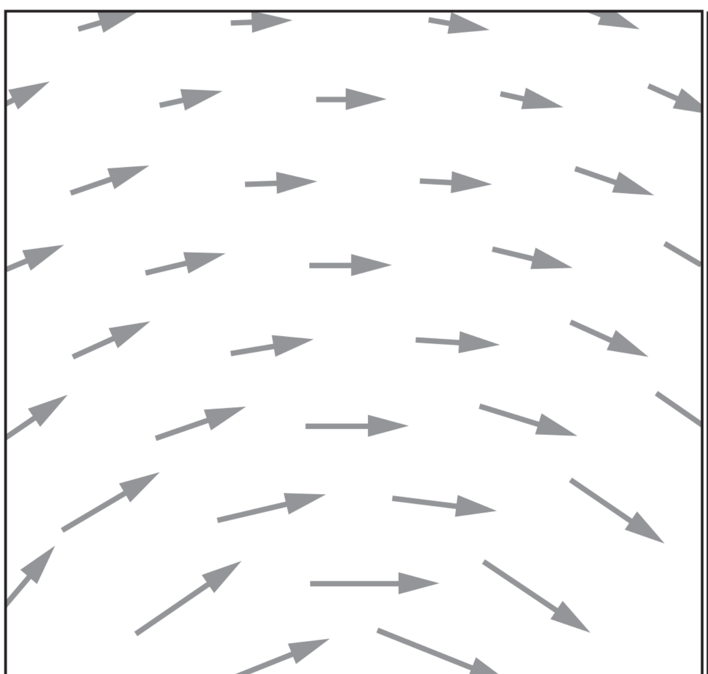
$\nabla \cdot \mathbf{F} = 0, \quad \nabla \times \mathbf{F} \neq 0$



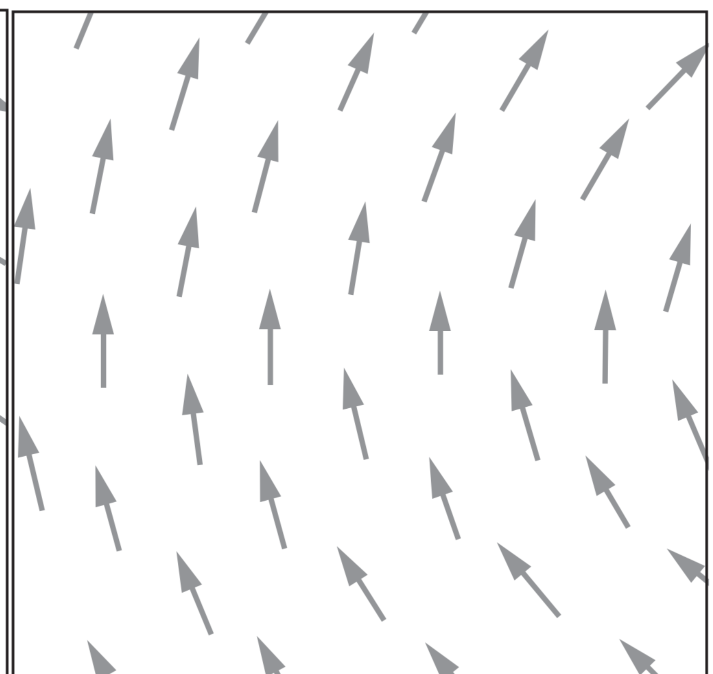
$\nabla \cdot \mathbf{F} \neq 0, \quad \nabla \times \mathbf{F} = 0$



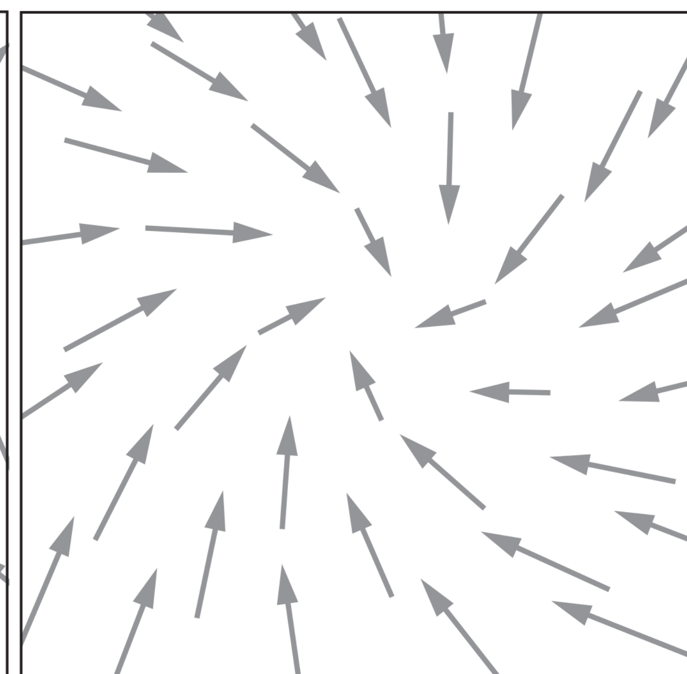
$\nabla \cdot \mathbf{F} = 0, \quad \nabla \times \mathbf{F} = 0$



$\nabla \cdot \mathbf{F} = 0, \quad \nabla \times \mathbf{F} = 0$



$\nabla \cdot \mathbf{F} = 0, \quad \nabla \times \mathbf{F} \neq 0$



$\nabla \cdot \mathbf{F} \neq 0, \quad \nabla \times \mathbf{F} \neq 0$

Product Rules

- Sum rule: $\frac{d}{dx}(f+g)=\frac{df}{dx}+\frac{dg}{dx}$ Multiplying by a constant: $\frac{d}{dx}(kf)=k\frac{df}{dx}$
- Product rule: $\frac{d}{dx}(fg)=f\frac{dg}{dx}+g\frac{df}{dx}$ Quotient rule: $\frac{d}{dx}\frac{f}{g}=\frac{g\frac{df}{dx}-f\frac{dg}{dx}}{g^2}$
- Similar relations hold for the vector derivatives:

$$\nabla(f+g)=\nabla f+\nabla g, \quad \nabla\cdot(\mathbf{A}+\mathbf{B})=\nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{B}, \quad \nabla\times(\mathbf{A}+\mathbf{B})=\nabla\times\mathbf{A}+\nabla\times\mathbf{B}$$

$$\nabla(kf)=k\nabla f, \quad \nabla\cdot(k\mathbf{A})=k\nabla\cdot\mathbf{A}, \quad \nabla\times(k\mathbf{A})=k\nabla\times\mathbf{A}$$

- 2 ways to construct a scalar as the product of 2 functions:

$$f\,g \quad (\text{product of 2 scalar functions})$$

$$\mathbf{A}\cdot\mathbf{B} \quad (\text{dot product of 2 vector functions})$$

- 2 ways to make a vector: $f\,\mathbf{A}$ (scalar times vector) $\mathbf{A}\times\mathbf{B}$ (cross product of 2 vectors)
- There are 6 product rules, 2 for gradients:

(i) $\nabla(fg)=f\nabla g+g\nabla f$

(ii) $\nabla(\mathbf{A}\cdot\mathbf{B})=\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})+(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}$

$$\sum_k \epsilon_{ijk}\,\epsilon_{mnk} = \delta_{im}\,\delta_{jn} - \delta_{in}\,\delta_{jm}\,, \qquad \partial_i \equiv \frac{\partial}{\partial\, x^i}$$

$$\mathbf{A}\times(\nabla\times\mathbf{B}) = \sum_{i,j,k} \epsilon_{ijk}\,\hat{\mathbf{x}}_i\,A_j\,(\nabla\times\mathbf{B})_k = \sum_{i,j,k} \epsilon_{ijk}\,\hat{\mathbf{x}}_i\,A_j \sum_{m,n} \epsilon_{kmn}\,\partial_m\,B_n$$

$$= \sum_{i,j,k,m,n} \epsilon_{ijk}\,\epsilon_{kmn}\,\hat{\mathbf{x}}_i\,A_j\,\partial_m\,B_n = \sum_{i,j,k,m,n} \epsilon_{ijk}\,\epsilon_{mnk}\,\hat{\mathbf{x}}_i\,A_j\,\partial_m\,B_n$$

$$= \sum_{i,j,m,n} (\delta_{im}\,\delta_{jn} - \delta_{in}\,\delta_{jm})\,\hat{\mathbf{x}}_i\,A_j\,\partial_m\,B_n = \sum_{i,j} \hat{\mathbf{x}}_i\,A_j\,(\partial_i\,B_j - \partial_j\,B_i)$$

$$(\mathbf{A}\cdot\nabla)\,\mathbf{B} = \sum_{i,j} A_j\,\partial_j\,(B_i\,\hat{\mathbf{x}}_i) = \sum_{i,j} \hat{\mathbf{x}}_i\,A_j\,\partial_j\,B_i$$

$$\Rightarrow \quad \mathbf{A}\times(\nabla\times\mathbf{B}) + (\mathbf{A}\cdot\nabla)\,\mathbf{B} = \sum_{i,j} \hat{\mathbf{x}}_i\,A_j\,\partial_i\,B_j$$

$$\Rightarrow \quad \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{B}\cdot\nabla)\,\mathbf{A} = \sum_{i,j} \hat{\mathbf{x}}_i\,B_j\,\partial_i\,A_j$$

$$\Rightarrow \quad \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\,\mathbf{B} + (\mathbf{B}\cdot\nabla)\,\mathbf{A}$$

$$= \sum_{i,j} \hat{\mathbf{x}}_i\,(A_j\,\partial_i\,B_j + B_j\,\partial_i\,A_j) = \sum_{i,j} \hat{\mathbf{x}}_i\,\partial_i\,(A_j\,B_j) = \nabla\,(\mathbf{A}\cdot\mathbf{B})$$

2 for divergences:

(iii) $\nabla\cdot(f\,\mathbf{A})=\nabla f\cdot\mathbf{A}+f\,\nabla\cdot\mathbf{A}$

(iv) $\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})$

2 for curls:

(v) $\nabla\times(f\,\mathbf{A})=\nabla f\times\mathbf{A}+f\,\nabla\times\mathbf{A}$

(vi) $\nabla\times(\mathbf{A}\times\mathbf{B})=(\mathbf{B}\cdot\nabla)\,\mathbf{A}-(\mathbf{A}\cdot\nabla)\,\mathbf{B}+(\nabla\cdot\mathbf{B})\,\mathbf{A}-(\nabla\cdot\mathbf{A})\,\mathbf{B}$

- The proofs come straight from the product rule for ordinary derivatives, eg,

$$\begin{aligned}\nabla\cdot(f\,\mathbf{A}) &= \partial_x(f\,A_x)+\partial_y(f\,A_y)+\partial_z(f\,A_z)\\ &= (A_x\,\partial_x f+f\,\partial_x A_x)+(A_y\,\partial_y f+f\,\partial_y A_y)+(A_z\,\partial_z f+f\,\partial_z A_z)\\ &= \nabla f\cdot\mathbf{A}+f\,\nabla\cdot\mathbf{A}\end{aligned}$$

- It is also possible to formulate 3 quotient rules:

$$\nabla\frac{f}{g}=\frac{g\,\nabla f-f\,\nabla g}{g^2},\quad \nabla\cdot\frac{\mathbf{A}}{g}=\frac{g\,\nabla\cdot\mathbf{A}-\mathbf{A}\cdot\nabla g}{g^2},\quad \nabla\times\frac{\mathbf{A}}{g}=\frac{g\,\nabla\times\mathbf{A}+\mathbf{A}\times\nabla g}{g^2}$$

2nd Derivatives

- By applying ∇ *twice*, we can construct 5 species of 2nd derivatives.

- The gradient ∇T is a *vector*, so we can take the *divergence* and *curl* of it:

(1) Divergence of gradient: $\nabla \cdot \nabla T$

(2) Curl of gradient: $\nabla \times \nabla T$

- The divergence $\nabla \cdot \mathbf{v}$ is a *scalar*—all we can do is take its *gradient*:

(3) Gradient of divergence: $\nabla (\nabla \cdot \mathbf{v})$

- The curl $\nabla \times \mathbf{v}$ is a *vector*, so we can take its *divergence* and *curl*:

(4) Divergence of curl: $\nabla \cdot (\nabla \times \mathbf{v})$

(5) Curl of curl: $\nabla \times (\nabla \times \mathbf{v})$

$$\nabla^2 T = \nabla \cdot \nabla T = \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) \cdot \left(\hat{\mathbf{x}} \frac{\partial T}{\partial x} + \hat{\mathbf{y}} \frac{\partial T}{\partial y} + \hat{\mathbf{z}} \frac{\partial T}{\partial z} \right) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

- This object, which we write as $\nabla^2 T$ for short, is called the **Laplacian** of T.

- The Laplacian of a scalar is a scalar. $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \sum_{k=1}^{3} \frac{\partial^2}{\partial x_k^2} = \sum_{k=1}^{3} \partial_k^2$

- The Laplacian of a vector, $\nabla^2 \mathbf{v}$: $\nabla^2 \mathbf{v} \equiv (\nabla^2 v_x)\,\hat{\mathbf{x}} + (\nabla^2 v_y)\,\hat{\mathbf{y}} + (\nabla^2 v_z)\,\hat{\mathbf{z}}$
- The curl of a gradient is always 0: $\nabla \times \nabla T = 0$
- Its proof hinges on the equality of cross derivatives: $\frac{\partial}{\partial x}\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\frac{\partial T}{\partial x}$
- $\nabla(\nabla \cdot \mathbf{v})$ seldom occurs in physical applications, $\nabla^2 \mathbf{v} = (\nabla \cdot \nabla)\,\mathbf{v} \neq \nabla(\nabla \cdot \mathbf{v})$
- The divergence of a curl, like the curl of a gradient, is always 0: $\nabla \cdot (\nabla \times \mathbf{v}) = 0$
- $\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$

Proof: Let $\partial_i \equiv \frac{\partial}{\partial x_i}$

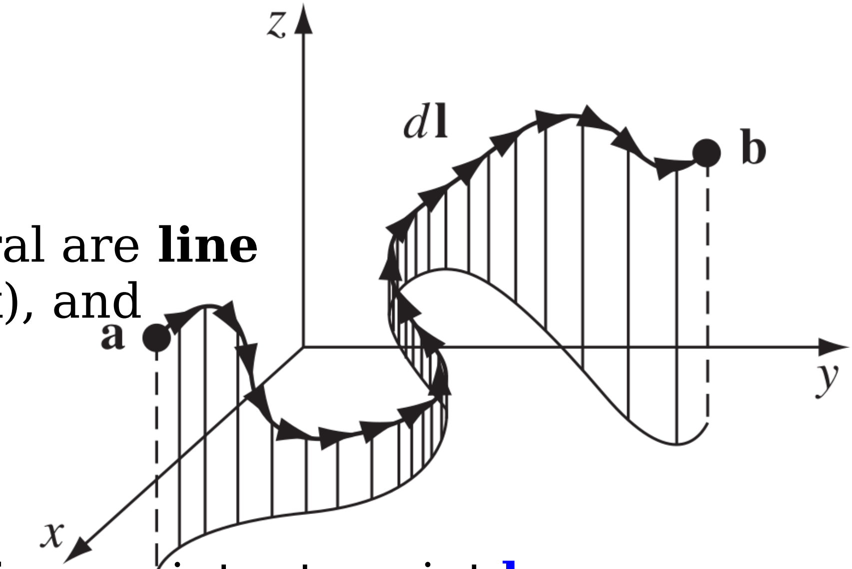
$$\nabla \times (\nabla \times \mathbf{v}) = \sum_{i,j,k} \epsilon_{ijk}\, \hat{\mathbf{x}}_i\, \partial_j (\nabla \times \mathbf{v})_k = \sum_{i,j,k} \epsilon_{ijk}\, \hat{\mathbf{x}}_i\, \partial_j \sum_{m,n} \epsilon_{kmn}\, \partial_m v_n$$

$$= \sum_{i,j,k,m,n} \epsilon_{ijk}\, \epsilon_{mnk}\, \hat{\mathbf{x}}_i\, \partial_j\, \partial_m v_n = \sum_{i,j,m,n} (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\, \hat{\mathbf{x}}_i\, \partial_j\, \partial_m v_n$$

$$= \Big(\sum_i \hat{\mathbf{x}}_i\, \partial_i\Big)\Big(\sum_j \partial_j v_j\Big) - \Big(\sum_j \partial_j \partial_j\Big) \sum_i v_i\, \hat{\mathbf{x}}_i = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

Integral Calculus

Line, Surface, and Volume Integrals



- In electrodynamics, the most important integral are **line** (or **path**) **integrals**, **surface integrals** (or **flux**), and **volume integrals**.

- **Line Integrals**: $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v} \cdot d\boldsymbol{\ell}$

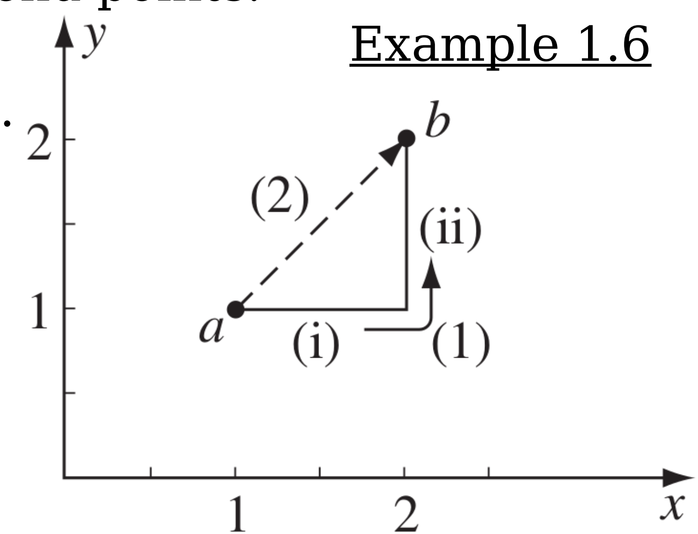
the integral is to be carried out along a path C from point **a** to point **b**.

- If the path forms a closed loop (ie, if **b=a**), it can be expressed as: $\oint \mathbf{v} \cdot d\boldsymbol{\ell}$

- One example of a line integral is the work done by a force **F**: $W = \int \mathbf{F} \cdot d\boldsymbol{\ell}$

- Ordinarily, the value of a line integral depends critically on the path, but there is an important special class of vector functions for which the line integral is *independent* of path and is determined entirely by the end points.

Example 1.6



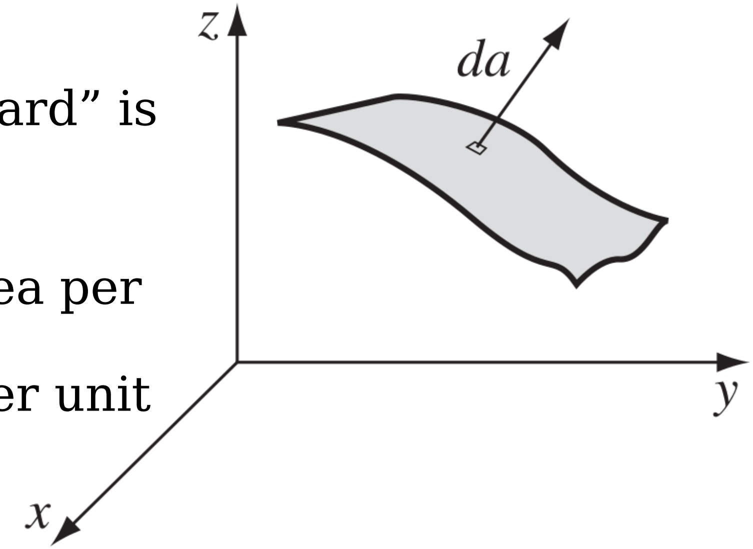
- A *force* that has this property is called **conservative**.

- **Surface Integrals**: $\int_{S} \mathbf{v} \cdot d\boldsymbol{a}$

the integral is over a specified surface S.

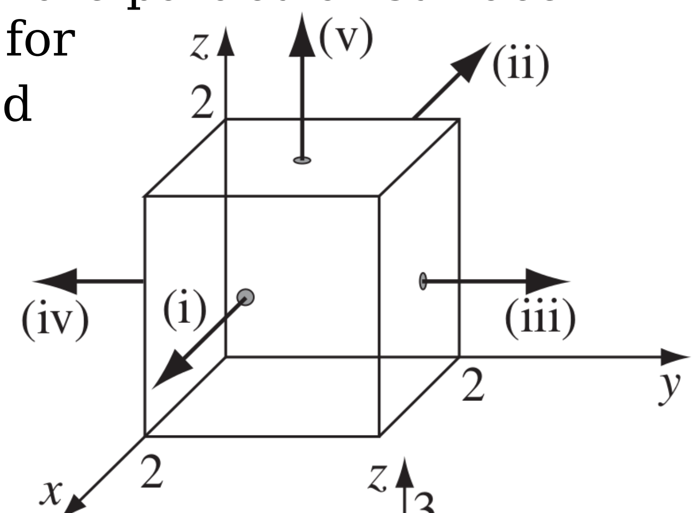
- If the surface is *closed*: $\oint \mathbf{v} \cdot d\boldsymbol{a}$

- For a closed surface, tradition dictates that "outward" is positive, but for open surfaces it's arbitrary.



- If $\mathbf{v}$ describes the flow of a fluid (mass per unit area per unit time), then $\int \mathbf{v} \cdot \mathrm{d}\boldsymbol{a}$ represents the total mass per unit time passing through the surface—hence "flux."

- Ordinarily, the value of a surface integral depends on the particular surface chosen, but there is a special class of vector functions for which it is *independent* of the surface and is determined entirely by the boundary line.

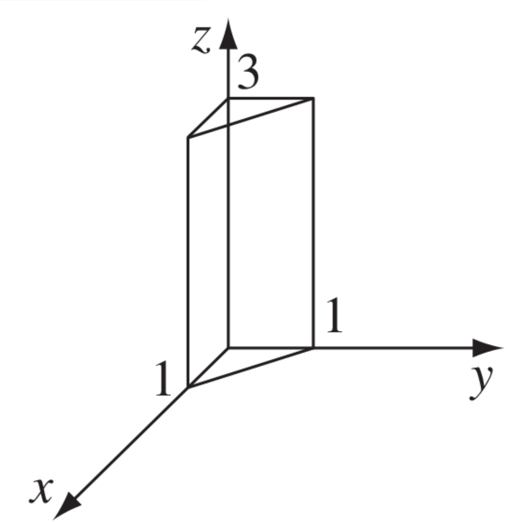


- **Volume Integrals**: $\int_{\mathcal{V}} T\, d\tau \quad \Leftarrow \quad \mathrm{d}\tau = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

Cartesian coordinates

- $$\int \mathbf{v}\, d\tau = \int \left(v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}}\right) d\tau$$
$$= \hat{\mathbf{x}} \int v_x\, \mathrm{d}\tau + \hat{\mathbf{y}} \int v_y\, \mathrm{d}\tau + \hat{\mathbf{z}} \int v_z\, \mathrm{d}\tau$$

because the unit vectors are constants, they come outside the integral.



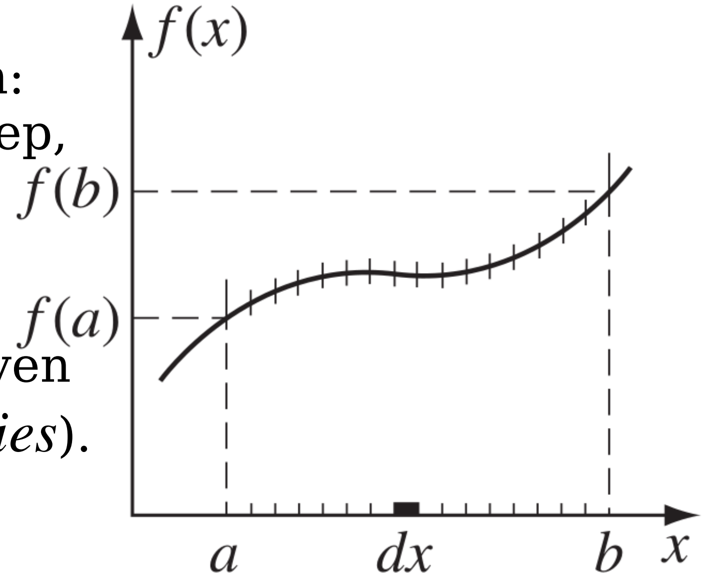
Example 1.7 Example 1.8

The Fundamental Theorem of Calculus

- Let $f(x)$ is a function of one variable, the **fundamental theorem of calculus**:

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a) \;\Leftarrow\; \int_a^b F\,dx = f(b) - f(a) \;\Leftarrow\; F = \frac{df}{dx}$$

- *Geometrical Interpretation*: $df = \frac{df}{dx}\,dx$ is the infinitesimal change in f when you go from x to $x+dx$. The fundamental theorem says that if you chop the interval from $\mathbf{a}$ to $\mathbf{b}$ into many tiny pieces, dx, and add up the increments df from each little piece, the result is equal to the total change in f: $f(\mathbf{b}) - f(\mathbf{a})$.

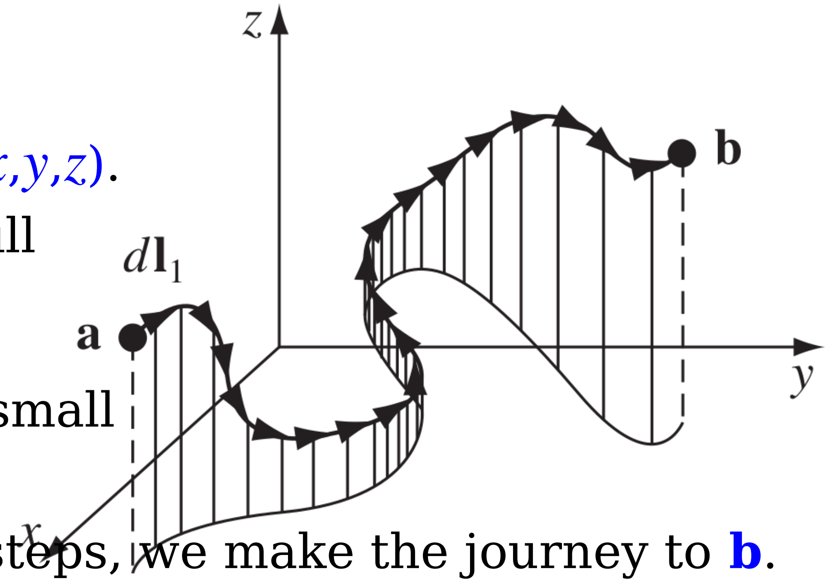


- 2 ways to determine the total change in the function: either subtract the values at the ends or go step-by-step, adding up all the tiny increments as you go. You'll get the same answer either way.

- So the *integral* of a *derivative* over some *region* is given by the *value of the function* at the end points (*boundaries*).

- In vector calculus there are 3 species of derivative (gradient, divergence, and curl), and each has its own "fundamental theorem," with essentially the same format.

The Fundamental Theorem for Gradients



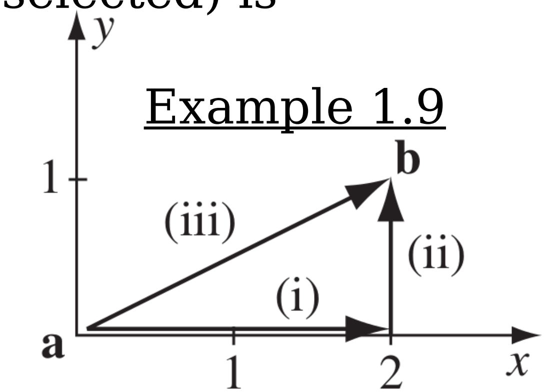
- Let we have a scalar function of 3 variables $T(x,y,z)$.

Starting at **a**, we move a small distance $d\boldsymbol{\ell}_1$. T will change by an amount $dT = \nabla T \cdot d\boldsymbol{\ell}_1$

- Now we move a little further, by an additional small displacement $d\boldsymbol{\ell}_2$; the incremental change in T will be $\nabla T \cdot d\boldsymbol{\ell}_2$. By proceeding by infinitesimal steps, we make the journey to **b**.

- The total change in T in going from **a** to **b** (along the path selected) is

$$\int_{\mathbf{a}}^{\mathbf{b}} \nabla T \cdot d\boldsymbol{\ell} = T(\mathbf{b}) - T(\mathbf{a}) \quad \Leftarrow \quad \text{fundamental theorem for gradients}$$



- The integral (a *line* integral) of a derivative (the *gradient*) is given by the value of T at the boundaries (**a** & **b**).

- Line integrals ordinarily depend on the path from **a** to **b**. But the rhs of the eqn makes no reference to the path—only to the end points.

- Gradients have the property that the line integrals are path independent:

Corollary 1: $\int_{\mathbf{a}}^{\mathbf{b}} \nabla T \cdot d\boldsymbol{\ell}$ is independent of the path taken from **a** to **b**.

Corollary 2: $\oint \nabla T \cdot d\boldsymbol{\ell} = 0$, since the beginning and end points are identical, and hence $T(\mathbf{b}) - T(\mathbf{a}) = 0$.

The Fundamental Theorem for Divergences

- The fundamental theorem for divergences states $$\int_{\mathcal{V}} \nabla\cdot\mathbf{v}\, d\tau = \oint_{\mathcal{S}} \mathbf{v}\cdot d\boldsymbol{a}$$

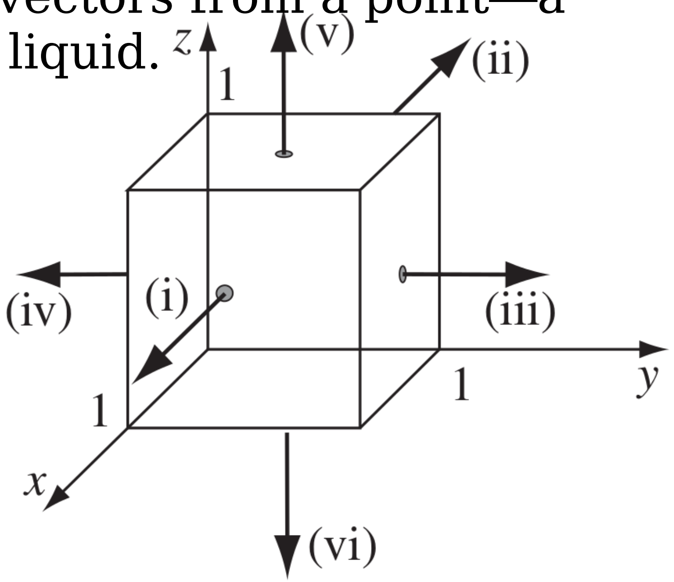
- It is called as **Gauss's theorem**, **Green's theorem**, the **divergence theorem**.

- The *integral* of a *derivative* (the *divergence*) over a *region* (*volume* V) is equal to the value of the function at the *boundary* (the *surface* S that bounds the volume).

- If $\mathbf{v}$ represents the flow of an incompressible fluid, then the flux of $\mathbf{v}$ is the total amount of fluid passing out through the surface, per unit time.

- The divergence measures the "spreading out" of the vectors from a point—a place of high divergence is like a "faucet," pouring out liquid.

- If we have a bunch of faucets in a region filled with incompressible fluid, an equal amount of liquid will be forced out through the boundaries of the region.



- 2 ways to determine how much is being produced:

(a) count up all the faucets, recording how much each puts out, or

(b) measure the flow at each point of the boundary, and add it all up:

$$\int (\text{faucets within the volume}) = \oint (\text{flow out through the surface})$$

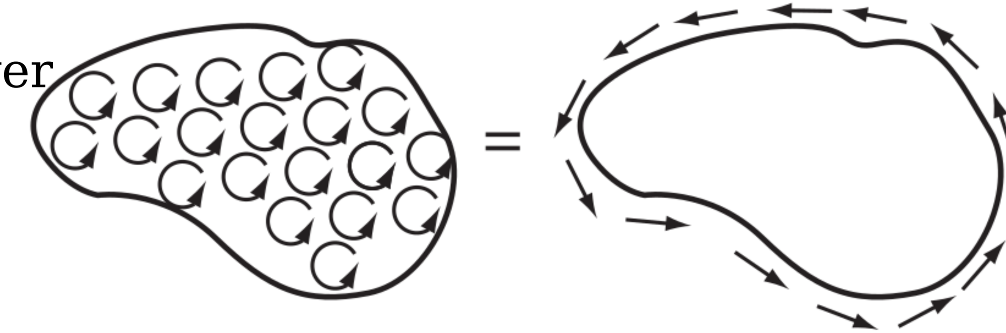
Example 1.10

The Fundamental Theorem for Curls

- The fundamental theorem for curls, also called **Stokes' theorem**:

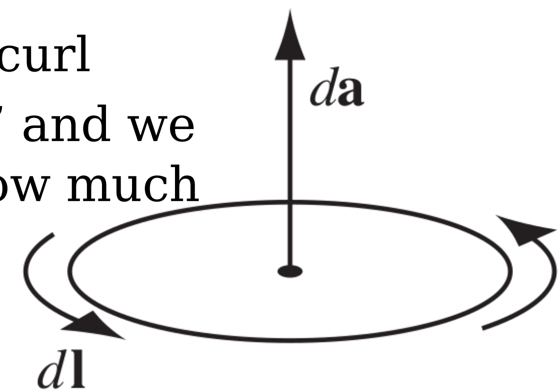
$$\int_S \nabla \times \mathbf{v} \cdot d\boldsymbol{a} = \oint_C \mathbf{v} \cdot d\boldsymbol{\ell}$$

- The *integral* of a *derivative* (the *curl*) over a *region* (surface S) is equal to the value of the function at the *boundary* (the perimeter of the surface, C).



- The curl measures the "twist" of $\mathbf{v}$; a region of high curl is a whirlpool.

- The integral of the curl over some surface (the *flux* of the curl *through* that surface) represents the "total amount of swirl," and we can determine that by going around the edge and finding how much the flow is following the boundary.



- $\oint \mathbf{v} \cdot d\boldsymbol{\ell}$ is sometimes called the **circulation** of $\mathbf{v}$.

- Consistency in Stokes' theorem is given by the right-hand rule: if your fingers point in the direction of the line integral, then your thumb is the direction of $d\boldsymbol{a}$.

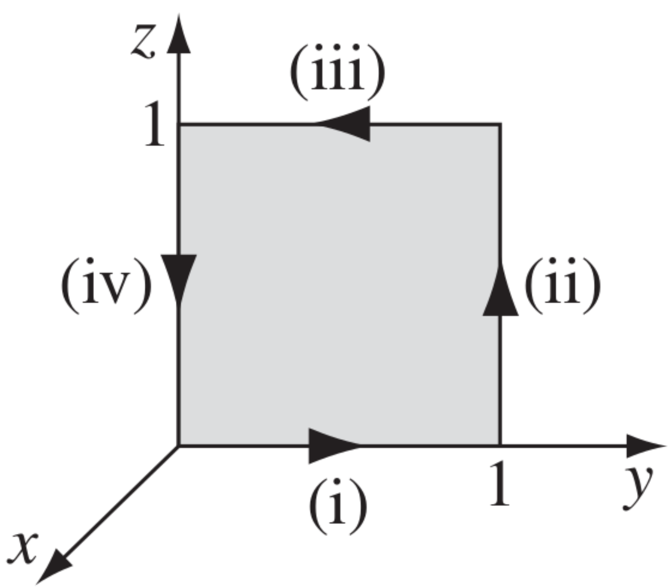
- Ordinarily, a flux integral depends critically on what surface you integrate over, but evidently this is *not* the case with curls.

- Stokes' theorem says that $\int \nabla \times \mathbf{v} \cdot d\boldsymbol{a}$ is equal to the line integral of $\mathbf{v}$ around the boundary, and the latter makes no reference to the surface you choose.

Corollary 1: $\int \nabla \times \mathbf{v} \cdot d\boldsymbol{a}$ depends only on the boundary line, not on the particular surface used.

Corollary 2: $\oint \nabla \times \mathbf{v} \cdot d\boldsymbol{a} = 0$ for any closed surface, since the boundary line, like the mouth of a balloon, shrinks down to a point, and hence the line integral vanishes.

<u>Example 1.11</u>

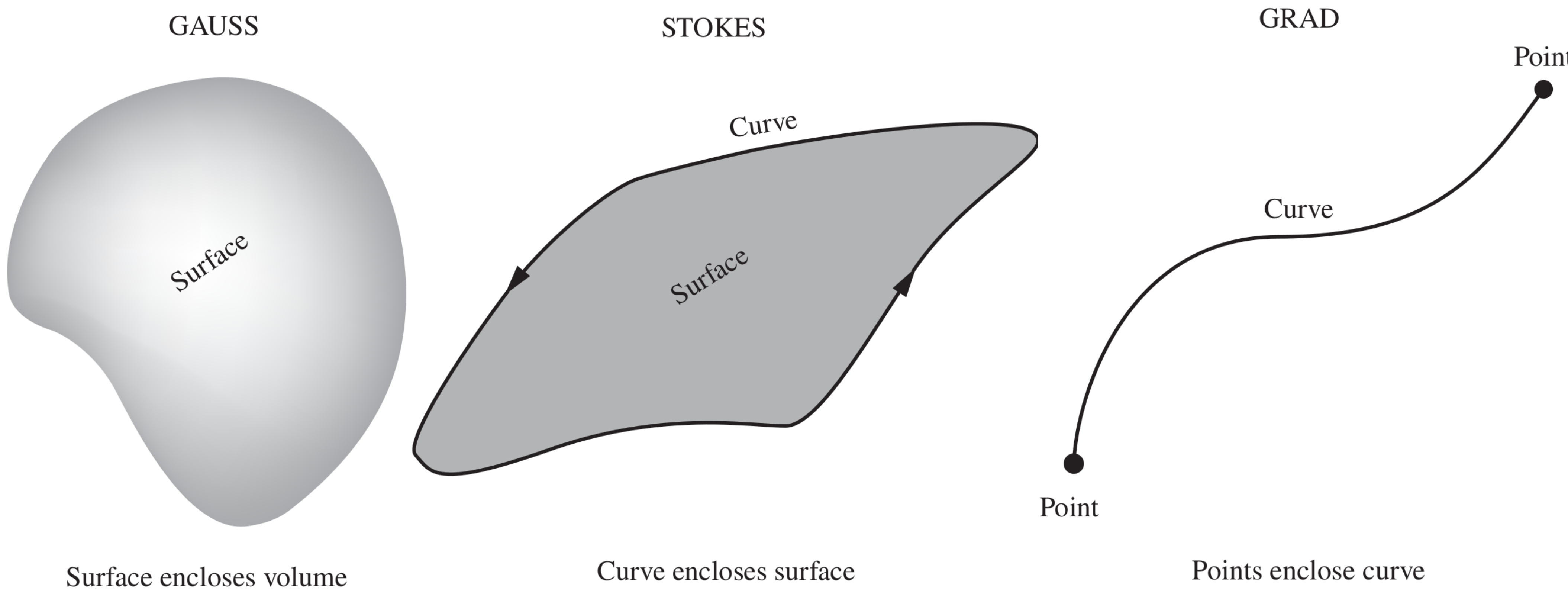


$$\int_{\mathcal{S}} \nabla \times \nabla T \cdot d\boldsymbol{a} = \oint_{\mathcal{C} = \partial \mathcal{S}} \nabla T \cdot d\boldsymbol{\ell} = 0$$

for arbitrary surface $\Rightarrow \nabla \times \nabla T = 0$

$$\int_{\mathcal{V}} \nabla \cdot (\nabla \times \mathbf{v}) \, d\tau = \oint_{\mathcal{S} = \partial \mathcal{V}} \nabla \times \mathbf{v} \cdot d\boldsymbol{a} = 0$$

for arbitrary volume $\Rightarrow \nabla \cdot (\nabla \times \mathbf{v}) = 0$



$$\int_{\text{volume}} \nabla\cdot\mathbf{F}\,\mathrm{d}\tau = \oint_{\text{surface}} \mathbf{F}\cdot\mathrm{d}\boldsymbol{a} \qquad \int_{\text{surface}} \nabla\times\mathbf{A}\cdot\mathrm{d}\boldsymbol{a} = \oint_{\text{curve}} \mathbf{A}\cdot\mathrm{d}\boldsymbol{\ell} \qquad \int_{\text{curve}} \nabla\phi\cdot\mathrm{d}\boldsymbol{\ell} = \phi_2 - \phi_1$$

In general, $\int_M \mathrm{d}\boldsymbol{\omega} = \int_{\partial M} \boldsymbol{\omega}$

Integration by Parts

- $$\frac{\mathrm{d}}{\mathrm{d}\,x}(f\,g)=f\,\frac{\mathrm{d}\,g}{\mathrm{d}\,x}+g\,\frac{\mathrm{d}\,f}{\mathrm{d}\,x}$$

$$\Rightarrow\quad \int_{\mathbf{a}}^{\mathbf{b}}\frac{\mathrm{d}}{\mathrm{d}\,x}(f\,g)\,\mathrm{d}\,x=(f\,g)\Big|_a^b=\int_{\mathbf{a}}^{\mathbf{b}} f\,\frac{\mathrm{d}\,g}{\mathrm{d}\,x}\,\mathrm{d}\,x+\int_{\mathbf{a}}^{\mathbf{b}} g\,\frac{\mathrm{d}\,f}{\mathrm{d}\,x}\,\mathrm{d}\,x$$

$$\Rightarrow\quad \int_{\mathbf{a}}^{\mathbf{b}} f\,\frac{\mathrm{d}\,g}{\mathrm{d}\,x}\,\mathrm{d}\,x=(f\,g)\Big|_a^b-\int_{\mathbf{a}}^{\mathbf{b}} g\,\frac{\mathrm{d}\,f}{\mathrm{d}\,x}\,\mathrm{d}\,x$$

Example 1.12

- It applies to the situation in which you are called upon to integrate the product of one function (f) and the derivative of another (g); it says you can transfer the derivative from g to f, at the cost of a minus sign and a boundary term.

- $$\nabla\cdot(f\,\mathbf{A})=f\,\nabla\cdot\mathbf{A}+\mathbf{A}\cdot\nabla\,f$$

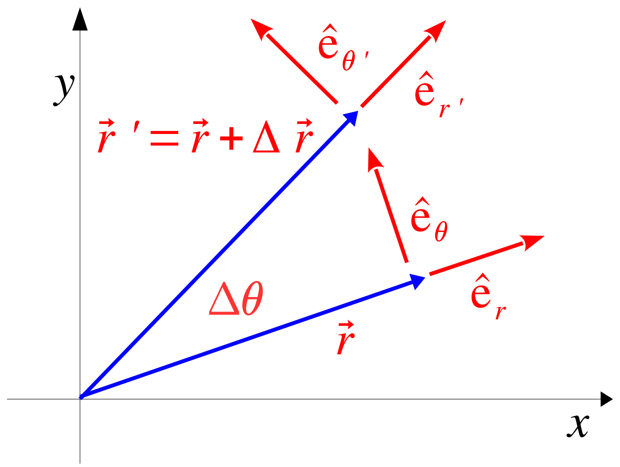
$$\Rightarrow\quad \int\nabla\cdot(f\,\mathbf{A})\,\mathrm{d}\,\tau=\int f\,\nabla\cdot\mathbf{A}\,\mathrm{d}\,\tau+\int\mathbf{A}\cdot\nabla\,f\,\mathrm{d}\,\tau=\oint f\,\mathbf{A}\cdot\mathrm{d}\,\boldsymbol{a}$$

$$\Rightarrow\quad \int_{\mathcal{V}} f\,\nabla\cdot\mathbf{A}\,\mathrm{d}\,\tau=\oint_{\mathcal{S}} f\,\mathbf{A}\cdot\mathrm{d}\,\boldsymbol{a}-\int_{\mathcal{V}}\mathbf{A}\cdot\nabla\,f\,\mathrm{d}\,\tau$$

- The integrand is the product f and the derivative (the *divergence*) of $\mathbf{A}$, and integration by parts licenses us to transfer the derivative from $\mathbf{A}$ to f (a *gradient*), at the cost of a minus sign and a boundary term (a surface integral).

Curvilinear Coordinates

Derivation for a polar coordinate



- In a polar coordinate, the unit vectors are

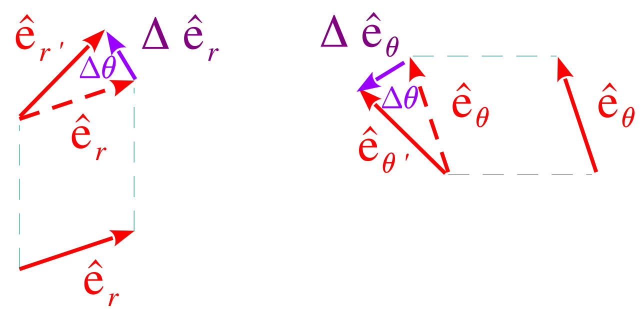
  $\hat{e}_r$ (radial direction) and $\hat{e}_\theta$ (tangential direction)

- The position vector can be written as $\vec{r} = r\,\hat{e}_r$

and the velocity is $\vec{u} = \dfrac{d}{dt}\vec{r} = \hat{e}_r \dfrac{dr}{dt} + r\dfrac{d}{dt}\hat{e}_r$

- Since the derivatives of the unit vectors in a polar coordinate are

$$\frac{d}{dt}\hat{e}_r = \omega\,\hat{e}_\theta, \qquad \frac{d}{dt}\hat{e}_\theta = -\omega\,\hat{e}_r \quad \Leftarrow \quad \omega = \frac{d\theta}{dt}$$



Therefore, $\vec{u} = \dot{r}\,\hat{e}_r + r\,\omega\,\hat{e}_\theta \quad \Leftarrow \quad \dot{r} = \dfrac{dr}{dt}$

- With r held constant $\vec{u} = r\,\omega\,\hat{e}_\theta = v\,\hat{e}_\theta$

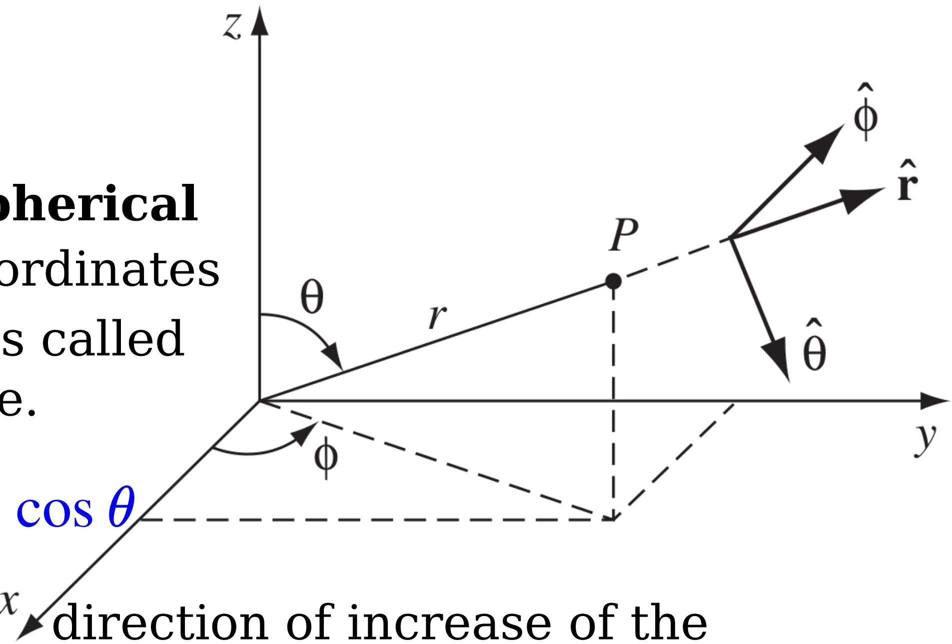
- The acceleration is $\vec{a} = \dfrac{d}{dt}\vec{u} = \ddot{r}\,\hat{e}_r + \dot{r}\dfrac{d}{dt}\hat{e}_r + (\dot{r}\,\omega + r\,\dot{\omega})\,\hat{e}_\theta + r\,\omega\dfrac{d}{dt}\hat{e}_\theta$

$$= \ddot{r}\,\hat{e}_r + \dot{r}\,\omega\,\hat{e}_\theta + (\dot{r}\,\omega + r\,\dot{\omega})\,\hat{e}_\theta - r\,\omega^2\,\hat{e}_r$$

$$\Rightarrow \qquad \vec{a} = \left(\ddot{r} - \frac{v^2}{r}\right)\hat{e}_r + (2\,\dot{r}\,\omega + r\,\alpha)\,\hat{e}_\theta \quad \Leftarrow \quad \alpha \equiv \dot{\omega} = \ddot{\theta}$$

- With r held constant $\vec{a} = -\dfrac{v^2}{r}\hat{e}_r + r\,\alpha\,\hat{e}_\theta = a_r\,\hat{e}_r + a_t\,\hat{e}_\theta$

Spherical Coordinates

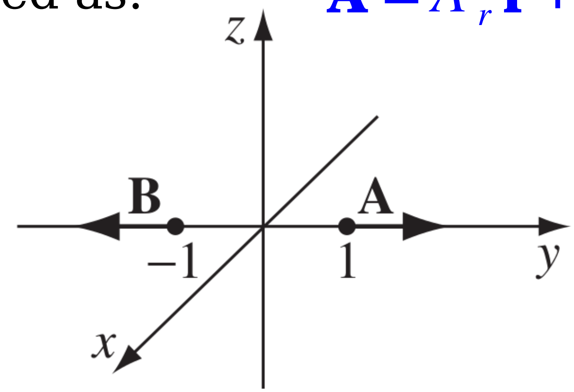


- Sometimes it is more convenient to use **spherical** coordinates (r, θ, ϕ) instead of Cartesian coordinates (x, y, z); r is the distance from the origin, θ is called the polar angle, and ϕ is the azimuthal angle.

- $x = r\sin\theta\cos\phi\,, \quad y = r\sin\theta\sin\phi\,, \quad z = r\cos\theta$

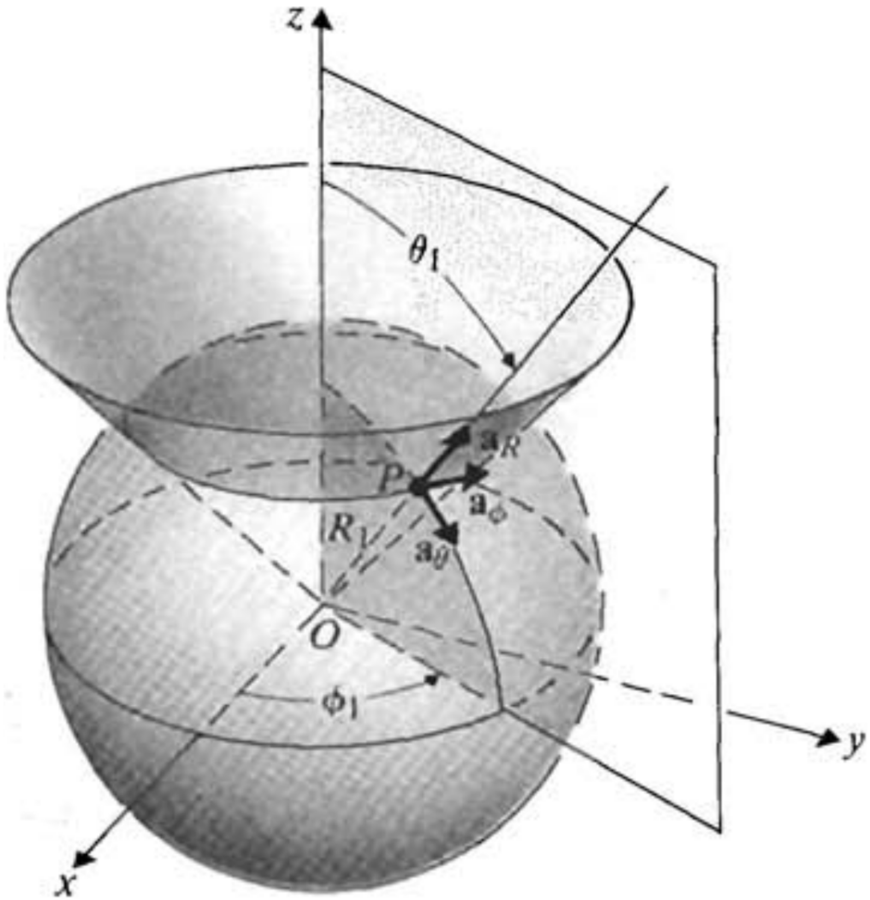
- 3 unit vectors, $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$, point in the direction of increase of the corresponding coordinates. They form an orthogonal (mutually perpendicular) basis set, and any vector $\mathbf{A}$ can be expressed as: $\mathbf{A} = A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\phi}}$

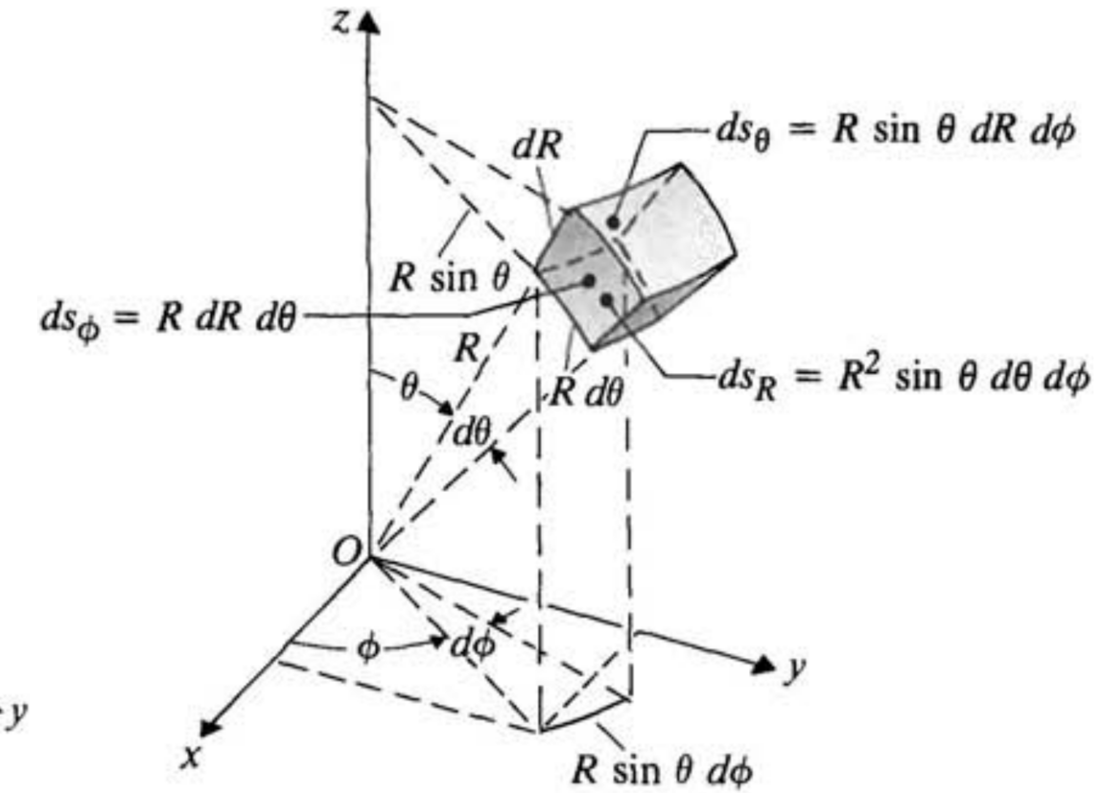
- $$\begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$



- *Warning*: $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$ are associated with a *particular point*, and they change direction as the point moves around (compared with Cartesian coordinates).

- One could take account of this by explicitly indicating the point of reference: $\hat{\mathbf{r}}(\theta, \phi)$, $\hat{\boldsymbol{\theta}}(\theta, \phi)$, $\hat{\boldsymbol{\phi}}(\theta, \phi)$

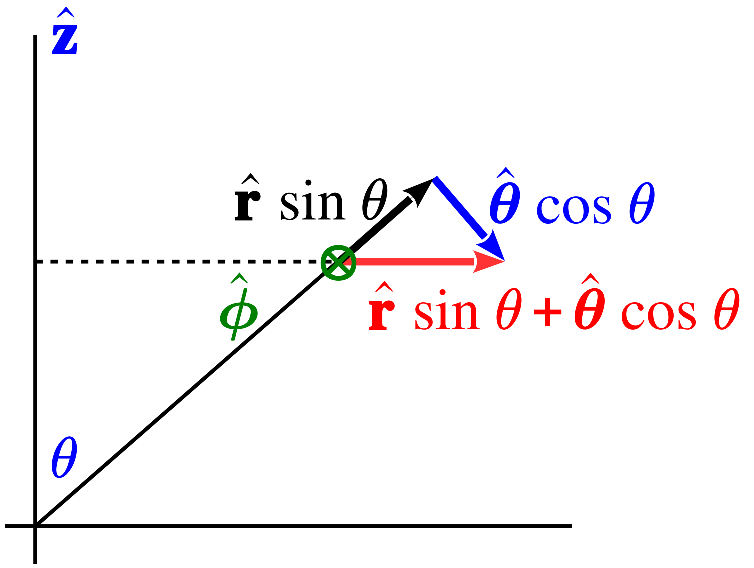




$$\begin{bmatrix} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{bmatrix} \Rightarrow \begin{bmatrix} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z} \\ \phi = \tan^{-1}\dfrac{y}{x} \end{bmatrix} \Rightarrow \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix} = \begin{bmatrix} \dfrac{\mathbf{r}}{r} \\ \hat{\boldsymbol{\phi}} \times \hat{\mathbf{r}} \\ \dfrac{\hat{\mathbf{z}} \times \hat{\mathbf{r}}}{\sin\theta} \end{bmatrix} \Rightarrow \hat{\mathbf{r}} \perp \hat{\boldsymbol{\theta}} \perp \hat{\boldsymbol{\phi}}$$

$$\Rightarrow \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix} \Rightarrow \hat{\mathbf{S}} = \mathbf{R}\,\hat{\mathbf{D}}$$

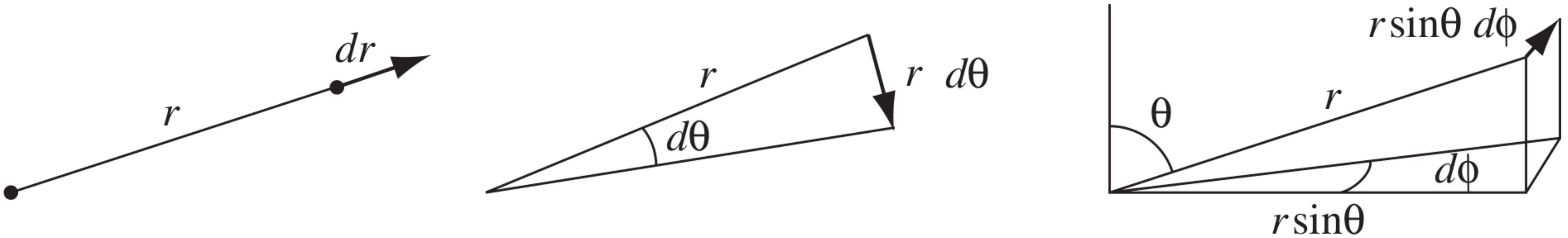
$$\Rightarrow \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix} \Rightarrow \hat{\mathbf{D}} = \mathbf{R}^T\,\hat{\mathbf{S}} = \mathbf{R}^{-1}\,\hat{\mathbf{S}}$$



$$\frac{\partial\hat{\mathbf{r}}}{\partial r} = 0\,, \quad \frac{\partial\hat{\boldsymbol{\theta}}}{\partial r} = 0\,, \quad \frac{\partial\hat{\boldsymbol{\phi}}}{\partial r} = 0$$

$$\Rightarrow \frac{\partial\hat{\mathbf{r}}}{\partial \theta} = \hat{\boldsymbol{\theta}}\,, \quad \frac{\partial\hat{\boldsymbol{\theta}}}{\partial \theta} = -\hat{\mathbf{r}}\,, \quad \frac{\partial\hat{\boldsymbol{\phi}}}{\partial \theta} = 0$$

$$\frac{\partial\hat{\mathbf{r}}}{\partial \phi} = \hat{\boldsymbol{\phi}}\sin\theta\,, \quad \frac{\partial\hat{\boldsymbol{\theta}}}{\partial \phi} = \hat{\boldsymbol{\phi}}\cos\theta\,, \quad \frac{\partial\hat{\boldsymbol{\phi}}}{\partial \phi} = -\hat{\mathbf{r}}\sin\theta - \hat{\boldsymbol{\theta}}\cos\theta$$



- Do not naively combine the spherical components of vectors associated with different points. Beware of differentiating a vector that is expressed in spherical coordinates, since the unit vectors themselves are functions of position. And do not take $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ outside an integral.

- An infinitesimal displacement in the $\hat{\mathbf{r}}$ direction is: $\mathrm{d}\,\ell_r = \mathrm{d}\,r$

- An infinitesimal element of length in the $\hat{\boldsymbol{\theta}}$ direction is: $\mathrm{d}\,\ell_\theta = r\,\mathrm{d}\,\theta$

- An infinitesimal element of length in the $\hat{\boldsymbol{\phi}}$ direction is: $\mathrm{d}\,\ell_\phi = r\sin\theta\,\mathrm{d}\,\phi$

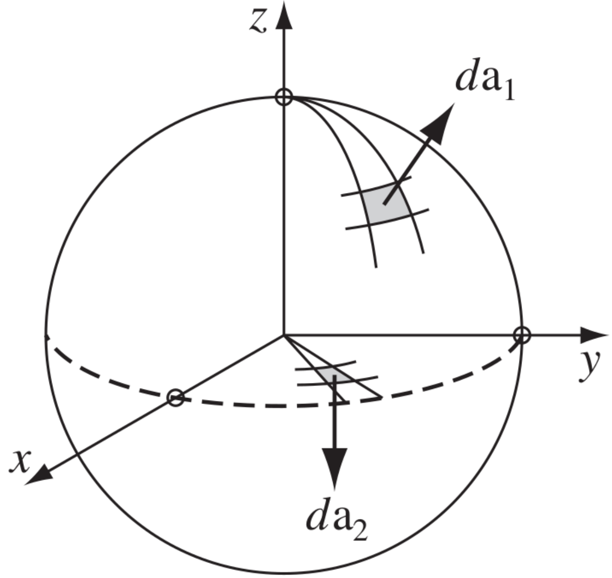
- Thus the general infinitesimal displacement is: $\mathrm{d}\,\boldsymbol{\ell} = dr\,\hat{\mathbf{r}} + r\,\mathrm{d}\,\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\mathrm{d}\,\phi\,\hat{\boldsymbol{\phi}}$

- The infinitesimal volume element in spherical coordinates is the product of the 3 infinitesimal displacements: $\mathrm{d}\,\tau = \mathrm{d}\,\ell_r\,\mathrm{d}\,\ell_\theta\,\mathrm{d}\,\ell_\phi = r^2\sin\theta\,\mathrm{d}\,r\,\mathrm{d}\,\theta\,\mathrm{d}\,\phi$

- Surface elements depend on the orientation of the surface. One has to analyze the geometry for any given case.

- For the surface of a sphere, r=const, whereas θ and ϕ change, so

$$\mathrm{d}\,\boldsymbol{a}_1 = \mathrm{d}\,\ell_\theta\,\mathrm{d}\,\ell_\phi\,\hat{\mathbf{r}} = r^2\sin\theta\,\mathrm{d}\,\theta\,\mathrm{d}\,\phi\,\hat{\mathbf{r}}$$



- If the surface lies in the xy plane,, so that $\theta=\pi/2$ while r and ϕ vary, then $d\mathbf{a}_2 = d\ell_r\, d\ell_\phi\, \hat{\boldsymbol{\theta}} = r\, dr\, d\phi\, \hat{\boldsymbol{\theta}}$

- $r\in[0,\ \infty),\quad \theta\in[0,\ \pi],\quad \phi\in[0,\ 2\pi]$

Example 1.13

- Now I would like to "translate" the vector derivatives (gradient, divergence, curl, and Laplacian) into r, θ, ϕ notation.

- Since $\nabla T = \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}$, one can do it in the hard way by translate

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi},\quad \frac{\partial}{\partial y} = \cdots,\quad \frac{\partial}{\partial z} = \cdots$$

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}),\quad \hat{\mathbf{y}} = \hat{\mathbf{y}}(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}),\quad \hat{\mathbf{z}} = \hat{\mathbf{z}}(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}),\quad \cdots$$

- We can do it in an easier way by

$$\ell_r \equiv r,\quad \ell_\theta \equiv r\,\theta\ [\text{with } r \text{ fixed}],\quad \ell_\phi \equiv (r\sin\theta)\,\phi\ [\text{with } r, \theta \text{ fixed}]$$

$$\Rightarrow\quad h_r = 1,\quad h_\theta = r,\quad h_\phi = r\sin\theta \quad \Leftarrow \quad h_i : \text{metric coefficients}$$

$$\Rightarrow\quad \nabla T = \hat{\mathbf{r}}\frac{\partial T}{\partial \ell_r} + \hat{\boldsymbol{\theta}}\frac{\partial T}{\partial \ell_\theta} + \hat{\boldsymbol{\phi}}\frac{\partial T}{\partial \ell_\phi} = \hat{\mathbf{r}}\frac{\partial T}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial T}{\partial \theta} + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial T}{\partial \phi}$$

Gradient: $$\nabla T = \left(\hat{\mathbf{r}} \frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\boldsymbol{\phi}}}{r \sin\theta} \frac{\partial}{\partial \phi} \right) T = \hat{\mathbf{r}} \frac{\partial T}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r} \frac{\partial T}{\partial \theta} + \frac{\hat{\boldsymbol{\phi}}}{r \sin\theta} \frac{\partial T}{\partial \phi}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 v_r \right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(v_\theta \sin\theta \right) + \frac{1}{r \sin\theta} \frac{\partial v_\phi}{\partial \phi}$$

$$= \frac{1}{r^2 \sin\theta} \left[\partial_r \left(v_r r^2 \sin\theta \right) + \partial_\theta \left(v_\theta r \sin\theta \right) + \partial_\phi \left(v_\phi r \right) \right]$$

Curl: $$\nabla \times \mathbf{v} = \frac{\hat{\mathbf{r}}}{r \sin\theta} \left[\frac{\partial}{\partial \theta} \left(v_\phi \sin\theta \right) - \frac{\partial v_\theta}{\partial \phi} \right] + \frac{\hat{\boldsymbol{\theta}}}{r} \left[\frac{1}{\sin\theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} \left(r v_\phi \right) \right]$$

$$+ \frac{\hat{\boldsymbol{\phi}}}{r} \left[\frac{\partial}{\partial r} \left(r v_\theta \right) - \frac{\partial v_r}{\partial \theta} \right] = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{r}} & r \hat{\boldsymbol{\theta}} & r \sin\theta \, \hat{\boldsymbol{\phi}} \\ \partial_r & \partial_\theta & \partial_\phi \\ v_r & r v_\theta & r \sin\theta \, v_\phi \end{vmatrix}$$

Laplacian: $$\nabla^2 T = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 T}{\partial^2 \phi}$$

$$\nabla \cdot \mathbf{v} = \left(\hat{\mathbf{r}} \frac{\partial}{\partial \ell_r} + \hat{\boldsymbol{\theta}} \frac{\partial}{\partial \ell_\theta} + \hat{\boldsymbol{\phi}} \frac{\partial}{\partial \ell_\phi} \right) \cdot \left(v_r \hat{\mathbf{r}} + v_\theta \hat{\boldsymbol{\theta}} + v_\phi \hat{\boldsymbol{\phi}} \right)$$

$$= \hat{\mathbf{r}} \cdot \frac{\partial}{\partial r} \left(v_r \hat{\mathbf{r}} + v_\theta \hat{\boldsymbol{\theta}} + v_\phi \hat{\boldsymbol{\phi}} \right) + \hat{\boldsymbol{\theta}} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \left(v_r \hat{\mathbf{r}} + v_\theta \hat{\boldsymbol{\theta}} + v_\phi \hat{\boldsymbol{\phi}} \right)$$

$$+ \hat{\boldsymbol{\phi}} \cdot \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \left(v_r \hat{\mathbf{r}} + v_\theta \hat{\boldsymbol{\theta}} + v_\phi \hat{\boldsymbol{\phi}} \right)$$

$$= \frac{\partial v_r}{\partial r} + 0 + 0 + \hat{\mathbf{r}} \cdot \left(v_r \frac{\partial \hat{\mathbf{r}}}{\partial r} + v_\theta \frac{\partial \hat{\boldsymbol{\theta}}}{\partial r} + v_\phi \frac{\partial \hat{\boldsymbol{\phi}}}{\partial r} \right)$$

$$+ 0 + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + 0 + \frac{\hat{\boldsymbol{\theta}}}{r} \cdot \left(v_r \frac{\partial \hat{\mathbf{r}}}{\partial \theta} + v_\theta \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta} + v_\phi \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \theta} \right)$$

$$+ 0 + 0 + \frac{1}{r \sin\theta} \frac{\partial v_\phi}{\partial \phi} + \frac{\hat{\boldsymbol{\phi}}}{r \sin\theta} \cdot \left(v_r \frac{\partial \hat{\mathbf{r}}}{\partial \phi} + v_\theta \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \phi} + v_\phi \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi} \right)$$

$$= \frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} + 0 + \frac{1}{r \sin\theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cos\theta}{r \sin\theta} + 0$$

$$= \frac{\partial v_r}{\partial r} + \frac{2 v_r}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta \cos\theta}{r \sin\theta} + \frac{1}{r \sin\theta} \frac{\partial v_\phi}{\partial \phi}$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 v_r \right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(v_\theta \sin\theta \right) + \frac{1}{r \sin\theta} \frac{\partial v_\phi}{\partial \phi}$$

For Curvilinear Coordinates

$$\mathrm{d}\,\boldsymbol{\ell} = \hat{\mathbf{n}}_1\, h_1\, \mathrm{d}\, u_1 + \hat{\mathbf{n}}_2\, h_2\, \mathrm{d}\, u_2 + \hat{\mathbf{n}}_3\, h_3\, \mathrm{d}\, u_3 \quad \Rightarrow \quad \mathrm{d}\,\tau = h_1\, h_2\, h_3\, \mathrm{d}\, u_1\, \mathrm{d}\, u_2\, \mathrm{d}\, u_3$$

$$\Rightarrow \quad \mathrm{d}\,\boldsymbol{a}_1 = \hat{\mathbf{n}}_1\, h_2\, h_3\, \mathrm{d}\, u_2\, \mathrm{d}\, u_3\,, \quad \mathrm{d}\,\boldsymbol{a}_2 = \hat{\mathbf{n}}_2\, h_3\, h_1\, \mathrm{d}\, u_3\, \mathrm{d}\, u_1\,, \quad \mathrm{d}\,\boldsymbol{a}_3 = \hat{\mathbf{n}}_3\, h_1\, h_2\, \mathrm{d}\, u_1\, \mathrm{d}\, u_2$$

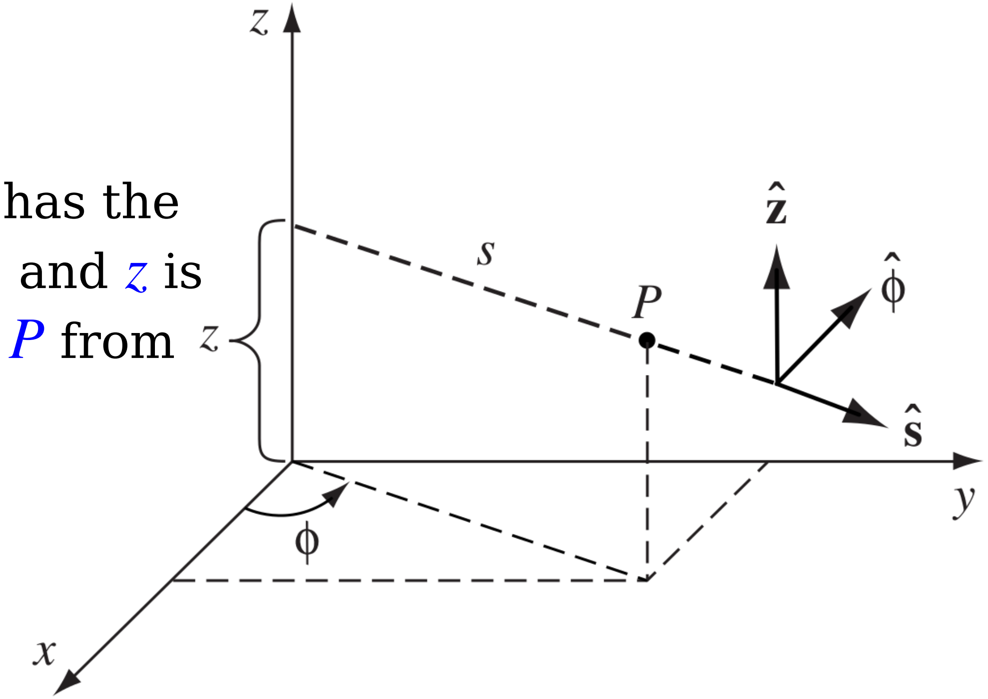
$$\nabla = \frac{\hat{\mathbf{n}}_1}{h_1}\,\frac{\partial}{\partial\, u_1} + \frac{\hat{\mathbf{n}}_2}{h_2}\,\frac{\partial}{\partial\, u_2} + \frac{\hat{\mathbf{n}}_3}{h_3}\,\frac{\partial}{\partial\, u_3} = \frac{\hat{\mathbf{n}}_1}{h_1}\,\partial_{u_1} + \frac{\hat{\mathbf{n}}_2}{h_2}\,\partial_{u_2} + \frac{\hat{\mathbf{n}}_3}{h_3}\,\partial_{u_3} = \sum_{i=1}^{3} \frac{\hat{\mathbf{n}}_j}{h_j}\,\frac{\partial}{\partial\, u_j}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial_{u_1}\left(A_1\, h_2\, h_3\right) + \partial_{u_2}\left(h_1\, A_2\, h_3\right) + \partial_{u_3}\left(h_1\, h_2\, A_3\right)}{h_1\, h_2\, h_3}$$

$$\nabla\times\mathbf{A} = \frac{1}{h_1\, h_2\, h_3}\begin{vmatrix} h_1\,\hat{\mathbf{n}}_1 & h_2\,\hat{\mathbf{n}}_2 & h_3\,\hat{\mathbf{n}}_3 \\ \partial_{u_1} & \partial_{u_2} & \partial_{u_3} \\ h_1\, A_1 & h_2\, A_2 & h_3\, A_3 \end{vmatrix} = \frac{1}{h_1\, h_2\, h_3}\sum_{i,\,j,\,k} \epsilon^{i\,j\,k}\, h_i\,\hat{\mathbf{n}}_i\,\frac{\partial\, h_k\, A_k}{\partial\, u_j}$$

$$\begin{aligned}\nabla^2 V \\ = \nabla\cdot\nabla\, V\end{aligned} = \frac{1}{h_1\, h_2\, h_3}\left[\partial_{u_1}\left(\frac{\partial_{u_1} V}{h_1}\, h_2\, h_3\right) + \partial_{u_2}\left(h_1\,\frac{\partial_{u_2} V}{h_2}\, h_3\right) + \partial_{u_3}\left(h_1\, h_2\,\frac{\partial_{u_3} V}{h_3}\right)\right]$$

Cylindrical Coordinates



- In the cylindrical coordinates (s, ϕ, z), ϕ has the same meaning as in spherical coordinates, and z is the same as Cartesian; s is the distance to P from the z axis.

- $x = s\cos\phi\,,\quad y = s\sin\phi\,,\quad z = z$

- $$\begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

- $d\ell_s = ds\,,\quad d\ell_\phi = s\,d\phi\,,\quad d\ell_z = dz$ $\Leftarrow\quad h_s = 1\,,\quad h_\phi = s\,,\quad h_z = 1$

  $\Rightarrow\quad d\boldsymbol{\ell} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}\,,\quad d\tau = s\,ds\,d\phi\,dz$

- $s \in [0,\ \infty)\,,\quad \phi \in [0,\ 2\pi]\,,\quad z \in (-\infty,\ \infty)$

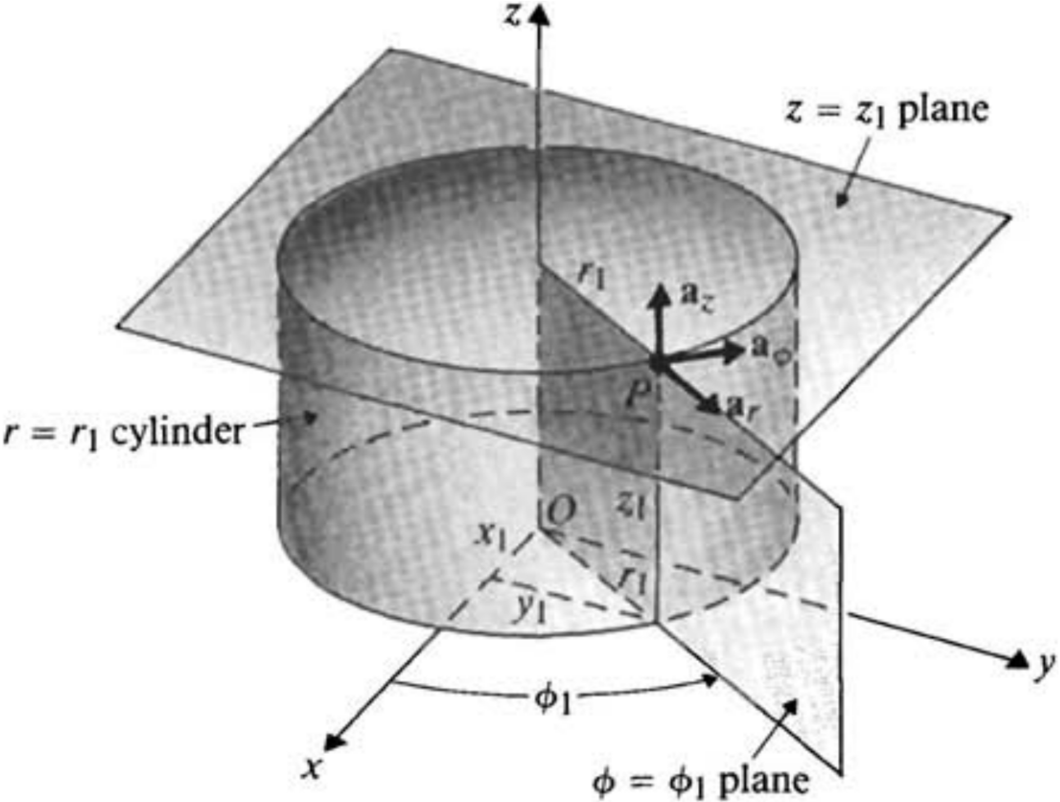
Gradient: $$\nabla T = \left(\hat{\mathbf{s}}\frac{\partial}{\partial s} + \frac{\hat{\boldsymbol{\phi}}}{s}\frac{\partial}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)T = \hat{\mathbf{s}}\frac{\partial T}{\partial s} + \frac{\hat{\boldsymbol{\phi}}}{s}\frac{\partial T}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}$$

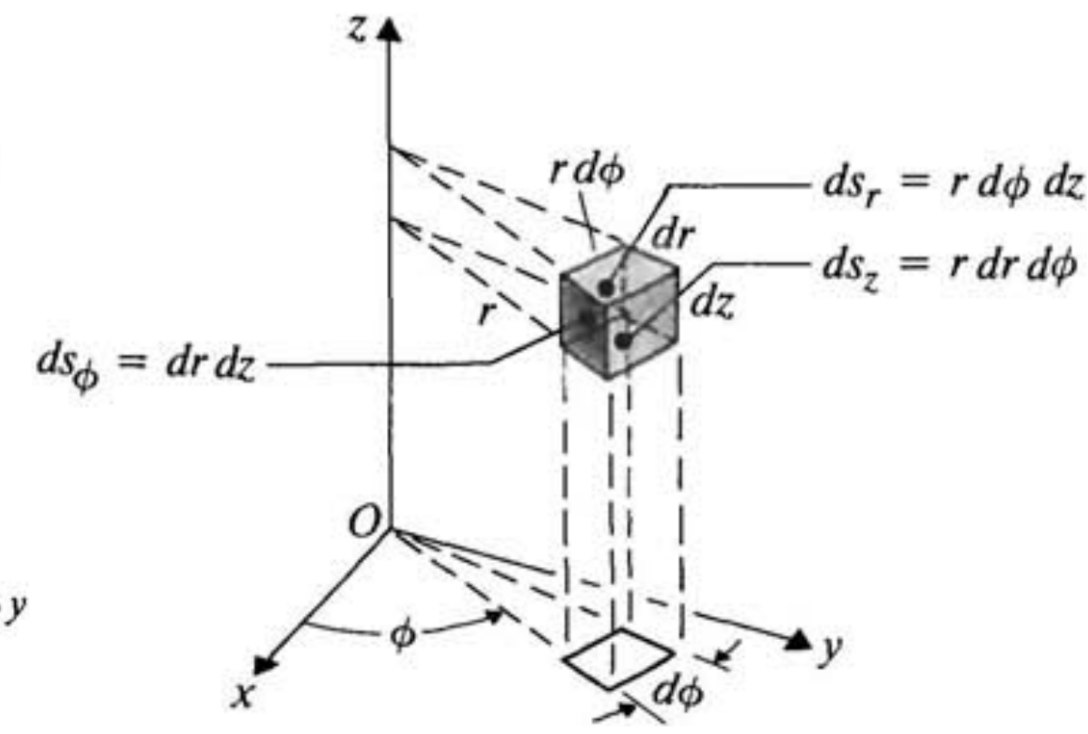
Divergence: $$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s\,v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} = \frac{1}{s}\left[\partial_s(s\,v_s) + \partial_\phi v_\phi + \partial_z(s\,v_z)\right]$$

$$Curl: \nabla \times \mathbf{v} = \left(\frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right) \hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right) \hat{\boldsymbol{\phi}} + \frac{1}{s} \left[\frac{\partial}{\partial s} (s\, v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{\mathbf{z}}$$

$$= \frac{1}{s} \begin{vmatrix} \hat{\mathbf{s}} & s\,\hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \partial_s & \partial_\phi & \partial_z \\ v_s & s\, v_\phi & v_z \end{vmatrix}$$

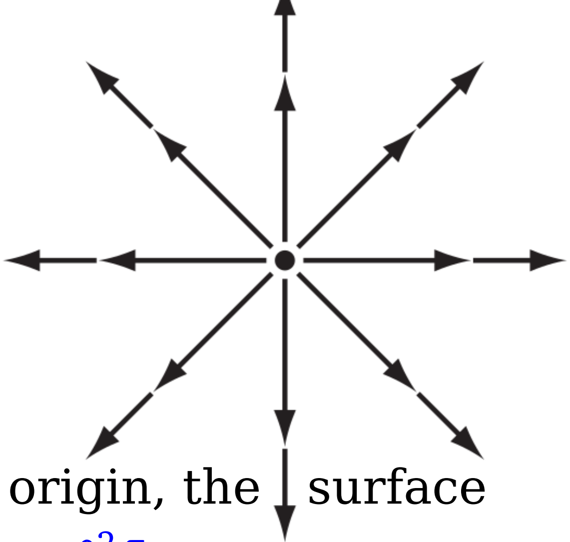
$$Laplacian: \nabla^2 T = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial T}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$





The Dirac Delta Function

The Divergence of $\hat{\mathbf{r}}/r^2$



- $\mathbf{v} = \hat{\mathbf{r}}/r^2$ is directed radially outward; it is likely to have a large positive divergence from it. But

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0 \quad \textbf{?} \qquad (?)$$

- If we integrate over a sphere of radius R, centered at the origin, the surface integral is

$$\int \mathbf{v} \cdot \mathrm{d}\boldsymbol{a} = \int \frac{\hat{\mathbf{r}}}{R^2} \cdot R^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi \,\hat{\mathbf{r}} = \int_0^{\pi} \sin\theta \,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi = 4\pi \qquad (\$)$$

- But the *volume* integral, $\int \nabla \cdot \mathbf{v} \,\mathrm{d}\tau = 0$, if we are really to believe Eq. (?). Does this mean that the divergence theorem is false?

- The source of the problem is at $r=0$, where $\mathbf{v}$ blows up. It is true that $\nabla \cdot \mathbf{v} = 0$ everywhere *except* the origin, but right *at* the origin the situation is complicated.

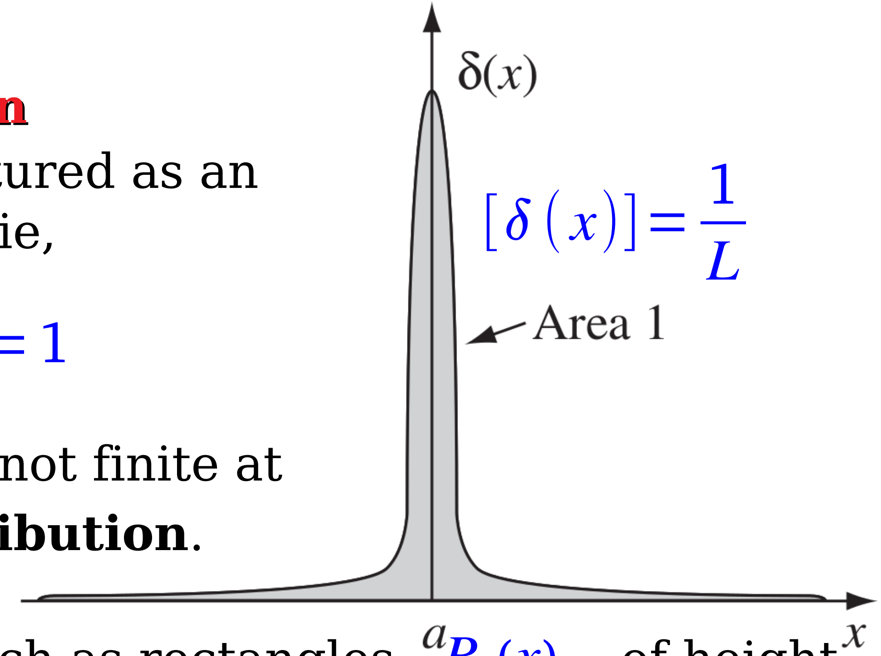
- The surface integral ($) is *independent of* R; if the divergence theorem is right (and it is), we should get $\int \nabla \cdot \mathbf{v} \,\mathrm{d}\tau = 4\pi$ for *any* sphere centered at the origin, no matter how small. So the entire contribution must come from the point $r=0$.

- Thus, $\nabla \cdot \mathbf{v}$ has the bizarre property that it vanishes everywhere except at one point, and yet its integral (over any volume containing that point) is 4π.

- This is where the **Dirac delta function** comes in.

The One-Dimensional Dirac Delta Function

- The 1d Dirac delta function, $\delta(x)$, can be pictured as an infinitely high, infinitesimally narrow "spike," ie,

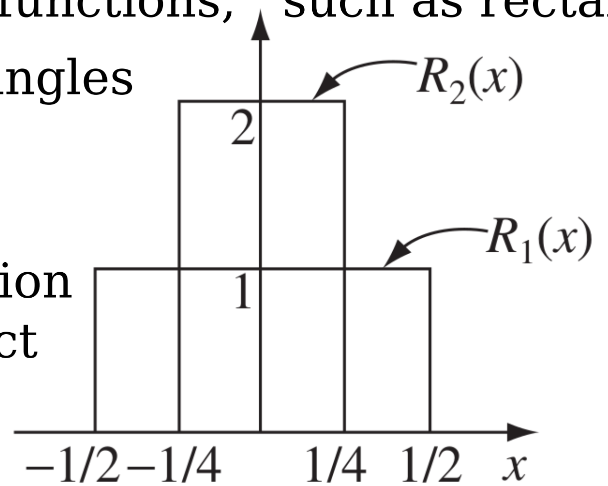


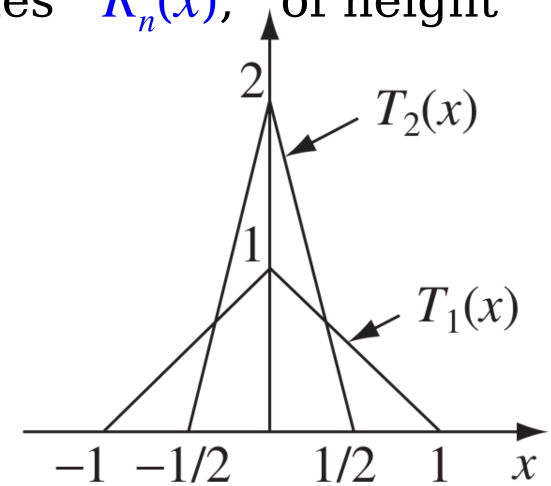
$$[\delta(x)] = \frac{1}{L}$$

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1$$

- $\delta(x)$ is not a function at all, since its value is not finite at $x=0$; so it is a **generalized function**, or **distribution**.

- It is the limit of a sequence of functions, such as rectangles $R_n(x)$, of height n and width $1/n$, or isosceles triangles $T_n(x)$, of height n and base $2/n$.





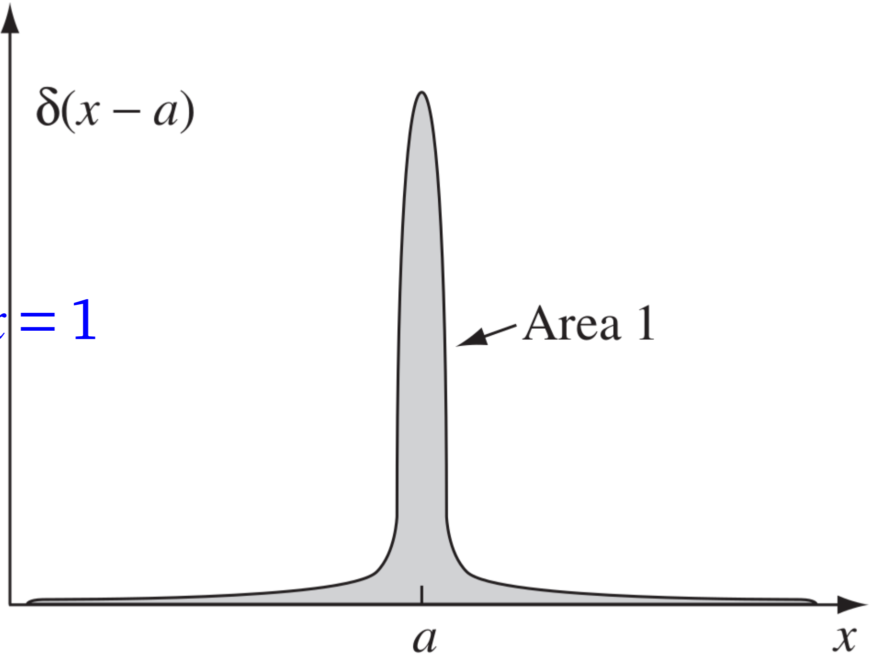
- If $f(x)$ is some "ordinary" function thus continuous, then the product $f(x)\delta(x)$ is 0 everywhere except at $x=0$,

$$\Rightarrow \quad f(x)\,\delta(x) = f(0)\,\delta(x) \quad \Rightarrow \int_{-\infty}^{\infty} f(x)\,\delta(x)\,\mathrm{d}x = f(0)\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = f(0)$$

- Under an integral, the delta function "picks out" the value of $f(x)$ at $x=0$.

- The integral need not run from $-\infty$ to $+\infty$; it is sufficient that the domain extend across the delta function, and $-\epsilon$ to $+\epsilon$ would do as well.

- we can shift the spike from $x=0$ to some other point, $x=a$:



$$\delta(x-a)=\begin{cases}0 & \text{if } x\neq a\\ \infty & \text{if } x=a\end{cases} \quad \text{with} \quad \int_{-\infty}^{\infty}\delta(x-a)\,\mathrm{d}x=1$$

$$\Rightarrow \quad f(x)\,\delta(x-a)=f(a)\,\delta(x-a)$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,\mathrm{d}x=f(a)$$

- Although δ isn't a legitimate function, *integrals* over δ are perfectly acceptable.

- It is best to think of the delta function as something that is *always intended for use under an integral sign*.

- In particular, 2 expressions involving delta functions (say, $D_1(x)$ and $D_2(x)$) are considered equal if, for all ("ordinary") functions $f(x)$,

$$\int_{-\infty}^{\infty} f(x)\,D_1(x)\,\mathrm{d}x=\int_{-\infty}^{\infty} f(x)\,D_2(x)\,\mathrm{d}x$$

<u>Example 1.14</u>, <u>Example1.15</u>

The Three-Dimensional Delta Function

- It is easy to generalize the delta function to 3d: $\delta^3(\mathbf{r}) = \delta(x)\,\delta(y)\,\delta(z)$

- This 3d delta function is 0 everywhere except at (0, 0, 0), where it blows up. Its volume integral is 1:

$$\int_{\text{all space}} \delta^3(\mathbf{r})\, d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\,\delta(y)\,\delta(z)\, dx\, dy\, dz = 1$$

$$\Rightarrow \int_{\text{all space}} f(\mathbf{r})\,\delta^3(\mathbf{r}-\mathbf{a})\, d\tau = f(\mathbf{a}) \quad \Leftarrow \quad [\delta^3(\mathbf{r})] = \frac{1}{L^3}$$

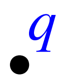
Integration with δ picks out the value of f at the location of the spike.

- The divergence of $\dfrac{\hat{\mathbf{r}}}{r^2}$ is 0 everywhere except at the origin, and yet its integral over any volume containing the origin is a constant (4π). These are precisely the defining conditions for the Dirac delta function;

$$\Rightarrow \quad \nabla\cdot\frac{\hat{\mathbf{r}}}{r^2} = 4\pi\,\delta^3(\mathbf{r}) \quad \Rightarrow \quad \nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 4\pi\,\delta^3(\vec{\mathscr{r}}) \quad \Leftarrow \quad \vec{\mathscr{r}} = \mathbf{r}-\mathbf{r}'$$

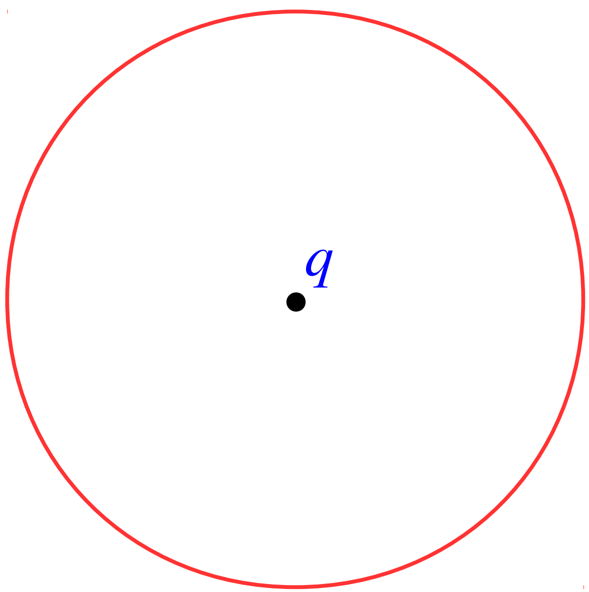
$$\Rightarrow \quad \nabla^2\frac{1}{\mathscr{r}} = -4\pi\,\delta^3(\vec{\mathscr{r}}) \quad \Leftarrow \quad \nabla\frac{1}{\mathscr{r}} = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \quad \Rightarrow \quad \nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\,\delta^3(\mathbf{r}-\mathbf{r}')$$

<u>Example 1.16</u>

- Useful in solving the problems with various boundaries.
- 

$$\Phi(\infty) \to 0 \quad \Leftarrow \quad \text{boundary condition}$$

$$\Rightarrow \quad \Phi(\mathbf{r}) = \frac{q}{r} \quad \Rightarrow \quad \nabla^2 \Phi(\mathbf{r}) = -4\pi q\, \delta^3(\mathbf{r})$$



$$\Phi'(R) = \Phi_0 \quad \Leftarrow \quad \text{boundary condition}$$

$$\Phi'(\mathbf{r}) = \Phi(\mathbf{r}) + F(\mathbf{r}) \quad \Leftarrow \quad \nabla^2 F(\mathbf{r}) = 0$$

$$\Rightarrow \quad \nabla^2 \Phi'(\mathbf{r}) = -4\pi q\, \delta^3(\mathbf{r})$$

- In general, $\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi q\, \delta^3(\mathbf{r} - \mathbf{r}') \quad \Leftarrow \quad G:\ \text{Green function}$

The Theory of Vector Fields

The Helmholtz Theorem

- Maxwell reduced the entire EM theory to 4 equations, specifying respectively the divergence and the curl of the **electric field** $\mathbf{E}$ and the **magnetic field** $\mathbf{B}$.

- To what extent is a vector function determined by its divergence and curl? Let

$$\begin{aligned} \nabla \cdot \mathbf{F} &= D \\ \nabla \times \mathbf{F} &= \mathbf{C} \quad \Leftarrow \quad \nabla \cdot \mathbf{C} = 0 \end{aligned} \quad \Rightarrow \quad \text{Can } \mathbf{F} \text{ be determined?}$$

- To solve a differential equation you must also be supplied with appropriate **boundary conditions**.

- In electrodynamics we typically require that the fields go to 0 “at infinity”.

- With the extra information, the **Helmholtz theorem** guarantees that the field is uniquely determined by its divergence and curl.

Potentials

- If the curl of a vector field ($\mathbf{F}$) vanishes (everywhere), then $\mathbf{F}$ can be written as the gradient of a **scalar potential** (Φ): $\nabla \times \mathbf{F} = 0 \quad \Leftrightarrow \quad \mathbf{F} = -\nabla \Phi$

Theorem 1

Curl-less (or "**irrotational**") **fields.** The following conditions are equivalent (that is, $\mathbf{F}$ satisfies one if and only if it satisfies all the others):

(a) $\nabla \times \mathbf{F} = 0$ everywhere.

(b) $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\boldsymbol{\ell}$ is independent of path, for any given end points.

(c) $\oint \mathbf{F} \cdot d\boldsymbol{\ell} = 0$ for any closed loop.

(d) $\mathbf{F}$ is the gradient of some scalar function: $\mathbf{F} = -\nabla \Phi$.

- The potential is not unique—any constant can be added to Φ with impunity, since this will not affect its gradient.

- If the divergence of a vector field ($\mathbf{F}$) vanishes (everywhere), then $\mathbf{F}$ can be expressed as the curl of a **vector potential** ($\mathbf{A}$): $\nabla \cdot \mathbf{F} = 0 \quad \Leftrightarrow \quad \mathbf{F} = -\nabla \times \mathbf{A}$

Theorem 2

Divergence-less (or "**solenoidal**") **fields**. The following conditions are equivalent:

(a) $\nabla \cdot \mathbf{F} = 0$ everywhere.

(b) $\int \mathbf{F} \cdot d\boldsymbol{a}$ is independent of surface, for any given boundary line.

(c) $\oint \mathbf{F} \cdot d\boldsymbol{a} = 0$ for any closed surface.

(d) $\mathbf{F}$ is the curl of some vector function: $\mathbf{F} = \nabla \times \mathbf{A}$.

- The vector potential is not unique—the gradient of any scalar function can be added to $\mathbf{A}$ without affecting the curl, since the curl of a gradient is 0.

- In *all* cases (*whatever* its curl and divergence may be) a vector field $\mathbf{F}$ can be written as the gradient of a scalar plus the curl of a vector:

$$\mathbf{F} = -\nabla \Phi + \nabla \times \mathbf{A} + \mathbf{C} \quad (\text{always}) \quad \Leftarrow \quad \mathbf{C} = \text{constant vector}$$

Selected problems: 6, 8, 13, 43, 47, 56