Chapter A Essential Concepts and Statics Review

A-1 *Introduction*

- **Objective**: To develop the relationships between the loads applied to a nonrigid (deformable) body and the internal forces and deformation induced in the body.
- **Principles and methods**:
  1. The equations of equilibrium and free-body diagram (by cutting through a member)
  2. Geometry of the body after the action of loads.
  3. The relationship between the loads and deformation.
  4. The size and shape of the member must be adjusted to keep the stress (force per unit area) below the strength of material to avoid failure.

A-2 *Support reactions (in Statics)*

If the support presents **translation** in a given direction (2D: 2; 3D: 3), then **a reaction force** must be developed on the member in that direction. Likewise, if **rotation** is prevented in a given direction (2D: 1; 3D; 3), **a reaction couple moment** must be exerted on the member in that direction.

(1) Two dimensional supports:

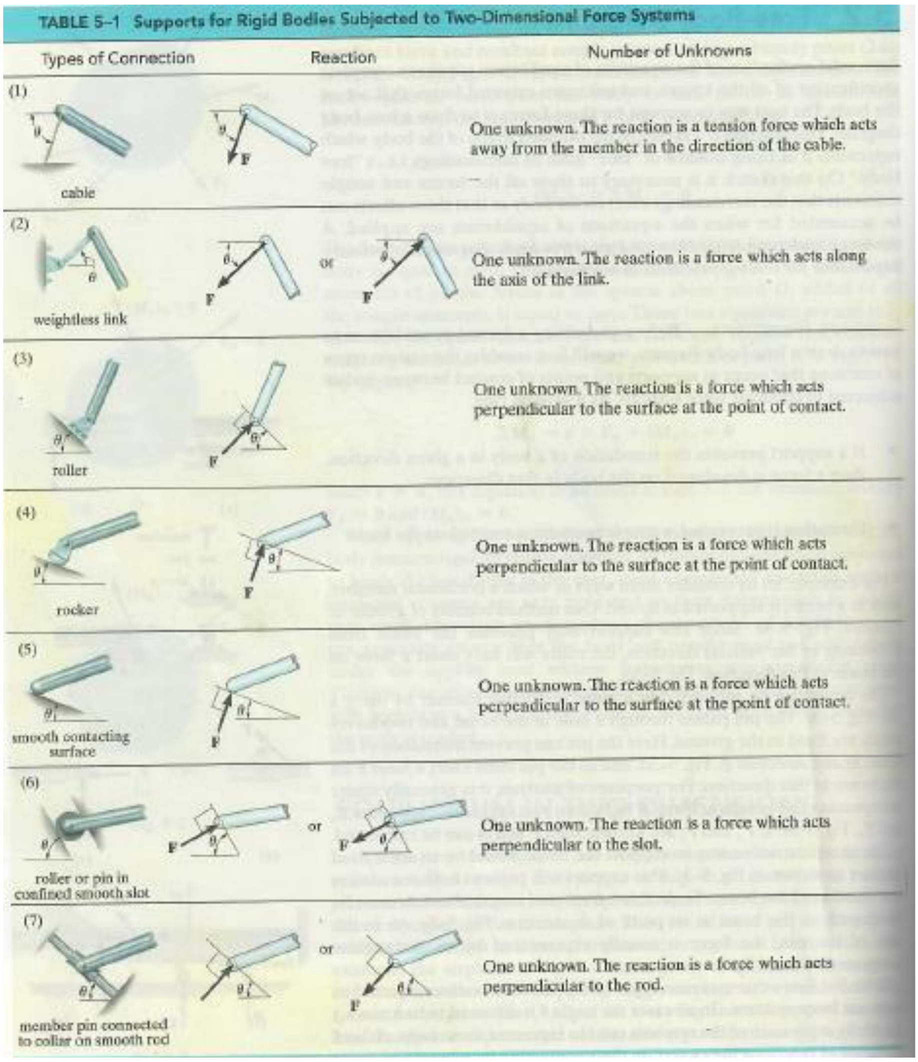
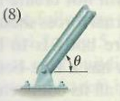
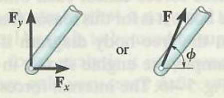
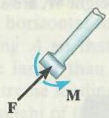
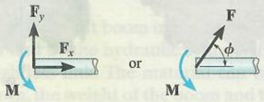


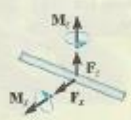
TABLE 5–1 Supports for Rigid Bodies Subjected to Two-Dimensional Force Systems

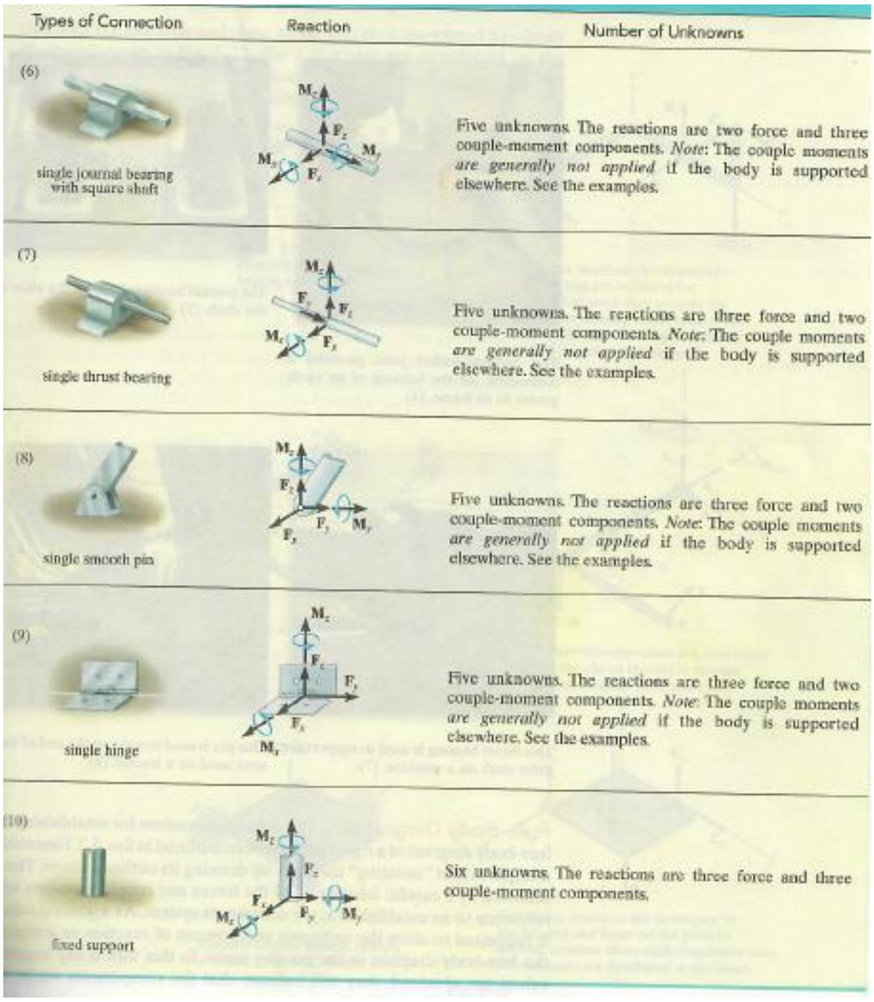
Types of Connection	Reaction	Number of Unknowns
(1) cable	θ, F	One unknown. The reaction is a tension force which acts away from the member in the direction of the cable.
(2) weightless link	θ, F or θ, F	One unknown. The reaction is a force which acts along the axis of the link.
(3) roller	θ, F	One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(4) rocker	θ, F	One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(5) smooth contacting surface	θ, F	One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(6) roller or pin in confined smooth slot	F, θ or F, θ	One unknown. The reaction is a force which acts perpendicular to the slot.
(7) member pin connected to collar on smooth rod	F, θ or F, θ	One unknown. The reaction is a force which acts perpendicular to the rod.

Types of Connection	Reaction	Number of Unknowns
(8)  smooth pin or hinge		Two unknowns. The reactions are two components of force, or the magnitude and direction ϕ of the resultant force. Note that ϕ and θ are not necessarily equal [usually not, unless the rod shown is a link as in (2)].
(9)  member fixed connected to collar on smooth rod		Two unknowns. The reactions are the couple moment and the force which acts perpendicular to the rod.
(10)  fixed support		Three unknowns. The reactions are the couple moment and the two force components, or the couple moment and the magnitude and direction ϕ of the resultant force.

(2) Three-dimensional supports

TABLE 5–2 Supports for Rigid Bodies Subjected to Three-Dimensional Force Systems

Types of Connection	Reaction	Number of Unknowns
(1)  cable		One unknown. The reaction is a force which acts away from the member in the known direction of the cable.
(2)  smooth surface support		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(3)  roller		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(4)  ball and socket		Three unknowns. The reactions are three rectangular force components.
(5)  single journal bearing		Four unknowns. The reactions are two force and two couple-moment components which act perpendicular to the shaft. Note: The couple moments are *generally not applied* if the body is supported elsewhere. See the examples.



Types of Connection	Reaction	Number of Unknowns
(6) single journal bearing with square shaft		Five unknowns. The reactions are two force and three couple-moment components. *Note*: The couple moments *are generally not applied* if the body is supported elsewhere. See the examples.
(7) single thrust bearing		Five unknowns. The reactions are three force and two couple-moment components. *Note*: The couple moments *are generally not applied* if the body is supported elsewhere. See the examples.
(8) single smooth pin		Five unknowns. The reactions are three force and two couple-moment components. *Note*: The couple moments *are generally not applied* if the body is supported elsewhere. See the examples.
(9) single hinge		Five unknowns. The reactions are three force and two couple-moment components. *Note*: The couple moments *are generally not applied* if the body is supported elsewhere. See the examples.
(10) fixed support		Six unknowns. The reactions are three force and three couple-moment components.

A-3 *Equilibrium of a rigid body (in Statics)*

$$\sum \boldsymbol{F} = 0; \quad \sum \boldsymbol{M_O} = 0$$

- 3-D problem:

$$\sum F_x = 0; \quad \sum F_y = 0; \quad \sum F_z = 0;$$

$$\sum (M_O)_x = 0; \quad \sum (M_O)_y = 0; \quad \sum (M_O)_z = 0.$$

There are, totally, 6 equilibrium equations.

- 2-D problem: (x-y plane)

$$\sum F_x = 0; \quad \sum F_y = 0; \quad \sum (M_O)_z = 0.$$

There are, totally, 3 equilibrium equations.

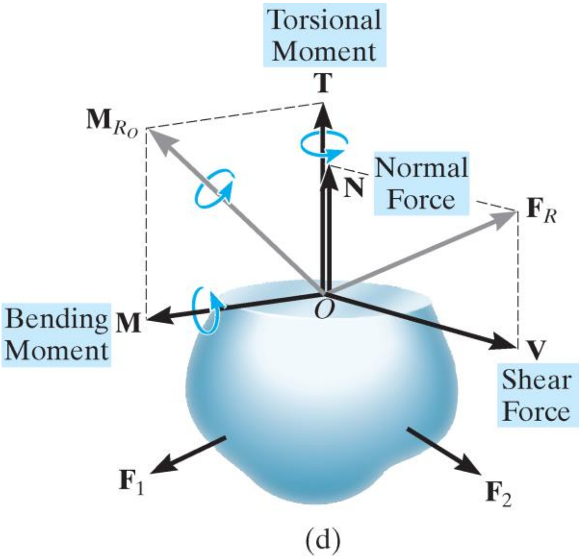
[Remarks]：

1. For 2-D problems under the action of concurrent external forces, only 2 equilibrium equations, i.e., $\sum F_x = 0$ and $\sum F_y = 0$, are required.
2. In statics, it is called "**statically determinate**" if the number of unknown forces is equal to the number of equilibrium equations; while it is called "**statically indeterminate**" if the number of

unknown forces is more than the number of equilibrium equations. **In statically indeterminate problems, the unknown forces cannot completely determined by equilibrium equations. However, with the aid of geometry condition of deformation, all the unknown forces can be completely determined.**

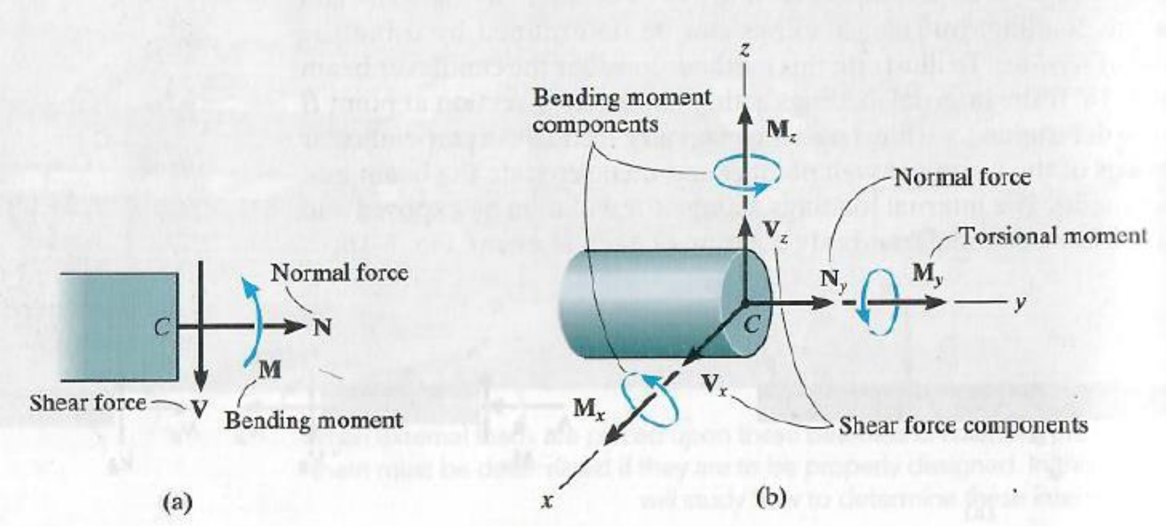
A-4 ***Internal forces (in Statics)***

In the study of mechanics of materials, it is necessary to examine the internal forces that exist throughout the interior by an imaginary cutting plane (or a section) of a body. It is apparent that the internal force system is dependent upon both the orientation and location of the section and can be determined by the equilibrium equations. Experience indicates that materials behave differently to forces trying to pull atoms apart than to forces trying to slide atoms past each other. For convenience a *xyz*-coordinate system is adopted in which x is perpendicular to the section and y and z lie in the section. The component of internal force, $\boldsymbol{F_R}$, perpendicular to the section, $(F_R)_x$ is called an **"axial force" or "normal force"**, generally denoted by symbol N. This force tends either to pull the body apart or to compress the body. The components of $\boldsymbol{F_R}$ that lie in the section, $(F_R)_y$ and $(F_R)_z$, are called the **"shear forces"**, generally denoted by V_y and V_z. These forces tend to slide one part of the body relative to other part. On the other hand, the component of internal couple, $\boldsymbol{M_{Ro}}$, perpendicular to the section, $(M_{Ro})_x$, is called a **"twisting couple" or "twisting moment", or "torque"**, generally denoted by symbol T. This couple tends to twist the body (just like to twistingly dry the clothes). The components of $\boldsymbol{M_{Ro}}$ that lie in the section, $(M_{Ro})_y$ and $(M_{Ro})_z$, are called the **"bending couples" or "bending moments"**, generally denoted by M_y and M_z. These couples tend to bend the body.



3-D problems

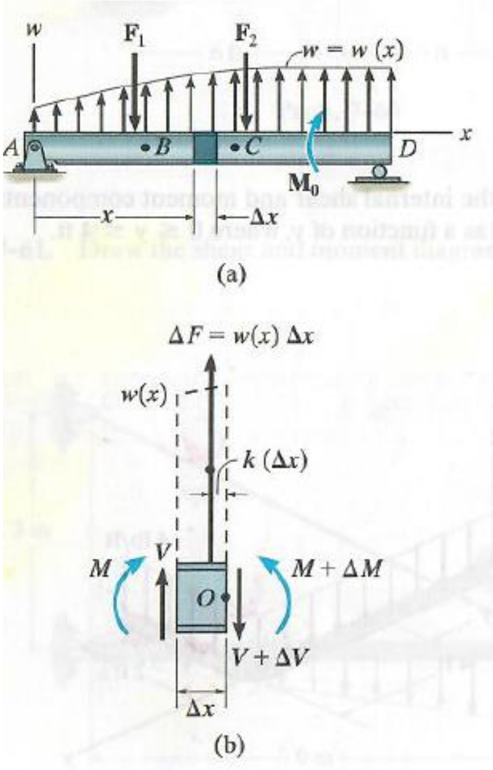
Internal forces in slender member:



2-D 3-D

Sign conventions: An internal force or couple component is defined as positive if the component is either in a positive coordinate direction when acting on a positive section or in the negative coordinate direction when acting on a negative section.

A-5 Relations between distributed load, shear, and moment (in Statics)



Relation Between the Distributed Load and Shear.
apply the force equation of equilibrium to the segment, then

$$+\uparrow \Sigma F_y = 0; \qquad V + w(x)\Delta x - (V + \Delta V) = 0$$
$$\Delta V = w(x)\Delta x$$

Dividing by Δx, and letting $\Delta x \to 0$, we get

$$\boxed{\frac{dV}{dx} = w(x)}$$

slope of shear diagram = distributed load intensity

If we rewrite the above equation in the form $dV = w(x)dx$ and perform an integration between any two points B and C on the beam, we see that

$$\Delta V = \int w(x)\, dx \qquad (7\text{–}2)$$

Change in shear = Area under loading curve

Relation Between the Shear and Moment. If we apply the moment equation of equilibrium about point O on the free-body diagram in Fig. 7–13*b*, we get

$$\circlearrowleft +\Sigma M_O = 0; \quad (M + \Delta M) - [w(x)\Delta x]\, k\Delta x - V\Delta x - M = 0$$
$$\Delta M = V\Delta x + k\, w(x)\Delta x^2$$

Dividing both sides of this equation by Δx, and letting $\Delta x \to 0$, yields

$$\frac{dM}{dx} = V \qquad (7\text{–}3)$$

Slope of moment diagram = Shear

In particular, notice that the maximum bending moment $|M|_{max}$ will occur at the point where the slope $dM/dx = 0$, since this is where the shear is equal to zero.

If Eq. 7–3 is rewritten in the form $dM = \int V\, dx$ and integrated between any two points B and C on the beam, we have

$$\Delta M = \int V\, dx \qquad (7\text{–}4)$$

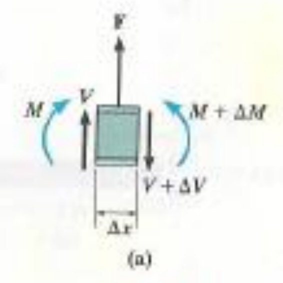
Change in moment = Area under shear diagram

As stated previously, the above equations do not apply at points where a *concentrated* force or couple moment acts. These two special cases create *discontinuities* in the shear and moment diagrams, and as a result, each deserves separate treatment.

Force. A free-body diagram of a small segment of the beam in Fig. 7–13*a*, taken from under one of the forces, is shown in Fig. 7–14*a*. Here force equilibrium requires

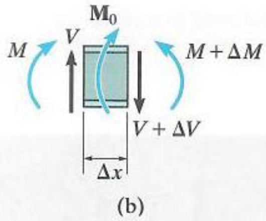
$$+\uparrow \Sigma F_y = 0; \qquad \Delta V = F \qquad (7\text{–}5)$$

Since the *change in shear is positive*, the shear diagram will "jump" *upward when* **F** *acts upward* on the beam. Likewise, the jump in shear (ΔV) is downward when **F** acts downward.



(a)

Fig. 7–14



(b)

Fig. 7–14 (cont.)

Couple Moment. If we remove a segment of the beam in Fig. 7–13*a* that is located at the couple moment $\mathbf{M}_0$, the free-body diagram shown in Fig. 7–14*b* results. In this case letting $\Delta x \to 0$, moment equilibrium requires

$$\circlearrowleft +\Sigma M = 0; \qquad \Delta M = M_0 \qquad (7\text{–}6)$$

Thus, the *change in moment is positive*, or the moment diagram will "jump" *upward if* $\mathbf{M}_0$ *is clockwise*. Likewise, the jump ΔM is downward when $\mathbf{M}_0$ is counterclockwise.

The examples which follow illustrate application of the above equations when used to construct the shear and moment diagrams. After working through these examples, it is recommended that you also go back and solve Examples 7.6 and 7.7 using this method.

Throughout the book the effects on a deformable body of the components of ***R*** and ***C*** will be examined in detail.

For instance:

- **Axial forces** : deformation and stresses induced by axial loading (chap. 4)
- **Twisting torque**: deformation and stresses induced by torsional loading
  (This element is called "***shaft***") (chap. 5)
- **Shear forces and bending moments**: deflection and stresses induced by flexural loading (this element is called "***beam***") (chaps. 6, 7, and 12)
- **Compressively axial force**: deformation and stresses induced by compressive loading (this element is called "***column***") (chap. 13)
  A phenomenon of geometrically unstable deformation "buckling" is introduced (not rupture or failure).

[Remarks]: The remained 4 chapters are:

- Stresses (chap. 1)
- Strains (chap. 2)
- Stress-strain relationships and material mechanical properties (chap. 3)
- Strain energy method and failure theories (chaps. 14 and 10.7)

Chapter 1.   Stress: Definitions and Concepts

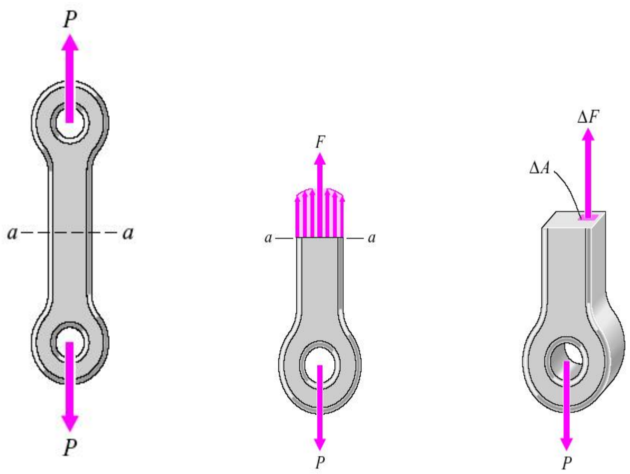
1-1 *Introduction*

Application of the equations of equilibrium to determine the forces exerted on a body by its supports or connections and the internal forces acting on a section is the first step in the solution of engineering problems. A second and equally important step is to determine the internal effect of the forces on the body, which is related to the behaviors of materials under the action of forces. Safety and economy in a design are two considerations for which an engineer must accept responsibility. **The intensity of the internal forces, called "stress", to which each part of a machine or structure is subjected and the deformation (the intensity deformation is called "strain") that each part experiences during the performance of its intended function should be able to be evaluated.** Then, by knowing the properties of the material (the relationships between stress and strain) from which the parts will be made, the engineer establishes the effective size and shape of the individual parts and the appropriate means of connecting them. In other words, a thorough mastery of the physical significance of "stress" and "strain" is paramount.

1-2 *Normal stress under axial loading*

"Stress" is the intensity of force. A body must be able to withstand the intensity of an internal force to avoid rupture or excessive deformation. Force intensity (stress) is force divided by the area over which the force is distributed

Stress = Force / Area   (1-1)



If the internal force is normal to the exposed section, this force intensity is called "normal stress" denoted by "σ". There are two kinds of force intensity. An average force intensity on a section is called "average normal stress", σ_{avg}

$$\sigma_{avg} = F/A \qquad (1\text{-}2)$$

While the stress at the point on the section to which ΔA converges is defined as

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \qquad (1\text{-}3)$$

Where ΔA and ΔF denote the small area on the exposed cross section and the resultant of normal component of the internal forces transmitted by this small area, respectively. **Generally, the normal stress at a point has more physical significance than the average normal stress.**

1-3 *Shearing stress in connections or mechanisms*

Loads applied to a structure or machine are generally transmitted to the individual members through connections or mechanisms. In all these connections or mechanisms, one of the most significant stresses induced is a shearing stress. As shown in Fig. 2-4, if only one cross section of the **bolt** is used to affect the load transfer between the members, the bolt is said to be in **"single shear"**. On the other hand, it can be observed that two cross sections of the pin are used to sustain the load transfer between the members, the pin is said to be in **"double shear"**. For single shear, $V = P$, while $2V = P$ for double shear. In other words, the shear force induced by double shear elements is smaller than single shear ones.

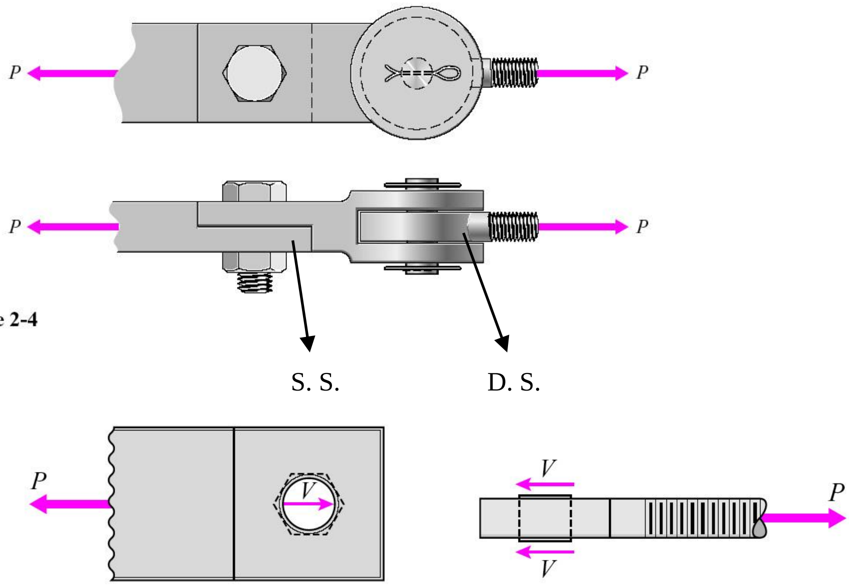


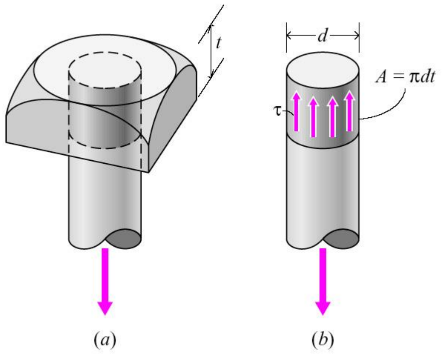
Figure 2-4

Similarly, an average shearing stress and shearing stress at a point of section are defined as, respectively:

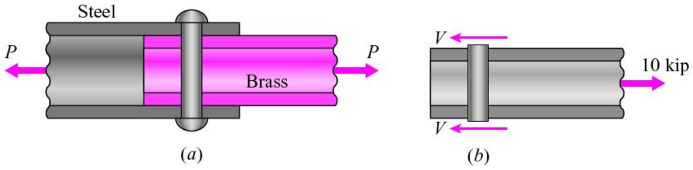
$$\tau_{avg} = V/A \qquad (1\text{-}4)$$

$$\tau = \lim_{\Delta A \to 0} \frac{\Delta V}{\Delta A} \qquad (1\text{-}5)$$

Another type of shear loading related to punch a hole in a metal plate, as shown in Fig. 2-7, is termed "punching shear".

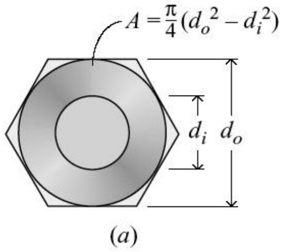


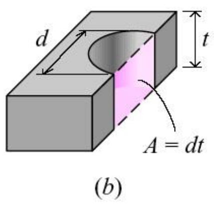
EX: The pin is either double-shear or fourfold-shear?



1-4 ***Bearing stress***

Bearing stresses (compressive normal stresses) occur on the surface of contact between two interacting members. Bearing stress is normal component of this contact stress. As shown in Fig. 2-8, bearing stresses are developed at the contact surfaces between the head of the bolt and the top plate, between the nut and the bottom plate, and between the shanks of bolts and the sides of the hole.





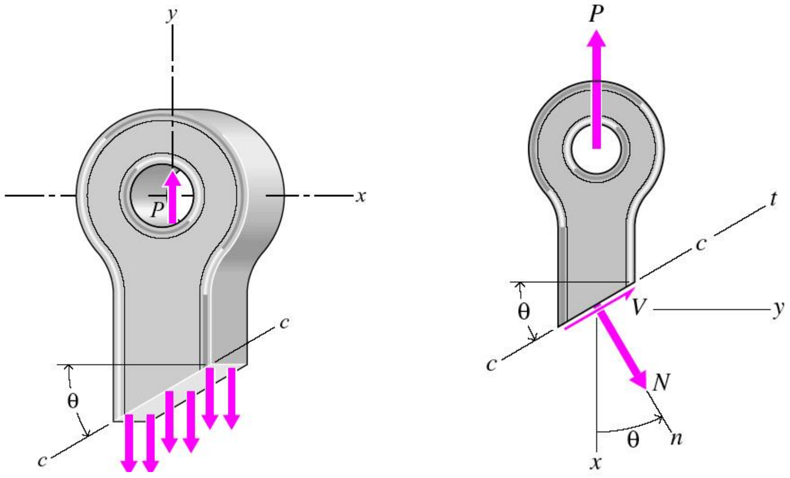
1-5 ***Units of stress***

The dimensions of stresses: FL^{-2}

- U.S. customary (**FPS**) system: **psi (pound per square inch); ksi = 1000 psi**
- **SI** system: **Pa (a Newton per square meter, N/m^2); MPa = 10^6 Pa; GPa =10^9 Pa**

1-6 Stresses on an inclined plane in an axially loaded member

As shown in Figs. 2-14~16, stresses on planes inclined to the axis of an axially loaded bar are considered. The axes and forces are all positive. The *x*-axis is the outward normal to a section perpendicular to the axis of the bar; and the n-axis is the outward normal to the inclined section. The angle θ is measured from a positive x-axis to a positive n-axis; a counterclockwise angle is positive. Positive y- and t-axes are located using the right-hand rule and a positive angle. The normal and shearing forces shown in Fig. 2-16 are all positive.



From equilibrium equations:

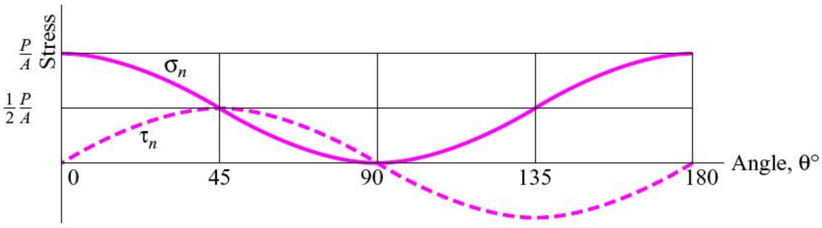
$$\sum F_n = 0 \Rightarrow N - P\cos\theta = 0 \Rightarrow N = P\cos\theta;$$

$$\sum F_t = 0 \Rightarrow V + P\sin\theta = 0 \Rightarrow V = -P\sin\theta$$

Since the relation between the area of axial cross section, A, and the area of inclined surface, A_n, is $A_n = A/\cos\theta$; therefore, the normal and shear stresses induced in the inclined surface under the assumption of uniform distribution are, respectively

$$\sigma_n = \frac{N}{A_n} = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta = \frac{P}{2A}(1+\cos 2\theta) \qquad (1\text{-}7)$$

$$\tau_n = \frac{V}{A_n} = \frac{-P\sin\theta}{A/\cos\theta} = -\frac{P}{A}\sin\theta\cos\theta = -\frac{P}{2A}\sin 2\theta \qquad (1\text{-}8)$$



As seen in Fig. 2-17, the magnitudes of σ_n and τ_n are the function of the angle θ. σ_n is maximum when θ is 0^o or 180^o; while τ_n is maximum when θ is 45^o or 135^o. Moreover, the magnitudes of the maximum normal and shearing stresses for axial tensile or compressive loading are

$$\sigma_{\max} = P/A; \qquad (1\text{-}9)$$

$$\tau_{\max} = P/2A. \qquad (1\text{-}10)$$

Laboratory experiments indicate that a **brittle material** loaded in tension will generally **fail in tension** on a transverse plane ($\theta = 0^o$), whereas a **ductile material** loaded in tension will **fail in shear on a 45º plane**. **(For brittle materials, $\sigma_{strength} < 2\tau_{strength}$; whereas for ductile materials, $\sigma_{strength} > 2\tau_{strength}$)**

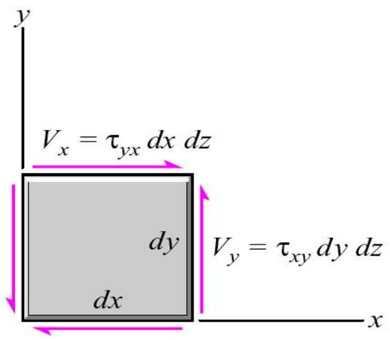
To define the normal stress and shear stress at an arbitrary point *P* in a body, for instance, on the *yz* plane that has an outward normal along *x*-axis, **these stresses should be designated by two subscripts. The first subscript designates the normal to the plane on which the stress acts and the second designates the coordinate axis to which the stress is parallel.** Therefore, the normal stress exerted on this plane is designated by σ_{xx} or σ_x, while the shear stresses acting on this plane which are parallel to *y* and *z* axes are designated by τ_{xy} and τ_{xz}, respectively.

The equality of shearing stresses on orthogonal planes can be demonstrated by applying the equations of equilibrium to the free-body diagram of a small rectangular block of thickness *dz*, shown in Fig. 2-19.

$$\sum M_z = 0 \Rightarrow \tau_{yx}(dxdz)dy = \tau_{xy}(dydz)dx$$

or $\tau_{yx} = \tau_{xy}$ (1-11)

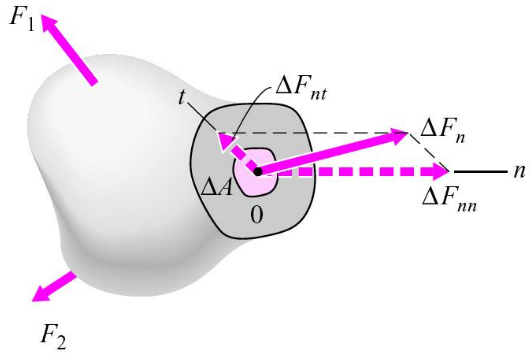
It means the **shearing stresses are symmetric**.

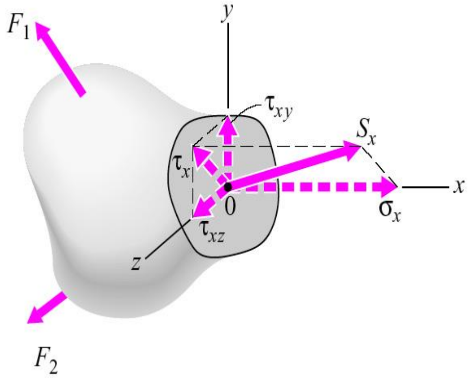


Free-body diagram

1-6 *Stress at a general point in an arbitrarily loaded member*

In complicated structural members or machine components, the stress distributions are generally not uniformly distributed on arbitrary internal planes; therefore, a more general concept of the state of stress at a point is needed. **The stresses at an arbitrary interior point *O* of a body in equilibrium can be determined as follows:**





(c)

1. We pass an imaginary "cutting plane" through point *O* of the body, and separate the body into two parts.
2. One of isolated parts of body is taken as free body, and all the external forces, which include external applied forces and distributive internal forces acting on this interior cutting plane with an outward normal ***n***, should be in equilibrium.
3. Generally, the distributions of internal forces acting on the section are not uniform. Any distributed force acting on a small area ΔA surrounding a point of interest can be replaced by a statically equivalent resultant force ΔF_n through *O* and a couple ΔM_n.
4. The resultant force ΔF_n can be resolved into components ΔF_{nn} normal to the plane and ΔF_{nt} tangent to the plane. A normal stress σ_n and a shearing stress τ_n are then defined as

$$\sigma_n = \lim_{\Delta A \to 0} \frac{\Delta F_{nn}}{\Delta A} \quad \text{and} \quad \tau_n = \lim_{\Delta A \to 0} \frac{\Delta F_{nt}}{\Delta A} \qquad (1\text{-}12)$$

5. In a Cartesian coordinate system, the stresses on planes passing through point *O* having outward normals in the *x*-, *y*-, and *z*-directions are usually chosen. Consider the plane *yz* having an outward normal in the *x*-direction. The resultant force ΔF_{nt} or ΔF_{xt} tangent to the *yz* plane can be resolved into components ΔF_{xy} and ΔF_{xz}, consequently, the related stress components are

$$\sigma_x = \sigma_{xx} = \lim_{\Delta A \to 0} \frac{\Delta F_{xx}}{\Delta A}; \quad \tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta F_{xy}}{\Delta A}; \quad \tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta F_{xz}}{\Delta A} \qquad (1\text{-}13)$$

Similarly, if the planes *xy* and *zx* having outward normals in the *z*- and *y*-direction, respectively, the corresponding stress components are (σ_z, τ_{zx}, τ_{zy}) and (σ_y, τ_{yz}, τ_{yx}), respectively. Therefore, **the stress state at point O can be expressed by a "tensor",** which is neither scalar nor vector mathematically; i.e.,

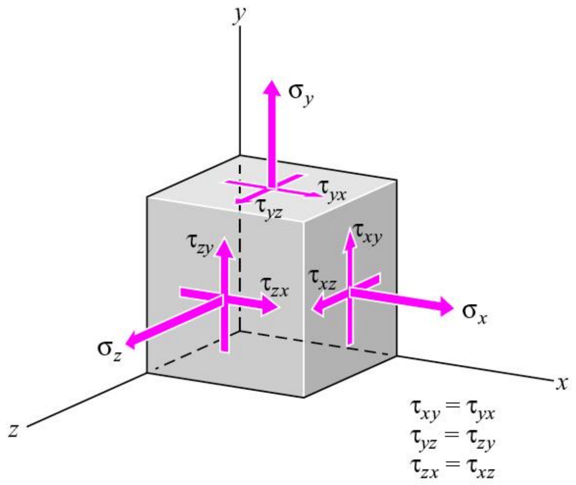
$$\underset{\sim}{\sigma} = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} \qquad (1\text{-}14)$$

Of the nine components of stress, **only six are independent** due to the symmetry of shearing stresses.

[Remarks]:

- **An infinite number of Cartesian coordinate systems can be selected, resulting in infinite number of stress tensors, which have different components just like the feature of vectors, but all of them are equivalent physically.**
- It is customary to show the stresses on positive and negative surfaces through a point using a small

cubic element as shown in Fig. 2-24.



1-7 ***Two-dimensional or plane stress***

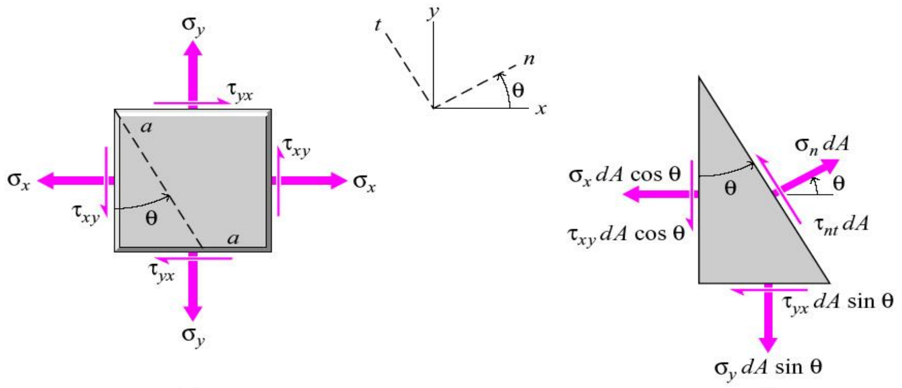
Plane stress (*xy*): A kind of stress state in which two parallel faces with outward normal in the z-direction of the small cubic element are free of stress; i.e.,

$\sigma_z = \tau_{zx} = \tau_{zy} = 0$. In other words, only three stress components, σ_x, σ_y, τ_{xy}, are non-zero. This stress state occurs at points within thin plates where the z-dimension of the body is small and the z-component of the external forces is zero. For convenience, this state of stress can be represented by the simple two-dimensional sketch, or a plane projection of the three-dimensional element.

Chapter 9. Stress Transformation

9-1 *The stress transformation equations for plane stress*

Equations relating the normal and shearing stresses σ_n and τ_{nt} on an arbitrary plane (whose normal ***n*** is oriented at an angle θ with respect to a reference x-axis; and a counterclockwise angle θ is positive) through a point and the known stresses σ_x, σ_y and τ_{xy} on the reference planes can be derived using the free-body diagram and the equations of equilibrium as shown in Fig. 2-27.



(a) (b)

$$\sum F_n = 0 \Rightarrow \sigma_n dA - \sigma_x (dA\cos\theta)\cos\theta - \sigma_y (dA\sin\theta)\sin\theta \quad - \tau_{yx}(dA\sin\theta)\cos\theta - \tau_{xy}(dA\cos\theta)\sin\theta = 0$$

or $\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$ (9-1)

By using relations of double angle, above equation becomes

$$\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta \qquad \text{(9-2)}$$

$$\sum F_t = 0 \Rightarrow \tau_{nt} dA + \sigma_x (dA\cos\theta)\sin\theta - \sigma_y (dA\sin\theta)\cos\theta$$

$$-\tau_{xy}(dA\cos\theta)\cos\theta + \tau_{yx}(dA\sin\theta)\sin\theta = 0$$

or $\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$ (9-3)

By using relations of double angle, above equation becomes

$$\tau_{nt} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta \qquad \text{(9-4)}$$

[Remarks]: When these equations are used, the sign conventions used in their development must be rigorously followed. Moreover, it can be found that

$$\left[\sigma_n - \left(\frac{\sigma_x + \sigma_y}{2}\right)\right]^2 + \tau_{nt}^{\ 2} = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2} \qquad \text{(9-5)}$$

It represents a cycle with center ($\frac{\sigma_x+\sigma_y}{2}, 0)$ **and radius** $R=\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}{}^2}$.

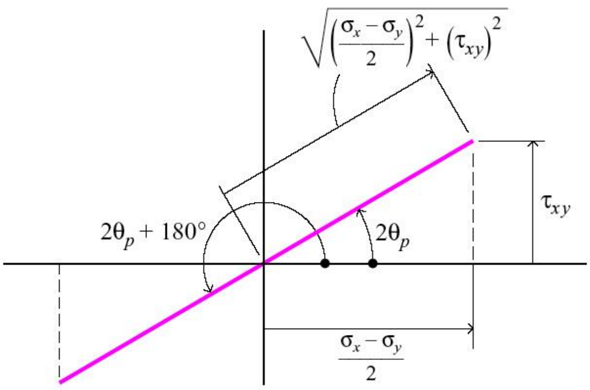
9-2 *Principal stresses* and *maximum shearing stress-plane stress*

As shown in previous section, the normal and shearing stresses σ_n and τ_{nt} vary with the angle θ. For design purpose, critical stresses at the point are usually the maximum tensile stress and the maximum shearing stress. Maximum and minimum values of occur at value of for which $d\sigma_n/d\theta$ is equal to zero. Differentiation of σ_n with respect to θ yields

$$\frac{d\sigma_n}{d\theta}=-(\sigma_x-\sigma_y)\sin 2\theta+2\tau_{xy}\cos 2\theta=0$$

$$\Rightarrow \tan 2\theta_p=\frac{2\tau_{xy}}{\sigma_x-\sigma_y} \quad \text{and} \quad \tau_{nt}(\theta_p)=-\frac{\sigma_x-\sigma_y}{2}\sin 2\theta_p+\tau_{xy}\cos 2\theta_p=0 \qquad (9\text{-}6)$$

Consequently, the shearing stress is zero on plane experiencing maximum and minimum values of normal stress. Therefore, planes free of shearing stress are known as ***principal planes***; normal stresses occurring on principle planes are known as ***principal stresses***. A third principal plane for plane stress state has an outward normal in the *z*-direction. For a given set of values of σ_x, σ_y, τ_{xy}, there are two values of $2\theta_p$ differing by 180 $^{\circ}$ and, consequently, two values of θ_p that are 90° apart. This proves that the principal planes are normal to each other.



Two principal stresses, as shown in Fig. 2-32, can be obtained as given by

$$\sigma_{p1,p2}=\frac{\sigma_x+\sigma_y}{2}\pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}{}^2} \qquad (9\text{-}7)$$

The third one is $\sigma_{p3}=\sigma_z=0$.

The maximum in-plane shearing stress τ_p occurs on planes located by values of θ where $d\tau_{nt}/d\theta$ is equal to zero.

$$\frac{d\tau_{nt}}{d\theta}=-(\sigma_x-\sigma_y)\cos 2\theta-2\tau_{xy}\sin 2\theta=0$$

$$\Rightarrow \tan 2\theta_\tau=-\frac{(\sigma_x-\sigma_y)}{2\tau_{xy}} \qquad (9\text{-}8)$$

Since two angles $2\theta_p$ and $2\theta_\tau$ differ by 90°, θ_p and θ_τ are 45° apart.

The maximum in-plane shearing stress τ_p are

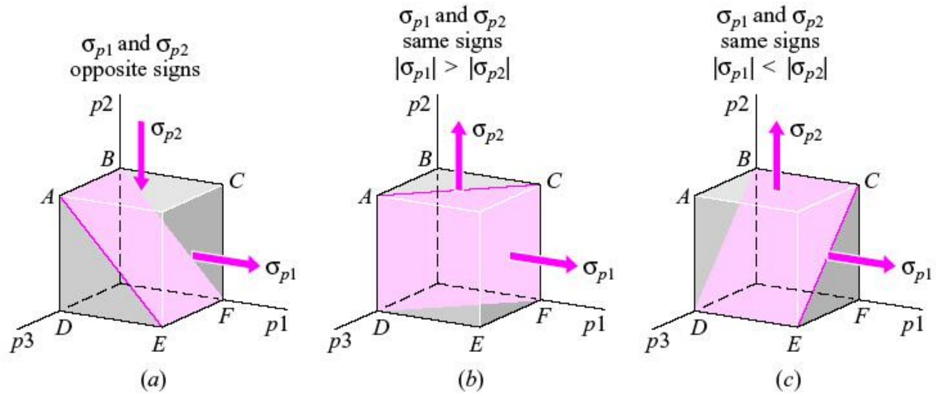
$$\tau_p = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}} = \pm\frac{(\sigma_{p1} - \sigma_{p2})}{2} = \frac{\sigma_{\max} - \sigma_{\min}}{2} \qquad (9\text{-}9)$$

When a state of plane stress exists, one of the principal stresses σ_{p3} is zero. The maximum shearing stress may be different from the maximum in-plane shearing stress and equal to:

(1) If $\sigma_{p1} > 0 > \sigma_{p2}$: The maximum shearing stress equals $(\sigma_{p1} - \sigma_{p2})/2$.

(2) If $\sigma_{p1} > \sigma_{p2} > 0$: The maximum shearing stress equals $(\sigma_{p1} - 0)/2$.

(3) If $0 > \sigma_{p2} > \sigma_{p1}$: The maximum shearing stress equals $(0 - \sigma_{p1})/2$.



9-3 Mohr's circle for plane stress

The German engineer Otto Mohr (1835-1918) developed a useful graphic interpretation of the transformation equations for plane stress.

It can be found that

$$\left[\sigma_n - \left(\frac{\sigma_x + \sigma_y}{2}\right)\right]^2 + \tau_{nt}^{\ 2} = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}$$

It represents a cycle with center ($\frac{\sigma_x + \sigma_y}{2}$,0) and radius $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$,

or denoted by $[\sigma_n - \sigma_{no}]^2 + \tau_{nt}^{\ 2} = R^2$, where $\sigma_{no} = (\sigma_x + \sigma_y)/2$.

As shown in Figure, we want to verify the stress transformation equations:

$\sigma_n = OC + CF \cos(2\theta_p - 2\theta) = OC + CA \cos 2\theta_p \cos 2\theta + CA \sin 2\theta_p \sin 2\theta$

where $CA \cos 2\theta_p = (\sigma_x - \sigma_y)/2$; $CA \sin 2\theta_p = \tau_{xy}$; and $OC = (\sigma_x + \sigma_y)/2 = \sigma_{avg}$. Therefore,

$$\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

Similarly,

$\tau_{nt} = CF \sin(2\theta_p - 2\theta) = CA \sin 2\theta_p \cos 2\theta - CA \cos 2\theta_p \sin 2\theta$

Therefore,

$$\tau_{nt} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

Moreover,

$$\sigma_{p1} = OD = OC + CD = OC + CA = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}\ ,$$

$$\sigma_{p2} = OE = OC - CE = OC - CA = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}\ .$$

$$\tau_p = CM = CA = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}\ ;\quad \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}\ .$$

It can be found all above relations, evaluated through Mohr's circle, are consistent with the stress transformation equations.

■ **The procedures to draw Mohr's circle are:**

(1) Choose a set of *x-y* reference frame, and identify the stress components σ_x, σ_y and τ_{xy} and list them with the proper sign.

(2) Draw a set of σ_n -τ_{nt} coordinate axes with σ_n and τ_{nt} positive to the right and upward (or downward), respectively.

(3) Plot the point (σ_x, -τ_{xy}) (or (σ_x, τ_{xy}) if τ_{nt} positive downward) and label it point *A*.

(4) Plot the point (σ_y, τ_{xy}) (or (σ_y, -τ_{xy}) if τ_{nt} positive downward) and label it point *G*.
[Remarks]: or plot the center *C*: (σ_{avg}, 0), where σ_{avg}=(σ_x +σ_y)/2.

(5) Draw a line between *A* and *G*. The intersection of this line and horizontal axis τ_{nt} establishes the center *C* and the radius *R* = *CA* = *CG*.
[Remarks]: or draw a line between *A* and *C*.

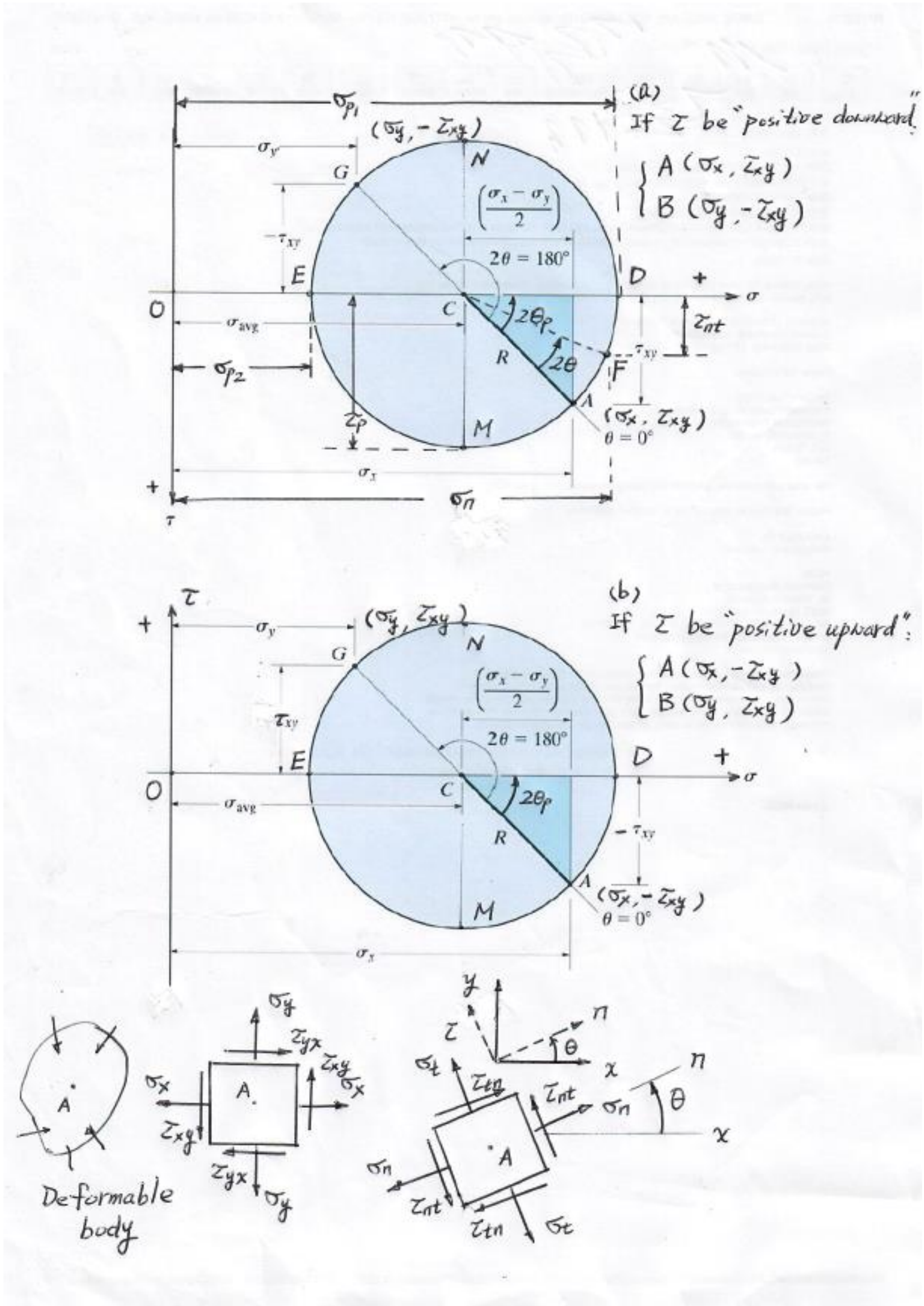
(6) Draw the circle with center *C* and the radius *R*.

Other points of interest in Mohr's circle are:

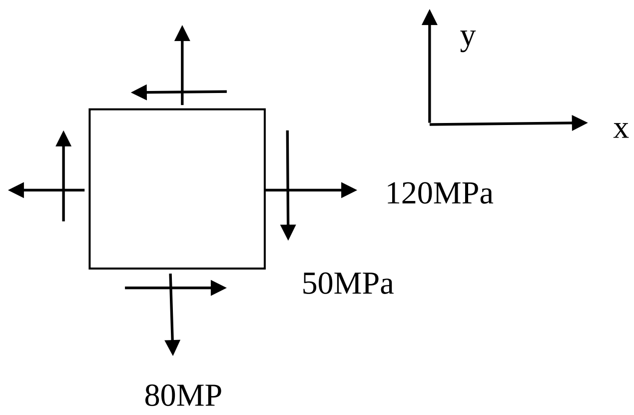
(1) Point *D*, which provides the maximum principal stress σ_{p1} = *OD*.

(2) Point *E*, which provides the maximum principal stress σ_{p2} = *OE*.

(3) Point *M* or *N*, which provides the maximum in-plane shearing stress τ_p = *CM*.



EX: If the stress state of a point A in a deformable body is as follows ($\sigma_z = \tau_{zx} = \tau_{zy} = 0$): plot the Mohr's circle, determine the principal stresses and maximum shear stress.



EXAMPLE 9.7

Due to the applied loading, the element at point A on the solid shaft in Fig. 9–18a is subjected to the state of stress shown. Determine the principal stresses acting at this point.

SOLUTION

Construction of the Circle. From Fig. 9–18a,

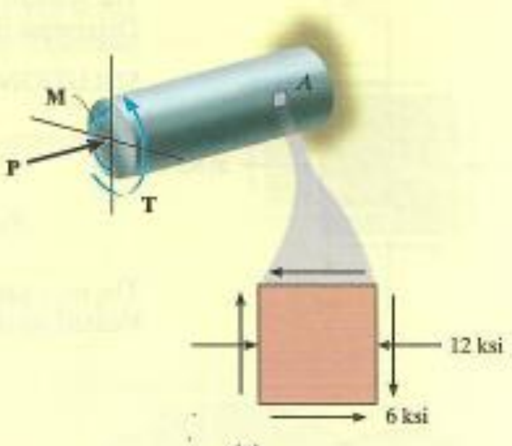
$$\sigma_x = -12 \text{ ksi} \qquad \sigma_y = 0 \qquad \tau_{xy} = -6 \text{ ksi}$$

The center of the circle is located on the σ axis at the point

$$\sigma_{avg} = \frac{-12 + 0}{2} = -6 \text{ ksi}$$

The reference point $A(-12, -6)$ and the center $C(-6, 0)$ are plotted in Fig. 9–18b. From the shaded triangle, the circle is constructed having a radius of

$$R = \sqrt{(12 - 6)^2 + (6)^2} = 8.49 \text{ ksi}$$



(a)

Principal Stress. The principal stresses are indicated by the coordinates of points B and D. We have, for $\sigma_1 > \sigma_2$,

$$\sigma_1 = 8.49 - 6 = 2.49 \text{ ksi} \qquad \textit{Ans.}$$

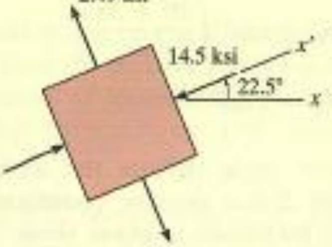
$$\sigma_2 = -6 - 8.49 = -14.5 \text{ ksi} \qquad \textit{Ans.}$$

The orientation of the element can be determined by calculating the angle $2\theta_{p_2}$ in Fig. 9–18b, which here is measured *counterclockwise* from CA to CD. It defines the direction θ_{p_2} of σ_2 and its associated principal plane. We have

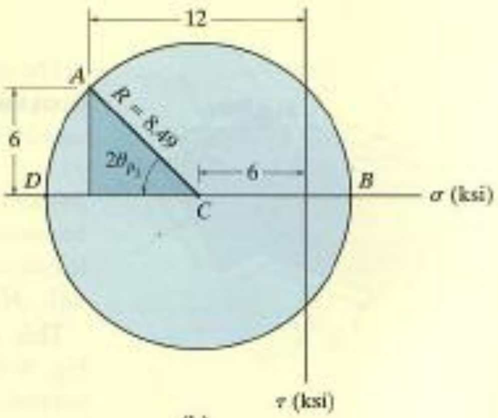
$$2\theta_{p_2} = \tan^{-1}\frac{6}{12 - 6} = 45.0^\circ$$

$$\theta_{p_2} = 22.5^\circ$$

The element is oriented such that the x' axis or σ_2 is directed 22.5° *counterclockwise* from the horizontal (x axis), as shown in Fig. 9–18c.



(c)



(b)

Fig. 9–18

EXAMPLE 9.8

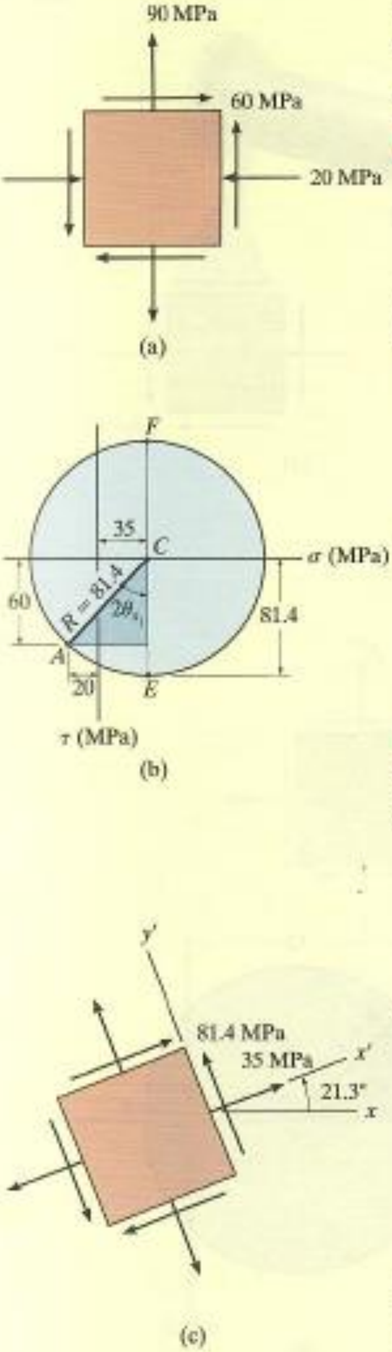


Fig. 9–19

The state of plane stress at a point is shown on the element in Fig. 9–19*a*. Determine the maximum in-plane shear stress at this point.

SOLUTION

Construction of the Circle. From the problem data,

$$\sigma_x = -20 \text{ MPa} \qquad \sigma_y = 90 \text{ MPa} \qquad \tau_{xy} = 60 \text{ MPa}$$

The σ, τ axes are established in Fig. 9–19*b*. The center of the circle C is located on the σ axis, at the point

$$\sigma_{avg} = \frac{-20 + 90}{2} = 35 \text{ MPa}$$

Point C and the reference point $A(-20, 60)$ are plotted. Applying the Pythagorean theorem to the shaded triangle to determine the circle's radius CA, we have

$$R = \sqrt{(60)^2 + (55)^2} = 81.4 \text{ MPa}$$

Maximum In-Plane Shear Stress. The maximum in-plane shear stress and the average normal stress are identified by point E (or F) on the circle. The coordinates of point $E(35, 81.4)$ give

$$\sigma_{avg} = 35 \text{ MPa} \qquad \textit{Ans.}$$

$$\tau_{\substack{max \\ in\text{-}plane}} = 81.4 \text{ MPa} \qquad \textit{Ans.}$$

The angle θ_{s_1}, measured *counterclockwise* from CA to CE, can be found from the circle, identified as $2\theta_{s_1}$. We have

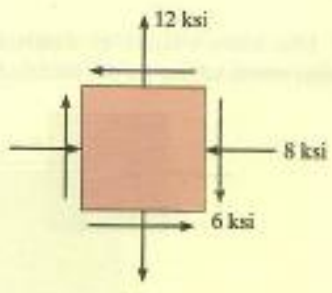
$$2\theta_{s_1} = \tan^{-1}\left(\frac{20 + 35}{60}\right) = 42.5^\circ$$

$$\theta_{s_1} = 21.3^\circ \qquad \textit{Ans.}$$

This *counterclockwise* angle defines the direction of the x' axis, Fig. 9–19*c*. Since point E has *positive* coordinates, then the average normal stress and the maximum in-plane shear stress both act in the *positive* x' and y' directions as shown.

EXAMPLE 9.9

The state of plane stress at a point is shown on the element in Fig. 9–20*a*. Represent this state of stress on an element oriented 30° counterclockwise from the position shown.



(a)

SOLUTION

Construction of the Circle. From the problem data,

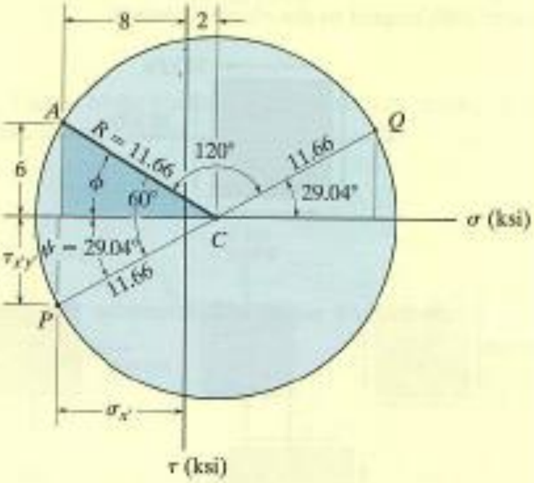
$$\sigma_x = -8 \text{ ksi} \qquad \sigma_y = 12 \text{ ksi} \qquad \tau_{xy} = -6 \text{ ksi}$$

The σ and τ axes are established in Fig. 9–20*b*. The center of the circle C is on the σ axis at the point

$$\sigma_{avg} = \frac{-8 + 12}{2} = 2 \text{ ksi}$$

The reference point for $\theta = 0°$ has coordinates $A(-8, -6)$. Hence from the shaded triangle the radius CA is

$$R = \sqrt{(10)^2 + (6)^2} = 11.66$$



(b)

Stresses on 30° Element. Since the element is to be rotated 30° *counterclockwise*, we must construct a radial line CP, $2(30°) = 60°$ *counterclockwise*, measured from CA $(\theta = 0°)$, Fig. 9–20*b*. The coordinates of point $P(\sigma_{x'}, \tau_{x'y'})$ must then be obtained. From the geometry of the circle,

$$\phi = \tan^{-1}\frac{6}{10} = 30.96° \qquad \psi = 60° - 30.96° = 29.04°$$

$$\sigma_{x'} = 2 - 11.66 \cos 29.04° = -8.20 \text{ ksi} \qquad \textit{Ans.}$$

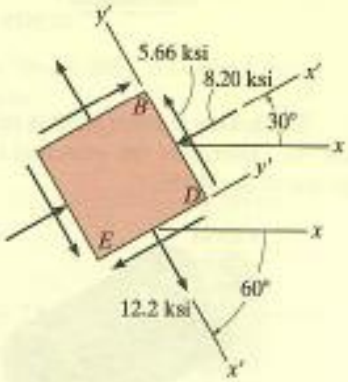
$$\tau_{x'y'} = 11.66 \sin 29.04° = 5.66 \text{ ksi} \qquad \textit{Ans.}$$

These two stress components act on face BD of the element shown in Fig. 9–20*c*, since the x' axis for this face is oriented 30° *counterclockwise* from the x axis.

The stress components acting on the adjacent face DE of the element, which is 60° *clockwise* from the positive x axis, Fig. 9–20*c*, are represented by the coordinates of point Q on the circle. This point lies on the radial line CQ, which is 180° from CP, or 120° *clockwise* from CA. The coordinates of point Q are

$$\sigma_{x'} = 2 + 11.66 \cos 29.04° = 12.2 \text{ ksi} \qquad \textit{Ans.}$$

$$\tau_{x'y'} = -(11.66 \sin 29.04) = -5.66 \text{ ksi} \quad \text{(check)} \qquad \textit{Ans.}$$



(c)

Fig. 9–20

NOTE: Here $\tau_{x'y'}$ acts in the $-y'$ direction, Fig. 9–20*c*.

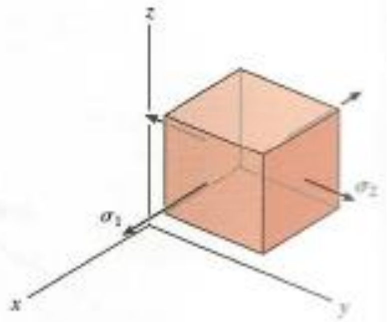
9.5 ABSOLUTE MAXIMUM SHEAR STRESS

Since the strength of a ductile material depends upon its ability to resist shear stress, it becomes important to find the ***absolute maximum shear stress*** in the material when it is subjected to a loading. To show how this can be done, we will confine our attention only to the most common case of plane stress,* as shown in Fig. 9–21*a*. Here *both* σ_1 and σ_2 are tensile. If we view the element in two dimensions at a time, that is, in the y–z, x–z, and x–y planes, Figs. 9–21*b*, 9–21*c*, and 9–21*d*, then we can use Mohr's circle to determine the maximum in-plane shear stress for each case. For example, Mohr's circle extends between 0 and σ_2 for the case shown in Fig. 9–21*b*. From this circle, Fig. 9–21*e*, the maximum in-plane shear stress is $\tau_{\substack{\max \\ \text{in-plane}}} = \sigma_2/2$. Mohr's circles for the other two cases are also shown in Fig. 9–21*e*. Comparing all three circles, the absolute maximum shear stress is

$$\boxed{\tau_{\substack{\text{abs} \\ \max}} = \frac{\sigma_1}{2}} \qquad (9\text{–}13)$$

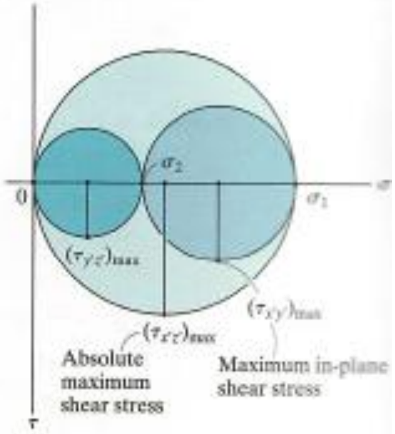
σ_1 and σ_2 have the same sign

It occurs on an element that is rotated 45° about the y axis from the element shown in Fig. 9–21*a* or Fig. 9–21*c*. It is this out of plane shear stress that will cause the material to fail, not $\tau_{\substack{\max \\ \text{in-plane}}}$.

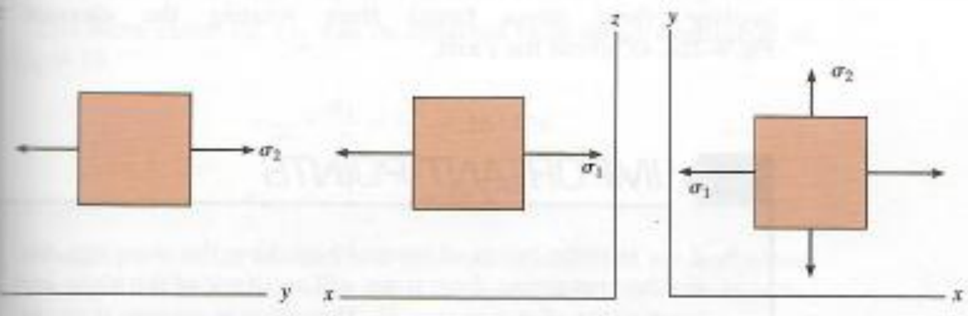


x–y plane stress

(a)



(e)



(b) (c) (d)

Fig. 9–21

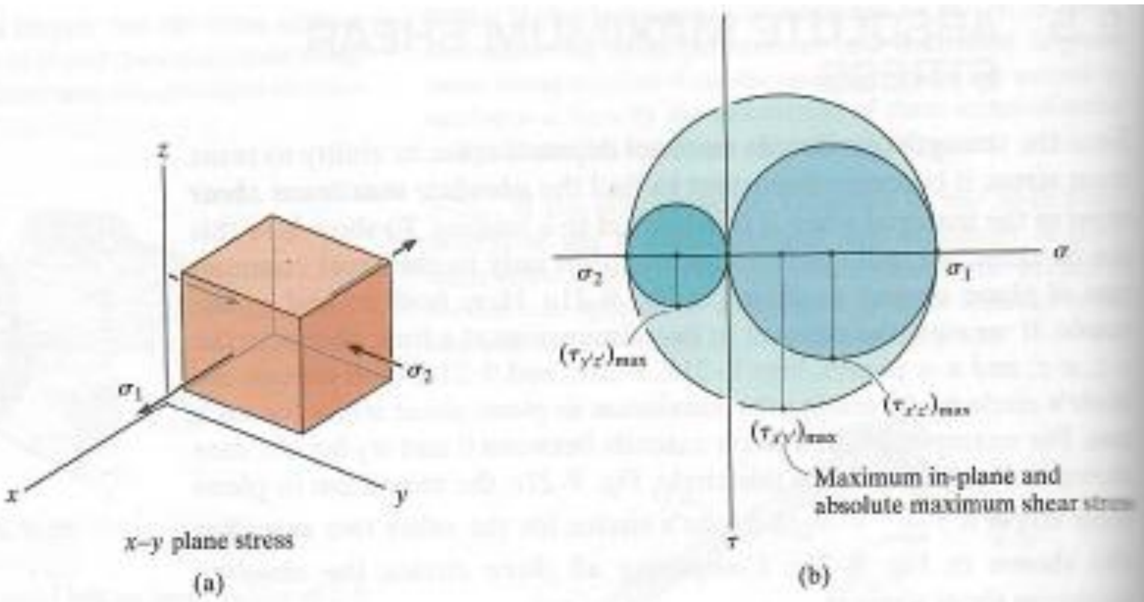


Fig. 9–22

In a similar manner, if one of the in-plane principal stresses has the *opposite sign* of the other, Fig. 9–22*a*, then the three Mohr's circles that describe the state of stress for the element when viewed from each plane are shown in Fig. 9–22*b*. Clearly, in this case

$$\boxed{\tau_{\substack{abs\\max}} = \frac{\sigma_1 - \sigma_2}{2}} \qquad (9\text{–}14)$$

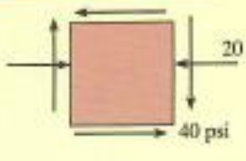
σ_1 and σ_2 have opposite signs

Here the absolute maximum shear stress is equal to the maximum in-plane shear stress found from rotating the element in Fig. 9–22*a*, 45° about the *z* axis.

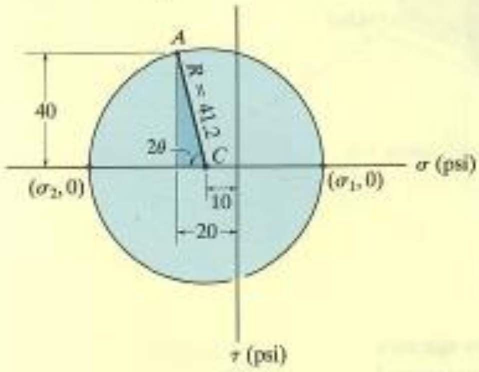
IMPORTANT POINTS

- If the in-plane principal stresses both have the same sign, the absolute maximum shear stress will occur out of the plane and has a value of $\tau_{\substack{abs\\max}} = \sigma_{max}/2$. This value is greater than the in-plane shear stress.
- If the in-plane principal stresses are of opposite signs, then the absolute maximum shear stress will equal the maximum in-plane shear stress; that is, $\tau_{\substack{abs\\max}} = (\sigma_{max} - \sigma_{min})/2$.

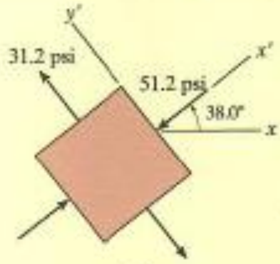
EXAMPLE 9.11



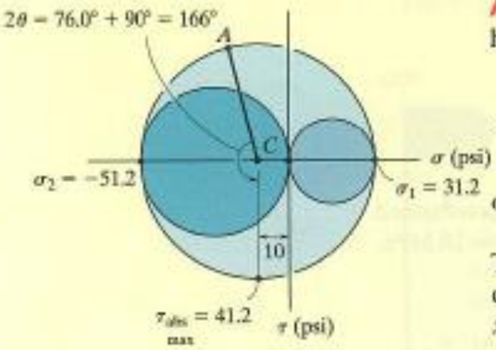
(a)



(b)



(c)



(d)

Fig. 9–24

Due to an applied loading, an element at a point on a machine shaft is subjected to the state of plane stress shown in Fig. 9–24*a*. Determine the principal stresses and the absolute maximum shear stress at the point.

SOLUTION

Principal Stresses.
The in-plane principal stresses can be determined from Mohr's circle. The center of the circle is on the σ axis at $\sigma_{avg} = (-20 + 0)/2 = -10$ psi. Plotting the reference point $A(-20, -40)$, the radius CA is established and the circle is drawn as shown in Fig. 9–24*b*. The radius is

$$R = \sqrt{(20 - 10)^2 + (40)^2} = 41.2 \text{ psi}$$

The principal stresses are at the points where the circle intersects the σ axis; i.e.,

$$\sigma_1 = -10 + 41.2 = 31.2 \text{ psi}$$

$$\sigma_2 = -10 - 41.2 = -51.2 \text{ psi}$$

From the circle, the *counterclockwise* angle 2θ, measured from CA to the $-\sigma$ axis, is

$$2\theta = \tan^{-1}\left(\frac{40}{20 - 10}\right) = 76.0^\circ$$

Thus,

$$\theta = 38.0^\circ$$

This *counterclockwise* rotation defines the direction of the x' axis and σ_2, Fig. 9–24*c*. We have

$$\sigma_1 = 31.2 \text{ psi} \quad \sigma_2 = -51.2 \text{ psi} \qquad \textit{Ans.}$$

Absolute Maximum Shear Stress. Since these stresses have opposite signs, applying Eq. 9–14 we have

$$\tau_{\substack{\text{abs}\\ \text{max}}} = \frac{\sigma_1 - \sigma_2}{2} = \frac{31.2 - (-51.2)}{2} = 41.2 \text{ psi} \qquad \textit{Ans.}$$

$$\sigma_{avg} = \frac{31.2 - 51.2}{2} = -10 \text{ psi}$$

These same results can also be obtained by drawing Mohr's circle for each orientation of an element about the x, y, and z axes, Fig. 9–24*d*. Since σ_1 and σ_2 are of *opposite signs*, then the absolute maximum shear stress as noted equals the maximum in-plane shear stress.

9-4 General state of stress at a point

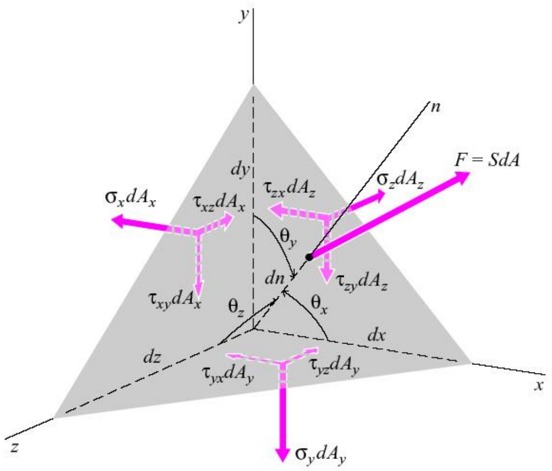


Figure 2-44

Consider the equilibrium of a small element for an arbitrary point O, as shown in Fig. 2-44, which has an oblique surface with outward normal $\boldsymbol{n}$ besides three orthogonal planes with normals in x-, y- and z-direction, respectively. The direction cosine of the oblique surface is denoted by $l = \cos\theta_x$, $m = \cos\theta_y$, and $n = \cos\theta_z$, respectively. Moreover, the relations of surface area are $dA_x = ldA$, $dA_y = mdA$, $dA_z = ndA$. The equilibrium of forces along x-, y-, and z-axis are, respectively,

$$\sum F_x = 0 \Rightarrow S_x dA - \sigma_x ldA - \tau_{yx} mdA - \tau_{zx} ndA = 0$$

$$\sum F_y = 0 \Rightarrow S_y dA - \sigma_y mdA - \tau_{xy} ldA - \tau_{zy} ndA = 0$$

$$\sum F_z = 0 \Rightarrow S_z dA - \sigma_z ndA - \tau_{xz} ldA - \tau_{yz} mdA = 0$$

or $S_x = \sigma_x l + \tau_{yx} m + \tau_{zx} n$

$$S_y = \tau_{xy} l + \sigma_y m + \tau_{zy} n \qquad (9\text{-}10)$$

$$S_z = \tau_{xz} l + +\tau_{yz} m + \sigma_z n$$

Since $\boldsymbol{S} \cdot \boldsymbol{n} = \sigma_{\boldsymbol{n}} \Rightarrow S_x l + S_y m + S_z n = \sigma_n$, where $\boldsymbol{n} = (l, m, n)$, $l^2 + m^2 + n^2 = 1$

Therefore, $\sigma_n = \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2\tau_{xy} lm + 2\tau_{yz} mn + 2\tau_{zx} nl \qquad (9\text{-}11)$

The corresponding shearing stress τ_{nt} satisfies the relation:

$$S^2 = \sigma_n^2 + \tau_{nt}^2$$

If the oblique plane is just the principal plane, i.e., $\tau_{nt} = 0$, and $S\,\boldsymbol{n} = \sigma_p\,\boldsymbol{n}$,

or $S_x = \sigma_p l$, $S_y = \sigma_p m$, $S_z = \sigma_p n$

The equations (9-10) can be rewritten as:

$$(\sigma_x - \sigma_p)l + \tau_{yx}m + \tau_{zx}n = 0\,,$$

$$\tau_{xy}l + (\sigma_y - \sigma_p)m + \tau_{zy}n = 0\,, \tag{9-12}$$

$$\tau_{xz}l + \tau_{yz}m + (\sigma_z - \sigma_p)n = 0\,;$$

or $$\begin{pmatrix} \sigma_x - \sigma_p & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \sigma_p & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_p \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = 0\,. \tag{9-13}$$

This set of equations has a nontrivial solution only if the determinant of the coefficients of *l*, *m*, and *n* is equal to zero. Thus,

$$\begin{vmatrix} \sigma_x - \sigma_p & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \sigma_p & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_p \end{vmatrix} = 0\,. \tag{9-14}$$

Expansion of the determinant yields the following cubic equation for determining the principal stresses:

$$\sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0\,, \tag{9-15}$$

where $I_1 = \sigma_x + \sigma_y + \sigma_z$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = \begin{vmatrix} \sigma_x & \tau_{yx} \\ \tau_{xy} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{zx} \\ \tau_{xz} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{zy} \\ \tau_{yz} & \sigma_z \end{vmatrix}$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{vmatrix}$$

The method to determine the principal stresses and their corresponding directions are called the problem of *eigenvalues* and *eigenvectors*. The principal stresses and principle directions are *eigenvalues* and *eigenvectors*, respectively. Since they are independent of the coordinate *x-y-z* selected, the coefficients of cubic equation I_1, I_2, I_3 and are also independent of the coordinate, they are called the first, the second, and the third *stress invariants*, respectively. After obtain the eigenvalue σ_{pi}, i = 1, 2, 3, the corresponding unknown eigenfunction (l_i, m_i, n_i) can be evaluated by using

$$\begin{pmatrix} \sigma_x - \sigma_{pi} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \sigma_{pi} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_{pi} \end{pmatrix} \begin{pmatrix} l_i \\ m_i \\ n_i \end{pmatrix} = 0 \tag{9-16}$$

Chapter 2 Strain: Definitions and Concepts

2-1 Introduction

In the design of structural elements or machine components, the deformations experienced by the body, as a result of the applied loads, often represent as important a design consideration as the stresses. Therefore, the nature of the deformations experienced by a real deformable body as a result of internal force or stress distributions will be studied, and methods to measure or compute deformations will be established.

2-2 ***Displacement, deformation, and strain***

2-2-1 ***Displacement***

When a system of loads is applied to a machine component or structural element, individual points of the body generally move. The movement of a point with respect to some reference frame can be represented by a vector quantity known as a ***displacement***. **The displacement of a point may be composed of rigid body translation, rigid body rotation, and *deformation*. The latter will cause the size and/or the shape of a body to be altered**, individual points of the body move relative to one another.

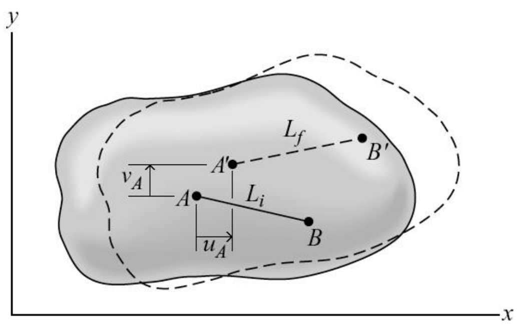


Figure 3-1

2-2-2 ***Deformation***

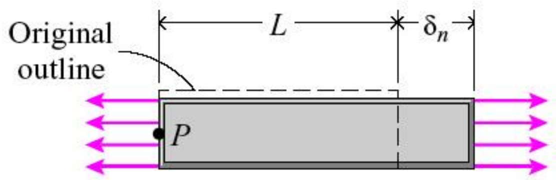
Deformation may be related to force or stress or to a change in temperature. Generally, it leads to the size and/or the shape of a body to be altered.

2-2-3 ***Strain***

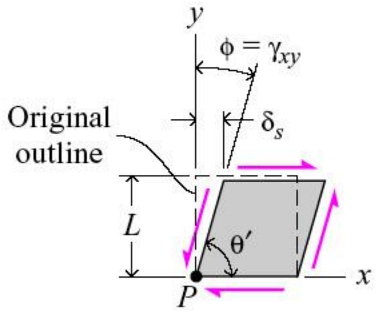
Strain is the quantity used to measure the intensity of a deformation (deformation per unit length) just as stress is used to measure the intensity of an internal force (force per unit area). Similar to stresses, two kinds of strains can also be classified:

- ***Normal strain* (ε):** measures the change in size (elongation or contraction of an arbitrary line segment) of a body during deformation.
- ***Shearing strain* (γ):** measures the change in shape (change in angle between two lines that are orthogonal in the undeformed state) of a body during deformation.

The deformation or strain may be the results of a stress, of a change in temperature, or of other physical processes such as grain growth and film growth.



(a)



(b)

2-2-4 ***Average axial strain***

The change in length of a simple bar under an axial load can be illustrated by a normal strain. If the length and axial deformation of the bar are L and δ_n, respectively,

Then the average axial strain can be expressed by

$$\varepsilon_{avg} = \frac{\delta_n}{L} \qquad (2\text{-}1)$$

2-2-5 ***Axial strain at a point***

If the deformation is nonuniform along the length, the average axial strain may be significantly different from the axial strain at an arbitrary point P along the bar. It is better to determine the strain by making the length over which the axial deformation along x axis is measured smaller and smaller, i.e.,

$$\varepsilon_x(P) = \lim_{\Delta L \to 0} \frac{\Delta \delta_n}{\Delta L} = \frac{d\delta_n}{dL} \qquad (2\text{-}2)$$

2-2-6 ***Shearing strain***

A deformation involving a change in shape can be illustrated by a shearing strain. If the length of angle and the deformation in a direction normal to the length are L and δ_S, respectively, then

$$\gamma_{avg} = \frac{\delta_S}{L} = \tan\phi \qquad (2\text{-}3)$$

Since δ_S / L is usually very small (typically $\delta_S / L < 0.001$), $\sin\phi \cong \tan\phi \cong \phi$,

where the angle ϕ is measured in radians. Therefore, $\gamma_{avg} = \phi = \delta_S / L$ is the decrease in the angle

between two reference lines x and y that are orthogonal in the undeformed state. The shearing strain at a point can be defined similarly as:

$$\gamma_{xy}(P) = \lim_{\Delta L \to 0} \frac{\Delta\delta_S}{\Delta L} = \frac{d\delta_S}{dL} = \phi = \frac{\pi}{2} - \theta' , \quad (2\text{-}4)$$

where θ' is the angle in the deformed state between two initially orthogonal reference lines.

2-2-7 ***Units of strains***

It is obvious that both normal and shearing strains are dimensionless quantities; however, normal strains are frequently expressed in units of inch per inch (in./in) or micro-inch per inch (μin./in.), while shearing stresses are expressed in radians or microradians. The micro- (10^{-6}) is denoted by μ.

2-2-8 ***Sign convention for strains***

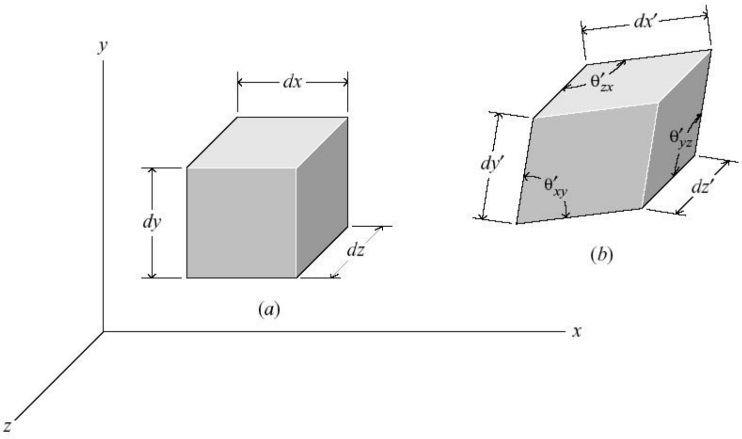
- **Normal strains**: **positive for elongation (tensile strains) and negative for contraction (compressive strains).**
- **Shearing strains**: **positive as the angle between reference lines decreases and negative as the angle increases.**

[Remarks]: For most of engineering materials in the elastic range, neither normal strains nor shearing strains seldom exceed values of 0.2%.

2-3 ***The state of strain at a point***

In many practical engineering problems involving the design of structural or machine elements, it is difficult to determine the distributions of stress solely by mathematical analysis; therefore, theoretical analysis supplemented by experimental measurement is generally required. Strains can be measured by several methods; however, except for the simplest cases with uniform distribution of stress, the stresses cannot be obtained directly. Therefore, the usual procedure is to measure the strains and calculate the state of stresses by using the stress-strain equations.

As shown in Fig. 3-6, the complete state of strain at an arbitrary point P in a body under load can be determined by considering the deformation associated with a small volume of material surrounding the point. This small volume can be assumed to have the shape of a rectangular parallelepiped with its faces oriented perpendicular to the reference axes *x*, *y*, and *z* in the undeformed state. Since the element of volume is very small, deformations are assumed to be uniform; therefore, it is reasonable that parallel planes remain plane and paralleling, and straight lines remain straight lines after deformation. **The strain is also a tensor quantity which has 6 independent components, just similar to stress tensor** and can be expressed in terms of the deformations:



$$\varepsilon_x = \frac{dx' - dx}{dx} = \frac{d\delta_x}{dx} = \frac{\partial u}{\partial x}, \Rightarrow dx' = (1+\varepsilon_x)dx$$

$$\varepsilon_y = \frac{dy' - dy}{dy} = \frac{d\delta_y}{dy} = \frac{\partial v}{\partial x}, \Rightarrow dy' = (1+\varepsilon_y)dy$$

$$\varepsilon_z = \frac{dz' - dz}{dz} = \frac{d\delta_z}{dz} = \frac{\partial w}{\partial z}, \Rightarrow dz' = (1+\varepsilon_z)dz \qquad (2\text{-}5)$$

$$\gamma_{xy} = \gamma_{yx} = \frac{\pi}{2} - \theta_{xy}^{'} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y},$$

$$\gamma_{yz} = \gamma_{zy} = \frac{\pi}{2} - \theta_{yz}^{'} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z},$$

$$\gamma_{zx} = \gamma_{xz} = \frac{\pi}{2} - \theta_{zx}^{'} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x},$$

where $d\delta_x$, $d\delta_y$ and $d\delta_z$ denote the change of length dx, dy and dz after deformation, respectively; while θ_{xy}', θ_{yz}' and θ_{xz}' denote the distorted angles of right angles on the plane xy, yz and xz, respectively; u, v and w denote the displacements of an arbitrary point along axes x, y and z, respectively.

Similarly, for two arbitrary orthogonal lines oriented in the n and t directions in the undeformed element, the corresponding strains are given by

$$\varepsilon_n = \frac{dn' - dn}{dn} = \frac{d\delta_n}{dx}, \quad \gamma_{nt} = \gamma_{tn} = \frac{\pi}{2} - \theta_{nt}^{'} \qquad (2\text{-}6)$$

Chap. 10 Strain Transformation

10-1 *The strain transformation equations for plane strain*

Consider the case of *x*-*y* plane strain, i.e., $\varepsilon_z = \gamma_{zx} = \gamma_{zy} = 0$

10-1-1 *Normal strain* ε_n

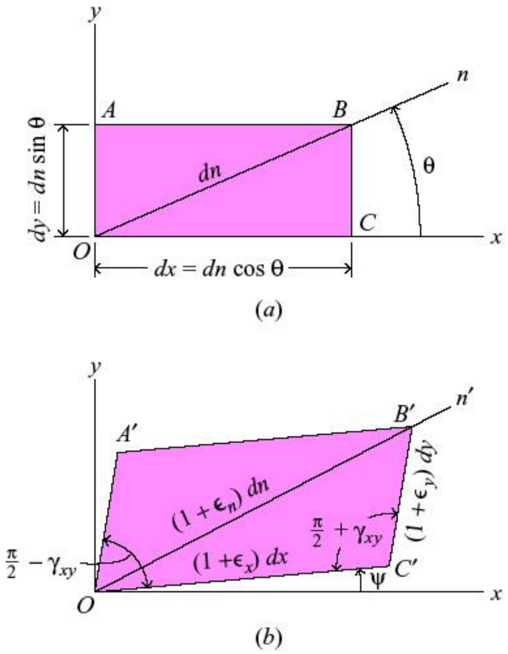


Figure 3-7

As shown in Fig. 3-7, the shaded rectangular (a) represents a small unstrained element of material, and the parallelepiped denotes the deformed element. The relations of strain components corresponding to the *x*-*y* and *n*-*t* coordinates will be derived. Consider the triangle *OC'B'*

$$(OB')^2 = (OC')^2 + (CB')^2 - 2(OC')(CB')\cos\left(\frac{\pi}{2} + \gamma_{xy}\right)$$

$$\Rightarrow [(1+\varepsilon_n)dn]^2 = [(1+\varepsilon_x)dx]^2 + [(1+\varepsilon_y)dy]^2$$

$$-2[(1+\varepsilon_x)dx][(1+\varepsilon_y)dy](-\sin\gamma_{xy})$$

By using the relations $dx = dn\cos\theta$ and $dy = dn\sin\theta$, and neglect the second order terms, such as ε^2 and $\varepsilon\gamma$ due to very small strains, as well as the approximation

$\sin\gamma \cong \gamma$

$$\Rightarrow \varepsilon_n = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \qquad (10\text{-}1)$$

10-1-2 *Shearing strain* γ_{nt}

As the material deforms, the *n*-direction rotates counterclockwise through an angle ϕ_n, as shown in Fig. 3-8. For the triangle *OC'B'*

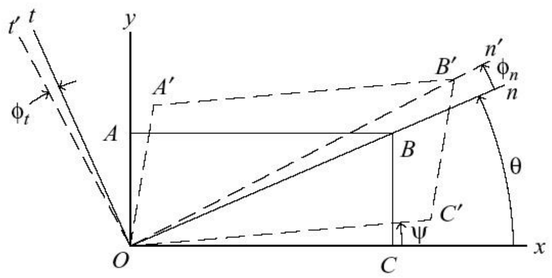


Figure 3-8

$B'C'\sin\angle OC'B' = OB'\sin\angle B'OC'$,

or in terms of the strains

$$(1+\varepsilon_y)dy\sin\left(\frac{\pi}{2}+\gamma_{xy}\right) = (1+\varepsilon_n)dn\sin[\theta+(\phi_n-\psi)]$$

$$\Rightarrow \phi_n = -(\varepsilon_x-\varepsilon_y)\sin\theta\cos\theta - \gamma_{xy}\sin^2\theta + \psi$$

$$\Rightarrow \phi_t = \phi_n\left(\theta+\frac{\pi}{2}\right) = -(\varepsilon_x-\varepsilon_y)\sin\left(\theta+\frac{\pi}{2}\right)\cos\left(\theta+\frac{\pi}{2}\right) - \gamma_{xy}\sin^2\left(\theta+\frac{\pi}{2}\right)+\psi$$

$$= (\varepsilon_x-\varepsilon_y)\sin\theta\cos\theta - \gamma_{xy}\cos^2\theta + \psi$$

$$\gamma_{nt} = \phi_n - \phi_t = -2(\varepsilon_x-\varepsilon_y)\sin\theta\cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\Rightarrow \frac{\gamma_{nt}}{2} = -\frac{(\varepsilon_x-\varepsilon_y)}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta \qquad (10\text{-}2)$$

It can be found that if the shearing strains γ_{ij} are replaced by ε_{ij} mathematically, ($\gamma_{ij}/2 = \varepsilon_{ij}$), then the relations of transformation for strains (3-8*a*) and (3-8*b*) are completely identical to stresses. In other words, the Mohr's cycle of strains is the same as the stresses.

10-2 ***Principal strains and maximum shear strain***

For case of plane strain, the in-plane principal directions, in-plane principal strains, and the maximum in-plane shear strain are

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x-\varepsilon_y}, \qquad (10\text{-}3)$$

$$\varepsilon_{p1},\varepsilon_{p2} = \frac{\varepsilon_x+\varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x-\varepsilon_y}{2}\right)+\left(\frac{\gamma_{xy}}{2}\right)^2}, \qquad (10\text{-}4)$$

$$\gamma_p = 2\sqrt{\left(\frac{\varepsilon_x-\varepsilon_y}{2}\right)^2+\left(\frac{\gamma_{xy}}{2}\right)^2}. \qquad (10\text{-}5)$$

10-3 *Mohr's circle for plane strain*

The equation of Mohr's circle for strain and its radius are

$$\left(\varepsilon_n - \frac{\varepsilon_x + \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{nt}}{2}\right)^2 = \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2 = R^2 \qquad (10\text{-}6)$$

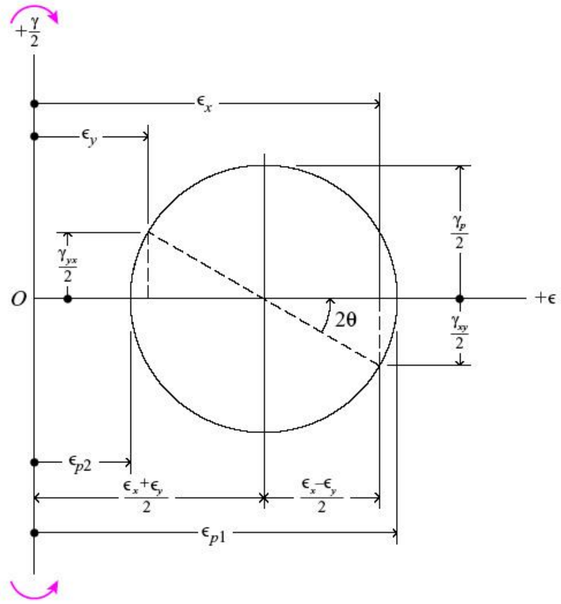


Figure 3-13

EXAMPLE 2.1

Determine the average normal strains in the two wires in Fig. 2–5 if the ring at A moves to A'.

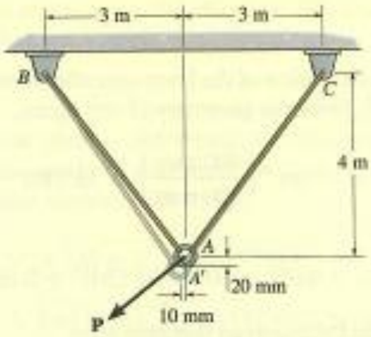


Fig. 2–5

SOLUTION

Geometry. The original length of each wire is

$$L_{AB} = L_{AC} = \sqrt{(3\text{ m})^2 + (4\text{ m})^2} = 5\text{ m}$$

The final lengths are

$$L_{A'B} = \sqrt{(3\text{ m} - 0.01\text{ m})^2 + (4\text{ m} + 0.02\text{ m})^2} = 5.01004\text{ m}$$

$$L_{A'C} = \sqrt{(3\text{ m} + 0.01\text{ m})^2 + (4\text{ m} + 0.02\text{ m})^2} = 5.02200\text{ m}$$

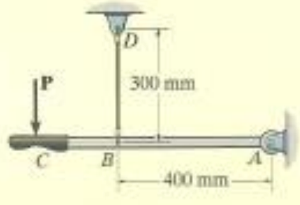
Average Normal Strain.

$$\epsilon_{AB} = \frac{L_{A'B} - L_{AB}}{L_{AB}} = \frac{5.01004\text{ m} - 5\text{ m}}{5\text{ m}} = 2.01(10^{-3})\text{ m/m} \qquad \textit{Ans.}$$

$$\epsilon_{AC} = \frac{L_{A'C} - L_{AC}}{L_{AC}} = \frac{5.02200\text{ m} - 5\text{ m}}{5\text{ m}} = 4.40(10^{-3})\text{ m/m} \qquad \textit{Ans.}$$

EXAMPLE 2.2

When force **P** is applied to the rigid lever arm ABC in Fig. 2–6a, the arm rotates counterclockwise about pin A through an angle of 0.05°. Determine the normal strain in wire BD.



(a)

SOLUTION I

Geometry. The orientation of the lever arm after it rotates about point A is shown in Fig. 2–6b. From the geometry of this figure,

$$\alpha = \tan^{-1}\left(\frac{400\text{ mm}}{300\text{ mm}}\right) = 53.1301°$$

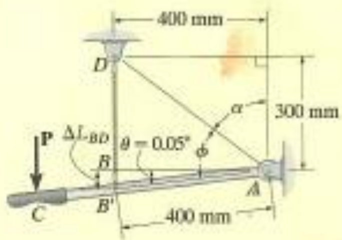
Then

$$\phi = 90° - \alpha + 0.05° = 90° - 53.1301° + 0.05° = 36.92°$$

For triangle ABD the Pythagorean theorem gives

$$L_{AD} = \sqrt{(300\text{ mm})^2 + (400\text{ mm})^2} = 500\text{ mm}$$

Using this result and applying the law of cosines to triangle $AB'D$,



(b)

Fig. 2–6

$$\begin{aligned} L_{B'D} &= \sqrt{L_{AD}^2 + L_{AB'}^2 - 2(L_{AD})(L_{AB'})\cos\phi} \\ &= \sqrt{(500\text{ mm})^2 + (400\text{ mm})^2 - 2(500\text{ mm})(400\text{ mm})\cos 36.92°} \\ &= 300.3491\text{ mm} \end{aligned}$$

Normal Strain.

$$\epsilon_{BD} = \frac{L_{B'D} - L_{BD}}{L_{BD}}$$

$$= \frac{300.3491\text{ mm} - 300\text{ mm}}{300\text{ mm}} = 0.00116\text{ mm/mm} \qquad \textit{Ans.}$$

SOLUTION II

Since the strain is small, this same result can be obtained by approximating the elongation of wire BD as ΔL_{BD}, shown in Fig. 2–6b. Here,

$$\Delta L_{BD} = \theta L_{AB} = \left[\left(\frac{0.05°}{180°}\right)(\pi\text{ rad})\right](400\text{ mm}) = 0.3491\text{ mm}$$

Therefore,

$$\epsilon_{BD} = \frac{\Delta L_{BD}}{L_{BD}} = \frac{0.3491\text{ mm}}{300\text{ mm}} = 0.00116\text{ mm/mm} \qquad \textit{Ans.}$$

EXAMPLE 2.3

The plate shown in Fig. 2–7*a* is fixed connected along *AB* and held in the horizontal guides at its top and bottom, *AD* and *BC*. If its right side *CD* is given a uniform horizontal displacement of 2 mm, determine (a) the average normal strain along the diagonal *AC*, and (b) the shear strain at *E* relative to the *x*, *y* axes.

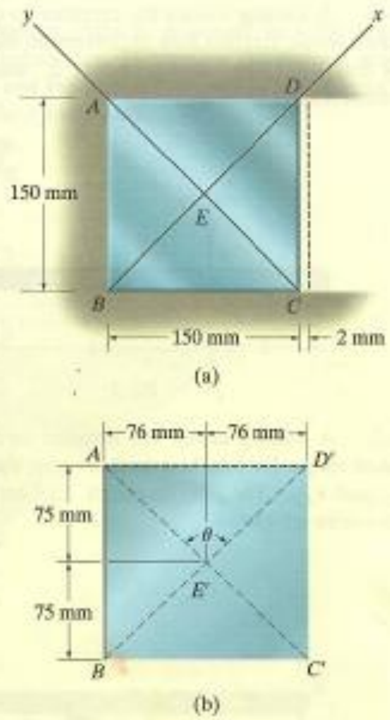


Fig. 2–7

SOLUTION

Part (a). When the plate is deformed, the diagonal *AC* becomes *AC'*, Fig. 2–7*b*. The lengths of diagonals *AC* and *AC'* can be found from the Pythagorean theorem. We have

$$AC = \sqrt{(0.150\text{ m})^2 + (0.150\text{ m})^2} = 0.21213\text{ m}$$

$$AC' = \sqrt{(0.150\text{ m})^2 + (0.152\text{ m})^2} = 0.21355\text{ m}$$

Therefore the average normal strain along *AC* is

$$(\epsilon_{AC})_{\text{avg}} = \frac{AC' - AC}{AC} = \frac{0.21355\text{ m} - 0.21213\text{ m}}{0.21213\text{ m}}$$

$$= 0.00669\text{ mm/mm} \qquad \textit{Ans.}$$

Part (b). To find the shear strain at *E* relative to the *x* and *y* axes, which are 90° apart, it is necessary to find the change in the angle at *E*. After deformation, Fig. 2–7*b*,

$$\tan\left(\frac{\theta}{2}\right) = \frac{76\text{ mm}}{75\text{ mm}}$$

$$\theta = 90.759^\circ = \left(\frac{\pi}{180^\circ}\right)(90.759^\circ) = 1.58404\text{ rad}$$

Applying Eq. 2–3, the shear strain at *E* is therefore the change in the angle *AED*,

$$\gamma_{xy} = \frac{\pi}{2} - 1.58404\text{ rad} = -0.0132\text{ rad} \qquad \textit{Ans.}$$

The *negative sign* indicates that the once 90° angle becomes larger.

NOTE: If the *x* and *y* axes were horizontal and vertical at point *E*, then the 90° angle between these axes would not change due to the deformation, and so $\gamma_{xy} = 0$ at point *E*.

*10.4 ABSOLUTE MAXIMUM SHEAR STRAIN

In Sec. 9.5 it was pointed out that in the case of plane stress, the abso… maximum shear stress in an element of material will occur *out of the pl…* when the principal stresses have the *same sign*, i.e., both are tensile or bo… are compressive. A similar result occurs for plane strain. For example, if the principal in-plane strains cause elongations, Fig. 10–13*a*, then the thr… Mohr's circles describing the normal and shear strain components for the element rotations about the *x*, *y*, and *z* axes are shown in Fig. 10–13*b*. By inspection, the largest circle has a radius $R = (\gamma_{xz})_{\max}/2$, and so

$$\boxed{\gamma_{\substack{\text{abs}\\ \max}} = (\gamma_{xz})_{\max} = \epsilon_1} \qquad (10\text{–}14)$$

ϵ_1 and ϵ_2 have the same sign

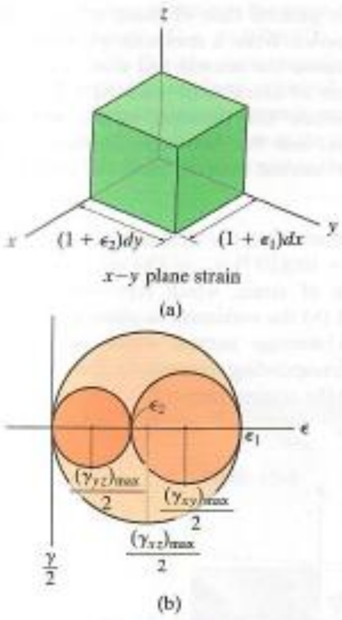


Fig. 10–13

This value gives the *absolute maximum shear strain* for the materi… Note that it is *larger* than the maximum in-plane shear strain, which is $(\gamma_{xy})_{\max} = \epsilon_1 - \epsilon_2$.

Now consider the case where one of the in-plane principal strains is of *opposite sign* to the other in-plane principal strain, so that ϵ_1 causes elongation and ϵ_2 causes contraction, Fig. 10–14*a*. The three Mohr's circles, which describe the strain components on the element rotat… about the *x*, *y*, *z* axes, are shown in Fig. 10–14*b*. Here

$$\boxed{\gamma_{\substack{\text{abs}\\ \max}} = (\gamma_{xy})_{\substack{\text{in-plane}\\ \max}} = \epsilon_1 - \epsilon_2} \qquad (10\text{–}15)$$

ϵ_1 and ϵ_2 have opposite signs

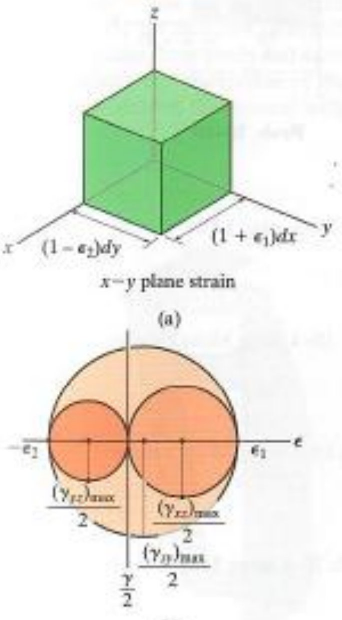


Fig. 10–14

IMPORTANT POINTS

- If the in-plane principal strains both have the same sign, the absolute maximum shear strain will occur out of plane and has a value of $\gamma_{\substack{\text{abs}\\ \max}} = \epsilon_{\max}$. This value is greater than the maximum in-plane shear strain.
- If the in-plane principal strains are of opposite signs, then the absolute maximum shear strain equals the maximum in-plane shear strain, $\gamma_{\substack{\text{abs}\\ \max}} = \epsilon_1 - \epsilon_2$.

EXAMPLE 10.7

The state of plane strain at a point has strain components of $\epsilon_x = -400(10^{-6})$, $\epsilon_y = 200(10^{-6})$, and $\gamma_{xy} = 150(10^{-6})$, Fig. 10–15*a*. Determine the maximum in-plane shear strain and the absolute maximum shear strain.

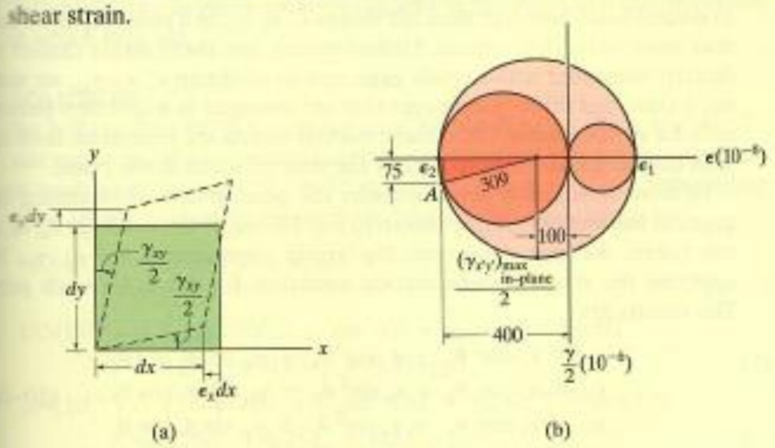


Fig. 10–15

SOLUTION

Maximum In-Plane Shear Strain. We will solve this problem using Mohr's circle. The center of the circle is at

$$\epsilon_{avg} = \frac{-400 + 200}{2}(10^{-6}) = -100(10^{-6})$$

Since $\gamma_{xy}/2 = 75(10^{-6})$, the reference point A has coordinates $(-400(10^{-6}), 75(10^{-6}))$, Fig. 10–15*b*. The radius of the circle is therefore

$$R = \left[\sqrt{(400 - 100)^2 + (75)^2}\right](10^{-6}) = 309(10^{-6})$$

From the circle, the in-plane principal strains are

$$\epsilon_1 = (-100 + 309)(10^{-6}) = 209(10^{-6})$$
$$\epsilon_2 = (-100 - 309)(10^{-6}) = -409(10^{-6})$$

Also, the maximum in-plane shear strain is

$$\gamma^{max}_{in\text{-}plane} = \epsilon_1 - \epsilon_2 = [209 - (-409)](10^{-6}) = 618(10^{-6}) \qquad \textit{Ans.}$$

Absolute Maximum Shear Strain. Since the *principal in-plane strains have opposite signs*, the maximum in-plane shear strain is *also* the absolute maximum shear strain; i.e.,

$$\gamma_{\substack{abs \\ max}} = 618(10^{-6}) \qquad \textit{Ans.}$$

The three Mohr's circles, plotted for element orientations about each of the x, y, z axes, are also shown in Fig. 10–15*b*.

EXAMPLE 10.4

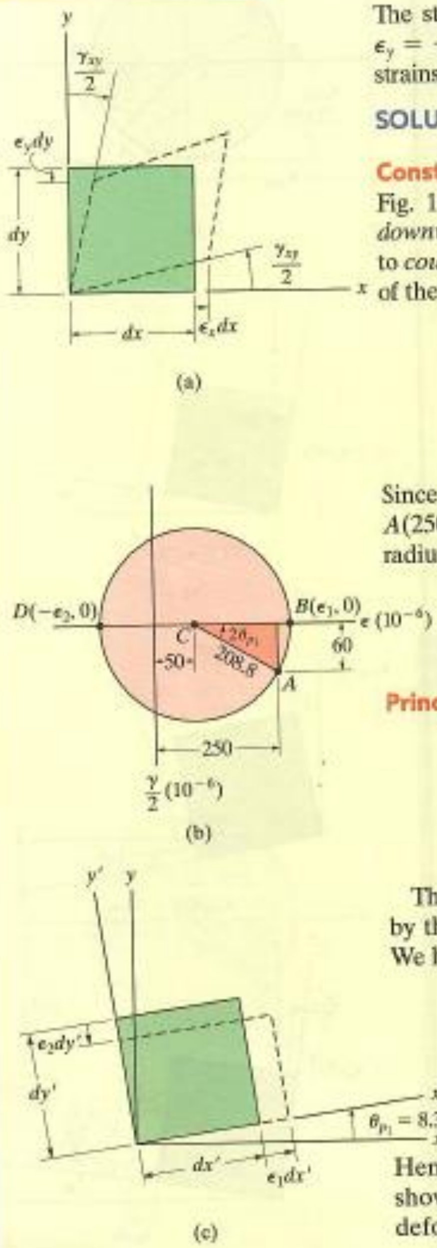


Fig. 10–10

The state of plane strain at a point has components of $\epsilon_x = 250(10^{-6})$, $\epsilon_y = -150(10^{-6})$, $\gamma_{xy} = 120(10^{-6})$, Fig. 10–10*a*. Determine the principal strains and the orientation of the element upon which they act.

SOLUTION

Construction of the Circle. The ϵ and $\gamma/2$ axes are established in Fig. 10–10*b*. Remember that the *positive* $\gamma/2$ axis must be directed *downward* so that *counterclockwise* rotations of the element correspond to *counterclockwise* rotation around the circle, and vice versa. The center of the circle C is located at

$$\epsilon_{avg} = \frac{250 + (-150)}{2}(10^{-6}) = 50(10^{-6})$$

Since $\gamma_{xy}/2 = 60(10^{-6})$, the reference point $A\ (\theta = 0°)$ has coordinates $A(250(10^{-6}), 60(10^{-6}))$. From the shaded triangle in Fig. 10–10*b*, the radius of the circle is

$$R = \left[\sqrt{(250-50)^2 + (60)^2}\,\right](10^{-6}) = 208.8(10^{-6})$$

Principal Strains. The ϵ coordinates of points B and D are therefore

$$\epsilon_1 = (50 + 208.8)(10^{-6}) = 259(10^{-6}) \qquad \textit{Ans.}$$

$$\epsilon_2 = (50 - 208.8)(10^{-6}) = -159(10^{-6}) \qquad \textit{Ans.}$$

The direction of the positive principal strain ϵ_1 in Fig. 10–10*b* is defined by the *counterclockwise* angle $2\theta_{p_1}$, measured from $CA\ (\theta = 0°)$ to CB. We have

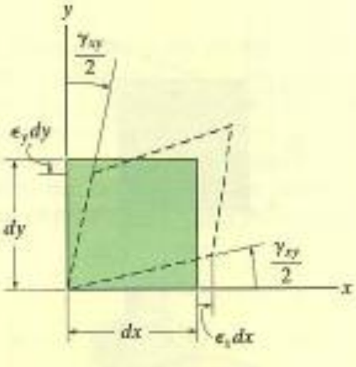
$$\tan 2\theta_{p_1} = \frac{60}{(250 - 50)}$$

$$\theta_{p_1} = 8.35° \qquad \textit{Ans.}$$

Hence, the side dx' of the element is inclined *counterclockwise* 8.35° as shown in Fig. 10–10*c*. This also defines the direction of ϵ_1. The deformation of the element is also shown in the figure.

EXAMPLE 10.5

The state of plane strain at a point has components of $\epsilon_x = 250(10^{-6})$, $\epsilon_y = -150(10^{-6})$, $\gamma_{xy} = 120(10^{-6})$, Fig. 10–11*a*. Determine the maximum in-plane shear strains and the orientation of the element upon which they act.



(a)

SOLUTION

The circle has been established in the previous example and is shown in Fig. 10–11*b*.

Maximum In-Plane Shear Strain. Half the maximum in-plane shear strain and average normal strain are represented by the coordinates of point *E* or *F* on the circle. From the coordinates of point *E*,

$$\frac{(\gamma_{x'y'})_{\substack{max \\ in\text{-}plane}}}{2} = 208.8(10^{-6})$$

$$(\gamma_{x'y'})_{\substack{max \\ in\text{-}plane}} = 418(10^{-6}) \qquad \textit{Ans.}$$

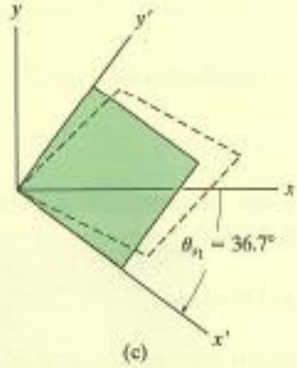
$$\epsilon_{avg} = 50(10^{-6})$$

To orient the element, we will determine the clockwise angle $2\theta_{s_1}$, measured from *CA* ($\theta = 0°$) to *CE*.

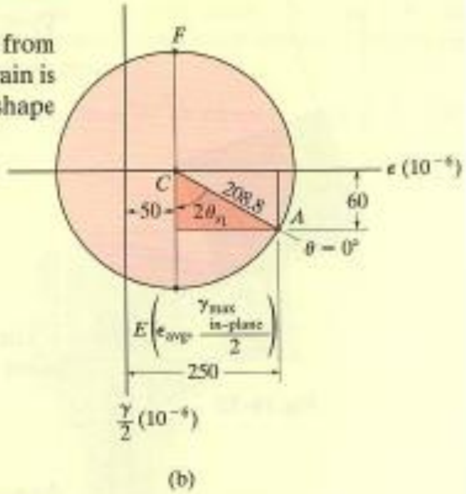
$$2\theta_{s_1} = 90° - 2(8.35°)$$

$$\theta_{s_1} = 36.7° \qquad \textit{Ans.}$$

This angle is shown in Fig. 10–11*c*. Since the shear strain defined from point *E* on the circle has a positive value and the average normal strain is also positive, these strains deform the element into the dashed shape shown in the figure.



(c)



(b)

Fig. 10–11

EXAMPLE 10.6

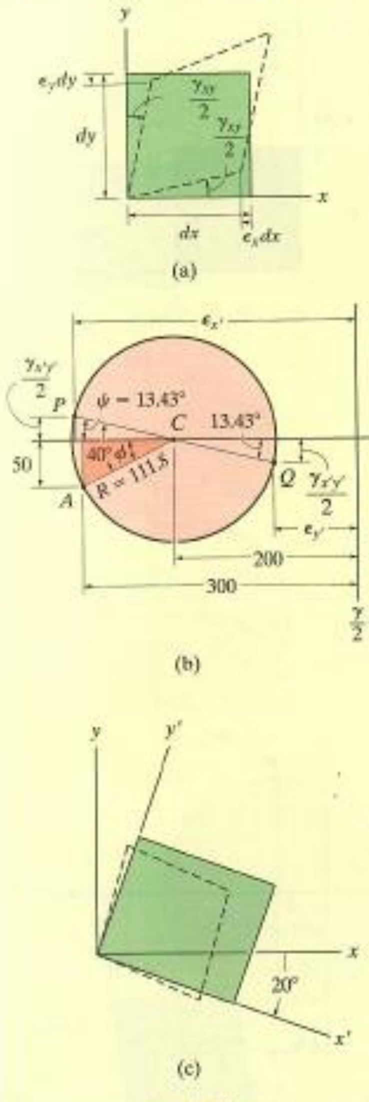


Fig. 10–12

The state of plane strain at a point has components of $\epsilon_x = -300(10^{-6})$, $\epsilon_y = -100(10^{-6})$, $\gamma_{xy} = 100(10^{-6})$, Fig. 10–12*a*. Determine the state of strain on an element oriented 20° clockwise from this position.

SOLUTION

Construction of the Circle. The ϵ and $\gamma/2$ axes are established in Fig. 10–12*b*. The center of the circle is at

$$\epsilon_{avg} = \left(\frac{-300 - 100}{2}\right)(10^{-6}) = -200(10^{-6})$$

The reference point A has coordinates $A(-300(10^{-6}), 50(10^{-6}))$, and so the radius CA, determined from the shaded triangle, is

$$R = \left[\sqrt{(300 - 200)^2 + (50)^2}\,\right](10^{-6}) = 111.8(10^{-6})$$

Strains on Inclined Element. Since the element is to be oriented 20° *clockwise*, we must consider the radial line CP, $2(20°) = 40°$ *clockwise*, measured from CA ($\theta = 0°$), Fig. 10–12*b*. The coordinates of point P are obtained from the geometry of the circle. Note that

$$\phi = \tan^{-1}\left(\frac{50}{(300 - 200)}\right) = 26.57°, \qquad \psi = 40° - 26.57° = 13.43°$$

Thus,

$$\epsilon_{x'} = -(200 + 111.8\cos 13.43°)(10^{-6}) = -309(10^{-6}) \qquad \textit{Ans.}$$

$$\frac{\gamma_{x'y'}}{2} = -(111.8\sin 13.43°)(10^{-6})$$

$$\gamma_{x'y'} = -52.0(10^{-6}) \qquad \textit{Ans.}$$

The normal strain $\epsilon_{y'}$ can be determined from the ϵ coordinate of point Q on the circle, Fig. 10–12*b*.

$$\epsilon_{y'} = -(200 - 111.8\cos 13.43°)(10^{-6}) = -91.3(10^{-6}) \qquad \textit{Ans.}$$

As a result of these strains, the element deforms relative to the x', y' axes as shown in Fig. 10–12*c*.

10-4 *Strain measurement and Rosette analysis*

In most experimental work involving strain measurement, the strains are measured on a free surface of a member where a state of plane stress exists. Electrical resistance strain gauges have been developed to provide accurate measurements of normal strain. Shear strains are difficult to measure directly than normal stresses. Generally, shear strains are obtained by measuring normal strains in two or three different directions.

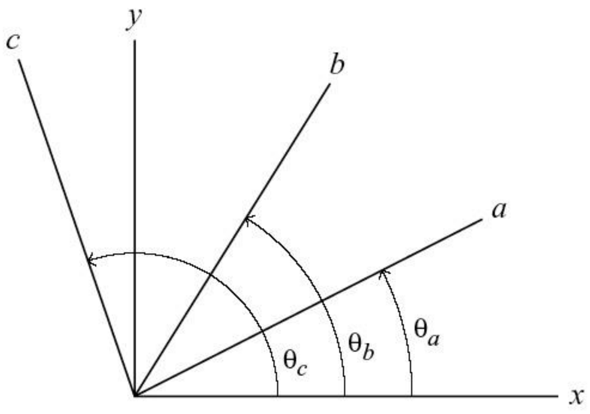


Figure 3-17

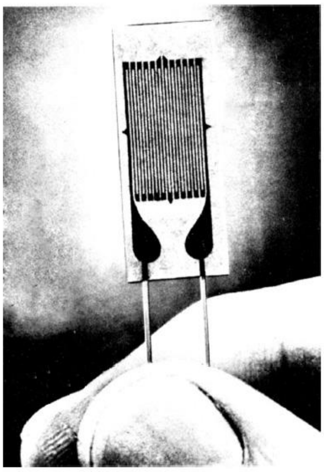


Figure 3-16

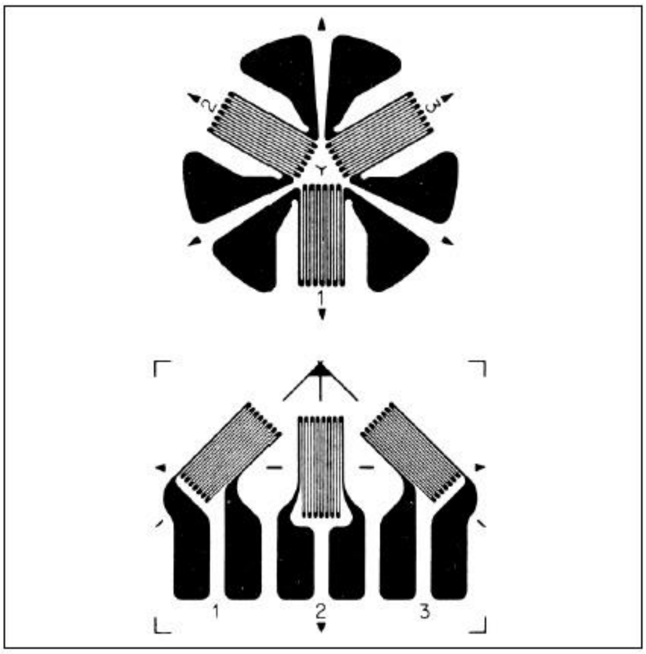


Figure 3-18

As shown in Fig. 3-17,

$$\varepsilon_a = \varepsilon_x \cos^2\theta_a + \varepsilon_y \sin^2\theta_a + \gamma_{xy}\sin\theta_a\cos\theta_a$$

$$\varepsilon_b = \varepsilon_x \cos^2\theta_b + \varepsilon_y \sin^2\theta_b + \gamma_{xy}\sin\theta_b\cos\theta_b$$

$$\varepsilon_c = \varepsilon_x \cos^2\theta_c + \varepsilon_y \sin^2\theta_c + \gamma_{xy}\sin\theta_c\cos\theta_c$$

After measuring three data of normal strains, three unknown strains ε_x, ε_y and γ_{xy} can be completely determined by three equations.

EXAMPLE 10.8

The state of strain at point A on the bracket in Fig. 10–17a is measured using the strain rosette shown in Fig. 10–17b. The readings from the gages give $\epsilon_a = 60(10^{-6})$, $\epsilon_b = 135(10^{-6})$, and $\epsilon_c = 264(10^{-6})$. Determine the in-plane principal strains at the point and the directions in which they act.

SOLUTION

We will use Eqs. 10–16 for the solution. Establishing an x axis, Fig. 10–17b, and measuring the angles counterclockwise from this axis to the centerlines of each gage, we have $\theta_a = 0°$, $\theta_b = 60°$, and $\theta_c = 120°$. Substituting these results, along with the problem data, into the equations gives

$$60(10^{-6}) = \epsilon_x \cos^2 0° + \epsilon_y \sin^2 0° + \gamma_{xy} \sin 0° \cos 0°$$
$$= \epsilon_x \qquad (1)$$
$$135(10^{-6}) = \epsilon_x \cos^2 60° + \epsilon_y \sin^2 60° + \gamma_{xy} \sin 60° \cos 60°$$
$$= 0.25\epsilon_x + 0.75\epsilon_y + 0.433\gamma_{xy} \qquad (2)$$
$$264(10^{-6}) = \epsilon_x \cos^2 120° + \epsilon_y \sin^2 120° + \gamma_{xy} \sin 120° \cos 120°$$
$$= 0.25\epsilon_x + 0.75\epsilon_y - 0.433\gamma_{xy} \qquad (3)$$

Using Eq. 1 and solving Eqs. 2 and 3 simultaneously, we get

$$\epsilon_x = 60(10^{-6}) \quad \epsilon_y = 246(10^{-6}) \quad \gamma_{xy} = -149(10^{-6})$$

These same results can also be obtained in a more direct manner from Eq. 10–17.

The in-plane principal strains will be determined using Mohr's circle. The center, C, is at $\epsilon_{avg} = 153(10^{-6})$, and the reference point on the circle is at $A[60(10^{-6}), -74.5(10^{-6})]$, Fig. 10–17$c$. From the shaded triangle, the radius is

$$R = \left[\sqrt{(153-60)^2 + (74.5)^2}\right](10^{-6}) = 119.1(10^{-6})$$

The in-plane principal strains are therefore

$$\epsilon_1 = 153(10^{-6}) + 119.1(10^{-6}) = 272(10^{-6}) \qquad \textit{Ans.}$$
$$\epsilon_2 = 153(10^{-6}) - 119.1(10^{-6}) = 33.9(10^{-6}) \qquad \textit{Ans.}$$
$$2\theta_{p_2} = \tan^{-1}\frac{74.5}{(153-60)} = 38.7°$$
$$\theta_{p_2} = 19.3° \qquad \textit{Ans.}$$

NOTE: The deformed element is shown in the dashed position in Fig. 10–17d. Realize that, due to the Poisson effect, the element is *also* subjected to an out-of-plane strain, i.e., in the z direction, although this value will not influence the calculated results.

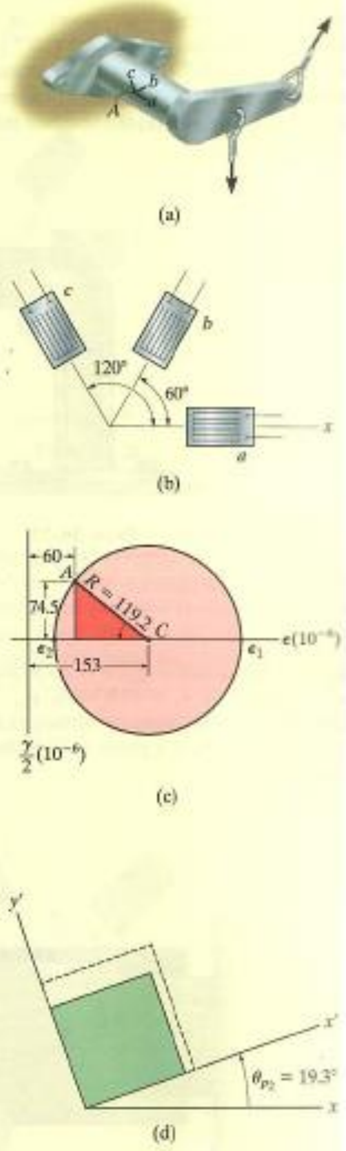


Fig. 10–17

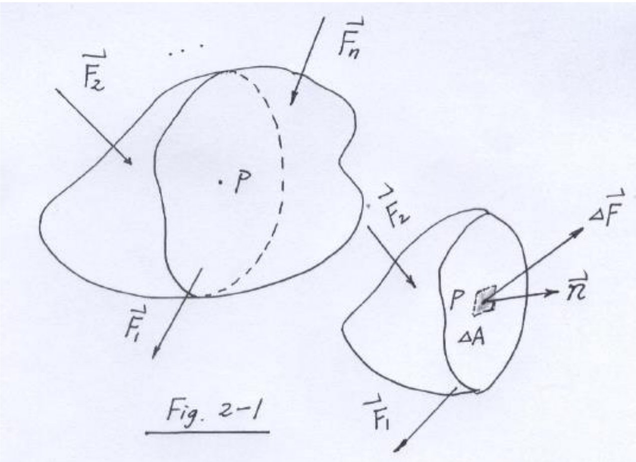
Appendix B: Basic issue of strain and stress (Theory of Elasticity)

2B-1 *Introduction*

Let us consider a deformable body subjected to the action of external loadings, the resulting deformations may be not uniform from point to point due to the non-uniform and multi-directional external forces and geometry of body. Therefore, it is, in general, necessary in elasticity to consider the overall behavior of the body from the properties of differentially small elements within the body by using three equilibrium equations, compatibility equations, and relations between forces and deformation described in the following sections.

2B-2 *Stresses*

Stress is simply an internal distributed force per unit area in a body and represents interaction between neighboring points under the action of external loadings. Consider an elastic body in equilibrium, subjected to the action of external forces $\boldsymbol{F_1}$, $\boldsymbol{F_2}$,…, $\boldsymbol{F_n}$, as shown in Fig. 2B-1. To determine the stress state at point P, it is necessary to expose a surface containing point P by decomposing the body to two portions by a plane passing through point P with normal unit vector $\boldsymbol{n}$. The external forces on the left-hand side portion of body, as well as the resultant forces acting on the cut surface shall be equilibrium. In the general case, the stress distribution will not uniform across the cut surface, and the stresses will be neither normal nor tangential to the surface at a given point. The stress distribution at a point, however, will have components in the normal and tangential directions giving rise to a normal stress (tensile or compressive) and a tangential stress (shear).



As shown in the Fig. 2B-1, let the resultant force acting on a small area ΔA around point P be $\Delta\boldsymbol{F}$.

We define the “stress vector (traction)” $\overset{n}{T}$ acting at point P an a plane with unit normal vector $\boldsymbol{n}$ as

$$\overset{n}{T} = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \qquad \text{(2B-1)}$$

If the cut plane intersecting point P is selected with unit vector $\boldsymbol{n}$ just parallel to the x axis of rectangular Cartesian frame of reference as shown in Fig. 2B-2. The stress vector through point P on this plane becomes

$$\overset{x}{T} = \lim_{\Delta A_x \to 0} \frac{\Delta F}{\Delta A_x} \qquad \text{(2B-2)}$$

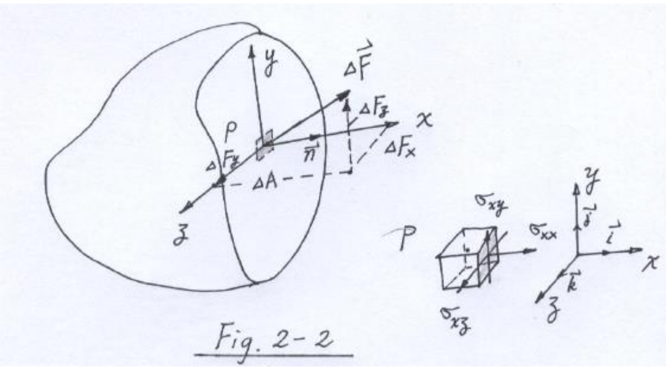


Fig. 2B-2

Since the resultant force acting on point P has three components in the x, y, and z directions, which are

ΔF_x, ΔF_y, and ΔF_z, respectively; i.e.,

$$\Delta F = \Delta F_x i + \Delta F_y j + \Delta F_z k \qquad \text{(2B-3)}$$

where ***i***, ***j***, and ***k*** are the unit vectors in x, y, and z directions, respectively.
By inserting Eqs. (2B-3) into (2B-2), we obtain

$$\overset{x}{T} = \lim_{\Delta A_x \to 0}\left(\frac{\Delta F_x}{\Delta A_x} i + \frac{\Delta F_y}{\Delta A_x} j + \frac{\Delta F_z}{\Delta A_x} k\right) = \overset{x}{T}_x i + \overset{x}{T}_y j + \overset{x}{T}_z k \qquad \text{(2B-4a)}$$

where $\overset{x}{T}_x = lim \dfrac{\Delta F_x}{\Delta A_x} \quad \Rightarrow \sigma_{xx}$ normal stress (2-4b)

$\overset{x}{T}_y = lim \dfrac{\Delta F_y}{\Delta A_x} \quad \Rightarrow \sigma_{xy}$ shear stress (2-4c)

$\overset{x}{T}_z = lim \dfrac{\Delta F_z}{\Delta A_x} \quad \Rightarrow \sigma_{xz}$ shear stress (2-4d)

Similarly, the cut plane intersecting point P can be selected with unit vector ***n*** just parallel to the y, and z axes, respectively. We then obtain the stress components as follows:

For the cut plane with normal along y axis:

$\overset{y}{T}_x = lim \dfrac{\Delta F_x}{\Delta A_y} \quad \Rightarrow \sigma_{yx}$ shear stress, (2B-4e)

$\overset{y}{T}_y = lim \dfrac{\Delta F_y}{\Delta A_y} \quad \Rightarrow \sigma_{yy}$ normal stress, (2B-4f)

$\overset{y}{T}_z = lim \dfrac{\Delta F_z}{\Delta A_y} \quad \Rightarrow \sigma_{yz}$ shear stress. (2B-4g)

For the cut plane with normal along z axis:

$\overset{z}{T}_x = lim \dfrac{\Delta F_x}{\Delta A_z} \quad \Rightarrow \sigma_{zx}$ shear stress , (2B-4h)

$\overset{z}{T}_y = lim \dfrac{\Delta F_y}{\Delta A_z} \quad \Rightarrow \sigma_{zy}$ shear stress , (2B-4i)

$\overset{z}{T}_z = lim \dfrac{\Delta F_z}{\Delta A_z} \quad \Rightarrow \sigma_{zz}$ normal stress , (2B-4j)

or symbolically expressed as

$$\overset{i}{T}_j = \lim_{\Delta A_i \to 0} \frac{\Delta F_j}{\Delta A_i} = \sigma_{ij}, \qquad i, j = x, y, z\,. \qquad \text{(2B-5)}$$

Therefore, the stress state at point P, expressed by totally 9 components as shown in Fig. 2B-3, can also be represented by a set of new notation

$$T^{i}_{j} = \sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \qquad (2B\text{-}6)$$

This is called the stress matrix or stress tensor. The first and the second subscript indices of stress tensor σ_{ij}, i.e, i and j, represent the normal direction of the plane on which stress acts and the direction of stress, respectively.

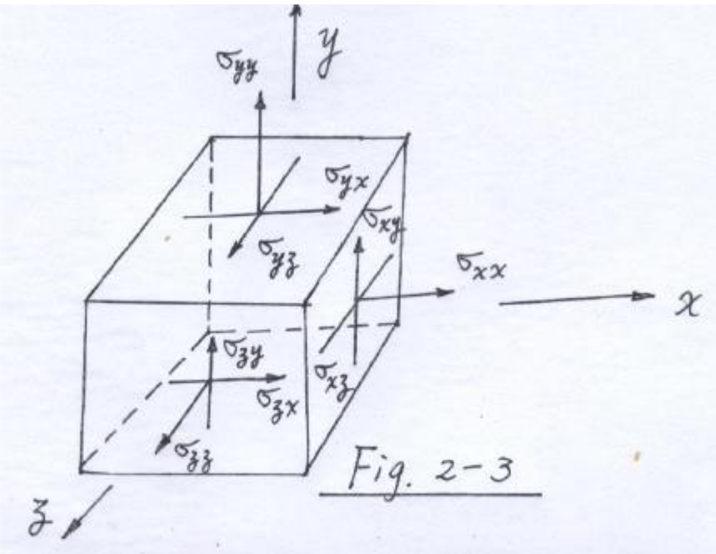


Fig. 2B-3

2B-3 ***Deformation and Strain in small deformation theory***

As a body is subjected to the action of external forces, it deforms such that an arbitrary point $P(x, y, z)$ in the body will undergo a **displacement** $\boldsymbol{u} = (u, v, w)$, and move to a new point $P'(x', y', z')$. They have the following relationship:

$$\begin{aligned} x' &= x + u \\ y' &= y + v \\ z' &= z + w \end{aligned} \qquad (2B\text{-}7)$$

In general, the *displacement u of a body* may be considered to be the summation of three items, i.e., *rigid-body displacement, rigid-body rotation, and deformation*. And only the deformation is related to the strains of the body, which may be classified into deformation due to size change (*dilatation* due to expansion or shrinkage of volume), and due to shape change (*distortion* of shape with no size change). Consider a continuous body, which undergoes a small geometrically compatible deformation (i.e., no voids or overlapping occur during deformation), an element of infinitesimal dimensions, Δx, Δy, and Δz, originating from point P_0 can be constructed where the initially undeformed infinitesimal rectangular element in the xy plane are indicated by $PBCD$, as shown in Fig. 2B-4.

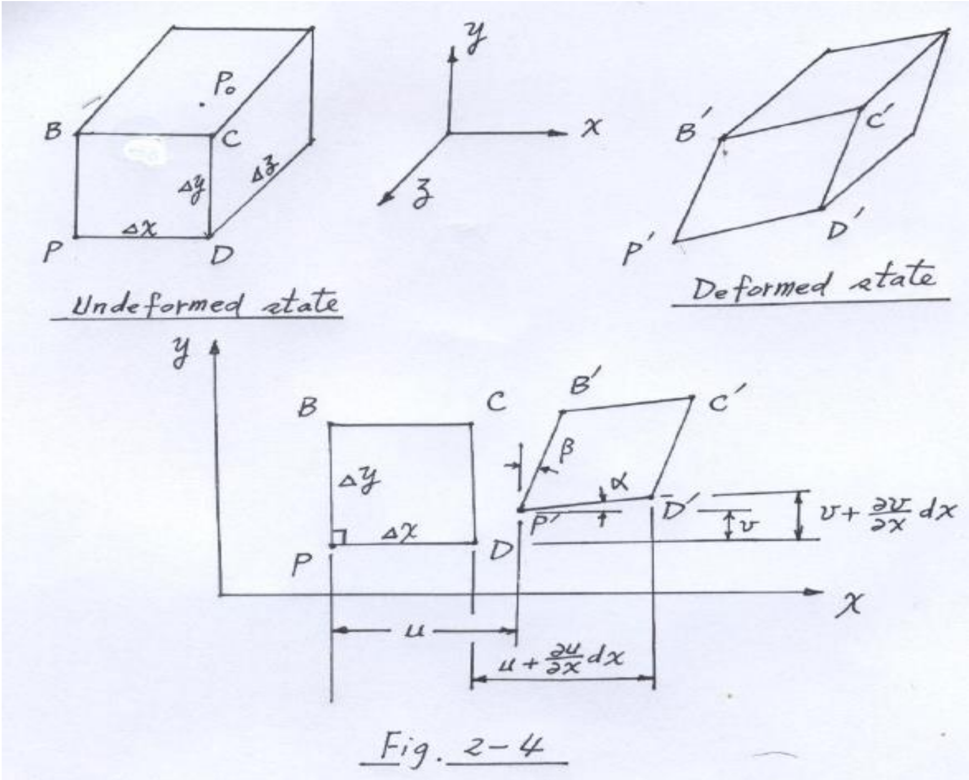


Fig. 2B-4

For a brief and clear description, let us focus the displacement of point P in the xy plane only. The displacement of point P can be described by continuous functions of x and y

$$u = u\,(x, y), \qquad v = v\,(x, y). \qquad \text{(2B-8)}$$

The functions can be expanded about point P in terms of a Taylor's series expansion. If u, $\partial u/\partial x$, $\partial^2 u/\partial x^2$, etc., are evaluated at point P, the displacement for point D, which is Δx from P, in the x direction will be

$$u_D = u + \frac{\partial u}{\partial x}\Delta x + \frac{1}{2!}\frac{\partial^2 u}{\partial x^2}(\Delta x)^2 + O((\Delta x)^3) \qquad \text{(2B-9)}$$

Likewise, if v, $\partial v/\partial x$, $\partial^2 v/\partial x^2$, etc., are evaluated for the point P, the displacement for point D in the y direction will be

$$v_D = v + \frac{\partial v}{\partial x}\Delta x + \frac{1}{2!}\frac{\partial^2 v}{\partial x^2}(\Delta x)^2 + O((\Delta x)^3) \qquad \text{(2B-10)}$$

If Δx is considered very small, it is satisfactory to neglect the terms higher than the first order. Thus

$$u_D = u + \frac{\partial u}{\partial x}\Delta x \quad \text{and} \quad v_D = v + \frac{\partial v}{\partial x}\Delta x \qquad \text{(2B-11)}$$

Similarly, if the displacements of point B, which is Δy from P, are also obtained from a Taylor's series expansion about point P, and Δy is considered very small, then

$$u_B = u + \frac{\partial u}{\partial y}\Delta y \quad \text{and} \quad v_B = v + \frac{\partial v}{\partial y}\Delta y \tag{2B-12}$$

Then the length of *P'D'* and *P'B'* can be written, respectively, as

$$P'D' = \left\{\left[(\Delta x + u_D) - u\right]^2 + (v_D - v)^2\right\}^{1/2} \cong \Delta x + (\partial u/\partial x)\Delta x \tag{2B-13a}$$

$$P'B' = \left\{\left[(\Delta y + v_B) - v\right]^2 + (u_B - u)^2\right\}^{1/2} \cong \Delta y + (\partial v/\partial y)\Delta y \tag{2B-13b}$$

The rate of change in elongation of PD and PB, defined as the *normal strain* ε_{xx} and ε_{yy}, respectively, is

$$\varepsilon_{xx} = \lim_{\Delta x \to 0} \frac{P'D' - PD}{PD} = \frac{\partial u}{\partial x} \tag{2B-14}$$

$$\varepsilon_{yy} = \lim_{\Delta y \to 0} \frac{P'B' - PB}{PB} = \frac{\partial v}{\partial y} \tag{2B-15}$$

As seen in Fig. 2B-4, the *reduction in angle DPB is defined as the shear strain* γ_{xy} in point P in the form

$$\gamma_{xy} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\angle BPD - \angle B'P'D'\right) = (\alpha + \beta) \tag{2B-16}$$

where $tan\alpha = \dfrac{(\partial v/\partial x)\Delta x}{\Delta x} = \dfrac{\partial v}{\partial x}$ and $tan\beta = \dfrac{(\partial u/\partial y)\Delta y}{\Delta y} = \dfrac{\partial u}{\partial y}$ (2B-17)

However, if the strains are very small, or the angles α and β are very small, then $tan\alpha \approx \alpha$ and $tan\beta \approx \beta$, the shear strain γ_{xy} can be represented by

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{yx} \tag{2-18a}$$

The different form of shear strain, ε_{xy}, defined as half of γ_{xy}, is generally used in mathematical description of elasticity and can be written as

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \varepsilon_{yx} \tag{2-18b}$$

The rigid-body rotation of a line segment at point P can be found from the average rotations of the line segments *PD* and *PB.* This can be accomplished by determining the rotation of the bisector of *PD* and *PB.*

The initial angle of the bisector of angle *BPD* relative to the *x* axis is π/4. The final angle that the bisector of angle *B'P'D'* makes with the *x* axis is:

$$\alpha+\frac{1}{2}\left[\frac{\pi}{2}-(\alpha+\beta)\right]=\frac{\pi}{4}+\frac{1}{2}(\alpha-\beta) \qquad \text{(2B-19)}$$

Then the rigid-body rotation about *z* axis, ω_{xy}, can be obtained by subtracting the initial angle of the bisector π/4 from the final angle as shown in Eq. (2B-19)

$$\omega_{xy}=\frac{1}{2}(\alpha-\beta)\cong\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=\omega_{yx} \qquad \text{(2B-20)}$$

Considering w to be the displacement of point P in the z direction and then performing a similar analysis in the *yz* and *zx* planes, it results in

$$\varepsilon_z=\frac{\partial w}{\partial z} \qquad \text{(2B-21)}$$

$$\gamma_{yz}=2\varepsilon_{yz}=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}=\gamma_{zy} \qquad \text{(2B-22)}$$

$$\gamma_{zx}=2\varepsilon_{zx}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=\gamma_{xz} \qquad \text{(2B-23)}$$

$$\omega_{yz}=\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)=-\omega_{zy} \qquad \text{(2B-24)}$$

$$\omega_{zx}=\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)=-\omega_{xz} \qquad \text{(2B-25)}$$

Therefore, the strain state at point *P*, expressed by totally 9 components can also be represented by a set of new notation in mathematical theory of elasticity

$$\varepsilon_{ij}=\begin{pmatrix}\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz}\\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz}\\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}\end{pmatrix} \qquad \text{(2B-26)}$$

This is called the *strain matrix or strain tensor*, in which *only 6 components are independent due to symmetry property of shear strain*. Moreover, the rigid-body rotation tensor is anti-symmetric. If we use the notation x_1, x_2, and x_3 to represent *x*, *y*, and *z*, respectively; and u_1, u_2, and u_3 to represent *u*, *v*, and *w*, respectively, then the relationship between strains and displacements becomes

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \qquad i, j = 1, 2, 3 \qquad \text{(2B-27)}$$

[Example]: If the displacement fields in a deformed body along *x*, *y*, and *z* directions are *u*, *v*, and *w*, respectively, which can be expressed by

$u(x, y, z) = a_1 + b_1x + c_1y + d_1 z$

$v(x, y, z) = a_2 - c_1x + c_2y + d_2 z$
$w(x, y, z) = a_3 + d_1x - d_2y + d_3 z$

where the parameters a_i, b_i, c_i, and d_i are all constants.
Please evaluate the strain tensor and illustrate the physical meaning of all the parameters a_i, b_i, c_i and d_i in the displacement functions.

2B-4 ***The equilibrium equations***

An elastic body under consideration is subjected to the action of external forces, which can be classified into *surface tractions and body forces (body moments)*. The surface tractions are the external forces acting on the surface of body in the unit of force per unit area; for instance, the wind and hydrostatic pressure acting on the surface of body. The body forces, such as gravitational force, electric and magnetic forces, etc., are the external forces whose magnitudes are proportional to the volume and are exerted on the interior of body in the unit of force per unit volume. The body moments are similar to body forces and are also exerted on the interior of body but in unit of moment per unit volume, such as materials with electric and magnetic dipoles. In this section, the equilibrium equations, formulated with respect to rectangular Cartesian frame of reference, are shown only. The equilibrium condition on an arbitrarily and infinitesimal element taken from interior of body, acting by stresses and body forces, as shown in Fig. 2B-5, is considered. The changes in stresses are replaced by Taylor's series expansion terms in which nonlinear higher order terms are neglected. If B_x, B_y, and B_z represent the body forces in x, y, and z direction, respectively, but without body forces, the force and moment equilibrium conditions in x direction are:

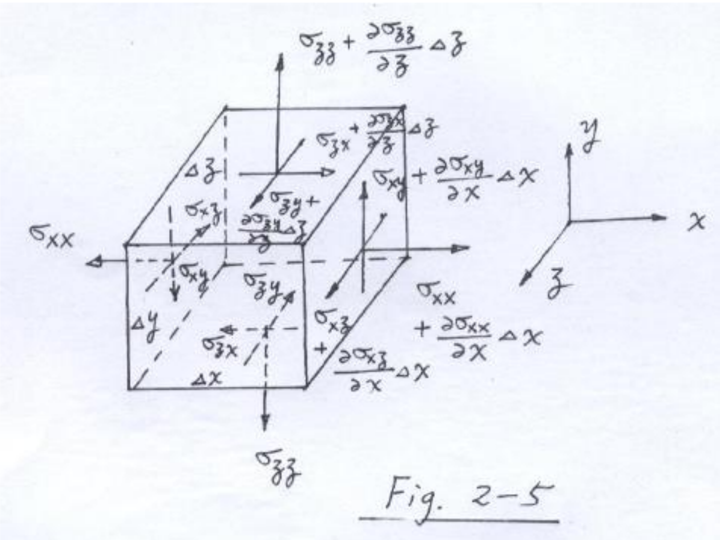


Fig. 2B-5

- $$\sum_i F_x = 0 \Rightarrow \quad \left(\sigma_{xx} + \frac{\partial \sigma_x}{\partial x}\Delta x\right)\Delta y \Delta z + \left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}\Delta x\right)\Delta z \Delta x + \left(\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z}\Delta z\right)\Delta x \Delta y$$

$$-\sigma_{xx}\Delta y \Delta z - \sigma_{yx}\Delta z \Delta x - \sigma_{zx}\Delta x \Delta y - B_x \Delta x \Delta y \Delta z = 0$$

Simplify this expression and let Δx, Δy, Δz approach zero, we obtain

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}+B_x=0 \qquad \text{(2B-28)}$$

- $$\sum_i M_x=0 \Rightarrow \quad \left(\sigma_{yz}+\frac{\partial\sigma_{yz}}{\partial y}\Delta y\right)\Delta x\Delta z\frac{\Delta y}{2}-\left(\sigma_{zy}+\frac{\partial\sigma_{zy}}{\partial z}\Delta z\right)\Delta x\Delta y\frac{\Delta z}{2}$$

$$+\sigma_{yz}\Delta x\Delta z\frac{\Delta y}{2}-\sigma_{zy}\Delta x\Delta y\frac{\Delta z}{2}=0$$

Simplify this expression and let Δx, Δy, Δz approach zero, we obtain

$$\sigma_{yz}=\sigma_{zy} \qquad \text{(2B-29)}$$

Similarly, summing forces and moments in the *y* and *z* directions, respectively, yields

$$\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zy}}{\partial z}+B_y=0 \qquad \text{(2B-30)}$$

$$\sigma_{zx}=\sigma_{xz} \qquad \text{(2B-31)}$$

$$\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}+B_z=0 \qquad \text{(2B-32)}$$

$$\sigma_{xy}=\sigma_{yx} \qquad \text{(2B-32)}$$

If we use the notation x_1, x_2, and x_3 to represent *x*, *y*, and *z*, respectively, then the force equilibrium equations, as shown in Eqs. (2B-28), (2B-30), and (2B-32), can be compactly expressed in tensor form:

$$\sigma_{ij,j}+B_i=0 \qquad i, j = 1, 2, 3 \qquad \text{(2B-33)}$$

where the repetition of an index in a term will denote a summation with respect to that index over its range; for instance, $\sigma_{1j,j}=\sigma_{11,1}+\sigma_{12,2}+\sigma_{13,3}$; the subscript "," means differentiating with respect to coordinate, for instance, $\sigma_{12,3}=\partial\sigma_{12}/\partial x_3$. Moreover, it is seen that the stresses are symmetric without the action of body moment, i.e.,

$$\sigma_{ij}=\sigma_{ji} \qquad i, j = 1, 2, 3 \qquad \text{(2B-34)}$$

Consequently, *only 6 stress components in stress tensor are independent with no body moment*.

[Remarks]: If the body under consideration moves with acceleration, the equilibrium equations shall be modified to *equations of motions* as follows:

$$\sigma_{ij,j} + B_i = \rho \ddot{u}_i \qquad i, j = 1, 2, 3 \qquad \text{(2-33a)}$$

where ρ is the density of body; the dot "." on the upper side of the variable means differentiating with respect to time.

2B-5 *Compatibility*

As we try to find a solution of the stress distribution in a body, the equilibrium equations of the body must be satisfied in case the body under consideration is in static equilibrium. At any given section, it may be possible to find many different sets of stress distributions which all satisfy the condition of equilibrium. **An acceptable stress distribution is one which ensures a continuous deformation distribution of the body.** This is the essential characteristic of **compatibility;** i.e., the stress distribution must be compatible with boundary conditions and a continuous distribution of deformations so that no "holes" or "overlapping" of specific points in the body occur. Sometimes, the stress distributions can be uniquely determined by the equilibrium equations and associated traction boundary conditions. If it is needed to evaluate the displacements of the body further, some compatibility relations must be used to assure the continuous deformation distribution of the body without generation of holes and overlapping. The six compatibility equations in terms of strains can be derived

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} \qquad \text{(2B-35)}$$

$$\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} = \frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} \qquad \text{(2B-36)}$$

$$\frac{\partial^2 \gamma_{zx}}{\partial z \partial x} = \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} \qquad \text{(2B-37)}$$

$$2\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \qquad \text{(2B-38)}$$

$$2\frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \qquad \text{(2B-39)}$$

$$2\frac{\partial^2 \varepsilon_{zz}}{\partial y \partial z} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} \right) \qquad \text{(2B-40)}$$

The first three equations are the in-plane dependent; while the last three equations are the out-of plane dependent.

[Remarks]: The compatibility equations (2B-35)~(2B-40) can be expressed by tensor notation as follows:

$$\varepsilon_{ij,kl} - \varepsilon_{ki,jl} - \varepsilon_{lj,ki} + \varepsilon_{kl,ji} = 0, \; i, j, k, l = 1, 2, 3$$

Above tensor equations can be written to 81 equations; but only 6 of these 81 equations are truly independent.

2B-6 *Cauchy's formula*

The Cauchy's formula provides the useful relations between tractions acting on a specific surface and the stresses at a specific point adjacent to the surface. Therefore, it is useful to evaluate the tractions acting on the surface if the stresses at this point are known; or, on the contrary, to evaluate the stresses at a point if the tractions acting on the surface through the point are given. Moreover, Cauchy's formula can also be applied to perform the stress transformation with respect to different coordinates and to evaluate the principal stresses and principal axes. Consider a tetrahedral element about a point *P*, which has three mutually orthogonal planar surfaces with normals in the *x*, *y*, and *z* directions, respectively, while the last surface is an arbitrary oblique plane with unit normal direction ***n***, as shown in Fig. 2B-6.

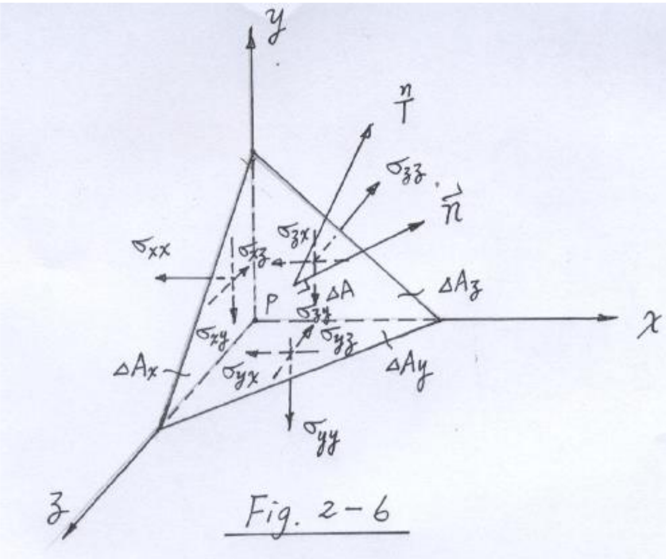


Fig. 2B-6

- $\sum_i F_x = 0 \Rightarrow$

$$\overset{n}{T}_x \Delta A - \sigma_{xx}\Delta A_x - \sigma_{yx}\Delta A_y - \sigma_{zx}\Delta A_z = 0$$

where $\Delta A_x = \Delta A \cdot n_x$, $\Delta A_y = \Delta A \cdot n_y$, and $\Delta A_z = \Delta A \cdot n_z$

Then we can obtain the Cauchy's formula in *x* direction by taking the limit *h* approaches zero (the point *P* is located on the oblique plane):

$$\overset{n}{T}_x = \sigma_{xx} n_x + \sigma_{yx} n_y + \sigma_{zx} n_z \qquad \text{(2B-41)}$$

Similarly, Cauchy's formula in *y* and *z* directions can also be obtained

- $\sum_i F_y = 0 \Rightarrow \overset{n}{T}_y = \sigma_{xy} n_x + \sigma_{yy} n_y + \sigma_{zy} n_z \qquad \text{(2B-42)}$

- $\sum_i F_z = 0 \Rightarrow \overset{n}{T}_z = \sigma_{xz}n_x + \sigma_{yz}n_y + \sigma_{zz}n_z$ (2B-43)

Cauchy's formula can also be compactly expressed in tensor form

$$\overset{n}{T}_i = \sigma_{ji}n_j\,, \qquad i, j = x, y, z \qquad (2B\text{-}44)$$

2B-7 Principal stresses (strains) and maximum shear stress

We have shown that the stress state at arbitrary point of the body can be expressed by a stress matrix or stress tensor which generally includes both normal and shear stress components. However, along some specific direction, we called the principal axis of stress, the stress matrix becomes diagonalized, i.e., all the shear stresses in off-diagonal portion become disappeared and only the normal stresses in the diagonal portion still remain. These three normal stresses are called principal stresses, which are independent of the directions of coordinates we select**.** Let us consider a point P in the interior of body whose stress state is σ_{ij}. According to Cauchy's formula, the traction vector acting on a specific surface passing through point P and with unit normal ***n*** can then be expressed as

$$\overset{n}{T}_i = \sigma_{ij}n_j\,, \quad i, j = x, y, z \quad \text{(or 1, 2, 3)} \qquad (2B\text{-}45)$$

If the unit normal ***n*** of the surface is parallel to one of the principal axes, the traction vector acting on the surface, represented by the principal stress σ, shall be normal to the surface, which can be expressed mathematically as

$$\overset{n}{T}_x = \sigma_{xx}n_x + \sigma_{xy}n_y + \sigma_{xz}n_z = \sigma n_x \qquad (2B\text{-}46a)$$

$$\overset{n}{T}_y = \sigma_{yx}n_x + \sigma_{yy}n_y + \sigma_{yz}n_z = \sigma n_y \qquad (2B\text{-}46b)$$

$$\overset{n}{T}_z = \sigma_{zx}n_x + \sigma_{zy}n_y + \sigma_{zz}n_z = \sigma n_z \qquad (2B\text{-}46c)$$

or expressed in matrix form

$$\begin{pmatrix} \sigma_{xx}-\sigma & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy}-\sigma & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}-\sigma \end{pmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = 0 \qquad (2B\text{-}47a)$$

[Remarks]: Equation (2-47) can also be derived in tensor notation as:

$$\overset{n}{T} = \sigma n \Rightarrow \overset{n}{T}_i = \sigma n_i \Rightarrow \sigma_{ij}n_j - \sigma n_i = 0 \Rightarrow \left(\sigma_{ij} - \sigma\delta_{ij}\right)n_j = 0 \qquad (2B\text{-}47b)$$

where the *symbol δ_{ij} is Kronecker delta, defined as δ_{ij} = 1 if i = j and δ_{ij} = 0 if* $i \neq j$. It is obvious that Eqs. (2-47b) in tensor form and (2-47a) in matrix form are identical.

The evaluation of principal stresses and principal axes as shown in Eq. (2B-47), becomes an eigenvalue problem which is of the form:

$$[A]\{s\} = \lambda\{s\} \qquad \text{(2B-48)}$$

Equation (2-48) can be placed in the form

$$([A] - \lambda[I])\{s\} = 0 \qquad \text{(2B-49)}$$

where $[I]$ is a unit matrix. Compared Eqs. (2B-49) to (2B-47), it is obvious to find that $[A] = [\sigma_{ij}]$, $\lambda = \sigma$, and $\{s\} = \{n_i\}$.
To avoid the trivial solution, i.e., {s} = {0}, [A]-λ[I] is forced to be singular. That is the determinant is set to zero as follows

$$\left|[A] - \lambda[I]\right| = 0 \qquad \text{(2B-50)}$$

Equation (2B-50) is called *the characteristic equation*, which yields n roots of λ which are the eigenvalues of the matrix [*A*], which, for instance, is $n \times n$ matrix. For each eigenvalue λ_i there is an associated eigenvector $\{s_i\}$ obtained from Eq. (2B-49). ***The corresponding three eigenvalues and eigenvectors are principal stresses and principal axes, respectively*.**
Generally the three principal stresses are denoted as σ_1, σ_2, and σ_3, where the ordering in magnitude is generally such that $\sigma_1 > \sigma_2 > \sigma_3$. If three principal stresses are distinct from each other in magnitude, there exists only one set of principal axes. If two of the principal stresses are equal, there will exist an infinite set of surfaces containing these principal stresses, where the normals of these surfaces are perpendicular to the direction of the third principal stress. If all three principal stresses are equal, a hydrostatic state of stress exists, and regardless of orientation all surfaces contain the same principal stresses with no shear stress. **The *maximum shear stress, τ_{max}, is equal to half of the difference maximum and minimum principal stresses*, i.e.,**

$$\tau_{\max} = (\sigma_1 - \sigma_3)/2, \text{ and } \tau_{\min} = -(\sigma_1 - \sigma_3)/2, \qquad \text{(2B-51)}$$

The solutions of principal stresses, σ, are independent of the coordinate system used to define the coefficients of the cubic equation for σ in the form

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \qquad \text{(2B-52)}$$

Therefore, the three coefficients of σ in Eq. (2B-52), I_1, I_2, and I_3, are constant and are normally referred to as the *stress invariants*. Thus

$$I_1 = \sigma_{ii} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \qquad \text{(2B-53a)}$$

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{zy} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{zx} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{vmatrix} = \left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}\right)/\,2$$

$$= \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \sigma_{yz}^2 - \sigma_{zx}^2 - \sigma_{xy}^2 \qquad \text{(2B-53b)}$$

$$I_3 = \left|\sigma_{ij}\right| = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{vmatrix} = e_{rst}\sigma_{r1}\sigma_{s2}\sigma_{s3} \qquad \text{(2B-53c)}$$

where e_{rst}, the *permutation symbol*, is defined as: it vanishes whenever the values of any two indices; e_{rst} = 1 when the subscripts permute like 1, 2, 3; and e_{rst} = −1 otherwise. The *Kronecker delta* can be related to *permutation symbol* by

$$e_{ijk}e_{ist} = \delta_{js}\delta_{kt} - \delta_{jt}\delta_{ks}\,.$$

[Example]: For the stress matrix given below, determine the principal stresses and corresponding principal axes.

$$(\sigma) = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} MPa$$

2B-8 *Stress and strain transformations*

Relationship of stress tensors between two different coordinates with same origin can be compactly derived by tensor notation. Consider a tetrahedral element about a point P having two different oblique planes with unit normals ***n*** and ***n’***, respectively, as shown in Fig. 2-8, the corresponding traction vectors on these two planes are $\overset{n}{T}$ and $\overset{n'}{T'}$, respectively. Hence,

$$\overset{n'}{T'}\cdot n = \overset{n'}{T_i}\, n_i = \sigma_{ji} n_j{}' n_i = \sigma_{ji} n_i n_j{}' = \overset{n}{T_j}\, n_j{}' = \overset{n}{T} \cdot n' \qquad \text{(2B-54)}$$

Suppose two different rectangular frames (x_1, x_2, x_3) and (x_1’, x_2’, x_3’) have the same origin *P* and the direction cosine between x_i’ and x_j is denoted by n_{ij}. If σ_{kl}’ represents the stress component of traction vector $\overset{n'}{T'}$ acting on the plane normal to the axis x_k’ along the direction of x_l’. In other words, the normals ***n’*** and ***n*** are parallel to the axes x_k’ and of x_l’, respectively. By using the Eq. (2-54), we can write

$$\sigma_{kl}{}' = \overset{n'}{T'}\cdot n = \overset{n}{T}\cdot n' = \sigma_{ji} n_{kj} n_{li} = n_{ki} n_{lj} \sigma_{ij} \qquad \text{(2B-55a)}$$

Similarly, we can also obtain

$$\sigma_{kl} = n_{ik} n_{jl} \sigma_{ij}' \qquad \text{(2B-55b)}$$

Equations (2-55a) can also be expressed as

$$\begin{pmatrix} \sigma_{xx}' & \sigma_{xy}' & \sigma_{xz}' \\ \sigma_{yx}' & \sigma_{yy}' & \sigma_{yz}' \\ \sigma_{zx}' & \sigma_{zy}' & \sigma_{zz}' \end{pmatrix} = \begin{pmatrix} n_{x'x} & n_{x'y} & n_{x'z} \\ n_{y'x} & n_{y'y} & n_{y'z} \\ n_{z'x} & n_{z'y} & n_{z'z} \end{pmatrix} \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} n_{x'x} & n_{x'y} & n_{x'z} \\ n_{y'x} & n_{y'y} & n_{y'z} \\ n_{z'x} & n_{z'y} & n_{z'z} \end{pmatrix}^T$$

(2B-56)

Since the coordinate transformation between *x'y'z'* system and *xyz* system can be expressed by

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{pmatrix} n_{x'x} & n_{x'y} & n_{x'z} \\ n_{y'x} & n_{y'y} & n_{y'z} \\ n_{z'x} & n_{z'y} & n_{z'z} \end{pmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \qquad \text{(2B-57)}$$

The displacements relative to *x'y'z'* system can also be related to that of the *xyz* system as

$$\begin{Bmatrix} u' \\ v' \\ w' \end{Bmatrix} = \begin{pmatrix} n_{x'x} & n_{x'y} & n_{x'z} \\ n_{y'x} & n_{y'y} & n_{y'z} \\ n_{z'x} & n_{z'y} & n_{z'z} \end{pmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \qquad \text{(2B-58)}$$

It can be proved in the lengthy procedures that

$$\begin{pmatrix} \varepsilon_{xx}' & \varepsilon_{xy}' & \varepsilon_{xz}' \\ \varepsilon_{yx}' & \varepsilon_{yy}' & \varepsilon_{yz}' \\ \varepsilon_{zx}' & \varepsilon_{zy}' & \varepsilon_{zz}' \end{pmatrix} = \begin{pmatrix} n_{x'x} & n_{x'y} & n_{x'z} \\ n_{y'x} & n_{y'y} & n_{y'z} \\ n_{z'x} & n_{z'y} & n_{z'z} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} \begin{pmatrix} n_{x'x} & n_{x'y} & n_{x'z} \\ n_{y'x} & n_{y'y} & n_{y'z} \\ n_{z'x} & n_{z'y} & n_{z'z} \end{pmatrix}^T$$

(2B-59)

Compared Eqs. (2-59) to (2-56), it is seen that the strain tensor performs the same coordinate transformation as that of the stress tensor.

2B-9 *Mohr's circles in two and three dimensions*

Let us start from a simpler two-dimensional case first. As shown in Fig. 2-9, a transformation of stresses is performed between *x'y'* system and *xy* system and *z'* keeps concided with *z*. Since the directional cosines are:

$$n_{x'x} = cos\theta,\ n_{x'y} = sin\theta,\ n_{x'z} = 0;\ n_{y'x} = -sin\theta,\ n_{y'y} = cos\theta,\ n_{y'z} = 0;$$

$$n_{z'x} = 0,\ n_{z'y} = 0,\ n_{z'z} = 1. \qquad \text{(2-60)}$$

By substituting Eqs. (2-60) into (2-56), we obtain

$$\sigma_{x'x'} = \sigma_{xx} cos^2\theta + \sigma_{yy} sin^2\theta + 2\sigma_{xy} cos\theta sin\theta \qquad \text{(2-61a)}$$

$$\sigma_{y'y'} = \sigma_{xx} \sin^2\theta + \sigma_{yy} \cos^2\theta - 2\sigma_{xy} \sin\theta\cos\theta \qquad \text{(2-61b)}$$

$$\sigma_{x'y'} = -\left(\sigma_{xx} - \sigma_{yy}\right)\sin\theta\cos\theta + \sigma_{xy}\left(\cos^2\theta - \sin^2\theta\right) \qquad \text{(2-61c)}$$

If we are only interested in obtaining the normal stress and shear stress (s, t) on a single surface, say the *x'* surface, then Eqs. (2-61a) to (2-61c) can be combined to

$$\sigma = \sigma_{xx} \cos^2\theta + \sigma_{yy} \sin^2\theta + 2\sigma_{xy} \cos\theta\sin\theta \qquad \text{(2-62a)}$$

$$\tau = -\left(\sigma_{xx} - \sigma_{yy}\right)\sin\theta\cos\theta + \sigma_{xy}\left(\cos^2\theta - \sin^2\theta\right) \qquad \text{(2-62b)}$$

These equations can be rewritten using the trigonometric identities

$$\sigma = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta + \sigma_{xy} \sin 2\theta \qquad \text{(2-63a)}$$

$$\tau = -\frac{\sigma_{xx} - \sigma_{yy}}{2} \sin 2\theta + \sigma_{xy} \cos 2\theta \qquad \text{(2-63b)}$$

Then, we obtain the equation of Mohr's circle

$$\left(\sigma - \sigma_{ave}\right)^2 + \tau^2 = R^2 \qquad \text{(2-64a)}$$

where $\sigma_{ave} = \dfrac{\sigma_{xx} + \sigma_{yy}}{2}$ and $R = \sqrt{\left(\dfrac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}{}^2}$. (2-64b)

The Mohr's circle can be shown as in Fig. 2-10.

From the Mohr's circle it can be seen that the maximum and minimum value of σ and τ are: $\sigma_{\substack{max \\ min}} = \sigma_{ave} \pm R$ and $\tau_{\substack{max \\ min}} = \pm R$ (2-65).

An element in a body undergoing a general state of three-dimensional stress can be transformed to an element containing only principal stresses σ_1, σ_2, and σ_3 acting along principal axes 1, 2, and 3, respectively. A transformation of stresses in the 12 plane depends only on σ_1 and σ_2; in the 23 plane depends only on σ_2 and σ_3; and in the 13 plane depends only on σ_1 and σ_3. This means that for each case, a plane stress analysis describes the state of stress in each of the three planes, and three Mohr's circles can be constructed to portray each case as shown in Fig. 2-11. Furthermore, it will shown that all possible states of stress (σ, τ) exist either on the circles or within the shaded area as shown in the figure.

For the element containing the principal stresses in the directions 1, 2, and 3, let n_1, n_2, and n_3 be the directional cosines of an arbitrary surface relative to the 1, 2, and 3 axes, respectively. The stress state (*σ, τ*) of the surface can be obtained by Cauchy's law

$$\sigma = \overset{n}{T} \cdot n = \sigma_i n_i^2 , \quad i = 1, 2, 3 \qquad \text{(2-66a)}$$

$$\sigma^2 + \tau^2 = \overset{n}{T} \cdot \overset{n}{T} = \left(\sigma_i n_i\right)^2 , \quad i = 1, 2, 3 \qquad \text{(2-66b)}$$

$$n \cdot n = 1 \qquad \text{(2-66c)}$$

The solution of $\boldsymbol{n}$ of above three equations can be obtained

$$n_1^2 = \frac{\tau^2 + (\sigma - \sigma_2)(\sigma - \sigma_3)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)}, \quad n_2^2 = \frac{\tau^2 + (\sigma - \sigma_3)(\sigma - \sigma_1)}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)},$$

$$n_3^2 = \frac{\tau^2 + (\sigma - \sigma_1)(\sigma - \sigma_2)}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)}. \qquad \text{(2-67)}$$

If the order of three principal stresses in magnitude is $\sigma_1 > \sigma_2 > \sigma_3$, then

Eq. (2-67) $\Rightarrow$ $\tau^2 + (\sigma - \sigma_2)(\sigma - \sigma_3) \geq 0$ (2-68a)

$$\tau^2 + (\sigma - \sigma_3)(\sigma - \sigma_1) \leq 0 \qquad \text{(2-68b)}$$

$$\tau^2 + (\sigma - \sigma_1)(\sigma - \sigma_2) > 0 \qquad \text{(2-68c)}$$

Since $\tau^2 + (\sigma - \sigma_i)(\sigma - \sigma_j) = 0$ represents a circle passing through the points (σ_i, 0) and (σ_j, 0), $\tau^2 + (\sigma - \sigma_i)(\sigma - \sigma_j) > 0$ and $\tau^2 + (\sigma - \sigma_i)(\sigma - \sigma_j) < 0$ show the regions beyond and within the circle, respectively. Consequently, it is seen the possible stress state shall be located either on the boundaries of three circles or within the shaded region shown in Fig. 2-11. Moreover, the maximum shear stress is equal to (σ_1–σ_3)/2 in the direction $n = \left(\pm 1/\sqrt{2} \quad 0 \quad \pm 1/\sqrt{2}\right)$.

[Remarks]: The method to determine the stress state (σ, τ), which directional cosine vector with respect to principal axes (n_1, n_2, n_3) is as follows:

(a) Let $n_i = cos\,\alpha_i$, plot a line passing through point (σ_1, 0) with an angle $\pi/2 + \alpha_1$ with the horizontal axis σ, which will intersect circle C_2 at point P.

(b) Plot a line passing through point (σ_3, 0) with an angle $\pi/2 - \alpha_3$ with the horizontal axis σ, which will intersect circle C_2 at point Q.

(c) Plot two circular arcs S_1 and S_2 based on the origin O_1, radius O_1P and the origin O_3, radius O_3Q, respectively; The intersection point R between these two arcs denotes the stress state of the surface in three-dimensional Mohr's circles.

Reference:

1. Y. C. Fung, *Foundations of Solid Mechanics*, 1965, Prentice-Hall International, Inc.
2. R. G. Budynas, *Advanced Strength and Applied Stress Analysis*, Second ed., 1999, McGRAW-HILL International Editions.
3. Some Elasticity books in Japanese version.

[Appendix 2A-1]: ***Formulation of compatibility equations***

Suppose point P_0 (x_{10}, x_{20}, x_{30}) be an arbitrary point in the domain of elastic body under consideration with displacement u_{i0} and rigid body rotation ω_{ij0}. Moreover, the domain is assumed to be ***single-connected*** and strains are ***C^2-level continuous*** in the domain V. The displacement and rigid body rotation of the other arbitrary point P' (x_1', x_2', x_3') in the domain V are u_i' and ω_{ij}', respectively. Let us connect the points P_0 and P' by using a curve C and then integrate the du_j along the curve

$$\int_{P_0}^{P'} du_j = u_j(x_1', x_2', x_3') - u_{j0} \qquad \text{(A-1)}$$

and $\int_{P_0}^{P'} du_j = \int_{P_0}^{P'} u_{j,k} dx_k = \int_{P_0}^{P'} (\varepsilon_{jk} + \omega_{jk}) dx_k = \int_{P_0}^{P'} \varepsilon_{jk} dx_k + \int_{P_0}^{P'} \omega_{jk} dx_k$. (A-2)

The second term in the R.H.S. of Eq. (A-2), by using the relation $dx_k' = 0$ and technique of integration by parts, can be changed to

$$\int_{P_0}^{P'} \omega_{jk} dx_k = \int_{P_0}^{P'} \omega_{jk} d(x_k - x_k') = (x_k' - x_{k0})\omega_{jk0} + \int_{P_0}^{P'} (x_k' - x_k)\omega_{jk,l} dx_l, \qquad \text{(A-3)}$$

where $\omega_{jk,l} = \dfrac{\partial}{\partial x_l}\left[\dfrac{1}{2}\left(\dfrac{\partial u_j}{\partial x_k} - \dfrac{\partial u_k}{\partial x_j}\right)\right] = \dfrac{1}{2}(u_{j,kl} - u_{k,jl}) + \dfrac{1}{2}(u_{l,jk} - u_{l,jk})$

$$= \frac{1}{2}(u_{j,l} + u_{l,j})_{,k} - \frac{1}{2}(u_{k,l} + u_{l,k})_{,j} = \varepsilon_{lj,k} - \varepsilon_{kl,j} \qquad \text{(A-4)}$$

Substituting Eqs. (A-2)~(A-4) into (A-1), we obtain

$$u_j(x_1', x_2', x_3') = u_{j0} + \omega_{jk0}(x_k' - x_{k0}) + \int_{P_0}^{P'} U_{jl} dx_l, \qquad \text{(A-5)}$$

where $U_{jl} = \varepsilon_{jl} + (x_k' - x_k)(\varepsilon_{lj,k} - \varepsilon_{kl,j})$ (A-6)

By using the hypothesis of continuum mechanics, the displacement at point P' shall definitely be a single value. In other words, the value of displacement at point P' shall be independent of the path C; or the last term at R.H.S. of Eq. (A-5), $U_{jl}dx_l$, shall be *exactly differentiable*. The corresponding sufficient and necessary conditions of *exact differentiation* can be written mathematically as

$$\Rightarrow U_{ji,l} - U_{jl,i} = 0 \qquad \text{(A-7)}$$

By inserting Eqs. (A-6) into (A-7), it yields

$$\Rightarrow [\varepsilon_{ij,l} - \delta_{kl}(\varepsilon_{ij,k} - \varepsilon_{ki,j}) - \varepsilon_{jl,i} + \delta_{ki}(\varepsilon_{lj,k} - \varepsilon_{kl,j})]$$

$$+ (x_k' - x_k)(\varepsilon_{ij,kl} - \varepsilon_{ki,jl} - \varepsilon_{lj,ki} + \varepsilon_{kl,ji}) = 0$$

$$\Rightarrow (\varepsilon_{ij,kl} - \varepsilon_{ki,jl} - \varepsilon_{lj,ki} + \varepsilon_{kl,ji}) = 0 \qquad \text{(A-8)}$$

Equation (A-8) can be broken down and written to 3^4 = 81 equations, in which only 6 equations, expressed by Eqs. (2B-35)~(2B-39) are independent.