

1.4 Exercise :

2. (a) $\lim_{x \rightarrow 2} [f(x) + g(x)] = \lim_{x \rightarrow 2} f(x) + \lim_{x \rightarrow 2} g(x) = 2 + 0 = 2$

(b) $\lim_{x \rightarrow 1} g(x)$ does not exist since its left- and right-hand limits are not equal, so the given limit does not exist.

(c) $\lim_{x \rightarrow 0} [f(x)g(x)] = \lim_{x \rightarrow 0} f(x) \cdot \lim_{x \rightarrow 0} g(x) = 0 \cdot 1.3 = 0$

(d) Since $\lim_{x \rightarrow -1} g(x) = 0$ and g is in the denominator, but $\lim_{x \rightarrow -1} f(x) = -1 \neq 0$, the given limit does not exist.

(e) $\lim_{x \rightarrow 2} x^3 f(x) = \left[\lim_{x \rightarrow 2} x^3 \right] \left[\lim_{x \rightarrow 2} f(x) \right] = 2^3 \cdot 2 = 16$

(f) $\lim_{x \rightarrow 1} \sqrt{3 + f(x)} = \sqrt{3 + \lim_{x \rightarrow 1} f(x)} = \sqrt{3 + 1} = 2$

11. $\lim_{x \rightarrow 5} \frac{x^2 - 6x + 5}{x - 5} = \lim_{x \rightarrow 5} \frac{(x - 5)(x - 1)}{x - 5} = \lim_{x \rightarrow 5} (x - 1) = 5 - 1 = 4$

14. $\lim_{x \rightarrow -1} \frac{x^2 - 4x}{x^2 - 3x - 4}$ does not exist since $x^2 - 3x - 4 \rightarrow 0$ but $x^2 - 4x \rightarrow 5$ as $x \rightarrow -1$.

18. $\lim_{h \rightarrow 0} \frac{\sqrt{1+h} - 1}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{1+h} - 1}{h} \cdot \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} = \lim_{h \rightarrow 0} \frac{(1+h) - 1}{h(\sqrt{1+h} + 1)} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{1+h} + 1)}$
 $= \lim_{h \rightarrow 0} \frac{1}{\sqrt{1+h} + 1} = \frac{1}{\sqrt{1+1} + 1} = \frac{1}{2}$

23. $\lim_{x \rightarrow 16} \frac{4 - \sqrt{x}}{16x - x^2} = \lim_{x \rightarrow 16} \frac{(4 - \sqrt{x})(4 + \sqrt{x})}{(16x - x^2)(4 + \sqrt{x})} = \lim_{x \rightarrow 16} \frac{16 - x}{x(16 - x)(4 + \sqrt{x})}$
 $= \lim_{x \rightarrow 16} \frac{1}{x(4 + \sqrt{x})} = \frac{1}{16(4 + \sqrt{16})} = \frac{1}{16(8)} = \frac{1}{128}$

25. $\lim_{x \rightarrow -4} \frac{\frac{1}{4} + \frac{1}{x}}{4 + x} = \lim_{x \rightarrow -4} \frac{\frac{x + 4}{4x}}{4 + x} = \lim_{x \rightarrow -4} \frac{x + 4}{4x(4 + x)} = \lim_{x \rightarrow -4} \frac{1}{4x} = \frac{1}{4(-4)} = -\frac{1}{16}$

26. $\lim_{x \rightarrow -4} \frac{\sqrt{x^2 + 9} - 5}{x + 4} = \lim_{x \rightarrow -4} \frac{(\sqrt{x^2 + 9} - 5)(\sqrt{x^2 + 9} + 5)}{(x + 4)(\sqrt{x^2 + 9} + 5)} = \lim_{x \rightarrow -4} \frac{(x^2 + 9) - 25}{(x + 4)(\sqrt{x^2 + 9} + 5)}$
 $= \lim_{x \rightarrow -4} \frac{x^2 - 16}{(x + 4)(\sqrt{x^2 + 9} + 5)} = \lim_{x \rightarrow -4} \frac{(x + 4)(x - 4)}{(x + 4)(\sqrt{x^2 + 9} + 5)}$
 $= \lim_{x \rightarrow -4} \frac{x - 4}{\sqrt{x^2 + 9} + 5} = \frac{-4 - 4}{\sqrt{16 + 9} + 5} = \frac{-8}{5 + 5} = -\frac{4}{5}$

38. $|x + 6| = \begin{cases} x + 6 & \text{if } x + 6 \geq 0 \\ -(x + 6) & \text{if } x + 6 < 0 \end{cases} = \begin{cases} x + 6 & \text{if } x \geq -6 \\ -(x + 6) & \text{if } x < -6 \end{cases}$

We'll look at the one-sided limits.

$$\lim_{x \rightarrow -6^+} \frac{2x + 12}{|x + 6|} = \lim_{x \rightarrow -6^+} \frac{2(x + 6)}{x + 6} = 2 \quad \text{and} \quad \lim_{x \rightarrow -6^-} \frac{2x + 12}{|x + 6|} = \lim_{x \rightarrow -6^-} \frac{2(x + 6)}{-(x + 6)} = -2$$

The left and right limits are different, so $\lim_{x \rightarrow -6} \frac{2x + 12}{|x + 6|}$ does not exist.

41. Since $|x| = -x$ for $x < 0$, we have $\lim_{x \rightarrow 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right) = \lim_{x \rightarrow 0^-} \left(\frac{1}{x} - \frac{1}{-x} \right) = \lim_{x \rightarrow 0^-} \frac{2}{x}$, which does not exist since the denominator approaches 0 and the numerator does not.

42. Since $|x| = x$ for $x > 0$, we have $\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{|x|} \right) = \lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{x} \right) = \lim_{x \rightarrow 0^+} 0 = 0$.

44. (a) (i) If $x \rightarrow 1^+$, then $x > 1$ and $g(x) = x - 1$. Thus, $\lim_{x \rightarrow 1^+} g(x) = \lim_{x \rightarrow 1^+} (x - 1) = 1 - 1 = 0$.

(ii) If $x \rightarrow 1^-$, then $x < 1$ and $g(x) = 1 - x^2$. Thus, $\lim_{x \rightarrow 1^-} g(x) = \lim_{x \rightarrow 1^-} (1 - x^2) = 1 - 1^2 = 0$.

Since the left- and right-hand limits of g at 1 are equal, $\lim_{x \rightarrow 1} g(x) = 0$.

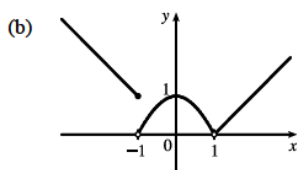
(iii) If $x \rightarrow 0$, then $-1 < x < 1$ and $g(x) = 1 - x^2$. Thus, $\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} (1 - x^2) = 1 - 0^2 = 1$.

(iv) If $x \rightarrow -1^-$, then $x < -1$ and $g(x) = -x$. Thus, $\lim_{x \rightarrow -1^-} g(x) = \lim_{x \rightarrow -1^-} (-x) = -(-1) = 1$.

(v) If $x \rightarrow -1^+$, then $-1 < x < 1$ and $g(x) = 1 - x^2$. Thus,

$$\lim_{x \rightarrow -1^+} g(x) = \lim_{x \rightarrow -1^+} (1 - x^2) = 1 - (-1)^2 = 1 - 1 = 0$$

(vi) $\lim_{x \rightarrow -1} g(x)$ does not exist because the limits in part (iv) and part (v) are not equal.



$$\begin{aligned} 49. \lim_{x \rightarrow 0} \frac{\sin 3x}{x} &= \lim_{x \rightarrow 0} \frac{3 \sin 3x}{3x} \quad [\text{multiply numerator and denominator by 3}] \\ &= 3 \lim_{3x \rightarrow 0} \frac{\sin 3x}{3x} \quad [\text{as } x \rightarrow 0, 3x \rightarrow 0] \\ &= 3 \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \quad [\text{let } \theta = 3x] \\ &= 3(1) \quad [\text{Equation 2}] \\ &= 3 \end{aligned}$$

$$\begin{aligned} 50. \lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 6x} &= \lim_{x \rightarrow 0} \left(\frac{\sin 4x}{x} \cdot \frac{x}{\sin 6x} \right) = \lim_{x \rightarrow 0} \frac{4 \sin 4x}{4x} \cdot \lim_{x \rightarrow 0} \frac{6x}{6 \sin 6x} \\ &= 4 \lim_{x \rightarrow 0} \frac{\sin 4x}{4x} \cdot \frac{1}{6} \lim_{x \rightarrow 0} \frac{6x}{\sin 6x} = 4(1) \cdot \frac{1}{6}(1) = \frac{2}{3} \end{aligned}$$

$$53. \lim_{x \rightarrow 0} \frac{\sin 3x}{5x^2 - 4x} = \lim_{x \rightarrow 0} \left(\frac{\sin 3x}{3x} \cdot \frac{3}{5x^2 - 4x} \right) = \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \cdot \lim_{x \rightarrow 0} \frac{3}{5x^2 - 4x} = 1 \cdot \left(\frac{3}{-4} \right) = -\frac{3}{4}$$

$$\begin{aligned} 54. \lim_{x \rightarrow 0} \frac{\sin 3x \sin 5x}{x^2} &= \lim_{x \rightarrow 0} \left(\frac{3 \sin 3x}{3x} \cdot \frac{5 \sin 5x}{5x} \right) = \lim_{x \rightarrow 0} \frac{3 \sin 3x}{3x} \cdot \lim_{x \rightarrow 0} \frac{5 \sin 5x}{5x} \\ &= 3 \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \cdot 5 \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} = 3(1) \cdot 5(1) = 15 \end{aligned}$$

55. Divide numerator and denominator by θ . ($\sin \theta$ also works.)

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta + \tan \theta} = \lim_{\theta \rightarrow 0} \frac{\frac{\sin \theta}{\theta}}{1 + \frac{\sin \theta}{\theta} \cdot \frac{1}{\cos \theta}} = \frac{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}}{1 + \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \lim_{\theta \rightarrow 0} \frac{1}{\cos \theta}} = \frac{1}{1 + 1 \cdot 1} = \frac{1}{2}$$

65. Since the denominator approaches 0 as $x \rightarrow -2$, the limit will exist only if the numerator also approaches

0 as $x \rightarrow -2$. In order for this to happen, we need $\lim_{x \rightarrow -2} (3x^2 + ax + a + 3) = 0 \Leftrightarrow$

$3(-2)^2 + a(-2) + a + 3 = 0 \Leftrightarrow 12 - 2a + a + 3 = 0 \Leftrightarrow a = 15$. With $a = 15$, the limit becomes

$$\lim_{x \rightarrow -2} \frac{3x^2 + 15x + 18}{x^2 + x - 2} = \lim_{x \rightarrow -2} \frac{3(x+2)(x+3)}{(x-1)(x+2)} = \lim_{x \rightarrow -2} \frac{3(x+3)}{x-1} = \frac{3(-2+3)}{-2-1} = \frac{3}{-3} = -1.$$

1.5 Exercise :

$$29. f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$$

By Theorem 5, since $f(x)$ equals the polynomial x^2 on $(-\infty, 1)$, f is continuous on $(-\infty, 1)$. By Theorem 7, since $f(x)$ equals the root function \sqrt{x} on $(1, \infty)$, f is continuous on $(1, \infty)$. At $x = 1$, $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^2 = 1$ and

$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \sqrt{x} = 1$. Thus, $\lim_{x \rightarrow 1} f(x)$ exists and equals 1. Also, $f(1) = \sqrt{1} = 1$. Thus, f is continuous at $x = 1$.

We conclude that f is continuous on $(-\infty, \infty)$.

$$31. f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 2x^2 & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$

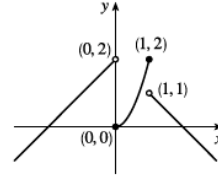
f is continuous on $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$ since on each of

these intervals it is a polynomial. Now $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (x + 2) = 2$ and

$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} 2x^2 = 0$, so f is discontinuous at 0. Since $f(0) = 0$, f is continuous from the right at 0. Also

$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 2x^2 = 2$ and $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2 - x) = 1$, so f is discontinuous at 1. Since $f(1) = 2$,

f is continuous from the left at 1.



$$33. f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

f is continuous on $(-\infty, 2)$ and $(2, \infty)$. Now $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (cx^2 + 2x) = 4c + 4$ and

$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x^3 - cx) = 8 - 2c$. So f is continuous $\Leftrightarrow 4c + 4 = 8 - 2c \Leftrightarrow 6c = 4 \Leftrightarrow c = \frac{2}{3}$. Thus, for f

to be continuous on $(-\infty, \infty)$, $c = \frac{2}{3}$.

$$34. f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leq x < 3 \\ 2x - a + b & \text{if } x \geq 3 \end{cases}$$

$$\text{At } x = 2: \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2^-} \frac{(x+2)(x-2)}{x-2} = \lim_{x \rightarrow 2^-} (x+2) = 2+2 = 4$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (ax^2 - bx + 3) = 4a - 2b + 3$$

$$\text{We must have } 4a - 2b + 3 = 4, \text{ or } 4a - 2b = 1 \quad (1).$$

$$\text{At } x = 3: \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (ax^2 - bx + 3) = 9a - 3b + 3$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (2x - a + b) = 6 - a + b$$

$$\text{We must have } 9a - 3b + 3 = 6 - a + b, \text{ or } 10a - 4b = 3 \quad (2).$$

Now solve the system of equations by adding -2 times equation (1) to equation (2).

$$-8a + 4b = -2$$

$$\frac{10a - 4b = 3}{2a = 1}$$

So $a = \frac{1}{2}$. Substituting $\frac{1}{2}$ for a in (1) gives us $-2b = -1$, so $b = \frac{1}{2}$ as well. Thus, for f to be continuous on $(-\infty, \infty)$,

$$a = b = \frac{1}{2}.$$

37. $f(x) = x^2 + 10 \sin x$ is continuous on the interval $[31, 32]$, $f(31) \approx 957$, and $f(32) \approx 1030$. Since $957 < 1000 < 1030$, there is a number c in $(31, 32)$ such that $f(c) = 1000$ by the Intermediate Value Theorem. *Note:* There is also a number c in $(-32, -31)$ such that $f(c) = 1000$.
39. $f(x) = x^4 + x - 3$ is continuous on the interval $[1, 2]$, $f(1) = -1$, and $f(2) = 15$. Since $-1 < 0 < 15$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^4 + x - 3 = 0$ in the interval $(1, 2)$.
41. $f(x) = \cos x - x$ is continuous on the interval $[0, 1]$, $f(0) = 1$, and $f(1) = \cos 1 - 1 \approx -0.46$. Since $-0.46 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x - x = 0$, or $\cos x = x$, in the interval $(0, 1)$.

1.6 Exercise :

14. $\lim_{x \rightarrow 5^-} \frac{6}{x-5} = -\infty$ since $(x-5) \rightarrow 0$ as $x \rightarrow 5^-$ and $\frac{6}{x-5} < 0$ for $x < 5$.
16. $\lim_{x \rightarrow \pi^-} \cot x = \lim_{x \rightarrow \pi^-} \frac{\cos x}{\sin x} = -\infty$ since the numerator is negative and the denominator approaches 0 through positive values as $x \rightarrow \pi^-$.
17. $\lim_{x \rightarrow 2\pi^-} x \csc x = \lim_{x \rightarrow 2\pi^-} \frac{x}{\sin x} = -\infty$ since the numerator is positive and the denominator approaches 0 through negative values as $x \rightarrow 2\pi^-$.
20. $\lim_{t \rightarrow -\infty} \frac{t^2 + 2}{t^3 + t^2 - 1} = \lim_{t \rightarrow -\infty} \frac{(t^2 + 2)/t^3}{(t^3 + t^2 - 1)/t^3} = \lim_{t \rightarrow -\infty} \frac{1/t + 2/t^3}{1 + 1/t - 1/t^3} = \frac{0 + 0}{1 + 0 - 0} = 0$
21. $\lim_{t \rightarrow \infty} \frac{\sqrt{t} + t^2}{2t - t^2} = \lim_{t \rightarrow \infty} \frac{(\sqrt{t} + t^2)/t^2}{(2t - t^2)/t^2} = \lim_{t \rightarrow \infty} \frac{1/t^{3/2} + 1}{2/t - 1} = \frac{0 + 1}{0 - 1} = -1$
23. $\lim_{x \rightarrow \infty} \frac{(2x^2 + 1)^2}{(x-1)^2(x^2 + x)} = \lim_{x \rightarrow \infty} \frac{(2x^2 + 1)^2/x^4}{[(x-1)^2(x^2 + x)]/x^4} = \lim_{x \rightarrow \infty} \frac{[(2x^2 + 1)/x^2]^2}{[(x^2 - 2x + 1)/x^2][(x^2 + x)/x^2]}$
 $= \lim_{x \rightarrow \infty} \frac{(2 + 1/x^2)^2}{(1 - 2/x + 1/x^2)(1 + 1/x)} = \frac{(2 + 0)^2}{(1 - 0 + 0)(1 + 0)} = 4$
24. $\lim_{x \rightarrow \infty} \frac{x+2}{\sqrt{9x^2+1}} = \lim_{x \rightarrow \infty} \frac{(x+2)/x}{\sqrt{9x^2+1}/\sqrt{x^2}} = \lim_{x \rightarrow \infty} \frac{1+2/x}{\sqrt{9+1/x^2}} = \frac{1+0}{\sqrt{9+0}} = \frac{1}{3}$
25. $\lim_{x \rightarrow \infty} (\sqrt{9x^2+x} - 3x) = \lim_{x \rightarrow \infty} \frac{(\sqrt{9x^2+x} - 3x)(\sqrt{9x^2+x} + 3x)}{\sqrt{9x^2+x} + 3x} = \lim_{x \rightarrow \infty} \frac{(\sqrt{9x^2+x})^2 - (3x)^2}{\sqrt{9x^2+x} + 3x}$
 $= \lim_{x \rightarrow \infty} \frac{(9x^2+x) - 9x^2}{\sqrt{9x^2+x} + 3x} = \lim_{x \rightarrow \infty} \frac{x}{\sqrt{9x^2+x} + 3x} \cdot \frac{1/x}{1/x}$
 $= \lim_{x \rightarrow \infty} \frac{x/x}{\sqrt{9x^2/x^2 + x/x^2} + 3x/x} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{9+1/x} + 3} = \frac{1}{\sqrt{9+0} + 3} = \frac{1}{3+3} = \frac{1}{6}$
28. Since $0 \leq \sin^2 x \leq 1$, we have $0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$. Now $\lim_{x \rightarrow \infty} 0 = 0$ and $\lim_{x \rightarrow \infty} \frac{1}{x^2} = 0$, so by the Squeeze Theorem,
 $\lim_{x \rightarrow \infty} \frac{\sin^2 x}{x^2} = 0$.
29. $\lim_{x \rightarrow \infty} \cos x$ does not exist because as x increases $\cos x$ does not approach any one value, but oscillates between 1 and -1 .
31. $\lim_{x \rightarrow \infty} (x - \sqrt{x}) = \lim_{x \rightarrow \infty} \sqrt{x}(\sqrt{x} - 1) = \infty$ since $\sqrt{x} \rightarrow \infty$ and $\sqrt{x} - 1 \rightarrow \infty$ as $x \rightarrow \infty$.