GEOMETRY (1) HOMEWORK 10

(DUE: 9:50AM 11/24)

Write up the solutions to all problems.

Follow the notation used in class.

Problem A. *Show that the Gaussian curvature is preserved under isometry. Is the converse true? Justify your answer. Moreover, is Gaussian curvature also preserved by conformal maps?*

Problem B. *Show that in an orthogonal parameterization,*

$$K = -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\}.$$

Problem C. *Compute the Christoffel symbols of S^2 in*

(a) spherical coordinates;
(b) Euclidean coordinates;
(c) polar coordinates.

Problem D. *Consider a surface of revolution*

$$(f(v)\cos u, f(v)\sin u, g(v)) \quad \text{with } f(v) \neq 0.$$

Compute all the Christoffel symbols and the Gaussian curvature.

Problem E. *Let S be a regular surface and $\mathbf{x}\colon U \to S$ be a parameterization. Recall that we have a bijection*

$$\Phi_U\colon \mathbf{x}(U) \times \mathbf{R}^2 \longrightarrow \coprod_{p\in\mathbf{x}(U)} T_pS, \qquad (p,(a,b)) \mapsto a\mathbf{x}_u(q) + b\mathbf{x}_v(q)$$

where $q \in U$ is a point such that $\mathbf{x}(q) = p$.

Suppose $\mathbf{y}\colon V \to S$ is another parameterization around p, i.e., $\mathbf{x}(U)\cap \mathbf{y}(V) \neq \varnothing$. We obtain another map

$$\Phi_V\colon \mathbf{y}(V) \times \mathbf{R}^2 \longrightarrow \coprod_{p\in\mathbf{y}(V)} T_pS, \qquad (p,(A,B)) \mapsto A\mathbf{y}_s(Q) + B\mathbf{y}_t(Q)$$

where $Q \in V$ is a point such that $\mathbf{y}(Q) = p$.

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Consider the composition

$$\Phi_V^{-1} \circ \Phi_U \colon \mathbf{x}(U) \cap \mathbf{y}(V) \times \mathbf{R}^2 \to \mathbf{x}(U) \cap \mathbf{y}(V) \times \mathbf{R}^2.$$

Then $(\Phi_V^{-1} \circ \Phi_U)(p, (a, b)) = (p, (A, B))$. *Let* $\psi = \mathbf{y}^{-1} \circ \mathbf{x}$. *Show that*

$$\begin{bmatrix} A \\ B \end{bmatrix} = (\mathrm{d}\psi)_q \begin{bmatrix} a \\ b \end{bmatrix}.$$

Problem F. *Let* S *be a surface and* $p \in S$. *Recall that two functions* f *and* g *locally defined near* p *are equivalent, written as* $f \sim g$, *if and only if*

$$f|_U = g|_U$$

for some open set $U \ni p$ *in* S. *Show that this is an equivalence relation. Hence* $\mathscr{C}_p^\infty(S)$ *is well-defined.*

Now consider $S = \mathbf{R}^2$. *Let* $p \in S$ *and* $X \in T_pS$ *is a tangent vector. Identify* T_pS *with* $\mathbf{R}^2$. *Show that* $X(f)$ *is the usual directional derivative of* f *along* X *at* p. *Here* $f \in \mathscr{C}_p^\infty(S)$.