GEOMETRY (1) HOMEWORK 9

(DUE: 9:50AM 11/17)

Write up the solutions to all problems.

Follow the notation used in class.

Definition 0.1. We have defined the notion of *differential of a function*. Recall that for a differentiable function $f\colon U \subset \mathbf{R}^n \to \mathbf{R}$, the differential at $p \in U$ is a map

$$(\mathrm{d}f)_p\colon T_p\mathbf{R}^n \to T_{f(p)}\mathbf{R}, \quad v \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ \alpha)(t)$$

where $\alpha(t)$ is a curve defined on $(-\epsilon, \epsilon)$ such that $\alpha(0) = p$ and $\alpha'(0) = v$.

Identifying $T_{f(p)}\mathbf{R}$ with $\mathbf{R}$, we can regard $(\mathrm{d}f)_p$ as a functional on the vector space $T_p\mathbf{R}^n$.

Problem A.

(a) Show that $(\mathrm{d}f)_p$ is linear. Moreover, if $(x_1, \ldots, x_n)$ denotes the coordinates on $\mathbf{R}^n$, show that

$$(\mathrm{d}f)_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)(\mathrm{d}x_i)_p$$

Let S be a regular surface and $\mathbf{x}\colon U \to S$ is a parameterization. Now regard (a, b) as a tangent vector at $q \in U$.

(b) Show that $(\mathrm{d}\mathbf{x})_q(a, b) = a\mathbf{x}_u + b\mathbf{x}_v$. Here $(\mathrm{d}\mathbf{x})_q\colon T_q\mathbf{R}^2 \to T_{\mathbf{x}(q)}S$ is the differential of $\mathbf{x}$ at $q \in U$.

Over $\mathbf{x}(U)$ the first fundamental form $\mathrm{I}_p\colon T_pS \to \mathbf{R}$ is expressed as

$$\mathrm{I}_p(a\mathbf{x}_u + b\mathbf{x}_v) = Ea^2 + 2Fab + Gb^2.$$

(c) Show that $\mathrm{I} \circ (\mathrm{d}\mathbf{x}) = E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2$.

Date: November 12, 2026.

Problem B. *Let* $\varphi\colon \mathbf{R}^2 \to \mathbf{R}^2$ *be given by*

$$\varphi(x, y) = (u(x, y), v(x, y))$$

where u, v *are differentiable functions. Assume that* u *and* v *satisfy the Cauchy–Riemann equations*

$$u_x = v_y, \quad u_y = -v_x.$$

Show that φ *is a local conformal map from* $\mathbf{R}^2 \setminus Q$ *into* $\mathbf{R}^2$ *where* $Q = \{(x, y) \mid u_x^2 + u_y^2 = 0\}$.

Hence a complex differentiable, or holomorphic, function f *is conformal at* z *provided that* $f'(z) \neq 0$. *(You don't have to prove this.)*

Problem C. *Let* S *be a surface of revolution. Prove that the rotations about its axis are isometries of* S.

Problem D. *Show that the catenoid and the helicoid are local isometric using the parameterizations*

$$\begin{cases} \mathbf{x}(u, v) = (\cosh v \cos u, \cosh v \sin u, v)\,, \\ \mathbf{y}(u, v) = (\sinh v \sin u, -\sinh v \cos u, u)\,. \end{cases}$$

Compute their Gaussian curvature.

Problem E. *Let* $w_1(x, y) = (1, 0)$ *and* $w_2(x, y) = (0, e^x)$ *be two smooth vector fields on* $\mathbf{R}^2$. *Show that there is no local reparameterization* $\mathbf{x}\colon U \to \mathbf{R}^2$ *such that* $\mathbf{x}_u = w_1 \circ \mathbf{x}$ *and* $\mathbf{x}_v = w_2 \circ \mathbf{x}$ *over* U.

Problem F. *Show that the Lemma for solving the planar problem implies the existence of isothermal coordinates.*

Hint. *Take a parameterization* $\mathbf{x}\colon U \to S$ *with coordinates* (u, v) *and write*

$$\mathrm{I} = E_0(u, v)\mathrm{d}u^2 + G_0(u, v)\mathrm{d}v^2.$$

Make a change of variable $x = \sqrt{E_0(0, 0)}u$ *and* $y = \sqrt{G_0(0, 0)}v$. *Then*

$$\mathrm{I} = E(x, y)\mathrm{d}x^2 + G(x, y)\mathrm{d}y^2, \text{ with } E(0, 0) = G(0, 0) = 1.$$

Set $a = \sqrt{G/E}$ *and* $b = a^{-1}$. *Lemma implies a solution* $s(x, y)$ *exists.*

Let $A = -a^{-1}s_y$ *and* $C = as_x$. *Consider*

$$t(x, y) = \int_0^x A(\xi, 0)\mathrm{d}\xi + \int_0^y C(x, \eta)\mathrm{d}\eta.$$

Show that (s, t) *is the new parameterization we are looking for.*