$$\mathbf{X}_{\ell m} = \frac{1}{\sqrt{\ell(\ell+1)}}\hat{\mathbf{L}}Y_{\ell m}(\theta,\phi),\ L_\pm = L_x \pm iL_y$$

$$\begin{aligned}
\frac{\mathrm{d}P^E_{\ell m}}{\mathrm{d}\Omega} &= \frac{Z_0}{2k^2}\left|a^E_{\ell m}\mathbf{X}_{\ell m}\times\hat{\mathbf{r}}\right|^2 = \frac{Z_0}{2k^2}\left|a^E_{\ell m}\right|^2(\mathbf{X}^*_{\ell m}\times\hat{\mathbf{r}})\cdot(\mathbf{X}_{\ell m}\times\hat{\mathbf{r}}) \\
&= \frac{Z_0}{2k^2}\left|a^E_{\ell m}\right|^2\mathbf{X}^*_{\ell m}\cdot[\hat{\mathbf{r}}\times(\mathbf{X}_{\ell m}\times\hat{\mathbf{r}})] \qquad \leftarrow \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b}) \\
&= \frac{Z_0}{2k^2}\left|a^E_{\ell m}\right|^2\mathbf{X}^*_{\ell m}\cdot[\mathbf{X}_{\ell m}(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})-\hat{\mathbf{r}}(\mathbf{X}_{\ell m}\cdot\hat{\mathbf{r}})] \qquad \leftarrow \hat{\mathbf{r}}\cdot\mathbf{X}_{\ell m} = \hat{\mathbf{r}}\cdot\frac{\hat{\mathbf{r}}\times\nabla}{i\sqrt{\ell(\ell+1)}}Y_{\ell m} = 0 \\
&= \frac{Z_0}{2k^2}\left|a^E_{\ell m}\right|^2\left|\mathbf{X}_{\ell m}\right|^2 \\
&= \frac{Z_0}{2k^2}\left|a^E_{\ell m}\right|^2\left|\frac{\hat{\mathbf{L}}}{\sqrt{\ell(\ell+1)}}Y_{\ell m}\right|^2 \\
&= \frac{Z_0}{2k^2\ell(\ell+1)}\left|a^E_{\ell m}\right|^2\left|\left[\frac{L_++L_-}{2}\hat{\mathbf{x}}+\frac{L_+-L_-}{2i}\hat{\mathbf{y}}+L_z\hat{\mathbf{z}}\right]Y_{\ell m}\right|^2 \\
&= \frac{Z_0\left|a^E_{\ell m}\right|^2}{2k^2\ell(\ell+1)}\left\{\begin{aligned}&\left|\frac{\sqrt{(\ell-m)(\ell+m+1)}Y_{\ell m+1}+\sqrt{(\ell-m+1)(\ell+m)}Y_{\ell m-1}}{2}\right|^2 \\ &+\left|\frac{\sqrt{(\ell-m)(\ell+m+1)}Y_{\ell m+1}-\sqrt{(\ell-m+1)(\ell+m)}Y_{\ell m-1}}{2i}\right|^2 \\ &+\left|mY_{\ell m}\right|^2\end{aligned}\right\} \\
&= \frac{Z_0\left|a^E_{\ell m}\right|^2}{2k^2\ell(\ell+1)}\left\{\begin{aligned}&\frac{1}{2}(\ell-m)(\ell+m+1)\left|Y_{\ell m+1}\right|^2+\frac{1}{2}(\ell-m+1)(\ell+m)\left|Y_{\ell m-1}\right|^2 \\ &+\frac{1}{4}\sqrt{(\ell^2-m^2)((\ell+1)^2-m^2)}\begin{pmatrix}Y^*_{\ell m+1}Y_{\ell m-1}+Y^*_{\ell m-1}Y_{\ell m+1} \\ -Y^*_{\ell m+1}Y_{\ell m-1}-Y^*_{\ell m-1}Y_{\ell m+1}\end{pmatrix} \\ &+m^2\left|Y_{\ell m}\right|^2\end{aligned}\right\} \\
&= \frac{Z_0\left|a^E_{\ell m}\right|^2}{4k^2\ell(\ell+1)}\left[\begin{aligned}&(\ell-m)(\ell+m+1)\left|Y_{\ell m+1}\right|^2+(\ell-m+1)(\ell+m)\left|Y_{\ell m-1}\right|^2 \\ &+2m^2\left|Y_{\ell m}\right|^2\end{aligned}\right]
\end{aligned}$$

Thus,

$$P_{\ell m} = \int\frac{\mathrm{d}P^E_{\ell m}}{\mathrm{d}\Omega}\mathrm{d}\Omega$$

$$
\begin{aligned}
&= \frac{Z_0 \left|a^E_{\ell m}\right|^2}{4k^2\ell(\ell+1)} \left[\begin{array}{l} (\ell-m)(\ell+m+1)\displaystyle\int |Y_{\ell m+1}|^2 \,\mathrm{d}\Omega + (\ell-m+1)(\ell+m)\displaystyle\int |Y_{\ell m-1}|^2 \,\mathrm{d}\Omega \\ +2m^2 \displaystyle\int |Y_{\ell m}|^2 \,\mathrm{d}\Omega \end{array} \right] \\
&= \frac{Z_0 \left|a^E_{\ell m}\right|^2}{4k^2\ell(\ell+1)} \left[(\ell-m)(\ell+m+1) + (\ell-m+1)(\ell+m) + 2m^2\right] \\
&= \frac{Z_0 \left|a^E_{\ell m}\right|^2}{4k^2\ell(\ell+1)} \left[2\ell(\ell+1) - 2m^2 + 2m^2\right] \\
&= \frac{Z_0 \left|a^E_{\ell m}\right|^2}{2k^2}
\end{aligned}
$$