Substituting $f_\ell(r) = \dfrac{u_\ell(r)}{\sqrt{r}}$ into the equation, $\left(\dfrac{\mathrm{d}^2}{\mathrm{d}r^2} + \dfrac{2}{r}\dfrac{\mathrm{d}}{\mathrm{d}r} + k^2 - \dfrac{\ell(\ell+1)}{r^2}\right) f_\ell(r) = 0$, the first two terms on the left hand side are

$$
\begin{aligned}
&\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\frac{u_\ell(r)}{\sqrt{r}} \\
&= \frac{1}{\sqrt{r}}\frac{\mathrm{d}^2 u_\ell(r)}{\mathrm{d}r^2} + 2\frac{\mathrm{d}r^{-1/2}}{\mathrm{d}r}\frac{\mathrm{d}u_\ell(r)}{\mathrm{d}r} + u_\ell(r)\frac{\mathrm{d}^2}{\mathrm{d}r^2}(\frac{1}{\sqrt{r}}) + \frac{2}{r}\left(\frac{1}{\sqrt{r}}\frac{\mathrm{d}u_\ell(r)}{\mathrm{d}r} + u_\ell(r)\frac{\mathrm{d}r^{-1/2}}{\mathrm{d}r}\right) \\
&= \frac{1}{\sqrt{r}}\frac{\mathrm{d}^2 u_\ell(r)}{\mathrm{d}r^2} + 2(-\frac{1}{2}\frac{1}{r\sqrt{r}})\frac{\mathrm{d}u_\ell(r)}{\mathrm{d}r} + u_\ell(r)(\frac{3}{4}\frac{1}{r^2\sqrt{r}}) + \frac{2}{r\sqrt{r}}\frac{\mathrm{d}u_\ell(r)}{\mathrm{d}r} - \frac{1}{r^2\sqrt{r}}u_\ell(r) \\
&= \frac{1}{\sqrt{r}}\left(\frac{\mathrm{d}^2 u_\ell(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u_\ell(r)}{\mathrm{d}r} - \frac{1}{4r^2}u_\ell(r)\right) \\
&= \frac{1}{\sqrt{r}}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{4r^2}\right)u_\ell(r).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
0 &= \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} + k^2 - \frac{\ell(\ell+1)}{r^2}\right) f_\ell(r) \\
&= \frac{1}{\sqrt{r}}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{4r^2}\right)u_\ell(r) + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right)\frac{u_\ell(r)}{\sqrt{r}} \\
&= \frac{1}{\sqrt{r}}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + k^2 - \frac{\ell(\ell+1)+\frac{1}{4}}{r^2}\right)u_\ell(r) \\
&= \frac{k^2}{\sqrt{r}}\left(\frac{\mathrm{d}^2}{\mathrm{d}(kr)^2} + \frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}(kr)} + 1 - \frac{(\ell+\frac{1}{2})^2}{(kr)^2}\right)u_\ell(r).
\end{aligned}
$$

The equation for $u_\ell(r)$ is

$$
\left(\frac{\mathrm{d}^2}{\mathrm{d}(kr)^2} + \frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}(kr)} + 1 - \frac{(\ell+\frac{1}{2})^2}{(kr)^2}\right)u_\ell(r) = 0,
$$

which is the same form as the Bessel equation,

$$
\frac{\mathrm{d}^2 R}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}R}{\mathrm{d}x} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0. \tag{3.77}
$$

The solution for R is the Bessel function of order ν, saying $J_\nu(x)$. The solution for $u_\ell(r)$ is the Bessel function of order $\ell + \dfrac{1}{2}$, saying $J_{\ell+\frac{1}{2}}(kr)$.